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LARGE AND MODERATE DEVIATION PRINCIPLES FOR AVERAGED STOCHASTIC APPROXIMATION METHOD FOR THE ESTIMATION OF A REGRESSION FUNCTION

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Abstract. In this paper we prove large deviations principles for the averaged stochastic approximation method for the estimation of a regression function introduced by Mokkadem et al. (2009). We show that the averaged stochastic approximation algorithm constructed using the weight sequence which minimize the asymptotic variance gives the same pointwise LDP as the Nadaraya-Watson kernel estimator. Moreover, we give a moderate deviations principle for these estimators. It turns out that the rate function obtained in the moderate deviations principle for the averaged stochastic approximation algorithm constructed using the weight sequence which minimize the asymptotic variance is larger than the one obtained for the Nadaraya-Watson estimator and the one obtained for the semi-recursive estimator.

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1. Introduction. Let $(X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)$ be independent, identically distributed pairs of random variables with joint density function $g(x, y)$, and let f denote the probability density of X. In order to construct a stochastic algorithm for the estimation of the regression function $r : x \mapsto \mathbb{E}(Y|X = x)$ at a point x such that $f(x) \neq 0$, Mokkadem et al. [9] defines an algorithm, which approximates the zero of the function $h : y \mapsto f(x)r(x) - f(x)y$. Following Robbins-Monro's procedure, this algorithm is defined by setting $r_0(x) \in \mathbb{R}$ and, for $n \geq 1$,

$$r_n(x) = r_{n-1}(x) + \gamma_n \mathcal{W}_n(x)$$

where $\mathcal{W}_n(x)$ is an "observation" of the function h at the point $r_{n-1}(x)$. To define $\mathcal{W}_n(x)$, Mokkadem et al. [9] follow the approach of Révész [11, 12] and Tsybakov [13], and introduces a kernel K (that is, a function satisfying $\int_{\mathbb{R}} K(x)dx = 1$) and a bandwidth (h_n) (that is, a sequence of positive real numbers that goes to zero), and sets

$$\mathcal{W}_n(x) = h_n^{-1} Y_n K(h_n^{-1}[x - X_n]) - h_n^{-1} K(h_n^{-1}[x - X_n]) r_{n-1}(x).$$

Then, the estimator r_n can be rewritten as

$$(1)\ r_n(x) = \left(1 - \gamma_n h_n^{-1} K\left(\frac{x - X_n}{h_n}\right)\right) r_{n-1}(x) + \gamma_n h_n^{-1} Y_n K\left(\frac{x - X_n}{h_n}\right).$$

Now, let the stepsize in (1) satisfy $\lim_{n\to\infty} n\gamma_n = \infty$, and let (q_n) be a positive sequence such that $\sum q_n = \infty$. The averaged stochastic approximation algorithm for the estimation of a regression function is defined by setting

$$\overline{r}_n(x) = \frac{1}{\sum_{k=1}^n q_k} \sum_{k=1}^n q_k r_k(x) \tag{2}$$

(where the $r_k(x)$ are given by the algorithm (1)).

Recently, large and moderate deviations results have been proved for the well-known nonrecursive Nadaraya-Watson's kernel regression estimator, first by Louani [5], and then by Joutard [4]. Mokkadem et al. [8] show that the rate function obtained in the moderate deviations principle for the semi-recursive estimator is larger than the one obtained for the Nadaraya-Watson estimator.

Let us first recall that a $\mathbb{R}^m$-valued sequence $(Z_n)_{n\geq 1}$ satisfies a large deviations principle (LDP) with speed (ν_n) and good rate function I if:

1. (ν_n) is a positive sequence such that $\lim_{n\to\infty} \nu_n = \infty$;

2. $I : \mathbb{R}^m \to [0, \infty]$ has compact level sets;

3. for every borel set $B \subset \mathbb{R}^m$,

$$\begin{aligned} -\inf_{x\in \mathring{B}} I(x) &\leq \liminf_{n\to\infty} \nu_n^{-1} \log \mathbb{P}[Z_n \in B] \\ &\leq \limsup_{n\to\infty} \nu_n^{-1} \log \mathbb{P}[Z_n \in B] \leq -\inf_{x\in\overline{B}} I(x), \end{aligned}$$

where $\mathring{B}$ and $\overline{B}$ denote the interior and the closure of B respectively. Moreover, let (v_n) be a nonrandom sequence that goes to infinity; if $(v_n Z_n)$ satisfies a LDP, then (Z_n) is said to satisfy a moderate deviations principle (MDP).

The first aim of this paper is to establish pointwise LDP for the averaged stochastic approximation algorithm (2). It turns out that the rate function depends on the bandwidth (h_n) and on the weight (q_n).

We show that using the bandwidths $(h_n) \equiv \left(cn^{-a}\right)$ with $c > 0$ and $a \in (1-\alpha, (4\alpha-3)/2)$ (with $\alpha \in (\frac{3}{4}, 1]$), and the weight $(q_n) = \left(c' n^{-q}\right)$ with $c' > 0$ and $q < \min\{1-2a, (1+a)/2\}$, the sequence $(\overline{r}_n(x) - r(x))$ satisfies a LDP with speed (nh_n) and the rate function defined as follows:

$$I_{a,q,x}(t) = \sup_{u\in\mathbb{R}} \{ut - \psi_{a,q,x}(u)\},$$

which is the Fenchel-Legendre transform of the function $\psi_{a,q,x}$ defined as follows:

$$\psi_{a,q,x}(u) = (1-q) \int_{[0,1]\times\mathbb{R}^2} s^{-a} \left(e^{us^{a-q} K(z) \frac{(y-r(x))}{f(x)}} - 1 \right) g(x,y)\, dsdzdy. \tag{3}$$

Noting that, in the special case $(q_n) = (h_n)$, which is the case when the weight (q_n) minimizes the asymptotic variance of $\overline{r}_n$ (see Mokkadem et al. [9]), we obtain the same rate function for the pointwise LDP as the one obtained for the Nadaraya-Watson estimator (see [5]).

Our second aim is to provide pointwise MDP for the averaged stochastic approximation algorithm (2). In this case, we consider a more general weight sequence defined as $q_n = \gamma(n)$ for all n, where γ is a regularly varying function with exponent $(-q)$, $q < \min\{1-2a, (1+a)/2\}$.

For any positive sequence (v_n) satisfying

$$\lim_{n\to\infty} v_n = \infty, \qquad \lim_{n\to\infty} \frac{v_n^2}{nh_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} v_n h_n^2 = 0 \tag{4}$$

and general bandwidths (h_n), we prove that the sequence

$$v_n\left(\overline{r}_n(x) - r(x)\right)$$

satisfies a LDP of speed $\left(nh_n/v_n^2\right)$ and good rate function $J_{a,q,x} : \mathbb{R} \to \mathbb{R}$ defined by

$$J_{a,q,x}(t) = \frac{1+a-2q}{(1-q)^2}\frac{f(x)}{\operatorname{Var}[Y|X=x]\int_{\mathbb{R}} K^2(z)\,dz}\frac{t^2}{2}. \tag{5}$$

Let us point out that when the weight (q_n) is chosen to be a regularly varying function with exponent $(-a)$ (e.g. $(q_n) = (h_n)$), which is the case when the weight (q_n) minimizes the asymptotic variance of $\overline{r}_n$ (see [9]), the factor $(1+a-2q)/(1-q)^2$ which is present in (5) can be reduced to $1/(1-a)$, and then we can write

$$J_{a,x}(t) = \frac{1}{(1-a)}\frac{f(x)}{\operatorname{Var}[Y|X=x]\int_{\mathbb{R}} K^2(z)\,dz}\frac{t^2}{2}. \tag{6}$$

Moreover, Louani [5] establish the moderate deviations behaviour for the Nadaraya-Watson ([6], [14]) estimator defined as

$$\widehat{r}_n(x) = \begin{cases} \dfrac{\widehat{m}_n(x)}{\widehat{f}_n(x)} & if \quad \widehat{f}_n(x) \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where

$$\widehat{m}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} Y_i K\left(\frac{x-X_i}{h_n}\right) \quad \text{and} \quad \widehat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right).$$

They prove that, for any positive sequence (v_n) satisfying (4), the sequence $v_n\left(\widehat{r}_n(x) - r(x)\right)$ satisfies a LDP with speed $\left(nh_n/v_n^2\right)$ and good rate function $\widehat{J}_x : \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{J}_x(t) = \frac{f(x)}{\operatorname{Var}[Y|X=x]\int_{\mathbb{R}} K^2(z)\,dz}\frac{t^2}{2}. \tag{8}$$

Recently, Mokkadem et al. [8] establish the moderate deviations behaviour for the semi-recursive version of the Nadaraya-Watson estimator defined as

$$\tilde{r}_n(x) = \begin{cases} \dfrac{\tilde{m}_n(x)}{\tilde{f}_n(x)} & if \quad \tilde{f}_n(x) \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where

$$\tilde{m}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{h_i}K\left(\frac{x-X_i}{h_i}\right) \quad \text{and} \quad \tilde{f}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_i}K\left(\frac{x-X_i}{h_i}\right).$$

They prove that, for any positive sequence (v_n) satisfying (4), the sequence $v_n(\tilde{r}_n(x) - r(x))$ satisfies a LDP with speed (nh_n/v_n^2) and good rate function $\tilde{J}_{a,x} : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{J}_{a,x}(t) = (1+a)\frac{f(x)}{\operatorname{Var}[Y|X=x]\int_{\mathbb{R}} K^2(z)\,dz}\frac{t^2}{2}. \tag{10}$$

Then, it follows from (6), (8) and (10), that the rate function obtained in the MDP of $\overline{r}_n$ defined with a weight (q_n) minimizing the asymptotic variance of $\overline{r}_n$ (e.g. $(q_n) = (h_n)$) is larger than the one obtained for the Nadaraya-Watson kernel estimator (7) and than the one obtained for the semi-recursive kernel estimator (9); this means that the averaged stochastic approximation algorithm $\overline{r}_n(x)$ defined with a weight (q_n), which is chosen to be a regularly varying function with exponent $(-a)$ (e.g. $(q_n) = (h_n)$) is more concentrated around $r(x)$ than the two others estimators (Nadaraya-Watson (7) and semi-recursive (9)).

2. Assumptions and main results. Let us first define the class of positive sequences that will be used in the statement of our assumptions.

Definition 1. *Let $\gamma \in \mathbb{R}$ and $(v_n)_{n\geq 1}$ be a nonrandom positive sequence. We say that $(v_n) \in \mathcal{GS}(\gamma)$, if*

$$\lim_{n\to\infty} n\left[1 - \frac{v_{n-1}}{v_n}\right] = \gamma. \tag{11}$$

Condition (11) was introduced by Galambos and Seneta [3] to define regularly varying sequences (see also [1]); it was used in [7] in the context of stochastic approximation algorithms. Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b \in \mathbb{R}$, $n^\gamma(\log n)^b$, $n^\gamma(\log\log n)^b$, and so on.

Let $g(s,t)$ denote the density of the couple (X,Y) (in particular $f(x) = \int_{\mathbb{R}} g(x,t)\,dt$), and set $a(x) = r(x)f(x)$.

2.1. Pointwise LDP for the averaged stochastic approximation algorithm. To establish pointwise LDP for $\overline{r}_n$, we need the following assumptions.

(L1) $K:\mathbb{R}\rightarrow\mathbb{R}$ is a nonnegative, continuous, bounded function satisfying $\int_{\mathbb{R}}K(z)\,dz=1$, $\int_{\mathbb{R}}zK(z)\,dz=0$ and $\int_{\mathbb{R}}z^2K(z)\,dz<\infty$.

(L2) *i)* $(\gamma_n)=\mathcal{GS}(-\alpha)$ with $\alpha\in(\frac{3}{4},1]$; $\lim_{n\to\infty}n\gamma_n\left(\ln\left(\sum_{k=1}^{n}\gamma_k\right)\right)^{-1}=\infty$.
ii) $(h_n)=\left(cn^{-a}\right)$ with $a\in(1-\alpha,(4\alpha-3)/2)$ and $c>0$.
iii) $(q_n)=\left(c'n^{-q}\right)$ with $q<\min\{1-2a,(1+a)/2\}$ and $c'>0$.

(L3) *i)* $g(s,t)$ is twice continuously differentiable with respect to s.
ii) For $q\in\{0,1,2\}$, $s\mapsto\int_{\mathbb{R}}t^qg(s,t)\,dt$ is a bounded function continuous at $s=x$.
For $q\in[2,3]$, $s\mapsto\int_{\mathbb{R}}|t|^q\,g(s,t)\,dt$ is a bounded function.
iii) For $q\in\{0,1\}$, $\int_{\mathbb{R}}|t|^q\left|\frac{\partial g}{\partial x}(x,t)\right|dt<\infty$, and $s\mapsto\int_{\mathbb{R}}t^q\frac{\partial^2 g}{\partial s^2}(s,t)\,dt$ is a bounded function continuous at $s=x$.

(L4) For any $u\in\mathbb{R}$, $t\rightarrow\int_{\mathbb{R}}\exp(uy)\,g(t,y)\,dy$ is continuous at x and bounded.

The proof of the following comment is given in [8].

Comment. Notice that $(L4)$ implies that $\forall m\geq 0,\forall\rho\geq 0$

$$\text{the function}\quad t\mapsto\int_{\mathbb{R}}|y|^m\exp(\rho\,|y|)\,g(t,y)\,dy\quad\text{is bounded.}\tag{12}$$

Before stating our results, we set

$$S_+=\{x\in\mathbb{R};K(x)>0\}\quad\text{and}\quad S_-=\{x\in\mathbb{R};K(x)<0\}$$

and for fixed $x\in\mathbb{R}$

$$T_+=\{y\in\mathbb{R};y-r(x)>0\}\quad\text{and}\quad T_-=\{y\in\mathbb{R};y-r(x)<0\}$$

Moreover, we set

$$O_+=(S_+\cap T_+)\cup(S_-\cap T_-)\quad\text{and}\quad O_-=(S_+\cap T_-)\cup(S_-\cap T_+)$$

The following proposition gives the properties of the functions $\psi_{a,q,x}$ and $I_{a,q,x}$; in particular, the behaviour of the rate function $I_{a,q,x}$.

Proposition 1 (Properties of $\psi_{a,q,x}$ and $I_{a,q,x}$)**.** *Let λ be the Lebesgue measure on $\mathbb{R}$ and let Assumptions (L1) and (L4) hold.*

(i) *$\psi_{a,q,x}$ is strictly convex, twice continuously differentiable on $\mathbb{R}$, and $I_{a,q,x}$ is a good rate function on $\mathbb{R}$.*

(ii) *If $\lambda(O_-) = 0$, $I_{a,q,x}(t) = +\infty$, when $t < 0$, and*

$$I_{a,q,x}(0) = \begin{cases} (1-q)/(1-a)\,\lambda(S_+)\,f(x) & \text{if} \quad \lambda(S_+ \cap T_+) > 0 \\ (1-q)/(1-a)\,\lambda(S_-)\,f(x) & \text{if} \quad \lambda(S_- \cap T_-) > 0 \end{cases}$$

$I_{a,q,x}$ is strictly convex on $\mathbb{R}$ and continuous on $(0, +\infty)$, and for any $t > 0$

$$I_{a,q,x}(t) = t\left(\psi'_{a,q,x}\right)^{-1}(t) - \psi_{a,q,x}\left(\left(\psi'_{a,q,x}\right)^{-1}(t)\right), \tag{13}$$

(iii) *If $\lambda(O_-) > 0$, then $I_{a,q,x}$ is finite and strictly convex on $\mathbb{R}$ and (13) holds for any $t \in \mathbb{R}$.*

We can now state the LDP for the averaged stochastic approximation algorithm (2).

Theorem 1 (Pointwise LDP for the averaged stochastic approximation algorithm)**.** *Let Assumptions* (L1)–(L4) *hold. Then, the sequence $(\overline{r}_n(x) - r(x))$ satisfies a LDP with speed (nh_n) and rate function defined as follows:*

$$I_{a,q,x}(t) = t\left(\psi'_{a,q,x}\right)^{-1}(t) - \psi_{a,q,x}\left(\left(\psi'_{a,q,x}\right)^{-1}(t)\right),$$

where $\psi_{a,q,x}$ is defined in (3)*.*

2.2. Pointwise MDP for the averaged stochastic approximation algorithm. Let (v_n) be a positive sequence; we assume that

(M1) $K : \mathbb{R} \to \mathbb{R}$ is a nonnegative, continuous, bounded function satisfying $\int_{\mathbb{R}} K(z)\,dz = 1$, $\int_{\mathbb{R}} zK(z)\,dz = 0$ and $\int_{\mathbb{R}} z^2 K(z)\,dz < \infty$.

(M2) *i)* $(\gamma_n) = \mathcal{GS}(-\alpha)$ with $\alpha \in (\frac{3}{4}, 1]$; $\lim_{n\to\infty} n\gamma_n \left(\ln\left(\sum_{k=1}^{n}\gamma_k\right)\right)^{-1} = \infty$.
ii) $(h_n) = \mathcal{GS}(-a)$ with $a \in (1-\alpha, (4\alpha - 3)/2)$.
iii) $(q_n) = \mathcal{GS}(-q)$ with $q < \min\{1-2a, (1+a)/2\}$.

(M3) *i)* $g(s,t)$ is twice continuously differentiable with respect to s.
ii) For $q \in \{0,1,2\}$, $s \mapsto \int_{\mathbb{R}} t^q g(s,t)\,dt$ is a bounded function continuous at $s = x$.
For $q \in [2,3]$, $s \mapsto \int_{\mathbb{R}} |t|^q\, g(s,t)\,dt$ is a bounded function.
iii) For $q \in \{0,1\}$, $\int_{\mathbb{R}} |t|^q \left|\frac{\partial g}{\partial x}(x,t)\right| dt < \infty$, and $s \mapsto \int_{\mathbb{R}} t^q \frac{\partial^2 g}{\partial s^2}(s,t)\,dt$ is a bounded function continuous at $s = x$.

(M4) For any $u \in \mathbb{R}$, $t \to \int_{\mathbb{R}} \exp(uy)\, g(t,y)\,dy$ is continuous at x and bounded.

(M5) *i)* $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} \frac{v_n^2}{nh_n} = 0$.
ii) $\lim_{n\to\infty} v_n h_n^2 = 0$

The following Theorem gives the pointwise MDP for the averaged stochastic approximation algorithm (2).

Theorem 2 (Pointwise MDP for the averaged stochastic approximation algorithm)**.** *Let Assumptions* $(M1)-(M5)$ *hold. Then, the sequence* $(v_n(\overline{r}_n(x)-r(x)))$ *satisfies a MDP with speed* $\left(nh_n/v_n^2\right)$ *and good rate function* $J_{a,q,x}$ *defined in* (5)*.*

3. Proofs. From now on, we set $n_0 \geq 3$ such that $\forall k \geq n_0$, $\gamma_k \leq (2\|f\|_\infty)^{-1}$ and $\gamma_k h_k^{-1}\|K\|_\infty \leq 1$. Moreover, we introduce the following notations:

$$\begin{aligned} Z_n(x) &= h_n^{-1} K\left(\frac{x-X_n}{h_n}\right), \\ W_n(x) &= h_n^{-1} Y_n K\left(\frac{x-X_n}{h_n}\right), \\ \eta_n(x) &= (Y_n - r(x))\, K\left(\frac{x-X_n}{h_n}\right), \end{aligned} \tag{14}$$

As explained in the introduction, we note that the stochastic approximation algorithm (1) can be rewritten as:

$$\begin{aligned} r_n(x) &= (1-\gamma_n Z_n(x))\, r_{n-1}(x) + \gamma_n W_n(x) \\ &= (1-\gamma_n f(x))\, r_{n-1}(x) + \gamma_n (f(x) - Z_n(x))\, r_{n-1}(x) + \gamma_n W_n(x). \end{aligned}$$

To establish the asymptotic behaviour of (r_n) and $(\overline{r}_n)$, we introduce the auxiliary stochastic approximation algorithm defined by setting $\rho_n(x) = r(x)$ for all $n \leq n_0 - 2$, $\rho_{n_0-1}(x) = r_{n_0-1}(x)$, and, for $n \geq n_0$,

$$\rho_n(x) \quad = \quad (1 - \gamma_n f(x))\, \rho_{n-1}(x) + \gamma_n (f(x) - Z_n(x))\, r(x) + \gamma_n W_n(x).$$

It follows that, for $n \geq n_0$,

$$\begin{aligned}\rho_n(x) - \rho_{n-1}(x) &= -\gamma_n f(x) [\rho_{n-1}(x) - r(x)] + \gamma_n [W_n(x) - r(x) Z_n(x)], \\ &= -\gamma_n f(x) [\rho_{n-1}(x) - r(x)] + \gamma_n h_n^{-1} \eta_n(x),\end{aligned}$$

and thus

$$\rho_{n-1}(x) - r(x) \quad = \quad \frac{h_n^{-1}}{f(x)} \eta_n(x) - \frac{1}{\gamma_n f(x)} [\rho_n(x) - \rho_{n-1}(x)].$$

Then, we can write that

$$\begin{aligned}\overline{\rho}_n(x) - r(x) &= \frac{1}{\sum_{k=1}^n q_k} \sum_{k=1}^n q_k [\rho_k(x) - r(x)] \\ &= \frac{1}{f(x)} T_n(x) - \frac{1}{f(x)} R_n^{(0)}(x)\end{aligned} \tag{15}$$

with

$$\begin{aligned}T_n(x) &= \frac{1}{\sum_{k=1}^n q_k} \sum_{k=n_0-1}^n q_k h_k^{-1} \eta_k(x), \\ R_n^{(0)}(x) &= \frac{1}{\sum_{k=n_0-1}^n q_k} \sum_{k=1}^n \frac{q_k}{\gamma_{k+1}} [\rho_{k+1}(x) - \rho_k(x)].\end{aligned}$$

Moreover, it was shown in [9], that under the assumptions (M1)–(M3), we have

$$\left| R_n^{(0)}(x) \right| = o\left(\sqrt{n^{-1} h_n^{-1}} + h_n^{-2} \right) \quad \text{a.s.,} \tag{16}$$

then, it follows from (15) and (16) that

$$\overline{\rho}_n(x) - \mathbb{E}[\overline{\rho}_n(x)] \quad = \quad \frac{1}{f(x)} \frac{1}{\sum_{k=1}^n q_k} \sum_{k=n_0-1}^n q_k h_k^{-1} (\eta_k(x) - \mathbb{E}[\eta_k(x)]).$$

Let (Ψ_n), (B_n) and $\left(\overline{\Delta}_n\right)$ be the sequences defined as

$$\Psi_n(x) \quad = \quad \frac{1}{f(x)} \frac{1}{\sum_{k=1}^n q_k} \sum_{k=n_0-1}^n q_k h_k^{-1} (\eta_k(x) - \mathbb{E}[\eta_k(x)]),$$

$$\begin{aligned} B_n(x) &= \mathbb{E}\left[\overline{\rho}_n(x)\right] - r(x), \\ \overline{\Delta}_n(x) &= \overline{r}_n(x) - \overline{\rho}_n(x). \end{aligned}$$

We have:

$$\overline{r}_n(x) - r(x) = \Psi_n(x) + B_n(x) + \overline{\Delta}_n(x). \tag{17}$$

Moreover, it was shown in [9], that $\overline{\Delta}_n(x)$ is negligible in front of $\overline{\rho}_n$. Then, it follows from (17), that the deviation behaviour of the sequence $(\overline{r}_n(x) - r(x))$ can be deduced from that of the sequence $(\overline{\rho}_n(x) - \rho(x))$ which is equal to $(\Psi_n(x) + B_n(x))$. Theorems 1 and 2 are then consequences of the following propositions.

Proposition 2 (Pointwise LDP and MDP for (Ψ_n))**.**

1. *Under the assumptions* (L1)–(L4)*, the sequence* $\overline{\rho}_n(x) - \mathbb{E}\left[\overline{\rho}_n(x)\right]$ *satisfies a LDP with speed* (nh_n) *and rate function* $I_{a,q,x}$*.*

2. *Under the assumptions* (M1)–(M5)*, the sequence* $(v_n \Psi_n(x))$ *satisfies a LDP with speed* $\left(nh_n/v_n^2\right)$ *and rate function* $J_{a,q,x}$*.*

Proposition 3 (Convergence rate of (B_n))**.** *Let Assumptions* (M1)–(M3) *hold. Then*

$$B_n(x) = O\left(h_n^2\right).$$

Set $x \in \mathbb{R}$; since the assumptions of Theorems 1 guarantee that $\lim_{n\to\infty} B_n(x) = 0$, then Theorem 1 is a straightforward consequence of the application of Proposition 2. Moreover, under the assumptions of Theorem 2, we have by application of Propostion 3, $\lim_{n\to\infty} v_n B_n(x) = 0$; Theorem 2 thus straightfully follows from the application of Part 2 of Proposition 2.

We now state a preliminary lemmas, which will be used in the proof of Proposition 2. For any $u \in \mathbb{R}$, set

$$\begin{aligned} \Lambda_{n,x}(u) &= \frac{v_n^2}{nh_n} \log \mathbb{E}\left[\exp\left(\frac{unh_n\Psi_n(x)}{v_n}\right)\right], \\ \Lambda_x^L(u) &= \psi_{a,q,x}(u), \\ \Lambda_x^M(u) &= \frac{u^2}{2}\frac{(1-q)^2}{1+a-2q}\frac{\operatorname{Var}\left[Y|X=x\right]}{f(x)}\int_{\mathbb{R}} K^2(z)\,dz. \end{aligned}$$

Lemma 1 (Pointwise convergence of $\Lambda_{n,x}$ when $v_n \equiv 1$)**.** *Let Assumptions* (L1)–(L4) *hold. Then, for all* $u \in \mathbb{R}$

$$\lim_{n\to\infty} \Lambda_{n,x}(u) = \Lambda_x^L(u).$$

Lemma 2 (Pointwise convergence of $\Lambda_{n,x}$ when $v_n \to \infty$)**.** *Let Assumptions* (M1–(M4) *hold. Then, for all* $u \in \mathbb{R}$

$$\lim_{n\to\infty} \Lambda_{n,x}(u) = \Lambda_x^M(u).$$

Our proofs are now organized as follows: Lemmas 1 and 2 are proved in Section 3.1, Proposition 2 in Section 3.2 and Proposition 3 in Section 3.3.

3.1. Proof of Lemmas 1 and 2. Set $u \in \mathbb{R}$, $u_n = u/v_n$ and $a_n = nh_n$. We have:

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{a_n} \log \mathbb{E}\left[\exp\left(u_n a_n \Psi_n(x)\right)\right] \\
&= \frac{v_n^2}{a_n} \log \mathbb{E}\left[\exp\left(\frac{u_n}{f(x)} \frac{a_n}{\sum_{k=1}^n q_k} \sum_{k=n_0-1}^{n} \frac{q_k}{h_k}\left(\eta_k(x) - \mathbb{E}\left[\eta_k(x)\right]\right)\right)\right] \\
&= \frac{v_n^2}{a_n} \sum_{k=n_0-1}^{n} \log \mathbb{E}\left[\exp\left(u_n \frac{a_n}{\sum_{k=1}^n q_k} \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right)\right] \\
&\quad - \frac{u}{f(x)} \frac{v_n}{\sum_{k=1}^n q_k} \sum_{k=n_0-1}^{n} \frac{q_k}{h_k} \mathbb{E}\left[\eta_k(x)\right].
\end{aligned}$$

By Taylor expansion, there exists $c_{k,n}$ between 1 and $\mathbb{E}\left[\exp\left(u_n \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right)\right]$ such that

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(u_n \frac{a_n}{\sum_{k=1}^n q_k} \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right)\right] \\
&\quad = \mathbb{E}\left[\exp\left(u_n \frac{a_n}{\sum_{k=1}^n q_k} \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right) - 1\right] \\
&\quad - \frac{1}{2c_{k,n}^2} \left(\mathbb{E}\left[\exp\left(u_n \frac{a_n}{\sum_{k=1}^n q_k} \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right) - 1\right]\right)^2
\end{aligned}$$

and $\Lambda_{n,x}$ can be rewriten as

$$\Lambda_{n,x}(u) = \frac{v_n^2}{a_n} \sum_{k=n_0-1}^{n} \mathbb{E}\left[\exp\left(u_n \frac{a_n}{\sum_{k=1}^n q_k} \frac{q_k}{h_k} \frac{\eta_k(x)}{f(x)}\right) - 1\right]$$

$$-\frac{1}{2}\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2$$

$$-\frac{u}{f(x)}\frac{v_n}{\sum_{k=1}^n q_k}\sum_{k=n_0-1}^{n}\frac{q_k}{h_k}\mathbb{E}\left[\eta_k(x)\right]. \tag{18}$$

Now, let us recall that, if $(b_n)\in\mathcal{GS}(-b^*)$ with $b^*<1$, then we have, for any fixed $k_0\geq 1$,

$$\lim_{n\to\infty}\frac{nb_n}{\sum_{k=k_0}^n b_k} = 1-b^*, \tag{19}$$

and

$$\sup_{k\leq n}\frac{b_n}{b_k}<\infty. \tag{20}$$

Moreover, since $\left(q_kh_k^{-1}\right)\in\mathcal{GS}\left(-\left(q-a\right)\right)$, it follows from (19) that

$$\left|u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\right| = O\left(\frac{u}{v_n}\frac{h_n}{h_k}\frac{q_k}{q_n}\right),$$

and from (20) that

$$\left|u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\right| = \begin{cases} O(1) & \text{when} \quad v_n\equiv 1\\ o(1) & \text{when} \quad v_n\to\infty\end{cases}$$

and thus, in the both cases, there exists $c>0$ such that

$$\left|u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\right| \leq c. \tag{21}$$

P r o o f o f L e m m a 2. A Taylor's expansion implies the existence of $c'_{k,n}$ between 0 and $u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}$ such that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\\
&= \frac{u_n}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\mathbb{E}\left[\eta_k(x)\right]+\frac{1}{2}\left(\frac{u_n}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\right)^2\mathbb{E}\left[\eta_k^2(x)\right]\\
&\quad+\frac{1}{6}\left(\frac{u_n}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\right)^3\mathbb{E}\left[\eta_k^3(x)\,e^{c'_{k,n}}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}\Lambda_{n,x}(u) &= \frac{u^2}{2(f(x))^2}\frac{a_n}{\left(\sum_{k=1}^n q_k\right)^2}\sum_{k=n_0-1}^{n}\frac{q_k^2}{h_k^2}\mathbb{E}\left[\eta_k^2(x)\right]\\ &+\frac{1}{6}\frac{u^2u_n}{(f(x))^3}\frac{a_n^2}{\left(\sum_{k=1}^n q_k\right)^3}\sum_{k=n_0-1}^{n}\frac{q_k^3}{h_k^3}\mathbb{E}\left[\eta_k^3(x)\exp\left(c'_{k,n}\right)\right]\\ &-\frac{1}{2}\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2.\end{aligned}$$

Let us note that under the assumption (M3), we have

$$\mathbb{E}\left[\eta_k^2(x)\right]=h_kVar\left[Y|X=x\right]f(x)\int_{\mathbb{R}}K^2(z)\,dz\left[1+o(1)\right].$$

Then, it follows that

$$\begin{aligned}\Lambda_{n,x}(u) &= \frac{u^2}{2}\frac{a_n}{\left(\sum_{k=1}^n q_k\right)^2}\sum_{k=n_0-1}^{n}\frac{q_k^2}{h_k}\frac{\mathrm{Var}\left[Y|X=x\right]}{f(x)}\int_{\mathbb{R}}K^2(z)\,dz\left[1+o(1)\right]\\ &+R_{n,x}^{(1)}(u)-R_{n,x}^{(2)}(u),\end{aligned}\tag{22}$$

with

$$\begin{aligned}R_{n,x}^{(1)}(u) &= \frac{1}{6}\frac{u^2u_n}{(f(x))^3}\frac{a_n^2}{\left(\sum_{k=1}^n q_k\right)^3}\sum_{k=n_0-1}^{n}\frac{q_k^3}{h_k^3}\mathbb{E}\left[\eta_k^3(x)\exp\left(c'_{k,n}\right)\right],\\ R_{n,x}^{(2)}(u) &= \frac{1}{2}\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2.\end{aligned}$$

Let us first show that

$$\lim_{n\to\infty}\left|R_{n,x}^{(1)}(u)\right|=0.$$

In view of $(M4)$ and (21), and since $|a-b|^3\leq 4\left(|a|^3+|b|^3\right)$, we have

$$\begin{aligned}&\mathbb{E}\left|\eta_k(x)^3\exp\left(c'_{k,n}\right)\right|\\ &\leq h_k\int_{\mathbb{R}^2}|y-r(x)|^3K^3(z)\exp\left(\frac{c}{f(x)}|y-r(x)|\,|K(z)|\right)g(x-zh_k,y)dydz\\ &\leq 4h_k\int_{\mathbb{R}}\exp\left(\frac{c}{f(x)}\left|r(x)\right|\left\|K\right\|_\infty\right)\end{aligned}$$

$$\times\left\{\int_{\mathbb{R}}|y|^{3}\exp\left(\frac{c}{f(x)}|y|\,\|K\|_{\infty}\right)g(x-zh_k,y)\,dy\right.$$
$$\left.+\,|r(x)|^{3}\int_{\mathbb{R}}\exp\left(\frac{c}{f(x)}|y|\,\|K\|_{\infty}\right)g(x-zh_k,y)\,dy\right\}K^{3}(z)\,dz$$
$$(23)=O(h_k).$$

Hence, it follows from (23) and (19), that

$$\begin{aligned}
&\left|\frac{u^{2}u_n}{(f(x))^{2}}\frac{a_n^{2}}{\left(\sum_{k=1}^{n}q_k\right)^{3}}\sum_{k=n_0-1}^{n}\frac{q_k^{3}}{h_k^{3}}\mathbb{E}\left[\eta_k^{3}(x)\,e^{c'_{k,n}}\right]\right|\\
&\quad= O\left(\frac{1}{v_n}\frac{a_n^{2}}{\left(\sum_{k=1}^{n}q_k\right)^{3}}\sum_{k=n_0-1}^{n}\frac{q_k^{3}}{h_k^{2}}\right)\\
&\quad= O\left(\frac{1}{v_n}\left(\frac{nq_n}{\sum_{k=1}^{n}q_k}\right)^{3}\frac{\sum_{k=n_0-1}^{n}q_k^{3}h_k^{-2}}{nq_n^{3}h_n^{-2}}\right)\\
&\quad= O\left(\frac{1}{v_n}\right)
\end{aligned}$$

which ensures that $\lim_{n\to\infty}\left|R_{n,x}^{(1)}(u)\right|=0$.
Let us now prove that

$$\lim_{n\to\infty}\left|R_{n,x}^{(2)}(u)\right|=0.$$

Noting that, under the assumption $(M3)$ we have

$$\begin{aligned}
\mathbb{E}(W_k(x)) &= a(x)+\frac{1}{2}h_k^{2}\int_{\mathbb{R}}y\frac{\partial^{2}g}{\partial x^{2}}(x,y)\,dy\int_{\mathbb{R}}z^{2}K(z)\,dz\,[1+o(1)],\\
\mathbb{E}(Z_k(x)) &= f(x)+\frac{1}{2}h_k^{2}\int_{\mathbb{R}}\frac{\partial^{2}g}{\partial x^{2}}(x,y)\,dy\int_{\mathbb{R}}z^{2}K(z)\,dz\,[1+o(1)].
\end{aligned}$$

Then, it follows from (14) that

$$\begin{aligned}
\mathbb{E}[\eta_k(x)] &= h_k\left[\mathbb{E}(W_k(x))-r(x)\,\mathbb{E}(Z_k(x))\right]\\
&= \frac{1}{2}h_k^{3}\left[\int_{\mathbb{R}}y\frac{\partial^{2}g}{\partial x^{2}}(x,y)\,dy-r(x)\int_{\mathbb{R}}\frac{\partial^{2}g}{\partial x^{2}}(x,y)\,dy\right]\\
&\quad\times\int_{\mathbb{R}}z^{2}K(z)\,dz\,[1+o(1)]\\
&= h_k^{3}m^{(2)}(x)\,f(x)\,[1+o(1)], \qquad (24)
\end{aligned}$$

where,

$$m^{(2)}(x)=\frac{1}{2f(x)}\left[\int_{\mathbb{R}}t\frac{\partial^2 g}{\partial x^2}(x,t)\,dt-r(x)\int_{\mathbb{R}}\frac{\partial^2 g}{\partial x^2}(x,t)\,dt\right]\int_{\mathbb{R}}z^2K(z)\,dz.$$

Moreover, in view of (19) and (24), we have

$$\begin{aligned}
&\left|\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2\right|\\
&\quad\le\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2\\
&\quad=\frac{v_n^2}{a_n}\sum_{k=n_0-1}^{n}\left(\mathbb{E}\left[u_n\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right]\right)^2(1+o(1))\\
&\quad=\frac{u^2}{(f(x))^2}a_n\sum_{k=n_0-1}^{n}\left(\frac{q_kh_k^{-1}}{\sum_{k=1}^n q_k}\mathbb{E}\left[\eta_k(x)\right]\right)^2(1+o(1))\\
&\quad=O\left(a_n\frac{\sum_{k=n_0-1}^n q_k^2h_k^4}{\left(\sum_{k=1}^n q_k\right)^2}\right)\\
&\quad=O\left(h_n^5\frac{\sum_{k=n_0-1}^n q_k^2h_k^4}{nq_n^2h_n^4}\left(\frac{nq_n}{\sum_{k=1}^n q_k}\right)^2\right)
\end{aligned}$$

$$=O\left(h_n^5\right) \tag{25}$$

which goes to 0 as $n\to\infty$. This proves that $\lim_{n\to\infty}\left|R_{n,x}^{(2)}(u)\right|=0$. Then, we obtain from (22) and (19), $\lim_{n\to\infty}\Lambda_{n,x}(u)=\Lambda_x^M(u)$. Which concludes the proof Lemma 2. □

Proof of Lemma 1. It follows from (18) that

$$\begin{aligned}
\Lambda_{n,x}(u) \;=\; &\frac{1}{a_n}\sum_{k=n_0-1}^{n}\mathbb{E}\left[\exp\left(u\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\\
&-\frac{1}{2a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2\\
&-\frac{u}{f(x)}\frac{1}{\sum_{k=1}^n q_k}\sum_{k=n_0-1}^{n}\frac{q_k}{h_k}\mathbb{E}\left[\eta_k(x)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a_n}\sum_{k=n_0-1}^{n} h_k \int_{\mathbb{R}^2}\left[\exp\left(\frac{u}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}(y-r(x))K(z)\right)-1\right]\\
&\quad \times g(x,y)\,dzdy\\
&\quad -R_{n,x}^{(3)}(u)-R_{n,x}^{(4)}(u)+R_{n,x}^{(5)}(u) \qquad (26)
\end{aligned}
$$

with

$$
\begin{aligned}
R_{n,x}^{(3)}(u) &= \frac{1}{2a_n}\sum_{k=n_0-1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}\frac{\eta_k(x)}{f(x)}\right)-1\right]\right)^2,\\
R_{n,x}^{(4)}(u) &= \frac{u}{f(x)}\frac{1}{\sum_{k=1}^n q_k}\sum_{k=n_0-1}^{n}\frac{q_k}{h_k}\mathbb{E}\left[\eta_k(x)\right],\\
R_{n,x}^{(5)}(u) &= \frac{1}{a_n}\sum_{k=n_0-1}^{n} h_k \int_{\mathbb{R}^2}\left[\exp\left(\frac{u}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}(y-r(x))K(z)\right)-1\right]\\
&\quad \times\left[g(x-zh_k,y)-g(x,y)\right]dzdy.
\end{aligned}
$$

It follows from (25), that $\lim_{n\to\infty}\left|R_{n,x}^{(3)}(u)\right|=0$.
Moreover, in view of (19) and (24), we have

$$
\begin{aligned}
\left|R_{n,x}^{(4)}(u)\right| &= O\left(\frac{1}{\sum_{k=1}^n q_k}\sum_{k=1}^{n} q_k h_k^2\right)\\
&= O\left(\frac{nq_n}{\sum_{k=1}^n q_k}\frac{\sum_{k=1}^n q_k h_k^2}{nq_n h_n^2}h_n^2\right)\\
&= O\left(h_n^2\right)
\end{aligned}
$$

which goes to 0 as $n\to\infty$.
Let us now prove that

$$
\lim_{n\to\infty}\left|R_{n,x}^{(5)}(u)\right|=0.
$$

Set $M>0$ and $\varepsilon>0$; we then have

$$
\begin{aligned}
R_{n,x}^{(5)}(u) = \frac{1}{a_n}\sum_{k=n_0-1}^{n} h_k &\int\limits_{\{|z|\leq M\}\times\mathbb{R}}\left[\exp\left(\frac{u}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}(y-r(x))K(z)\right)-1\right]\\
&\times\left[g(x-zh_k,y)-g(x,y)\right]dzdy\\
&+\frac{1}{a_n}\sum_{k=n_0-1}^{n} h_k \int\limits_{\{|z|>M\}\times\mathbb{R}}\left[\exp\left(\frac{u}{f(x)}\frac{a_n}{\sum_{k=1}^n q_k}\frac{q_k}{h_k}(y-r(x))K(z)\right)-1\right]
\end{aligned}
$$

$$\times \left[g\left(x - zh_k, y\right) - g\left(x, y\right)\right] dzdy$$
$$= I + II.$$

Using (21), and since for any $t \in \mathbb{R}$, $\left|e^t - 1\right| \leq |t|\, e^{|t|}$, we have

$$\begin{aligned}
|II| \quad &\leq \quad \frac{|u|}{f(x)} \sum_{k=n_0-1}^{n} \frac{q_k}{\sum_{k=1}^{n} q_k} \int_{\{|z|>M\}\times\mathbb{R}} |y - r(x)|\, |K(z)| \\
&\quad \times \exp\left(\frac{c}{f(x)} |y - r(x)|\, |K(z)|\right) |g(x - zh_k, y) - g(x, y)|\, dzdy \\
&\leq \quad \frac{|u|}{f(x)} \sum_{k=n_0-1}^{n} \frac{q_k}{\sum_{k=1}^{n} q_k} \int_{\{|z|>M\}} |K(z)| \\
&\quad \times \left[\int_{\mathbb{R}} |y - r(x)| \exp\left(\frac{c}{f(x)} |y - r(x)|\, |K(z)|\right) g(x - zh_k, y)\, dy\right] dz \\
&\quad + \frac{|u|}{f(x)} \sum_{k=n_0-1}^{n} \frac{q_k}{\sum_{k=1}^{n} q_k} \int_{\{|z|>M\}} |K(z)| \\
&\quad \times \left[\int_{\mathbb{R}} |y - r(x)| \exp\left(\frac{c}{f(x)} |y - r(x)|\, |K(z)|\right) g(x, y)\, dy\right] dz \\
&\leq \quad A \int_{\{|z|>M\}} |K(z)|\, dz,
\end{aligned}$$

where A is a constant; this last inequality follows from (12) and from the fact that K is bounded.
Now, since K is integrable, we can choose M such that

$$|II| \leq \frac{\varepsilon}{2}.$$

Now, for I, we write

$$\begin{aligned}
I \quad &= \quad \frac{1}{a_n} \sum_{k=n_0-1}^{n} h_k \int_{\{|z|\leq M\}\times\mathbb{R}} \exp\left(\frac{u}{f(x)} \frac{a_n}{\sum_{k=1}^{n} q_k} \frac{q_k}{h_k} (y - r(x))\, K(z)\right) \\
&\quad \times \left[g(x - zh_k, y) - g(x, y)\right] dzdy \\
&\quad - \frac{1}{a_n} \sum_{k=n_0-1}^{n} h_k \int_{\{|z|\leq M\}\times\mathbb{R}} \left[g(x - zh_k, y) - g(x, y)\right] dzdy
\end{aligned}$$

In view of $(M4)$, (12), (19), the dominated convergence theorem ensure that both integrals converge to 0. We deduce that for n large enough,

$$|I| \leq \frac{\varepsilon}{2},$$

which ensures that $\lim_{n\to\infty}\left|R_{n,x}^{(5)}(u)\right|=0$.
Then, it follows from (26), and (19) and from some analysis considerations that

$$\begin{aligned}
&\lim_{n\to\infty}\Lambda_{n,x}(u)\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=n_0-1}^{n}\left(\frac{k}{n}\right)^{-a}\\
&\quad\times\int_{\mathbb{R}^2}\left[\exp\left((1-q)\left(\frac{k}{n}\right)^{a-q}\frac{u}{f(x)}(y-r(x))K(z)\right)-1\right]g(x,y)\,dzdy\\
&= (1-q)\int_{[0,1]\times\mathbb{R}^2}s^{-a}\left(\exp\left(us^{a-q}K(z)\frac{(y-r(x))}{f(x)}\right)-1\right)g(x,y)\,dsdzdy\\
&= \Lambda_x^L(u)
\end{aligned}$$

and thus Lemma 1 is proved. □

3.2. Proof of Proposition 2. To prove Proposition 2, we apply Proposition 1, Lemmas 1 and 2 and the following result (see [10]).

Lemma 3. *Let (Z_n) be a sequence of real random variables, (ν_n) a positive sequence satisfying $\lim_{n\to\infty}\nu_n=+\infty$, and suppose that there exists some convex non-negative function Γ defined on $\mathbb{R}$ such that*

$$\Gamma(u)=\lim_{n\to\infty}\frac{1}{\nu_n}\log\mathbb{E}\left[\exp\left(u\nu_nZ_n\right)\right],\quad\forall u\in\mathbb{R},$$

If the Legendre function Γ^* of Γ is a strictly convex function, then the sequence (Z_n) satisfies a LDP *of speed (ν_n) and good rate fonction Γ^*.*

In our framework, when $v_n\equiv 1$, we take $Z_n=\overline{\rho}_n(x)-\mathbb{E}(\overline{\rho}_n(x))$, $\nu_n=nh_n$ with $h_n=cn^{-a}$ where $c>0$ and $a\in(1-\alpha,(4\alpha-3)/2)$ (with $\alpha\in(3/4,1]$), and the weight $(q_n)=\left(c'n^{-q}\right)$ with $c'>0$ and $q<\min\{1-2a,(1+a)/2\}$, and $\Gamma=\Lambda_x^L$. In this case, the Legendre transform of $\Gamma=\Lambda_x^L$ is the rate function $I_{a,q,x}(t)$ which is strictly convex by Proposition 1. Otherwise, when, $v_n\to\infty$, we take $Z_n=v_n\left(\overline{\rho}_n(x)-\mathbb{E}\left[\overline{\rho}_n(x)\right]\right)$, $\nu_n=nh_n/v_n^2$ and $\Gamma=\Lambda_x^M$; Γ^* is then the quadratic rate function $J_{a,q,x}$ defined in (5) and thus Proposition 2 follows.

3.3. Proof of Proposition 3. It follows from (15), (16), (19) and (24), that

$$\mathbb{E}\left[\overline{\rho}_n(x)\right]-r(x) = \frac{1}{f(x)}\mathbb{E}\left[T_n(x)\right]$$

$$\begin{aligned}
&= \frac{1}{f(x)} \frac{1}{\sum_{k=1}^{n} q_k} \sum_{k=n_0-1}^{n} \frac{q_k}{h_k} \mathbb{E}\left[\eta_k(x)\right] \\
&= \frac{1}{f(x)} \frac{\sum_{k=n_0-1}^{n} q_k h_k^2}{\sum_{k=1}^{n} q_k} m^{(2)}(x) f(x) \left[1 + o(1)\right] \\
&= h_n^2 \frac{1-q}{1-q-2a} m^{(2)}(x) \left[1 + o(1)\right] \\
&= O\left(h_n^2\right).
\end{aligned}$$

3.4. Proof of Proposition 1.

- Since $\left|e^t - 1\right| \le |t| e^{|t|}$ $\forall t \in \mathbb{R}$, it follows from (12) and (L1), that

$$\begin{aligned}
|\psi_{a,q,x}(u)| &\le (1-q) \int_{[0,1]\times\mathbb{R}^2} s^{-a} \left| \exp\left(us^{a-q} K(z) \frac{(y - r(x))}{f(x)} \right) - 1 \right| \\
&\quad \times g(x,y)\, dsdzdy \\
&\le (1-q) \frac{|u|}{f(x)} \int_{[0,1]\times\mathbb{R}^2} s^{-q} \left|y - r(x)\right| \left|K(z)\right| \\
&\quad \times \exp\left(|u| \frac{|y - r(x)|}{f(x)} \|K\|_\infty \right) g(x,y)\, dsdzdy \\
&\le \frac{|u|}{f(x)} \int_{\mathbb{R}^2} \left|y - r(x)\right| \left|K(z)\right| \\
&\quad \times \exp\left(|u| \frac{|y - r(x)|}{f(x)} \|K\|_\infty \right) g(x,y)\, dzdy \\
&= \frac{|u|}{f(x)} \int_{\mathbb{R}} \left|K(z)\right| dz \int_{\mathbb{R}} \left|y - r(x)\right| \\
&\quad \times \exp\left(|u| \frac{|y - r(x)|}{f(x)} \|K\|_\infty \right) g(x,y)\, dy \\
&< \infty
\end{aligned}$$

which ensures the existence of $\psi_{a,q,x}$. It is straightforward to check that $\psi_{a,q,x}$ is twice differentiable, with

$$\begin{aligned}
\psi'_{a,q,x}(u) &= (1-q) \int_{[0,1]\times\mathbb{R}^2} s^{-q} K(z) \frac{(y - r(x))}{f(x)} \\
&\quad \times \exp\left(us^{a-q} K(z) \frac{(y - r(x))}{f(x)} \right) g(x,y)\, dsdzdy \\
\psi''_{a,q,x}(u) &= (1-q) \int_{[0,1]\times\mathbb{R}^2} s^{a-2q} \left(K(z)\right)^2 \left(\frac{(y - r(x))}{f(x)} \right)^2
\end{aligned}$$

$$\times \exp\left(us^{a-q}K\left(z\right)\frac{\left(y-r\left(x\right)\right)}{f\left(x\right)}\right)g\left(x,y\right)dsdzdy.$$

Since $\psi''_{a,q,x}\left(u\right)>0\ \forall u\in\mathbb{R}$, $\psi'_{a,q,x}$ is increasing on $\mathbb{R}$, and $\psi_{a,q,x}$ is strictly convex on $\mathbb{R}$. It follows that its Cramer transform $I_{a,q,x}$ is a good rate function on $\mathbb{R}$ (see [2]) and (i) of Proposition 1 is proved.

- Let us now assume that $\lambda\left(O_-\right)=0$. We then have

$$\lim_{u\to-\infty}\psi'_{a,q,x}\left(u\right)=0 \quad \text{and} \quad \lim_{u\to+\infty}\psi'_{a,q,x}\left(u\right)=+\infty$$

so that the range of $\psi'_{a,q,x}$ is $(0,+\infty)$. Moreover

$$\lim_{u\to-\infty}\psi_{a,q,x}\left(u\right) = \begin{cases} -\left(1-q\right)/\left(1-a\right)\lambda\left(S_+\right)f\left(x\right) & \text{if } \lambda\left(S_+\cap T_+\right)>0\\ -\left(1-q\right)/\left(1-a\right)\lambda\left(S_-\right)f\left(x\right) & \text{if } \lambda\left(S_-\cap T_-\right)>0\end{cases}$$

(which can be $-\infty$). This implies in particular that

$$I_{a,q,x}\left(0\right) = \begin{cases} \left(1-q\right)/\left(1-a\right)\lambda\left(S_+\right)f\left(x\right) & \text{if } \lambda\left(S_+\cap T_+\right)>0\\ \left(1-q\right)/\left(1-a\right)\lambda\left(S_-\right)f\left(x\right) & \text{if } \lambda\left(S_-\cap T_-\right)>0\end{cases}$$

Now, when $t<0$, $\lim_{u\to-\infty}\left(ut-\psi_{a,q,x}\left(u\right)\right)=+\infty$ and $I_{a,q,x}\left(t\right)=+\infty$. Since $\psi'_{a,q,x}$ is increasing with range $(0,+\infty)$, when $t>0$, $\sup_u\left(ut-\psi_{a,q,x}\left(u\right)\right)$ is reached for $u_0\left(t\right)$ such that $\psi_{a,q,x}\left(u_0\left(t\right)\right)=t$, i.e. for $u_0\left(t\right)=\left(\psi'_{a,q,x}\right)^{-1}\left(t\right)$; this prove (13). (Note that, since $\psi''_{a,q,x}\left(t\right)>0$, the function $t\mapsto u_0\left(t\right)$ is differentiable on $(0,+\infty)$). Now, differentiating (13), we have

$$\begin{aligned} I'_{a,q,x}\left(t\right) &= u_0\left(t\right)+tu'_0\left(t\right)-\psi'_{a,q,x}\left(u_0\left(t\right)\right)u'_0\left(t\right)\\ &= \left(\psi'_{a,q,x}\right)^{-1}\left(t\right)+tu'_0\left(t\right)-tu'_0\left(t\right)\\ &= \left(\psi'_{a,q,x}\right)^{-1}\left(t\right).\end{aligned}$$

Since $\left(\psi'_{a,q,x}\right)^{-1}$ is an increasing function on $(0,+\infty)$, it follows that $I_{a,q,x}$ is strictly convex on $(0,+\infty)$ (and differentiable). Thus (ii) is proved.

- We Assume that $\lambda\left(O_-\right)>0$. In this case, $\psi'_{a,q,x}$ can be rewritten as

$$\psi'_{a,q,x}\left(u\right) = \left(1-q\right)\int_{[0,1]\times\left(\mathbb{R}^2\cap O_+\right)}s^{-q}K\left(z\right)\frac{\left(y-r\left(x\right)\right)}{f\left(x\right)}$$

$$\times \exp\left(us^{a-q}K(z)\frac{(y-r(x))}{f(x)}\right)g(x,y)\,dsdzdy$$
$$+(1-q)\int_{[0,1]\times(\mathbb{R}^2\cap O_-)} s^{-q}K(z)\frac{(y-r(x))}{f(x)}$$
$$\times \exp\left(us^{a-q}K(z)\frac{(y-r(x))}{f(x)}\right)g(x,y)\,dsdzdy$$

and we have

$$\lim_{u\to-\infty}\psi'_{a,q,x}(u)=-\infty \quad \text{and} \quad \lim_{u\to+\infty}\psi'_{a,q,x}(u)=+\infty$$

so that the range of $\psi'_{a,q,x}$ is $\mathbb{R}$ in this case. The proof of (*iii*) follows the same lines as previously, except that, in the present case, $\left(\psi'_{a,q,x}\right)^{-1}$ is defined on $\mathbb{R}$, and not only on $(0,+\infty)$.

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