

# FIXED POINTS AND DETERMINING SETS FOR HOLOMORPHIC SELF-MAPS OF A HYPERBOLIC MANIFOLD

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ABSTRACT. We study fixed point sets for holomorphic automorphisms (and endomorphisms) on hyperbolic manifolds. The main object of our interest is to determine the number and configuration of fixed points that forces an automorphism (endomorphism) to be the identity. These questions have been examined in a number of papers for bounded domains in  $\mathbb{C}^n$ . Here we extend these results to a finite dimensional hyperbolic manifold. In some important cases such extension is not obvious. A bounded domain can be equipped with an invariant Riemannian (Bergman) metric, and one can use differential geometry technics to obtain results. Such a metric is not always available on a general hyperbolic manifold. To overcome this obstacle we introduce locally a different invariant Hermitian metric.

## 0. INTRODUCTION

Let  $M$  be a hyperbolic manifold.  $H(M, M)$  is the set of holomorphic maps from  $M$  to  $M$ , i.e., the set of endomorphisms of  $M$ . A special case of endomorphisms are automorphisms of  $M$ ,  $Aut(M) \subset H(M, M)$ .

**Definition 0.1.** *A set  $K \subset M$  is called a determining subset of  $M$  with respect to  $Aut(D)$  ( $H(M, M)$  resp.) if, whenever  $g$  is an automorphism (endomorphism resp.) such that  $g(k) = k \forall k \in K$ , then  $g$  is the identity map of  $M$ .*

The notion of a determining set was first introduced in [FK1]. That paper was an attempt to find a higher dimensional analog of the following result of classical function theory [PL]: if  $f : M \rightarrow M$  is a conformal self-mapping of a plane domain  $M$  which fixes three distinct points then  $f(\zeta) = \zeta$ .

This one-dimensional result is true even for endomorphisms of bounded domains  $D \subset \subset \mathbb{C}$ . To prove this one needs to first use the well known theorem, stating that if an endomorphism of  $D$  fixes two distinct points, then it is an automorphism; and then use the above [PL] theorem.

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Determining sets (for automorphisms and endomorphisms) in case of bounded domains in  $\mathbb{C}^n$  have been further investigated in the following papers [FK2], [KK], [Vi1], [Vi2], [FM].

Let  $W_s(M)$  denote the set of  $s$ -tuples  $(x_1, \dots, x_s)$ , where  $x_j \in M$ , such that  $\{x_1, \dots, x_s\}$  is a determining set with respect to  $Aut(M)$ . Similarly,  $\widehat{W}_s(M)$  denotes the set of  $s$ -tuples  $(x_1, \dots, x_s)$  such that  $\{x_1, \dots, x_s\}$  is a determining set with respect to  $H(M, M)$ . So  $\widehat{W}_s(M) \subseteq W_s(M) \subseteq M^s$ . We now introduce two numbers  $s_0(M)$  and  $\widehat{s}_0(M)$ . In case  $Aut(M) = id$ ,  $s_0(M) = 0$ , otherwise  $s_0(M)$  is the least integer  $s$ , such that  $W_s(M) \neq \emptyset$ . The symbol  $\widehat{s}_0(M)$  denotes the least integer  $s$  such that  $\widehat{W}_s(M) \neq \emptyset$ . Hence,  $s_0(M) \leq \widehat{s}_0(M)$ .

In [Vi2] the estimate  $\widehat{s}_0(D) \leq n + 1$  was established for all bounded domains in  $\mathbb{C}^n$ . In Section 2 we generalize this by proving the same inequality for hyperbolic manifolds of dimension  $n$ . This certainly implies same inequality for *automorphisms* of a hyperbolic manifold  $M$ ,  $s_0(M) \leq n + 1$ . However for automorphisms much more information can be provided.

When  $M = D$  a bounded domain in  $\mathbb{C}^n$ , the above estimate can be refined (see [FM]) to  $s_0(D) \leq n$  for domains that are not biholomorphic to the unit ball  $B^n \subset \mathbb{C}^n$  (i.e. the only bounded domains in  $\mathbb{C}^n$  for which  $s_0(D) = n + 1$  are those biholomorphic to the ball). Also,  $s_0(D)$  depends on how large the group  $Aut(D)$  is, and corresponding inequalities have been proved in [FM]. In this paper (Sections 3,4) we generalize all these estimates on  $s_0(M)$  for any hyperbolic manifold.

If a positive integer  $s \geq s_0(M)$ , then  $W_s(D) \neq \emptyset$ , so there are  $s$  points such that if an automorphism of  $M$  fixes these points it will fix any point of  $M$ . Now the question arises whether the choice of these  $s$  points is generic. The answer is positive ([Vi1],[FM]) for any bounded domain  $D$  in  $\mathbb{C}^n$ :  $W_s(D) \subseteq D^s$  is open and dense if not empty. Again we prove here that the same is true for any hyperbolic manifold (section 5). The proof (in [FM]) of this last statement for a bounded domain used the Bergman metric on that domain. Such a Riemannian metric is not always available on a hyperbolic manifold. To overcome this obstacle we had to construct for any point  $x \in M$  an invariant (with respect to  $Aut(M)$ ) Hermitian metric in a neighborhood (open but not necessarily connected) of that point (see Lemma 1.7).

Similar properties for the determining sets of endomorphisms in general do not hold. We address related questions in the concluding part of section 5.

## 1. PRELIMINARY STATEMENTS

Throughout this section  $M$  denotes a hyperbolic manifold of finite dimension,  $Aut(M)$  is its group of holomorphic automorphisms.

**Lemma 1.1.**  *$Aut(M)$  is a normal family.*

Various versions of this statement have been used before. However, we cannot find a direct reference to this result in the literature. Therefore a brief proof is presented here.

*Proof.* It suffices to prove that if  $x_0 \in M$ , if  $f_j \in Aut(M)$  is a sequence such that the closure  $Q$  of the set  $\{f_j(x_0) : j \in \mathbb{N}\}$  is compact, and if  $K$  is a compact subset of  $M$ , then

$$S := \cup_{j=1}^{\infty} f_j(K) \subset\subset M.$$

Let  $d(\cdot, \cdot)$  denote the Kobayashi distance. For  $x \in M, r > 0$  let  $b(x, r) = \{y \in M : d(x, y) < r\}$ . Let  $\psi(x) = \sup\{r > 0 : b(x, r) \subset\subset M\}$ . Now we set

$$m = \max\{d(x_0, x) : x \in K\}, \quad \delta = \min\{\psi(x) : x \in K\},$$

and

$$P = \{x \in M : d(x, Q) \leq m, \psi(x) \geq \delta\}.$$

Then  $P$  is compact and  $S \subset P$ . □

Now we note the following. Let  $a \in M, f : M \rightarrow M$  a holomorphic map such that  $f(a) = a$ . Consider a small Kobayashi ball  $b = b(a, \varepsilon)$  that is biholomorphic to a bounded domain in  $\mathbb{C}^n$ , and whose closure is compact in  $M$ . Since the Kobayashi distance is non-increasing under holomorphic maps, we have  $f : b \rightarrow b$ . If  $f \in Aut(M)$ , then  $f|_b \in Aut(b)$ . The following three statements (cf. [Vi1]) hold for bounded domains in  $\mathbb{C}^n$ ; by using this remark one can prove them for any hyperbolic manifold.

**Lemma 1.2.** *Let  $a \in M, f : M \rightarrow M$  a holomorphic map such that  $f(a) = a$  and  $f'(a) = id$ . Then  $f = id$ .*

**Lemma 1.3.** *Let  $a \in M, f \in Aut(M)$  and  $f(a) = a$ . Then all the eigenvalues of  $f'(a)$  are of modulus one, and the matrix  $f'(a)$  is diagonalizable.*

**Corollary 1.4.** *In the assumption of the above Lemma, if  $f \neq id$ , one can find an appropriate power  $k$  such that the  $k$ -th iteration of  $f$ ,  $f^k = h \in Aut(M)$  will have the following properties:  $h(a) = a, h'(a)$  has at least one eigenvalue with non-positive real part.*

Let  $z \in M$ . Below we use the notion of an isotropy group  $I_z(M) = \{g \in \text{Aut}(M) : g(z) = z\}$ .

**Lemma 1.5** (H. Cartan). (*[Ca1, p.80]*) *Let  $D \subset \subset \mathbb{C}^n$ , let  $z \in D$ , and let  $I_z = I_z(D)$  be the isotropy subgroup at  $z$  of the automorphism group of  $D$ . Then there exists a holomorphic map  $\phi : D \rightarrow \mathbb{C}^n$  such that  $\phi(z) = 0$ ,  $\phi'(z) = id$ , and for all  $f \in I_z$  one has  $\phi \circ f = f'(z) \circ \phi$ .*

As in [Vi1, thm 2.3], for the proof of this Lemma, we define  $\phi : D \rightarrow \mathbb{C}^n$  by

$$\phi(\zeta) = \int_{G_z} f'(z)^{-1}(f(\zeta) - z) d\mu(f),$$

where  $d\mu$  is the Haar measure on  $I_z$ . Then  $\phi(z) = 0$ ,  $\phi'(z) = id$  (and therefore  $\phi$  is locally biholomorphic), and  $\phi \circ g = g'(z) \circ \phi$  for each  $g \in I_z$ .

Let  $M$  again be a hyperbolic manifold,  $x \in M$ ,  $T_x M$  the tangent space of  $M$  at  $x$ ,  $I_x = I_x(M)$  is the isotropy subgroup fixing  $x$ . The compact group  $I_x$  acts on  $T$  as differential maps: for  $g \in I_x$ ,  $v \in T$ ,  $g_*(v) = dg(x)v$ . Since the above Lemma can be considered in a small neighborhood of  $x$ , and  $T$  is isomorphic to  $\mathbb{C}^n$  the following statement holds.

**Lemma 1.6.** *For any point  $x \in M$  there exists a small neighborhood  $V \ni x$ , such that there is an injective holomorphic map  $\phi : V \rightarrow T$  such that  $g_* \circ \phi = \phi \circ g$  for  $g \in I_x$ , and  $d\phi(x) = id$ , the identity map of  $T = T_x M$ .*

Finally we will introduce an Hermitian invariant metric on a neighborhood of any point in  $M$ .

**Lemma 1.7.** *Let  $M$  be a hyperbolic manifold, let  $G = \text{Aut}(M)$ , and let  $x \in M$ . Then there is a neighborhood  $U$  of  $x$  such that  $G(U) = U$ , and a  $C^\infty$  Hermitian metric on  $U$  that is invariant under  $G$ .*

*Proof.* Since  $M$  is hyperbolic, the automorphism group  $G$  is a Lie group (see [Ko]) and the isotropy group  $I_x$  is a compact subgroup of  $G$ . The orbit  $G(x)$  is an embedded submanifold of  $M$ . There is a neighborhood  $V$  of  $x$  in  $M$  such that  $G(V)$  is naturally homotopic to  $G(x)$ . Let  $T = T_x M$  be the tangent space of  $M$  at  $x$ . Then  $T$  is a complex vector space and is isomorphic to  $\mathbb{C}^n$ . The elements of the compact group  $I_x$  act on  $T$  as differential maps: for  $g \in I_x$ ,  $g_*(v) = dg(x)v$ . Let  $h$  be a Hermitian metric on  $T$  invariant under  $I_x$ . By Lemma 1.6, if  $V$  is sufficiently small, there is an injective holomorphic map  $\phi : V \rightarrow T$  such that  $g_* \circ \phi = \phi \circ g$  for  $g \in I_x$ , and  $d\phi(x) = id$ , the identity

map of  $T = T_x M$ . The (real) subspace  $P$  of  $T$  consisting of vectors tangent to  $G(x)$  is invariant under  $I_x$ . So the orthogonal complement (with respect to the real part of  $h$ )  $Q$  of  $P$  is also invariant under  $I_x$ . Let  $S_1 = \{v \in Q : \|v\| < \delta\}$ , where  $\|\cdot\|$  is the norm induced by the Hermitian metric  $h$ , and choose  $\delta > 0$  so small that  $S_1 \subset\subset \phi(V)$ . Note that  $S_1$  is invariant under  $I_x$ . Let  $S = \phi^{-1}(S_1)$ . Then  $I_x(S) = S$ . Furthermore, for  $g \in G$ ,  $g(S) \cap S \neq \emptyset$  iff  $g \in I_x$ . The tube  $G(S)$  is diffeomorphic to the normal bundle of  $G(x)$  in  $M$  and to the twisted product  $G \times_{I_x} S$ . The pull-back  $h_0 = (\phi|_S)^* h$  is a Hermitian metric on the restriction to  $S$  of the tangent bundle  $TM$ . Now we define a Hermitian metric  $h_1$  on  $U = G(S)$  as follows. If  $y \in U$  and  $u, v \in T_y$ , then there is a  $g \in G$  such that  $g(y) \in S$ , and we define  $h_1(u, v) = h_0(g_* u, g_* v)$ . One can see that  $h_1$  is well-defined, since if  $g(y), g'(y) \in S$ , then  $g'g^{-1} \in I_x$ . Now  $h_1$  is a  $C^\infty$  metric on  $U$  that is invariant under  $G$ .  $\square$

## 2. AN ESTIMATE FOR $\widehat{s}_0(M)$

We need the following lemma (Thm. 5.2 in [Vi2])

**Lemma 2.1.** *Let  $D$  be a bounded domain in  $\mathbb{C}^n$ ,  $a \in D$ . Then there is an open  $U \subset D^n$  such that  $(a, \dots, a) \in \overline{U}$  and for all  $(z_1, \dots, z_n) \in U$ ,  $(a, z_1, \dots, z_n) \in \widehat{W}_{n+1}(D)$ .*

**Theorem 2.2.** *Let  $M$  be a hyperbolic manifold of complex dimension  $n$ . Then  $\widehat{s}_0(M) \leq n + 1$ .*

*Proof.* Pick a point  $a \in M$ . Let  $f : M \rightarrow M$  be a holomorphic map such that  $f(a) = a$ . Consider a small Kobayashi ball  $b = b(a, \varepsilon)$  whose closure is compact in  $M$ , and such that  $b$  is biholomorphic to a bounded domain  $D$  in  $\mathbb{C}^n$ ; let  $h : b \rightarrow D$  be such a biholomorphic map. Note that since the Kobayashi distance is non-increasing under holomorphic maps, we have  $f : b \rightarrow b$ , and therefore  $g = h \circ f \circ h^{-1} : D \rightarrow D$ . By using the preceding lemma, one can pick  $n$  points  $z_1, \dots, z_n \in D$ , such that  $Z = (h(a), z_1, \dots, z_n) \in \widehat{W}_{n+1}(D)$ . Consider the set of  $n + 1$  points  $h^{-1}(Z) = (a, h^{-1}(z_1), \dots, h^{-1}(z_n)) \subset b$ . If our function  $f \in H(M, M)$  (in addition to  $a$ ) is also fixing all points  $h^{-1}(z_j)$ , i.e.  $f|_{h^{-1}(Z)} = id$ , then  $g|_Z = id$  and therefore  $g = id$ . We conclude that  $f|_b = id$ , and consequently  $f = id$ . So,  $h^{-1}(Z) \in \widehat{W}_{n+1}(M)$ , and therefore  $\widehat{s}_0(M) \leq n + 1$ .  $\square$

## 3. ESTIMATES FOR $s_0(M)$

The goal of this section is to provide estimates for  $s_0(M)$  for a hyperbolic manifold  $M$ ,  $\dim(M) = n$ .

Since  $s_0(M) \leq \widehat{s}_0(M)$  theorem 2.2 implies

*For any hyperbolic manifold  $M$  of complex dimension  $n$ ,  $s_0(M) \leq n+1$ .*

*Remark.* In the next section we prove a refined inequality  $s_0(M) \leq n$  for  $M$  not biholomorphic to the unit ball in  $\mathbb{C}^n$ .

If  $H$  is (isomorphic to) a subgroup of the unitary group  $U(n)$ , let  $k(H)$  denote the least number  $k$  of vectors  $u_1, \dots, u_k$  such that if  $h \in H$  and if  $h(u_j) = u_j$  for  $j = 1, \dots, k$  then  $h = id$ . For  $x \in M$  the isotropy group  $I_x(M)$  is isomorphic to the group of its differentials at  $x$ , and these differentials are unitary with respect to the locally defined Hermitian inner product (the existence of which was proved in Lemma 1.7) on the tangent space  $T_x(M)$ . So  $I_x(M)$  is isomorphic to a subgroup of  $U(n)$ .

**Theorem 3.1.**  $s_0(M) \leq 1 + \min\{k(I_x(M)) : x \in M\}$ .

*Proof.* Choose  $x \in M$  so that  $k(I_x(M)) = \min\{k(I_x(M)) : x \in M\}$ . Denote that number by  $k$ . Let  $u_1, \dots, u_k$  be vectors in  $T_x M$  such that if  $h \in I_x(M)$  and if  $dh(x)(u_j) = u_j$  for  $j = 1, \dots, k$  then  $dh = id$  (hence  $h = id$ ). For each  $u_j$ , let  $x_j$  be a point on the geodesic through  $x$  in the direction  $u_j$ , so close to  $x$  that the geodesic is the unique length minimizing geodesic from  $x$  to  $x_j$ . Let  $f$  be an automorphism of  $M$  fixing  $x, x_1, \dots, x_k$ . Then  $df(x)$  fixes  $u_1, \dots, u_k$ . It follows that  $df(z) = id$  and  $f = id$ . Therefore,  $s_0(M) \leq 1 + \min\{k(I_x(M)) : x \in M\}$ .  $\square$

Let  $G$  be a subgroup of  $Aut(M)$ . By  $s_0(M, G)$  we denote the minimum number of distinct points in  $M$  such that if  $g \in G$ , and  $g$  fixes all these points, then  $g = id$ . So,  $s_0(M) = s_0(M, Aut(M))$ .

**Lemma 3.2.** *Let  $M$  be a hyperbolic manifold, let  $G$  be a subgroup of  $Aut(M)$ , and let  $q = \dim G$ . If  $q \geq 1$ , then  $s_0(M, G) \leq q$ . If  $q = 0$ , then  $s_0(M, G) \leq 1$ .*

*Proof.* First we consider the case where  $q \leq 1$ . Let  $e$  denote the identity element of  $G$ , and let  $Q = G \setminus \{e\}$ . For each  $g \in Q$ , the set  $\{x \in M : g(x) = x\}$  is an analytic set of  $M$  of dimension  $\leq 2n - 2$ . The set  $W_1 := \{(g, x) \in Q \times M : g(x) = x\}$  is an analytic set of  $Q \times M$  of dimension  $\leq (2n - 2) + q \leq 2n - 1 < \dim M$ . Let  $W$  denote the set of fixed points of nontrivial elements of  $G$ . Since  $W = \pi(W_1)$ , where  $\pi : Q \times M \rightarrow M$  is the projection, and since  $\dim W_1 < \dim M$ , we see that  $W \neq M$ . Therefore,  $s_0(M, G) \leq 1$ .

Now we assume that  $q \geq 2$ . There must be an orbit  $Q$  of  $G$  of positive dimension. Let  $x \in Q$ , and let  $H := G_x$  be the subgroup of  $G$

consisting of elements  $g$  satisfying  $g(x) = x$ . Then  $\dim H < \dim G$ . By induction hypothesis,  $s_0(M, H) \leq \dim G - 1$ . Therefore,  $s_0(M, G) \leq 1 + s_0(M, H) \leq \dim G$ .  $\square$

As a corollary we get

**Theorem 3.3.** *If  $\dim(\text{Aut}(M)) \geq 1$ , then  $s_0(M) \leq \dim(\text{Aut}(M))$ . If  $\dim(\text{Aut}(M)) = 0$ , then  $s_0(M) \leq 1$ .*

#### 4. A CHARACTERIZATION OF THE BALL IN $\mathbb{C}^n$

This section is devoted to the proof of the following statement.

**Theorem 4.1.** *Let  $M$  be a hyperbolic manifold of dimension  $n$ .  $s_0(M) = n + 1$  if and only if  $M$  is biholomorphic to the unit ball  $B^n$  in  $\mathbb{C}^n$ .*

The estimate  $s_0(B^n) = n + 1$  can be easily verified (see for example [FM]).

The rest of this section will be devoted to the proof that  $s_0(M) = n + 1$  implies that  $M$  is biholomorphic to the unit ball. To prove this we need the following two lemmas.

**Lemma 4.2.** *Let  $M$  be a hyperbolic manifold and  $x \in M$ . Suppose that the isotropy group  $I_x$  is transitive on the (real) directions at  $x$ . Then  $M$  is biholomorphic to the unit ball in  $\mathbb{C}^n$ .*

*Proof.* Since  $I_x$  is transitive on the directions at  $x$ , the group  $\text{Aut}(M)$  is not finite. Since the automorphism group of a compact hyperbolic manifold must be finite (see [Ko, p. 70]), we see that  $M$  is noncompact. By the main theorem in [GK],  $M$  is biholomorphic to  $\mathbb{C}^n$ .  $\square$

For a subgroup  $H$  of the unitary group  $U(n)$  we use the notion  $k(H)$  introduced at the beginning of section 3. The following Lemma was proved in ([FM], Lemma 1.4).

**Lemma 4.3.** *If  $H$  is a subgroup of  $U(n)$  with  $n \geq 2$  and if  $H$  is not transitive on  $S^{2n-1}$  then  $k(H) \leq n - 1$ .*

We are now ready to prove the remaining portion of Theorem 4.1 (i.e.,  $s_0(M) = n + 1$  implies that  $M$  is biholomorphic to the unit ball).

*Proof.* So, let  $s_0(M) = n + 1$ . If  $n = 1$  the statement ( $M$  is biholomorphic to the unit disc  $B^1$ ) is true. Indeed, if  $M$  is not biholomorphic to the disc or the annulus, its automorphism group is discrete. For each element  $g \in \text{Aut}(M)$ ,  $g \neq id$  the set of fixed points is discrete. Therefore there is a point  $x \in M$  that is not a fixed point of any nontrivial automorphism. This point will then form a determining set, and so,

$s_0(M) \leq 1$ . For the annulus  $s_0(M) = 1$ . Therefore if  $s_0(M) = 2$ ,  $M$  is biholomorphic to the unit disc.

Consider now the case where  $n \geq 2$ . Let  $z \in M$ . Suppose that  $M$  is not biholomorphic to  $B^n$ . Then  $I_z(M)$  is not transitive on the directions at  $z$ , by Lemma 4.2. Since  $I_z(M)$  is (isomorphic to) a subgroup of  $U(n)$ , by Lemma 4.3,  $k(I_z(M)) \leq n - 1$ . It follows (see Theorem 3.1) that  $s_0(M) \leq 1 + k(I_z(M)) \leq n$  if  $M$  is not biholomorphic to  $B^n$ .  $\square$

## 5. DETERMINING SETS $W_s(M)$ ARE OPEN AND DENSE

Our aim in this section is to prove the following theorem.

**Theorem 5.1.** *Let  $M$  be a hyperbolic manifold and  $s \geq 1$ . Then  $W_s(M) \subset M^s$  is open; if in addition  $W_s(M) \neq \emptyset$ , then  $W_s(M)$  is dense in  $M^s$ .*

Denote  $W = W_s(M)$ . First we prove that  $W \subset M^s$  is open.

*Proof.* Suppose  $W$  is not open. Then one can find a sequence of  $s$ -tuples  $Z_j = (x_1, \dots, x_s^j) \in M^s$  that converges to  $Z = (x_1, \dots, x_s) \in M^s$  and such that  $Z_j$  is not a determining set for  $M$ , and  $Z$  is. For each  $j$  there is an  $f_j \in \text{Aut}(M)$ ,  $f_j|_{Z_j} = \text{id}$ , but  $f_j \neq \text{id}$ . By Corollary 1.4 (replacing  $f_j$  by an appropriate iteration of  $f_j$  if needed) we may assume that the real part of at least one eigenvalue of  $f_j'(x_1^j)$  is non-positive. Switching again to a subsequence, if necessary, we find a sequence of automorphisms whose limit (see Lemma 1.1) is  $g \in \text{Aut}(M)$ , such that  $g|_Z = \text{id}$ , and one of the eigenvalues of  $g'(x_1)$  is non-positive. Therefore  $g \neq \text{id}$  which contradicts the original assumption that  $Z$  is a determining set for  $M$ .  $\square$

*Remark.* The above proof of the theorem for a bounded domain is given in [Vi1, Thm 3.1]. One can also prove Theorem 5.1 by using the idea of [FM, Lemma 2.3].

Now suppose that  $W \neq \emptyset$ . We need to prove that  $W$  is dense in  $M^s$ .

First we introduce some notation. If  $G$  is a subgroup of  $\text{Aut}(M)$ ,  $W_s(M, G)$  denotes the set of  $s$ -tuples  $(x_1, \dots, x_s)$ , where  $x_j \in M$ , such that each element  $g \in G$  satisfying  $g(x_j) = x_j$  for  $j = 1, \dots, s$  has to be the identity.

Let  $\rho_x(\cdot, \cdot)$  denote the metric introduced in Lemma 1.7 for a point  $x \in M$ . Let  $b(x, r)$  denote the ball with center  $x$  and radius  $r$  in that metric. Let  $\bar{b}(x, r)$  be the closure of  $b(x, r)$  in  $M$ .



**Lemma 5.2.** *Suppose that  $G$  is a subgroup of  $\text{Aut}(M)$ . If  $W_1(M, G) \neq \emptyset$  then  $W_1(M, G)$  is dense in  $M$ .*

*Proof.* In this proof, let  $W = W_1(M, G)$ . Suppose that  $W$  is not dense in  $M$ . Then the closure  $K$  of  $W$  in  $M$  is not equal to  $M$ . Let  $p$  be a boundary point of  $K$  in  $M$ . Denote  $\rho(\cdot, \cdot) = \rho_p(\cdot, \cdot)$ . Choose  $r > 0$  such that the closure of  $b(p, 4r)$  in  $U$  is compact, where  $U$  is a neighborhood from Lemma 1.7 (chosen for the point  $p$ ), and such that each pair of points of  $b(p, 4r)$  is connected by a unique length-minimizing geodesic segment in that metric. There exist points  $z, w$  such that  $\rho(z, p) < r$ ,  $\rho(w, p) < r$ ,  $w \in W$ , and  $z \notin K$ . Note that the orbit of  $w$ ,  $G(w) \subset W$ . Let  $Q = G(w) \cap \bar{b}(p, 4r)$ . Then  $Q$  is compact and  $Q \subset W$ . Let  $u$  be a point of  $Q$  nearest to  $z$ . Then  $u$  is also a point of  $G(w)$  nearest to  $z$ , and  $R := \rho(z, u) \leq \rho(z, w) < 2r$ . Choose a point  $y$  on the unique length-minimizing geodesic segment from  $z$  to  $u$  such that  $y \notin K$  and  $y \neq z$ . For each point  $x$  of  $G(w)$ , we see that

$$\rho(z, y) + \rho(y, x) \geq \rho(z, x) \geq \rho(z, u),$$

and that the two equalities hold simultaneously only if  $x = u$ . Hence,  $\rho(z, y) + \rho(y, x) > \rho(z, u) = R$  for each  $x \in G(w)$ ,  $x \neq u$ . It follows that  $\rho(y, x) > R - \rho(z, y) = \rho(y, u)$  for each  $x \in G(w)$ ,  $x \neq u$ . Therefore,  $u$  is the unique point of  $G(w)$  nearest to  $y$ . Since  $y \notin K$ , there is a nontrivial  $g \in G$  such that  $g(y) = y$ . Now  $\rho(y, u) = \rho(g(y), g(u)) = \rho(y, g(u))$  forces  $g(u) = u$ . Since  $u \in W$ , the map  $g$  must be the identity, contradicting the fact that  $g$  is not trivial. Therefore,  $W_1(M, G)$  is dense in  $M$ .  $\square$

*Proof of Theorem 5.1.* We have already proved that  $W_s(M)$  is open in  $M^s$ . Suppose now that  $W_s(M) \neq \emptyset$ . For  $g \in \text{Aut}(M)$  let  $Q_s(g)$  denote the mapping

$$Q_s(g) : M^s \rightarrow M^s, \quad Q_s(g)(z_1, \dots, z_s) = (g(z_1), \dots, g(z_s)).$$

Let  $G = \{Q_s(g) : g \in \text{Aut}(M)\}$ . Then  $G \subset \text{Aut}(M^s)$ , and  $W_1(M^s, G) = W_s(M)$ . By the previous lemma,  $W_s(M)$  is dense in  $M^s$ .

By using the same approach as in Theorem 5.1 in [Vi2] one can establish the following

**Theorem 5.3.** *If  $M$  is a taut manifold then  $\widehat{W}_s(M)$  is open in  $M^s$  for all  $s \geq 1$ .*

In general  $\widehat{W}_s(M)$  does not have to be open in  $M^s$  (see [FM]), nor be dense in  $M^s$  (cf. [Vi2],[FM]).

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