ISOMETRIES FOR THE CARATHÉODORY METRIC

MARCO ABATE AND JEAN-PIERRE VIGUÉ

1. Introduction

The following problem has been studied by many authors. Let D_1 and D_2 be two bounded domains in complex Banach spaces and let $f\colon D_1 \to D_2$ be a holomorphic map such that $f'(a)$ is a surjective isometry for the Carathéodory infinitesimal metric at a point a of D_1. The problem is to know whether f is an analytic isomorphism of D_1 onto D_2. For example, J.-P. Vigué [8] proved this is the case when D_1 and D_2 are two bounded domains in $\mathbb{C}^n$ and D_1 is convex. Similar results have been obtained when D_2 is convex using the Kobayashi infinitesimal metric (I. Graham [3] and L. Belkhchicha [1]). We have to remark that all these results are based on the theorem of L. Lempert ([5] et [6]; one can also consult M. Jarnicki and P. Pflug [4]) on the equality of Kobayashi and Carathéodory metrics on a bounded convex domain in $\mathbb{C}^n$. J.-P. Vigué [9] proved the first results on this subject in the case of bounded domains in complex Banach spaces.

Now, we can study the same problem dropping the hypothesis that $f'(a)$ is surjective. So, we only suppose that $f'(a)$ is an isometry for the Carathéodory infinitesimal metric. Does this imply that $f(D_1)$ is a complex analytic closed submanifold of D_2 and that f is an analytic isomorphism of D_1 onto $f(D_1)$?

Some results have been obtained by J.-P. Vigué [10] and P. Mazet [7] assuming that D_1 and D_2 are open unit balls in complex Banach spaces, that $a = 0$, and that the image of $f'(0)$ contains enough complex extremal points of the boundary of D_2. Under these hypotheses they proved that f is linear equal to $f'(0)$. This result shows that $f(D_1)$ is an analytic submanifold of D_2 and that f is an analytic isomorphism of D_1 onto $f(D_1)$.

Of course, if we do not suppose the existence of complex extremal points in the image of $f'(0)$, the map f has no reason to be linear. However, one can hope that $f(D_1)$ still is a complex analytic submanifold of D_2. In this paper we shall be able to prove such a result for maps of unit balls of complex Banach spaces, under some additional hypotheses on the Banach spaces involved.

The authors would like to thank the referee for his/her useful remarks.

2. The main results

We shall prove the following theorem:

Theorem 1. *Let* $(E_j, \|\|_j)$ *be complex Banach spaces and let* $B_j = \{x \in E_j \mid \|x\|_j < 1\}$*, for* $j = 1, 2$*. Let* $f\colon B_1 \to B_2$ *be a holomophic mapping with* $f(0) = 0$ *and* $\|f'(0)(X)\|_2 = \|X\|_1$ *for all* $X \in E_1$*. Then the following statements are equivalent :*

1. *there exists a direct decomposition* $E_2 = f'(0)(E_1) \oplus F$ *such that the corresponding projection* $\pi\colon E_2 \to f'(0)(E_1)$ *has norm* 1*;*
2. $f(B_1)$ *is a closed complex direct submanifold of* B_2*, the map* f *is a biholomorphism of* B_1 *onto* $f(B_1)$*, and there exists a holomorphic retraction of* B_2 *onto* $f(B_1)$*.*

To apply this theorem, we give the following definition:

Definition 1. *We say that a pair* (E_1, E_2) *of complex Banach spaces has the property* (V) *if for every linear isometry* $L\colon E_1 \to E_2$ *there exists a direct decomposition* $E_2 = L(E_1) \oplus F$ *such that the corresponding projection* $\pi\colon E_2 \to L(E_1)$ *has norm 1.*

From theorem 1 and definition 1, we deduce the following

Theorem 2. *Assume that the pair* (E_1, E_2) *of complex Banach spaces has the property* (V)*, and let* B_1 *and* B_2 *be their open unit balls. Let* $f\colon B_1 \to B_2$ *be a holomorphic map such that*

1. $f(0) = 0$*, and* $f'(0)$ *is an isometry for the Carathéodory infinitesimal metric,*

   or
2. B_1 *and* B_2 *are homogeneous, and there exists* $a \in B_1$ *such that* $f'(a)$ *is an isometry for the Carathéodory infinitesimal metric.*

Then $f(B_1)$ *is a closed complex direct submanifold of* B_2*, the map* f *is a biholomorphism of* B_1 *onto* $f(B_1)$*, and there exists a holomorphic retraction of* B_2 *onto* $f(B_1)$*.*

Now, we clearly need examples of pairs of complex Banach spaces satisfying property (V). The first (easy) example is given by Hilbert spaces.

Proposition 1. *Let* E_2 *a complex Hilbert space. Then, for every complex Banach space* E_1 *the pair* (E_1, E_2) *has property* (V)*.*

More interesting is the following theorem:

Theorem 3. *Let I be a set and let $l^\infty(I)$ be the complex Banach space of bounded sequences indexed by I, with the usual norm. Let E_2 be any Banach space. Then, the pair $\big(l^\infty(I), E_2\big)$ has property* (V).

Other pairs enjoying property (V) can be constructed using suitable subspaces of $\ell^\infty(I)$. For instance, let $c_0(I) \subset \ell^\infty(I)$ be the subspace given by the elements $(a_i)_{i\in I} \in \ell^\infty(I)$ such that for every $\varepsilon > 0$ there exists a finite subset $K \subseteq I$ so that $|a_i| < \varepsilon$ when $i \notin K$. Then:

Theorem 4. *For any sets I, J the pair $\big(c_0(I), c_0(J)\big)$ has property* (V).

Applying Theorem 2 and 3 with I finite, we get in particular a new result in the finite-dimensional case:

Corollary 1. *Let $f\colon \Delta^n \to D$ be a holomorphic map between a polydisk $\Delta^n \subset \mathbb{C}^n$ and an open convex circular bounded domain $D \subset \mathbb{C}^N$ (i.e., D is the unit ball for a suitable norm in $\mathbb{C}^N$). We also assume $n \leq N$, and that D is homogeneous (for instance, $D = \Delta^N$, B^N or a bounded symmetric domain). Assume that there exists $a \in \Delta^n$ such that $f'(a)$ is an isometry for the Carathéodory infinitesimal metrics. Then $f(\Delta^n)$ is a closed complex submanifold of D, the map f is a biholomorphism onto its image, and $f(\Delta^n)$ is a holomorphic retract of D.*

Before proving these results, we need to recall some facts.

3. Some classical results

The definition and the main properties of Carathéodory and Kobayashi infinitesimal metrics E_D and F_D on a bounded domain D are given in the book of T. Franzoni et E. Vesentini [2] (see also the book of M. Jarnicki and P. Pflug [4]).

Let B be the open unit ball of a complex Banach space E. It is well known that

$$E_B(0,x) = F_B(0,x) = \|x\| \ .$$

Furthermore, every biholomorphism $f\colon D_1 \to D_2$ between domains in complex Banach spaces is an isometry for the Carathéodory and Kobayashi infinitesimal metrics.

Finally, let us recall that, the open unit balls B of the complex Banach spaces $c_0(I)$ and $l^\infty(I)$ are homogeneous. Indeed, it is easy to check that, for every $a \in B$, the map $\varphi_a\colon B \to B$ given by

$$\forall i \in I \qquad\qquad \varphi_a(f)_i = \frac{f_i + a_i}{1 + \overline{a_i} f_i} \ ,$$

is an analytic automorphism of B.

Another example of homogeneous unit ball is given by the open unit ball B of the space $C(S, \mathbb{C})$ of continuous complex functions on a compact space S, because for every $a \in B$ the map $\varphi_a \colon B \to B$ given by

$$\varphi_a(f) = \frac{f + a}{1 + \overline{a} f}$$

is a biholomorphism of B.

4. Proof of Theorems 1 and 2

To begin, let us prove theorem 1.

Proof of Theorem 1. First, if $r : B_2 \longrightarrow f(B_1)$ is a holomorphic retraction, $r'(0)$ is a projection of norm ≤ 1 for the Carathéodory infinitesimal metrics, and,

as the Carathéodory infinitesimal metric at the origin is equal to the given norm, we get $\|r'(0)\| = 1$. This proves that (2) implies (1).

To prove that (1) implies (2), let us consider

$$\varphi = \pi \circ f \colon B_1 \to f'(0)(E_1) \ .$$

We have $\varphi(0) = 0$, $\varphi(B_1) \subseteq f'(0)(E_1) \cap B_2$ (because π has norm 1), and $\varphi'(0) = \pi \circ f'(0) = f'(0)$. So $\varphi'(0)$ is a linear isometry from E_1 onto $f'(0)(E_1)$. Using Cartan's uniqueness theorem (see [2]), one easily proves that φ is a linear isometry from B_1 onto $B_2' = f'(0)(E_1) \cap B_2$.

Finally, let $\psi \colon B_2' \to F$ be defined by

$$\psi(y) = (\mathrm{id} - \pi)\bigl(f(\varphi^{-1}(y))\bigr) \ .$$

Then the set $f(B_1)$ is the graph of ψ, the map $(\pi, \psi \circ \pi) \colon B_2 \to f(B_1)$ is a holomorphic retraction of B_2 onto $f(B_1)$, and $\varphi^{-1} \circ \pi|_{f(B_1)} \colon f(B_1) \to B_1$ is a holomorphic inverse of f, and the theorem is proved. $\square$

Now, we can prove Theorem 2.

Proof of Theorem 2. First, let us remark that, in case (2), by precomposing f with an analytic automorphism of B_1 and post-composing it with an analytic automorphism of B_2, we can assume that $f(0) = 0$ and that 0 is precisely the point a such that $f'(0)$ is an isometry for the Carathéodory infinitesimal metrics. Thus without loss of generality in both cases we can assume that $f'(0)$ is an isometry for the norms of E_1 and E_2.

Since (E_1, E_2) satisfies the property (V), there exists a direct decomposition $E_2 = f'(0)(E_1) \oplus F$ such that the corresponding projection $\pi \colon E_2 \to f'(0)(E_1)$ has norm 1 and we can apply Theorem 1. $\square$

5. Examples of pair of Banach spaces with property (V)

Now we have to give examples of pair of complex Banach spaces satisfying property (V). Proposition 1 (the case of Hilbert spaces) is easy and left as an exercise. Let us now give the

Proof of Theorem 3. We suppose that $E_1 = l^\infty(I)$ and we consider an isometry $L\colon \ell^\infty(I) \to E_2$. Let $G\colon L(E_1) \to l^\infty(I)$ be the inverse of L. So, G is a linear map of norm 1; for every $i \in I$, let G_i be the i-component of G. Then G_i is a linear form from $L(E_1)$ to $\mathbb{C}$ of norm 1. By the Hahn-Banach Theorem, we can extend G_i to a linear form $H_i\colon E_2 \to \mathbb{C}$ still of norm 1. Setting $H = (H_i)_{i\in I}$ we obtain a linear map $H\colon E_2 \to l^\infty(I)$ of norm 1 extending G. Then it is clear that $L\circ H$ is a projection of E_2 onto $L(l^\infty(I))$ of norm 1, and taking $F = \mathrm{Ker}(L \circ H)$ the theorem is proved. $\square$

Proof of Theorem 4. Let $L\colon c_0(I) \to c_0(J)$ be an isometry, and let $(e^k)_{k\in I}$ be the canonical basis of $c_0(I)$. Since L is an isometry, for every $k \in I$ there exists $j(k) \in J$ such that $|L(e^k)_{j(k)}| = 1$. Now, if we consider an element $v = (v_i)_{i\in I}$ of $c_0(I)$ such that $v_k = 0$, then $L(v)_{j(k)} = 0$. In fact, suppose that $L(v)_{j(k)} \neq 0$. For $\lambda \in \mathbb{C}$ small enough, we have $\|e^k + \lambda v\| = 1$. But

$$L(e^k + \lambda v)_{j(k)} = L(e^k)_{j(k)} + \lambda L(v)_{j(k)} = e^{i\theta} + \lambda L(v)_{j(k)} \ .$$

Therefore if $L(v)_{j(k)} \neq 0$, there exists $\lambda \in \mathbb{C}$ small enough such that the modulus of $L(e^k + \lambda v)_{j(k)}$ is greater than 1, and thus $\|L(e^k + \lambda v)\| > 1$, contradiction. It follows that the map $k \mapsto j(k)$ is injective.

Let $M = \{j(k) \mid k \in I\} \subseteq J$, and let $\pi\colon c_0(J) \to c_0(M)$ be the canonical projection. The previous argument shows

that $\pi \circ L(e^k) = \lambda_k e^{j(k)}$ with $|\lambda_k| = 1$ for all $k \in I$; it is then easy

to check that $\varphi = \pi\circ L\colon c_0(I) \to c_0(M)$ is a linear surjective isometry, and that $L \circ \varphi^{-1} \circ \pi\colon c_0(J) \to L\big(c_0(I)\big)$ is a linear projection of norm 1 of $c_0(J)$ onto $L\big(c_0(I)\big)$, as required. $\square$

It might be interesting to remark that the same proof yields that a pair (E_1, E_2) of complex Banach spaces satisfies property (V) if each E_j has a Schauder basis (e_j^k) such that

$$\Big\| \sum_k \lambda_k e_j^k \Big\|_{E_j} = \sup_k |\lambda_k| \ .$$

6. Final remarks

Not all pairs of complex Banach spaces have property (V); so we do not know whether Theorem 2 holds in general.

For example, the Banach spaces $c_0(\mathbb{N})$ is not complemented in $l^\infty(\mathbb{N})$, and so the pair $\big(c_0(\mathbb{N}),\, l^\infty(\mathbb{N})\big)$ does not have property (V).

It is also possible to build finite dimensional examples. Take $E_2 = (\mathbb{C}^3, \|\cdot\|_\infty)$, so that the unit ball of E_2 is the open polydisk Δ^3. If $L\colon \mathbb{C}^2 \to \mathbb{C}^3$ is given by $L(x,y) = (x, y, x+y)$, then $B = L^{-1}(\Delta^3)$ is the open unit ball in $\mathbb{C}^2$ for a norm $\|\cdot\|$; set $E_1 = (\mathbb{C}^2, \|\cdot\|)$. We claim that the pair (E_1, E_2) does not satisfy property (V). By construction, $L\colon E_1 \to E_2$ is a linear isometry. The set $(1,0,1) + (\{0\} \times \Delta \times \{0\})$ is contained in the boundary of Δ^3. Since $(1,0,1) \in L(\partial B)$, it is easy to check that if there exists a projection π of norm 1 from $\mathbb{C}^3$ onto $L(\mathbb{C}^2)$, then π must vanish on $\{0\} \times \mathbb{C} \times \{0\}$. Considering the point $(0,1,1)$, we analogously see that π must vanish on $\Delta \times \{0\} \times \{0\}$; and thus such a projection cannot exist.

References

[1] L. Belkhchicha. Caractérisation des isomorphismes analytiques sur la boule-unité de $\mathbb{C}^n$ pour une norme. Math. Z. **215** (1994), 129–141.

[2] T. Franzoni and E. Vesentini. Holomorphic maps and invariant distances. Notas de Matematica [Mathematical Notes], 69. North-Holland Publishing Co., Amsterdam-New York, 1980.

[3] I. Graham. Holomorphic mappings into strictly convex domains which are Kobayashi isometries at one point. Proc. Amer. Math. Soc. **105** (1989), 917–921.

[4] M. Jarnicki and P. Pflug. Invariant distances and metrics in complex analysis. de Gruyter Expositions in Mathematics, 9. Walter de Gruyter and Co., Berlin, 1993.

[5] L. Lempert. La métrique de Kobayashi et la représentation des domaines sur la boule. Bull. Soc. Math. France **109** (1981), 427–474.

[6] L. Lempert. Holomorphic retracts and intrinsic metrics in convex domains. Anal. Math. **8** (1982), 257–261.

[7] P. Mazet. Principe du maximum et lemme de Schwarz à valeurs vectorielles. Canad. Math. Bull. **40** (1997), 356–363.

[8] J.-P. Vigué. Sur les points fixes d'applications holomorphes. C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), 927–930.

[9] J.-P. Vigué. Sur la caractérisation des isomorphismes analytiques entre domaines bornés d'un espace de Banach complexe. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 145–155.

[10] J.-P. Vigué. Un lemme de Schwarz pour les boules-unités ouvertes. Canad. Math. Bull. **40** (1997), no. 1, 117–128.

Marco Abate, Dipartimento di Matematica, Università di Pisa,Largo Pontecorvo 5, 56127 Pisa, Italy.

e-mail : abate@dm.unipi.it

J.-P. V., LMA, Université de Poitiers, CNRS, Mathématiques, SP2MI, BP 30179, 86962 FUTUROSCOPE.

e-mail : vigue@math.univ-poitiers.fr