Integrable Hamiltonian systems associated to families of curves and their bi-Hamiltonian structure

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Abstract

In this paper we show how there is associated an integrable Hamiltonian system to a certain set of algebraic-geometric data. Roughly speaking these data consist of a family of algebraic curves, parametrized by an affine algebraic variety B, a subalgebra $\mathcal{C}$ of $\mathcal{O}(B)$ and a polynomial $\varphi(x,y)$ in two variables. The phase space is constructed geometrically from the family of curves and has a natural projection onto B; the regular functions on B lead to an algebra of functions in involution and the level sets of the moment map are symmetric products of algebraic curves.

While completely transparant from the geometrical point of view, a slight change of these integrable Hamiltonian systems is needed in order to explicitly realize these integrable Hamiltonian systems. Thus, we associate to the same data another integrable Hamiltonian system and show how they relate to the first one: there is a birational map between them (which is regular in one direction) which is (in the regular direction) a morphism of integrable Hamiltonian systems. Both the Poisson structure and the functions in involution are found by performing an Euclidean division of two polynomials, so that when the data are explicitly given, all ingredients of the integrable Hamiltonian system can be easily computed from it in an explicit way.

In the same spirit we also construct a large class of integrable bi-Hamiltonian systems. They depend on the extra datum of a polynomial $\psi(x,y)$ in two variables, which specifies a deformation of our family of curves. Our construction shows clearly how and why (certain) symmetries in the family of curves lead to a bi-Hamiltonian structure for the corresponding integrable Hamiltonian system.

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1. Introduction

In [Van2] we have shown that there is associated a (finite-dimensional) integrable Hamiltonian system to the following data

$$\begin{aligned} &1) \text{ an integer } d \geq 1,\\ &2) \text{ a polynomial } \varphi(x,y) \in \mathbf{C}[x,y] \setminus \{0\},\\ &3) \text{ a polynomial } F(x,y) \in \mathbf{C}[x,y] \setminus \mathbf{C}[x]. \end{aligned} \tag{1.1}$$

Explicitly, the phase space is $\mathbf{C}^{2d}$, which is viewed as the space of pairs of polynomials $(u(\lambda), v(\lambda))$ where

$$\begin{aligned} u(\lambda) &= \lambda^d + u_1\lambda^{d-1} + \cdots + u_{d-1}\lambda + u_d,\\ v(\lambda) &= \qquad\; v_1\lambda^{d-1} + \cdots + v_{d-1}\lambda + v_d, \end{aligned} \tag{1.2}$$

(thus the coefficients u_i and v_i of $u(\lambda)$ and $v(\lambda)$ serve as coordinates on $\mathbf{C}^{2d}$); the Poisson bracket on $\mathbf{C}^{2d}$, dictated by $\varphi(x,y)$, is given by

$$\begin{aligned} &\{u(\lambda), u_j\}^\varphi = \{v(\lambda), v_j\}^\varphi = 0,\\ &\{u(\lambda), v_j\}^\varphi = \{u_j, v(\lambda)\}^\varphi = \varphi(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \bmod u(\lambda), \qquad 1 \leq j \leq d; \end{aligned} \tag{1.3}$$

finally d independent functions $H_1, \dots, H_d$, in involution with respect to $\{\cdot\,,\cdot\}^\varphi$, are computed from

$$F(\lambda, v(\lambda)) \bmod u(\lambda) = H_1\lambda^{d-1} + \cdots + H_{d-1}\lambda + H_d. \tag{1.4}$$

In this paper we wish to explain and generalize this construction by using the concepts and tools which we introduced in [Van3]. The *concepts* we use here are those of an integrable Hamiltonian system on an affine Poisson variety and morphisms between such systems. In short an affine Poisson variety $(M, \{\cdot\,,\cdot\})$ is an affine algebraic variety M whose ring of regular functions $\mathcal{O}(M)$ is equipped with a Poisson bracket $\{\cdot\,,\cdot\}$; an integrable Hamiltonian system on it is given by an involutive subalgebra $\mathcal{A} \subset \mathcal{O}(M)$ (i.e., $\{\mathcal{A}, \mathcal{A}\} = 0$) which has the right size; a morphism between such systems is a morphism between their phase spaces which is compatible with the Poisson brackets and the involutive subalgebras. The *tools* we use consist of some basic theorems which allow to construct new integrable systems from old ones. See Section 2 below and Ch. II of [Van3].

The present construction generalizes the previous one in two aspects. The first one is that we replace the third datum in (1.1) by a d-parameter family of polynomials, i.e., by a polynomial $\mathcal{F}(x,y,b)$, the parameter b belonging to a d-dimensional affine variety B; in geometric terms the third datum in (1.1) is that of an algebraic curve (embedded in $\mathbf{C}^2$), which we replace here by an (effective) deformation family of algebraic curves. The second aspect which makes the present construction more general is that there is an extra datum, which is that of a subalgebra $\mathcal{C}$ of $\mathcal{O}(B)$. Essentially this extra datum specifies the algebra of Casimirs of the Poisson structure, which was trivial in our previous construction (it corresponds to the trivial choice $\mathcal{C} = \mathbf{C}$).

The phase space of the generalized integrable systems is given (possibly up to a divisor which needs to be removed) by the affine variety

$$M_\alpha = \{(u(\lambda), v(\lambda), b) \mid \mathcal{F}(\lambda, v(\lambda), b) \bmod u(\lambda) = 0\} \subset \mathbf{C}^{2d} \times B, \tag{1.5}$$

which is fibered over B with projection map $p_\alpha\colon M_\alpha \to B$. A Poisson bracket $\{\cdot\,,\cdot\}_\alpha$ on M_α is determined as before by φ giving (1.2) and all other brackets are computed from these by using the equation(s) $\mathcal{F}(\lambda, v(\lambda), b) = 0$. In particular it follows that $\{p_\alpha^*\mathcal{O}(B), p_\alpha^*\mathcal{O}(B)\}_\alpha = 0$, yielding that $(M, \{\cdot\,,\cdot\}_\alpha, p_\alpha^*\mathcal{O}(B))$ is an integrable Hamiltonian system. When $B = \mathbf{C}^d$ and $\mathcal{F}$ takes the form

$$\mathcal{F}(x,y,b) = F(x,y) - (b_1 x^{d-1} + \cdots + b_{d-1}x + b_d)$$

we recover our original construction.

Our construction is clarified by the construction of a slightly different integrable Hamiltonian system, associated to the same data. Namely, starting from the family of curves $\mathcal{F}(x,y,b) = 0$ we first construct the corresponding family of d-fold symmetric products of these curves, where $d = \dim B - \dim \mathcal{C}$. This family can be described globally as the quotient of the affine algebraic variety

$$\Gamma_\alpha^{(d)} = \left\{((x_1,y_1),\ldots,(x_d,y_d),b) \in \left(\mathbf{C}^2\right)^d \times B \mid \mathcal{F}(x_i,y_i,b) = 0\right\},$$

by the symmetric group S_d (which acts by permuting the d copies of $\mathbf{C}^2$). On $\Gamma_\alpha^{(d)}$ we use φ to construct a bracket for which all elements of $\mathcal{C}$ are Casimirs; moreover this bracket is S_d-invariant, hence it passes to the quotient. Strictly speaking these brackets are not regular but rational, we will explain in the text in detail how to deal with this, i.e., how to remove a divisor from $\Gamma_\alpha^{(d)}$ in order to make it into a genuine affine Poisson variety. As for integrability, the regular functions on $\mathcal{O}(B)$ give as before an integrable algebra, via the natural projection on the space B which parametrizes the family. This algebra is also S_d-invariant so we get an integrable Hamiltonian system on the quotient space $\Gamma_\alpha^{(d)}/S_d$.

The two integrable Hamiltonian system which we associate to the same set of data in this way are very closely related: they are almost isomorphic. More precisely there exists a regular map

$$\chi: M_\alpha \to \Gamma_\alpha^{(d)}/S_d$$

which is a morphism of the corresponding integrable Hamiltonian systems on these spaces. Moreover this map has a rational inverse, so the geometry of these integrable Hamiltonian systems is slightly different; for example, while the fibers of the moment map on $\Gamma_\alpha^{(d)}/S_d$ are symmetric products of algebraic curves, the fibers of the moment map on M_α are only affine parts of these; also, while no physical systems are known to be isomorphic to the ones on $\Gamma_\alpha^{(d)}/S_d$, many are actually isomorphic to the ones on M_α (sometimes up to a cover). Since each has its proper virtue, we found it important to give both constructions and to compare the results carefully. This will be done in Sections 3 and 4.

Our construction is easily adapted to produce many integrable bi-Hamiltonian systems. The data are slightly different from the ones above: B should be $\mathbf{C}^d$, but instead there is a new datum $\psi(x,y)$, a (general) polynomial in two variables (for simplicity we also take $\mathcal{C} = \mathbf{C}$ but that is not essential). Then the phase space is (up to a divisor)

$$M_\delta = \{(u(\lambda), v(\lambda), b_1, b_2) \mid \mathcal{F}(\lambda, v(\lambda), b_1 + \psi(x,y)b_2) \bmod u(\lambda) = 0\} \subset \mathbf{C}^{2d} \times B \times B. \qquad (1.6)$$

Remark that there are now two projections onto B, say p_1 and p_2. On M_δ we put two Poisson structures, the first one is the one which corresponds to φ (as above), the other one is the one which corresponds to the product $\varphi\psi$, let us denote these Poisson structures by $\{\cdot\,,\cdot\}_1$ and $\{\cdot\,,\cdot\}_2$. For the algebra of Casimirs (given by $\mathcal{C}$ in our previous construction), we choose for the first one $p_2^*\mathcal{O}(B)$ and for the second one $p_1^*\mathcal{O}(B)$. Then for any linear function β on B (recall that $B = \mathbf{C}^d$) we have

$$\{\cdot, p_1^*\beta\}_1 = \{\cdot, p_2^*\beta\}_2,$$

showing that these integrable Hamiltonian vector fields are bi-Hamiltonian (i.e., Hamiltonian with respect to two different Poisson structures). This will be explained in Paragraph 5.1. For some special polynomials ψ our construction breaks down. Since some of these cases are not without interest (from a certain point of view they are even more interesting/natural than the general ones), we will also show how to get a bi-Hamiltonian structure in these cases. That will be done in Paragraph 5.2.

2. Integrable Hamiltonian systems on affine Poisson varieties

In this section we recall from [Van3] the basic definitions and properties of integrable Hamiltonian systems on affine Poisson varieties.

2.1. Basic definitions

For an affine (algebraic) variety M (which is, as in [Har], assumed to be irreducible) we denote its ring of regular functions by $\mathcal{O}(M)$. A Lie algebra structure $\{\cdot\,,\cdot\}$ on $\mathcal{O}(M)$ is called a Poisson bracket if for any $f \in \mathcal{O}(M)$ the map

$$X_f = \{\cdot\,, f\} \colon \mathcal{O}(M) \to \mathcal{O}(M)$$

is a derivation, i.e., it satisfies the Leibniz rule. It leads to the concept of an affine Poisson variety.

Definition 2.1 An *affine Poisson variety* is a pair $(M, \{\cdot\,,\cdot\})$ where M is an affine variety and $\{\cdot\,,\cdot\}$ is a Poisson bracket on its ring of regular functions. The derivations $\{\cdot\,, f\}$ for $f \in \mathcal{O}(M)$ are called *Hamiltonian vector fields.*

As in the theory of Lie algebras, the main objects in this theory are the center and the maximal abelian subalgebras for the bracket; in the context of affine Poisson varieties the center of the bracket is called the algebra of Casimirs and a maximal abelian subalgebra is called an integrable algebra.

Definition 2.2 Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety. If f is an element of $\mathcal{O}(M)$ whose associated Hamiltonian vector field is zero the f is called a *Casimir;* the Casimirs form a subalgebra of $\mathcal{O}(M)$ called the *algebra of Casimirs,* which we denote by $\operatorname{Cas}(M)$.

Definition 2.3 Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety. Functions $f, g \in \mathcal{O}(M)$ are said to be *in involution* if $\{f, g\} = 0$; a subalgebra $\mathcal{A}$ of $\mathcal{O}(M)$ is said to be *involutive* if all its elements are in involution, $\{\mathcal{A}, \mathcal{A}\} = 0$. An involutive subalgebra $\mathcal{A} \subset \mathcal{O}(M)$ is said to be *integrable* if

1) $\dim \mathcal{A} = \frac{1}{2}(\dim M + \dim \operatorname{Cas}(M))$;
2) $\mathcal{A}$ is complete, i.e., $f \in \mathcal{A} \Leftrightarrow \{f, \mathcal{A}\} = 0$.

If $\mathcal{A}$ is integrable then $(M, \{\cdot\,,\cdot\}, \mathcal{A})$ is called an *integrable Hamiltonian system.*

Completeness forces any integrable algebra to contain the algebra of Casimirs. The string of inclusions

$$\operatorname{Cas}(M) \subset \mathcal{A} \subset \mathcal{O}(M)$$

leads to a commutative triangle of morphisms.

$$\begin{tikzcd} M \arrow[d, "\pi_{\mathcal{A}}"'] \arrow[dr, "\pi_{\operatorname{Cas} M}"] & \\ \operatorname{Spec} \mathcal{A} \arrow[r, "\pi"'] & \operatorname{Spec} \operatorname{Cas}(M) \end{tikzcd}$$

The map $\pi_{\mathcal{A}} \colon M \to \operatorname{Spec} \mathcal{A}$ is called the *moment map* and $\operatorname{Spec} \mathcal{A}$ is called the *base space*; the Hamiltonian vector fields $\{\cdot\,, f\}$, $f \in \mathcal{A}$ are tangent to the fibers of the moment map and span (at a general point of the each fiber) its tangent space. In this paper $\mathcal{A}$ will always be finitely generated so that $\operatorname{Spec} \mathcal{A}$ can be seen as an affine variety.

We will encounter in this paper also bi-Hamiltonian vector fields and bi-Hamiltonian hierarchies, which we define as follows. Let $\{\cdot\,,\cdot\}_1$ and $\{\cdot\,,\cdot\}_2$ be two (compatible[1]) Poisson brackets on M. Then every vector field which is Hamiltonian with respect to both brackets is called a *bi-Hamiltonian vector field* and a sequence of functions $\{f_i \mid i \in \mathbf{Z}\}$ is called a *bi-Hamiltonian hierarchy* (w.r.t. the brackets $\{\cdot\,,\cdot\}_1$ and $\{\cdot\,,\cdot\}_2$) if

$$\{\cdot\,, f_i\}_2 = \{\cdot\,, f_{i+1}\}_1, \qquad (i \in \mathbf{Z}).$$

Finally we also recall the notion of a morphism between integrable Hamiltonian systems.

Definition 2.4 Let $(M_1, \{\cdot\,,\cdot\}_1, \mathcal{A}_1)$ and $(M_2, \{\cdot\,,\cdot\}_2, \mathcal{A}_2)$ be two integrable Hamiltonian systems, then a *morphism* $\phi\colon (M_1, \{\cdot\,,\cdot\}_1, \mathcal{A}_1) \to (M_2, \{\cdot\,,\cdot\}_2, \mathcal{A}_2)$ is a regular map $\phi\colon M_1 \to M_2$ with the following properties

1) ϕ is a Poisson morphism, i.e., $\phi^*\{f,g\}_2 = \{\phi^* f, \phi^* g\}_1$ for all $f,g \in \mathcal{O}(M_2)$;
2) $\phi^* \operatorname{Cas}(M_2) \subset \operatorname{Cas}(M_1)$;
3) $\phi^* \mathcal{A}_2 \subset \mathcal{A}_1$.

If ϕ is moreover an isomorphism then ϕ^{-1} automatically satisfies 1), 2) and 3) and ϕ is called an *isomorphism* of integrable Hamiltonian systems.

Schematically, regularity of the map and 2) and 3) can be represented by the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{Cas}(M_2) & \subset & \mathcal{A}_2 & \subset & \mathcal{O}(M_2) \\ {\scriptstyle \phi^*}\big\downarrow & & {\scriptstyle \phi^*}\big\downarrow & & \big\downarrow{\scriptstyle \phi^*} \\ \operatorname{Cas}(M_1) & \subset & \mathcal{A}_1 & \subset & \mathcal{O}(M_1) \end{array} \tag{2.1}$$

By dualizing this diagram a commutative diagram between the corresponding Spec's is easily obtained.

2.2. Basic constructions and propositions

In order to show that a given subalgebra $\mathcal{A} \subset \mathcal{O}(M)$ on affine Poisson variety $(M, \{\cdot\,,\cdot\})$ is integrable one needs to verify that it is involutive, complete and of the right dimension. In this paper involutivity will be obvious from a direct computation, in other cases one often relies on the construction of an r-matrix representing the Poisson structure. Alternatively, involutivity is also obvious when $\mathcal{A}$ admits one or several bi-Hamiltonian hierarchies, as in Section 5 below. We recall the argument briefly in the following proposition, which goes essentially back to Lenard and Magri.

Proposition 2.5 *All functions f_i of a bi-Hamiltonian hierarchy $\{f_i \mid i \in \mathbf{Z}\}$ are in involution with respect to both Poisson brackets (hence with respect to any linear combination). If one of these functions is a Casimir (for either of the structures) then all these f_i are also in involution with the elements of any other bi-Hamiltonian hierarchy (w.r.t. these brackets).*

[1] Some authors impose the natural condition of compatibility of the brackets (i.e., the sum of the two brackets is also Poisson) but this is inessential for this paper, it is for example not used for the proof of Proposition 2.5.

Proof

If $\{f_i \mid i \in \mathbf{Z}\}$ forms a hierarchy, then for any $i < j$

$$\begin{aligned}\{f_i, f_j\}_1 &= \{f_i, f_{j-1}\}_2,\\ &= \{f_{i+1}, f_{j-1}\}_1,\\ &= \dots,\\ &= \{f_j, f_i\}_1,\end{aligned}$$

so $\{f_i, f_j\}_1 = 0$ by skew symmetry. They are also in involution with respect to the second bracket since $\{f_i, f_j\}_2 = \{f_i, f_{j+1}\}_1$. In the same way, if $\{g_j \mid j \in \mathbf{Z}\}$ is another bi-Hamiltonian hierarchy and f_k is a Casimir, say of $\{\cdot\,,\cdot\}_1$ then for any $i, j \in \mathbf{Z}$

$$\{f_i, g_j\}_1 = \{f_k, g_{i+j-k}\}_1 = 0,$$

so all functions of the hierarchy $\{f_i \mid i \in \mathbf{Z}\}$ are in involution with all functions of any other bi-Hamiltonian hierarchy. ∎

The computation of the dimension of $\mathcal{A}$ often turns out to be quite hard. Usually this is done by computing the co-dimension of the general fiber of the moment map $\pi_{\mathcal{A}}: M \to \operatorname{Spec} \mathcal{A}$, which is, of course, equal to the dimension of $\mathcal{A}$. A close investigation of the fibers of the moment map is also essential to the verification that $\mathcal{A}$ is complete, as is given in the following proposition (for a proof see [Van3]).

Proposition 2.6 *Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety and $\mathcal{A}$ an involutive subalgebra of $\mathcal{O}(M)$ of dimension*

$$\dim \mathcal{A} = \frac{\dim M + \dim \operatorname{Cas} M}{2}.$$

If the fibers of $\pi_{\mathcal{A}}: M \to \operatorname{Spec}(\mathcal{A})$ have the following two properties,
1) the general fiber is irreducible,
2) the fibers over all closed points have the same dimension,
then $\mathcal{A}$ is complete, hence integrable.

We will use two constructions which allow to construct new affine Poisson varieties or integrable Hamiltonian systems from old ones: taking quotients and removing divisors. This is given by the following two propositions (proofs are given in [Van3]).

Proposition 2.7 *Let $(M, \{\cdot\,,\cdot\}_M)$ be an affine Poisson variety and G a finite group. If there is given a regular Poisson action of G on M, i.e., an action $\chi: G \times M \to M$ such that χ is a Poisson morphism, then M/G has a unique structure of an affine Poisson variety $(M/G, \{\cdot\,,\cdot\}_0)$ for which $\pi: M \to M/G$ is a Poisson morphism. If $\mathcal{A}$ is an integrable algebra on $(M, \{\cdot\,,\cdot\}_M)$ then $\mathcal{A}^G$, the algebra of G-invariant functions in $\mathcal{A}$, is an integrable algebra on $(M/G, \{\cdot\,,\cdot\}_0)$ and*

$$\pi: (M, \{\cdot\,,\cdot\}_M, \mathcal{A}) \to \left(M/G, \{\cdot\,,\cdot\}_0, \mathcal{A}^G\right)$$

is a morphism of integrable Hamiltonian systems.

Proposition 2.8 *Let $(M, \{\cdot\,,\cdot\})$ be an affine Poisson variety and $f \in \mathcal{O}(M)$ a regular function which is not constant. Then there exists an affine Poisson variety $(N, \{\cdot\,,\cdot\}_N)$ and a Poisson morphism $N \to M$ which is dominant, having the complement (in M) of the zero locus of f as image.*

The last proposition needs some comments. As we will see below, many interesting brackets on some natural affine varieties (e.g., on $\mathbf{C}^n$) are not regular but rational, having their poles along some fixed divisor. So one might feel forced to work in the larger category of affine Poisson varieties with brackets on their field of rational functions. However, if the brackets of all regular functions on M have their poles on a single divisor $\mathcal{D}$ (which may be reducible), it is obvious that if f and g are any two rational functions on M, which have their poles along $\mathcal{D}$ only then their bracket $\{f,g\}$ will also have its poles along $\mathcal{D}$. Thus, instead of considering the algebra of rational functions, we may work as well with the (smaller and easier to handle) algebra of rational functions on M which have their poles on $\mathcal{D}$ only. This algebra is the algebra of regular functions on an affine variety, which may be identified with $M \setminus \mathcal{D}$; loosely speaking we will say that $M \setminus \mathcal{D}$ is an affine variety. Then the content of Proposition 2.8 is that, even when considering such rational brackets, we stay in the category of affine Poisson varieties.

3. Integrable Hamiltonian systems associated to families of curves

In this section we show how there is associated an integrable Hamiltonian system to the following data

$$
\begin{aligned}
&1)\ \text{a polynomial } \varphi(x,y)\in \mathbf{C}[x,y]\setminus\{0\},\\
&2)\ \text{an affine variety } B,\\
&3)\ \text{a closed immersion } \mathcal{F}: B\to \mathbf{C}[x,y],\\
&4)\ \text{a subalgebra } \mathcal{C} \text{ of } \mathcal{O}(B).
\end{aligned}
\tag{3.1}
$$

These data will be supposed fixed throughout this section. It is useful to denote $\alpha=(\varphi,B,\mathcal{F},\mathcal{C})$ and $d=\dim B-\dim \mathcal{C}$.

To the morphism $\mathcal{F}: B\to\mathbf{C}[x,y]$ there is naturally associated an element of $\mathcal{O}(B)[x,y]$, which we denote by $\mathcal{F}(x,y,b)$. In turn $\mathcal{F}(x,y,b)$ determines a family of algebraic curves $\mathcal{F}(x,y,b)=0$, parametrized by B. This family can be seen as a hypersurface in $\mathbf{C}^2\times B$ which we will denote by Γ_α. Since $\mathcal{F}$ is a closed immersion (i.e., it restricts to an isomorphism of B onto its image, which is closed[2]) the fibers of the projection map $\Gamma_\alpha\to B$ are precisely the algebraic curves of the family. $\mathcal{F}$ is determined by the family of algebraic curves up to a (multiplicative) constant; since our construction will only depend on $\mathcal{F}$ up to a constant, we may rephrase 3) geometrically as the datum of a family of algebraic curves, parametrized (effectively) by B.

We start by describing a natural integrable system associated to α, the one of interest for us will be obtained as a quotient of it (by a finite group). To construct the phase space we make the d-fold fiber product of $\Gamma_\alpha\to B$, i.e., we consider the affine variety

$$
\Gamma_\alpha^{(d)}=\left\{((x_1,y_1),\dots,(x_d,y_d),b)\in\left(\mathbf{C}^2\right)^d\times B \mid \mathcal{F}(x_i,y_i,b)=0\right\},
\tag{3.2}
$$

with its projection onto B,

$$
\pi_\alpha:\Gamma_\alpha^{(d)}\to B.
$$

The dimension of $\Gamma_\alpha^{(d)}$ is $\dim B+d$, the dimension of the fibers being d. We construct in the following proposition a Poisson structure $\left\{\cdot\,,\cdot\right\}_\alpha$ on $\Gamma_\alpha^{(d)}$. Strictly speaking this Poisson structure is rational (i.e., the bracket of regular functions is rational), but since all poles are along the zero locus D_α of some regular function, we may, as explained in and after Proposition 2.8 make $\left(\Gamma_\alpha^{(d)}\setminus D_\alpha,\left\{\cdot\,,\cdot\right\}_\alpha\right)$ into a genuine affine Poisson variety.

Proposition 3.1 *The brackets*

$$
\left\{x_i,x_j\right\}_\alpha=\left\{y_i,y_j\right\}_\alpha=0,\qquad \left\{y_i,x_j\right\}_\alpha=\varphi(x_j,y_i)\delta_{ij},
\tag{3.3}
$$

and

$$
\left\{\cdot\,,c\right\}_\alpha=0,\qquad \textit{for all } c\in\mathcal{C},
\tag{3.4}
$$

define a Poisson bracket on the affine variety $\Gamma_\alpha^{(d)}\setminus D_\alpha$, where D_α is the divisor of some regular function on $\Gamma_\alpha^{(d)}$.

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[2] To be very precise, closed means here Zariski closed in a finite-dimensional subspace A_{mn} of $\mathbf{C}[x,y]$ consisting of polynomials of degree at most m in x and n in y.

Proof

Let $\mathcal{C}'$ denote the subalgebra of $\mathcal{O}\left(\Gamma_\alpha^{(d)}\right)$ generated by $\mathcal{C}, x_i$ and y_i $(i = 1,\dots,d)$. Then the above brackets define a Poisson structure on $\mathcal{C}'$, since the Jacobi identity is verified. Since $\mathcal{F}$ is a closed immersion,

$$\dim \mathcal{C}' = \dim \mathcal{C} + 2d = \dim B + d = \dim \Gamma_\alpha^{(d)},$$

so that every regular (or rational) function on $\Gamma_\alpha^{(d)}$ is contained in the integral closure of $\mathcal{C}'$ in the field of fractions of $\mathcal{O}\left(\Gamma_\alpha^{(d)}\right)$. Thus, if $F \in \mathcal{O}\left(\Gamma_\alpha^{(d)}\right)$ then there exists a (not necessarily monic) polynomial $P(u) \in \mathcal{C}'[u]$ such that $P(F) = 0$. This allows to determine the bracket of any two functions F, G on $\mathcal{O}\left(\Gamma_\alpha^{(d)}\right)$ as follows. Let $P(u) = \sum p_i u^i$ and $Q(u) = \sum q_j u^j$ denote their minimal defining polynomials, $\deg P(u) = n$, $\deg Q(u) = m$. If $\{\cdot\,,\cdot\}_\alpha$ extends to $\Gamma_\alpha^{(d)}$ then for $f \in \mathcal{C}'$ we have

$$0 = \{P(F), f\}_\alpha = \sum_{i=0}^{n} \left\{p_i F^i, f\right\}_\alpha = \sum_{i=0}^{n} \{p_i, f\}_\alpha F^i + \sum_{i=1}^{n} p_i i F^{i-1} \{F, f\}_\alpha,$$

and the bracket $\{F, f\}_\alpha$ derives from it; remark that it is rational, its poles being along the divisor of zeroes of the regular function $\sum p_i i F^{i-1}$. Using this, one computes $\{F, G\}_\alpha$ from

$$0 = \{F, Q(G)\}_\alpha = \sum_{j=0}^{m} \left\{F, q_j G^j\right\}_\alpha = \sum_{j=0}^{m} \{F, q_j\}_\alpha G^j + \sum_{j=1}^{m} q_j j G^{j-1} \{F, G\}_\alpha;$$

it has also its poles along the zero locus of a regular function on $\Gamma_\alpha^{(d)}$. Since $\mathcal{O}\left(\Gamma_\alpha^{(d)}\right)$ is finitely generated, only a finite number of divisors will appear; let us call the minimal[3] divisor which contains these divisors D_α. Since D_α is (up to multiplicities) the divisor of zeroes of a regular function on $\Gamma_\alpha^{(d)}$ we may conclude using Proposition 2.8 that $\left(\Gamma_\alpha^{(d)} \setminus D_\alpha, \{\cdot\,,\cdot\}_\alpha\right)$ is an affine Poisson variety. ∎

The proposition shows that there exists a divisor D_α on $\Gamma_\alpha^{(d)}$ such that $\left(\Gamma_\alpha^{(d)} \setminus D_\alpha, \{\cdot\,,\cdot\}_\alpha\right)$ is an affine Poisson variety. We now show how the computation of this divisor can be implemented. Choose any generators $\beta_1, \dots, \beta_s$ for $\mathcal{O}(B)$ and let us use as the same notation for the corresponding generators $\pi_\alpha^* \beta_i$ of $\pi_\alpha^* \mathcal{O}(B)$. Applying naively the construction of the preceding paragraph to compute D_α would consist in determining for each generator β_j its defining polynomial and taking the bracket with all x_i and y_i. These defining polynomials can be computed from the relations $\mathcal{F}(x_i, y_i, b) = 0$, but this gives rise to long calculations. We show how to avoid their explicit calculation by computing the brackets $\{x_i, \beta_j\}_\alpha$ and $\{y_i, \beta_j\}_\alpha$ directly from these relations.

Let us first start with the easiest case, in which $\mathcal{C} = \mathbf{C}$ (no Casimirs) and $s = \dim B$ (so that B is isomorphic to $\mathbf{C}^d$). From the defining equations $\mathcal{F}(x_i, y_i, b) = 0$ we find

$$\left\{\begin{aligned} 0 &= \{\mathcal{F}(x_i, y_i, b), x_j\}_\alpha = \frac{\partial \mathcal{F}}{\partial y}(x_i, y_i, b)\varphi(x_j, y_i)\delta_{ij} + \sum_{k=1}^{s} \frac{\partial \mathcal{F}}{\partial \beta_k}(x_i, y_i, b)\{\beta_k, x_j\}_\alpha, \\ 0 &= \{\mathcal{F}(x_i, y_i, b), y_j\}_\alpha = -\frac{\partial \mathcal{F}}{\partial x}(x_i, y_i, b)\varphi(x_j, y_i)\delta_{ij} + \sum_{k=1}^{s} \frac{\partial \mathcal{F}}{\partial \beta_k}(x_i, y_i, b)\{\beta_k, y_j\}_\alpha. \end{aligned}\right. \tag{3.5}$$

[3] Since the multiplicities of the irreducible components of this divisor are only relevant for the divisor being the divisor of zeroes of a regular function, we may take them at this point all equal to 1.

To deduce the brackets $\{\beta_k, x_j\}_\alpha$ and $\{\beta_k, y_j\}_\alpha$ from it, we use the fact that $\mathcal{F}$ is an immersion. If we denote the natural function on $\mathbf{C}[x,y]$ which picks the i-th coefficient (fixing some ordering) by γ_i then $\mathcal{F}$ being immersive implies that the rank of the big matrix $\left(\frac{\partial}{\partial\beta_j}\mathcal{F}\circ\gamma_i\right)_{ij}$ is equal to d. By elementary operations on this matrix it is seen that this rank is the same as the rank of the square matrix

$$\left(\frac{\partial\mathcal{F}}{\partial\beta_j}(x_i,y_i,b)\right)_{1\leq i,j\leq d} \tag{3.6}$$

for general x_i, y_i i.e., on a Zariski open subset of $\Gamma_\alpha^{(d)}$. It follows that the rank of (3.6) is maximal on a Zariski open subset, which is explicitly given as the complement of the divisor D_α; of course D_α is just the zero locus of the determinant of the matrix (3.6). Thus, the brackets $\{x_i, \beta_j\}_\alpha$ and $\{y_i, \beta_j\}_\alpha$ have their poles along the divisor D_α only, which is easily computed as

$$D_\alpha \colon \det\left(\frac{\partial\mathcal{F}}{\partial\beta_j}(x_i,y_i,b)\right) = 0. \tag{3.7}$$

Remark that removing the divisor D_α from $\Gamma_\alpha^{(d)}$ has the effect of removing a divisor from every fiber of the moment map.

In the general case one also has to take into account the algebra $\mathcal{C}$, whose elements are to be Casimirs and the relations which hold between the generators. Let us pick any set of generators $\gamma_1,\dots,\gamma_m$ for $\mathcal{C}$ and any maximal set of independent relations $\mathcal{R}_1,\dots,\mathcal{R}_n$. Taking the brackets with x_j (and similarly with y_j) we get as in (3.5),

$$\begin{cases} 0 = \{\gamma_i, x_j\}_\alpha = \displaystyle\sum_{k=1}^{s}\frac{\partial\gamma_i}{\partial\beta_k}\{\beta_k, x_j\}_\alpha, \\ 0 = \{\mathcal{R}_i, x_j\}_\alpha = \displaystyle\sum_{k=1}^{s}\frac{\partial\mathcal{R}_i}{\partial\beta_k}\{\beta_k, x_j\}_\alpha, \end{cases} \tag{3.8}$$

where i runs from 1 to m in the first line and from 1 to n in the second one. (3.8) and the first line of (3.5) are combined by using the matrices

$$\mathcal{M}_\alpha = \begin{pmatrix} \frac{\partial\mathcal{F}}{\partial\beta} \\ \frac{\partial\gamma}{\partial\beta} \\ \frac{\partial\mathcal{R}}{\partial\beta} \end{pmatrix} \qquad \text{and} \qquad \mathcal{N}_\alpha = \begin{pmatrix} \frac{\partial\mathcal{F}}{\partial y}\varphi \\ 0 \\ 0 \end{pmatrix},$$

of size $(d+m+n)\times s$ resp. $(d+m+n)\times d$. We have introduced the matrices

$$\left(\frac{\partial\mathcal{F}}{\partial\beta}\right)_{ij} = \frac{\partial\mathcal{F}}{\partial\beta_j}(x_i,y_i,b), \qquad \left(\frac{\partial\mathcal{F}}{\partial y}\varphi\right)_{ij} = \frac{\partial\mathcal{F}}{\partial y}(x_i,y_i,b)\varphi(x_j,y_i)\delta_{ij},$$

and

$$\left(\frac{\partial\gamma}{\partial\beta}\right)_{ij} = \frac{\partial\gamma_i}{\partial\beta_j}, \qquad \left(\frac{\partial\mathcal{R}}{\partial\beta}\right)_{ij} = \frac{\partial\mathcal{R}_i}{\partial\beta_j}.$$

Then (3.8) and the first line of (3.5) are equivalent to

$$\mathcal{M}_\alpha \begin{pmatrix} \{\beta_1, x_1\}_\alpha & \dots & \{\beta_1, x_d\}_\alpha \\ \vdots & & \vdots \\ \{\beta_s, x_1\}_\alpha & \dots & \{\beta_s, x_d\}_\alpha \end{pmatrix} = -\mathcal{N}_\alpha. \tag{3.9}$$

We claim that the rank of $\mathcal{M}_\alpha$ is s on a Zariski open subset. As we have noticed, the first d rows are independent; since the next m rows correspond to generators of $\mathcal{C}$, there are $\dim \mathcal{C}$ independent rows among them, and similarly the last n rows contain $s - \dim B$ independent rows. Moreover rows in the three different blocks can not be dependent: the first ones depend on x_i, y_i while the others do not and the last rows cannot be dependent since the Casimirs are independent of the relations. It follows that

$$\operatorname{Rk} \mathcal{M}_\alpha = d + \dim \mathcal{C} + s - \dim B = s.$$

Since the brackets $\{y_i, \beta_j\}_\alpha$ are computed using the same matrix $\mathcal{M}_\alpha$ and as we will show that all brackets $\{\beta_i, \beta_j\}_\alpha$ are zero, we may conclude that the divisor D_α to be removed can be explicitly computed as the zero locus of one of the determinants which is not identically zero. Of course this divisor is not unique, it depends on the chosen determinant.

In the following proposition we show that $\pi_\alpha^* \mathcal{O}(B)$ is involutive and of maximum dimension, leading to an integrable algebra on $\left(\Gamma_\alpha^{(d)} \setminus D_\alpha, \{\cdot\,,\cdot\}_\alpha\right)$.

Proposition 3.2 *If for general $b \in B$ the polynomial $\mathcal{F}(x, y, b) = 0$ is irreducible then*

$$\left(\Gamma_\alpha^{(d)} \setminus D_\alpha, \{\cdot\,,\cdot\}_\alpha, \pi_\alpha^* \mathcal{O}(B)\right)$$

is an integrable Hamiltonian system.

Proof

We first show that $\pi_\alpha^* \mathcal{O}(B)$ is involutive with respect to $\{\cdot\,,\cdot\}_\alpha$. We use the obvious equality

$$\{F(x_i, y_i, b), F(x_j, y_j, b)\}_\alpha = 0$$

and write $\mathcal{F}(i)$ as a shorthand for $\mathcal{F}(x_i, y_i, b)$. If $i \neq j$ then this bracket expands in terms of any system of generators $\beta_1, \ldots, \beta_s$ for $\mathcal{O}(B)$ (which we identify as before with the corresponding generators $\pi_\alpha^* \beta_i$ of $\pi_\alpha^* \mathcal{O}(B)$) as

$$\sum_{k=1}^{s} \frac{\partial \mathcal{F}}{\partial x}(i) \frac{\partial \mathcal{F}}{\partial \beta_k}(j) \{x_i, \beta_k\}_\alpha + \sum_{k=1}^{s} \frac{\partial \mathcal{F}}{\partial y}(i) \frac{\partial \mathcal{F}}{\partial \beta_k}(j) \{y_i, \beta_k\}_\alpha - (i \leftrightarrow j)$$
$$+ \sum_{k,l=1}^{s} \frac{\partial \mathcal{F}}{\partial \beta_k}(i) \frac{\partial \mathcal{F}}{\partial \beta_l}(j) \{\beta_k, \beta_l\}_\alpha = 0,$$

where $(i \leftrightarrow j)$ denotes the two terms obtained by interchanging i and j in the first two terms. Now for $i \neq j$

$$\sum_{k=1}^{s} \frac{\partial \mathcal{F}}{\partial \beta_k}(j) \{x_i, \beta_k\}_\alpha = \{x_i, \mathcal{F}(x_j, y_j, b)\}_\alpha = 0,$$

so the first term vanishes; similarly the next three terms vanish and we are left with

$$\sum_{k,l=1}^{s} \frac{\partial \mathcal{F}}{\partial \beta_k}(i) \{\beta_k, \beta_l\}_\alpha \frac{\partial \mathcal{F}}{\partial \beta_l}(j) = 0, \tag{3.10}$$

for all $i \neq j$. By skew symmetry of the bracket, (3.10) is actually valid for all i and j. If $s = \dim B$ then the matrix $\frac{\partial \mathcal{F}}{\partial \beta_k}(i)$ is invertible and we find that $\{\beta_k, \beta_l\}_\alpha = 0$ for all k, l. Otherwise we use $\mathcal{C}$ and the relations $\mathcal{R}_i$ as before to obtain

$$\mathcal{M}_\alpha \left(\{\beta_i, \beta_j\}_\alpha\right) \mathcal{M}_\alpha^t = 0.$$

Since $\mathcal{M}_\alpha$ has rank s we are at the same conclusion: $\pi_\alpha^* \mathcal{O}(B)$ is involutive with respect to $\{\cdot\,,\cdot\}_\alpha$.

Let us count dimensions:

$$\begin{aligned}
\dim \Gamma_\alpha^{(d)} &= \dim B + d, \\
\dim \mathrm{Cas}\left(\Gamma_\alpha^{(d)}\right) &= \dim \mathcal{C} = \dim B - d, \\
\dim \pi_\alpha^* \mathcal{O}(B) &= \dim \mathcal{O}(B) = d,
\end{aligned}$$

where we used in the last line that π_α is surjective. Put together they lead to

$$\dim \pi_\alpha^* \mathcal{O}(B) = \frac{1}{2}\left(\dim \Gamma_\alpha^{(d)} + \dim \mathrm{Cas}\left(\Gamma_\alpha^{(d)}\right)\right),$$

so that $\pi_\alpha^* \mathcal{O}(B)$ has the right dimension in order to be integrable. Completeness of $\pi_\alpha^* \mathcal{O}(B)$ follows from the assumption that the general curve in the family is irreducible: under this assumption, the general fiber of π_α is also irreducible and since all fibers over points of B have the same dimension d, we have according to Proposition 2.6, that $\pi_\alpha^* \mathcal{O}(B)$ is complete. It follows that $\pi_\alpha^* \mathcal{O}(B)$ is integrable. ∎

If for general $b \in B$ the curve $\mathcal{F}(x, y, b) = 0$ is not irreducible then we still get an integrable Hamiltonian system by completing $\pi_\alpha^* \mathcal{O}(B)$ (i.e., by replacing it with its integral closure in its field of of fractions), but it may be hard to obtain a (more) explicit description of this algebra (see [Van3] for comments and details).

By now we have associated an integrable Hamiltonian system associated to the data (3.1). Using Proposition 2.7 we construct from it the quotient which is the integrable Hamiltonian system we were aiming at. The group which is acting on the integrable systems is the symmetry group S_d (of d letters). First of all, it acts on $\Gamma_\alpha^{(d)}$ in by permuting the d copies of $\mathbf{C}^2$ and the quotient

$$\mathrm{Sym}^d\, \Gamma_\alpha = \Gamma_\alpha^{(d)} / S_d$$

is an affine variety. Clearly the projection map $\pi_\alpha \colon \Gamma_\alpha^{(d)} \to B$ factorizes via $\mathrm{Sym}^d\, \Gamma_\alpha$ and, since no confusion can arise, we will use the same notation for the corresponding map

$$\pi_\alpha \colon \mathrm{Sym}^d\, \Gamma_\alpha \to B;$$

its fibers are now d-fold symmetric products of the fibers of $\Gamma_\alpha \to B$ (which are algebraic curves); in particular they are non-singular if the underlying curve is non-singular. The brackets (3.3) and (3.4) are clearly S_d-invariant, so that the divisor D_α and the Poisson structure $\{\cdot\,,\cdot\}_\alpha$ (on $\Gamma_\alpha^{(d)} \setminus D_\alpha$) are S_d-invariant — since S_d is a finite group, another way to formulate the latter is that the action of S_d on $\Gamma_\alpha^{(d)} \setminus D_\alpha$ is a Poisson action. A trivial application of Proposition 2.7 leads to the following corrolary of Propositions 3.1 and 3.2.

Proposition 3.3 *The brackets (3.3) and (3.4) define a Poisson strucure* $\{\cdot\,,\cdot\}_\alpha$ *on* $\mathrm{Sym}^d\, \Gamma_\alpha \setminus \mathcal{D}_\alpha$, *where* $\mathcal{D}_\alpha = D_\alpha / S_d$. *The fiber over* $b \in B$ *of the moment map* $\mathrm{Sym}^d\, \Gamma_\alpha \setminus \mathcal{D}_\alpha \to B$ *is (if non-empty)* *isomorphic to an affine part of the d-fold symmetric product of the curve $\mathcal{F}(x,y,b)=0$. Moreover* $\left(\operatorname{Sym}^d\Gamma_\alpha\setminus\mathcal{D}_\alpha,\{\cdot\,,\cdot\}_\alpha,\pi_\alpha^*\mathcal{O}(B)\right)$ *is an integrable Hamiltonian system and the projection map*

$$\left(\Gamma_\alpha^{(d)}\setminus D_\alpha,\{\cdot\,,\cdot\}^\varphi,\pi_\alpha^*\mathcal{O}(B)\right)\to\left(\operatorname{Sym}^d\Gamma_\alpha\setminus\mathcal{D}_\alpha,\{\cdot\,,\cdot\}^\varphi,\pi_\alpha^*\mathcal{O}(B)\right)$$

is a morphism of integrable Hamiltonian system.

∎

For an explicit given α an explicit description of these integrable Hamiltonian system can in principle be given. To obtain it, one looks for a description of the ring of regular functions on $\Gamma_\alpha^{(d)}$ which are S_d invariant (i.e., exhibit generators and a complete set of relations), which is often difficult to obtain. Thus, in practice the above description is already at the level of the phase space not very explicit (remark however that an explicit description of the Poisson bracket and the integrable algebra would follow at once from an explicit description of the phase space). We will come back to this in the next section.

To close this section we which to point out how the different Poisson structures $\{\cdot\,,\cdot\}_\alpha$ for varying φ are related via S_d-invariant vector fields which have the property that the Lie derivative of the Poisson brackets are also Poisson brackets[4]. In view of the formula

$$\{\cdot\,,\cdot\}_{\varphi+\psi}=\{\cdot\,,\cdot\}_\varphi+\{\cdot\,,\cdot\}_\psi$$

our space of Poisson structures is linearly generated by the Poisson brackets associated to monomials x^iy^j and we will exhibit the symmetries only for these. To do this, we define two vector fields X and Y as follows.

$$\begin{aligned} Xx_i&=-1, & Yx_i&=0,\\ Xy_i&=0, & Xy_i&=1. \end{aligned}$$

Let us compute the Lie derivatives $L_X\{\cdot\,,\cdot\}_\varphi$ and $L_Y\{\cdot\,,\cdot\}_\varphi$ for $\varphi=x^my^n$, evaluated on x_i and y_j (all other brackets are obviously zero).

$$\begin{aligned} L_X\{\cdot\,,\cdot\}_{x^my^n}(x_i,y_j)&=L_X\{x_i,y_j\}_{x^my^n}-\{L_Xx_i,y_j\}_{x^my^n}-\{x_i,L_Xy_j\}_{x^my^n},\\ &=-X(x_i^my_j^n\delta_{ij}),\\ &=mx_i^{m-1}y_j^n\delta_{ij} \end{aligned}$$

so we see that the Lie derivative of $\{\cdot\,,\cdot\}_{x^my^n}$ with respect to X is proportional to $\{\cdot\,,\cdot\}_{x^{m-1}y^n}$. Similarly we find that the Lie derivative of $\{\cdot\,,\cdot\}_{x^my^n}$ with respect to Y is proportional to $\{\cdot\,,\cdot\}_{x^my^{n-1}}$. Remark also that vector fields X and Y (hence their Lie derivatives) commute. Thus we get the following diagram representing the Poisson structures associated to x^my^n and their relations.

$$\begin{tikzcd}
\{\cdot\,,\cdot\}_1 & \{\cdot\,,\cdot\}_x \arrow[l, "X"'] & \{\cdot\,,\cdot\}_{x^2} \arrow[l, "X"'] & \cdots \arrow[l, "X"'] \\
\{\cdot\,,\cdot\}_y \arrow[u, "Y"'] & \{\cdot\,,\cdot\}_{xy} \arrow[l, "X"'] \arrow[u, "Y"'] & \{\cdot\,,\cdot\}_{x^2y} \arrow[l, "X"'] \arrow[u, "Y"'] & \cdots \arrow[l, "X"'] \\
\{\cdot\,,\cdot\}_{y^2} \arrow[u, "Y"'] & \{\cdot\,,\cdot\}_{xy^2} \arrow[l, "X"'] \arrow[u, "Y"'] & \{\cdot\,,\cdot\}_{x^2y^2} \arrow[l, "X"'] \arrow[u, "Y"'] & \cdots \arrow[l, "X"'] \\
\cdots \arrow[u, "Y"'] & \cdots \arrow[u, "Y"'] & \cdots \arrow[u, "Y"'] &
\end{tikzcd}$$

[4] Since only φ is relevant for the present discussion we will suppose now all data fixed, except for φ and we will denote the bracket corresponding to φ by $\{\cdot\,,\cdot\}_\varphi$.

4. A concrete realization

We now pass to the construction of a slightly different integrable Hamiltonian system, associated to the same data $\alpha = (\varphi, B, \mathcal{F}, \mathcal{C})$ as in (3.1). While the integrable Hamiltonian systems constructed in Proposition 3.3 have the advantage of being most natural and their geometry is completely transparant at all levels, they are not totally explicit; for example the phase space $\operatorname{Sym}^d \Gamma_\alpha$ is defined as the quotient $\Gamma_\alpha^{(d)}/S_d$ and has a quite complicated structure (many generators and relations). Moreover no classical integrable system is of this form, while many turn out to be birational to such systems; the ones we will construct now are birational to the ones of Section 3, they are totally explicit and turn out to be closely related to (i.e., isomorphic to, or isomorphic to a quotient of) most known examples of (finite-dimensional) integrable Hamiltonian systems. We first treat the general case and then show how the integrable Hamiltonian system introduced in [Van2] are obtained as a special case of it.

4.1. The general case

Consider the following affine variety

$$M_\alpha = \left\{ (u(\lambda), v(\lambda), b) \in \mathbf{C}^{2d} \times B \mid \mathcal{F}(\lambda, v(\lambda), b) \bmod u(\lambda) = 0 \right\},$$

($\mathbf{C}^{2d}$ is viewed here as the space of pairs of polynomials as in (1.2)) and denote the natural projection map on B by p_α. There is a natural map

$$\chi \colon M_\alpha \to \operatorname{Sym}^d \Gamma_\alpha$$

which is defined by

$$(u(\lambda), v(\lambda), b) \mapsto ((x_1, v(x_1)), \ldots, (x_d, v(x_d)), b),$$

where x_i are the (not necessarily distinct) roots of $u(\lambda)$, i.e., $u(\lambda) = \prod_{i=1}^d (\lambda - x_i)$. Remark that this map is a (well-defined) morphism since the coordinate ring of $\operatorname{Sym}^d \Gamma_\alpha$ contains all functions which are symmetric in (x_i, y_i); moreover a point of $\mathcal{F}(\lambda, v(\lambda), b) = 0 \bmod u(\lambda)$ is clearly mapped into a point for which $\mathcal{F}(x_i, y_i, b) = 0$, $(i = 1, \ldots, d)$ since each x_i is a root of $u(\lambda)$. The morphism χ is actually a birational isomorphism with inverse

$$((x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d)) \mapsto (u(\lambda), v(\lambda)) = \left(\prod_{i=1}^d (\lambda - x_i), \sum_{i=1}^d y_i \prod_{j \neq i} \frac{\lambda - x_j}{x_i - x_j} \right). \tag{4.1}$$

which is defined away from Δ/S_d where Δ denotes the diagonal

$$\Delta = \{ ((x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d)) \mid x_i = x_j \text{ for some } i \neq j \}.$$

Having a birational map we can transfer the Poisson structure $\{\cdot, \cdot\}_\alpha$ on $\operatorname{Sym}^d \Gamma_\alpha \setminus \mathcal{D}_\alpha$ to the complement of a divisor in M_α. Recalling from Section 3 that the Poisson structure on $\operatorname{Sym}^d \Gamma_\alpha \setminus \mathcal{D}_\alpha$ was completely determined by the brackets

$$\{x_i, x_j\}_\alpha, \ \{y_i, y_j\}_\alpha \ \text{and} \ \{x_i, y_j\}_\alpha$$

upon using the relations which define the phase space and the elements of $\mathcal{C}$, we see that the corresponding Poisson structure (i.e., the one which makes χ into a Poisson morphism) on M_α is completely determined by the brackets

$$\{u_i, u_j\}_\alpha, \ \{v_i, v_j\}_\alpha \text{ and } \{u_i, v_j\}_\alpha$$

(we use the same notation for the Poisson structure on both spaces), upon using the relations which define M_α and the elements of $\mathcal{C}$. This is a more economical way to determine the Poisson structure on M_α than by using the birational map since, as we will see, the divisor $\mathcal{E}_\alpha$ to be removed might be smaller than expected when using the birational map (see Paragraph 4.2); moreover, transferring the integrable algebra via a birational map is very delicate (one may lose completeness).

Proposition 4.1 *There is a Poisson structure $\{\cdot,\cdot\}_\alpha$ on $M_\alpha \backslash \mathcal{E}_\alpha$ (where $\mathcal{E}_\alpha$ is some divisor) which makes χ into a Poisson morphism and $(M_\alpha \setminus \mathcal{E}_\alpha, \{\cdot,\cdot\}_\alpha, p_\alpha^* \mathcal{O}(B))$ is an integrable Hamiltonian system.*

Proof

Let us compute the brackets $\{u_i, u_j\}_\alpha$, $\{v_i, v_j\}_\alpha$ and $\{u_i, v_j\}_\alpha$ which make χ into a Poisson morphism. Clearly $\{u_i, u_j\}_\alpha = 0$. If $1 \le j \le d$, then

$$\begin{aligned}
\{u_j, v(\lambda)\}_\alpha &= (-1)^j \left\{ \sum_{i_1<i_2<\cdots<i_j} x_{i_1} x_{i_2} \cdots x_{i_j}, \sum_{l=1}^d y_l \prod_{k\neq l} \frac{\lambda - x_k}{x_l - x_k} \right\}_\alpha, \\
&= (-1)^j \sum_{i_1<i_2<\cdots<i_j} \sum_{l=1}^d \{x_{i_1} x_{i_2} \cdots x_{i_j}, y_l\}_\alpha \prod_{k\neq l} \frac{\lambda - x_k}{x_l - x_k}, \\
&= (-1)^{j-1} \sum_{i_1<i_2<\cdots<i_j} \sum_{t=1}^j x_{i_1} x_{i_2} \cdots \widehat{x_{i_t}} \cdots x_{i_j} \varphi(x_{i_t}, y_{i_t}) \prod_{k\neq i_t} \frac{\lambda - x_k}{x_{i_t} - x_k}, \\
&= (-1)^{j-1} \sum_{l \notin \{i_1<i_2<\cdots<i_{j-1}\}} x_{i_1} x_{i_2} \cdots x_{i_{j-1}} \varphi(x_l, y_l) \prod_{k\neq l} \frac{\lambda - x_k}{x_l - x_k}, \\
&= (-1)^{j-1} \sum_{l=1}^d \varphi(x_l, y_l) \prod_{k\neq l} \frac{\lambda - x_k}{x_l - x_k} (-1)^{j-1} \sum_{m=0}^{j-1} x_l^m u_{j-m-1}, \\
&= \sum_{l=1}^d \sum_{m=0}^{j-1} x_l^m u_{j-m-1} \varphi(x_l, y_l) \prod_{k\neq l} \frac{\lambda - x_k}{x_l - x_k}.
\end{aligned}$$

Substituting $\lambda = x_l$ in the right hand side one sees that $\{u_j, v(\lambda)\}_\alpha$ is the (unique) polynomial in λ of degree less than d, which takes at $\lambda = x_l$ the value $\sum_{m=0}^{j-1} x_l^m u_{j-m-1}\varphi(x_l, v(x_l))$, for $l = 1, \ldots, d$. As the x_l are the zeros of $u(\lambda)$ and since $y_l = v(x_l)$ the same is true for $\sum_{m=0}^{j-1} \lambda^m u_{j-m-1}\varphi(\lambda, v(\lambda)) \bmod u(\lambda)$, and we find

$$\begin{aligned} \{u_j, v(\lambda)\}_\alpha &= \sum_{m=0}^{j-1} \lambda^m u_{j-m-1}\varphi(\lambda, v(\lambda)) \bmod u(\lambda), \\ &= \varphi(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \bmod u(\lambda). \end{aligned} \tag{4.2}$$

By a similar (but simpler) computation it follows that $\{v_i, v_j\}_\alpha = 0$.

Since χ is a Poisson morphism, $p_\alpha^*\mathcal{O}(B)$ is an involutive algebra. Since p_α is surjective,

$$\dim p_\alpha^*\mathcal{O}(B) = \dim B = \frac{1}{2}(\dim B + d + \dim B - d) = \frac{1}{2}(\dim M_\alpha + \dim \mathcal{C}).$$

To see that $p_\alpha^*\mathcal{O}(B)$ is complete, remark that χ maps every fiber of p_α to a fiber of π_α, i.e.,

$$\begin{tikzcd} M_\alpha \arrow[rr, "\chi"] \arrow[dr, "p_\alpha"'] & & \mathrm{Sym}^d\,\Gamma_\alpha \arrow[dl, "\pi_\alpha"] \\ & B & \end{tikzcd}$$

is commutative. Since χ is regular with rational inverse it restricts to a birational map on each fiber of the moment map p_α so all fibers of p_α have the same dimension (since the ones of π_α do) and the general fiber of p_α is irreducible (same reason). It follows from Proposition 2.6 that $p_\alpha^*\mathcal{O}(B)$ is complete, hence integrable.

The integrable vector fields $\{\cdot, \beta_k\}_\alpha$ are computed as in Section 3 by using the defining relations $\mathcal{F}(\lambda, v(\lambda), b) = 0 \bmod u(\lambda)$ and $\mathcal{C}$: since these relations were sufficient to determine the brackets on $\mathrm{Sym}^d\,\Gamma_\alpha$ they are also sufficient here. The (minimal) divisor on which they fail to be regular is denoted by $\mathcal{E}_\alpha$ (it may be empty, see Paragraph 4.2). Thus we have shown that $(M_\alpha \setminus \mathcal{E}_\alpha, \{\cdot,\cdot\}_\alpha, p_\alpha^*\mathcal{O}(B))$ is an integrable Hamiltonian system. The symmetries X and Y which we discussed in the preceding section can be transferred to the space M_α but we will not discuss this here. ∎

4.2. The easiest case

We now show how the integrable Hamiltonian system introduced in [Van2] (Section 2) are a special case of the systems of Paragraph 4. We start from the following data

$$\begin{aligned} &1)\ \text{an integer } d \geq 1, \\ &2)\ \text{a polynomial } \varphi(x,y) \in \mathbf{C}[x,y] \setminus \{0\}, \\ &3)\ \text{a polynomial } F(x,y) \in \mathbf{C}[x,y] \setminus \mathbf{C}[x], \end{aligned} \tag{4.3}$$

and associate to it corresponding data of the form (3.1), namely we choose

$$\begin{aligned} &1)\ B = \mathbf{C}[\beta_1, \ldots, \beta_d] \cong \mathbf{C}^d, \\ &2)\ \varphi(x,y) \text{ as above}, \\ &3)\ \mathcal{F}(x,y,b) = F(x,y) - (b_1 x^{d-1} + \cdots + b_{d-1}x + b_d), \\ &4)\ \mathcal{C} = \mathbf{C}, \end{aligned} \tag{4.4}$$

(in item 3 $b_i = \beta_i(b)$ for $b \in B$). The main observation to be made here is that with this choice M_α is isomorphic to $\mathbf{C}^{2d}$ (and $\mathcal{E}_\alpha$ is empty).

Lemma 4.2 *The projection map $\mathbf{C}^{2d} \times B \to \mathbf{C}^{2d}$ restricts to an isomorphism $M_\alpha \to \mathbf{C}^{2d}$.*

Proof

For $\mathcal{F}$ as above we have that

$$M_\alpha = \{(u(\lambda), v(\lambda), b) \mid \mathcal{F}(\lambda, v(\lambda), b) \bmod u(\lambda) = b_1\lambda^{d-1} + \cdots + b_{d-1}\lambda + b_d)\}.$$

Since $u(\lambda)$ is monic, $\mathcal{F}(\lambda, v(\lambda), b) \bmod u(\lambda)$ is a polynomial in u_i, v_i (and λ), hence the map $(u(\lambda), v(\lambda), b) \to (u(\lambda), v(\lambda))$ has a regular inverse. ∎

Thus in the present case the phase space is just $\mathbf{C}^{2d}$, the Poisson structure is given by $\{u_i, u_j\}_\alpha = \{v_i, v_j\}_\alpha$ and the brackets (4.2). The involutive algebra on $\mathbf{C}^{2d}$ is the polynomial algebra $\mathcal{A}_{F,d}$ generated by the d coefficients of

$$F(\lambda, v(\lambda)) \bmod u(\lambda).$$

In conclusion, for any data (4.3), $(\mathbf{C}^{2d}, \{\cdot\,,\cdot\}_\alpha, \mathcal{A}_{F,d})$ is an integrable Hamiltonian system and it coincides with the ones introduced in [Van2] (Section 2).

5. Integrable bi-Hamiltonian systems associated to families of curves

The above constructions lead at once to the construction of many bi-Hamiltonian systems, namely we will associate one to the following data

$$\begin{aligned}
&1)\ \text{a polynomial } \varphi(x,y) \in \mathbf{C}[x,y] \setminus \{0\},\\
&2)\ \text{an affine space } B = \mathbf{C}^d,\ (d \geq 1),\\
&3)\ \text{a closed immersion } \mathcal{F}: B \to \mathbf{C}[x,y],\\
&4)\ \text{a general polynomial } \psi \in \mathbf{C}[x,y].
\end{aligned} \tag{5.1}$$

The meaning of ψ being *general* will be explained in Paragraph 5.1 below; a special case of interest (in which $\psi(x,y)$ fails to be general in that sense) will be discussed in the second paragraph. Throughout this section the above data are fixed and we denote $\delta = (\varphi, B, \mathcal{F}, \psi)$. A subalgebra $\mathcal{C}$ of $\mathcal{O}(B)$, specifying the Casimirs, could also be chosen, but we take it to be trivial (i.e., $\mathcal{C} = \mathbf{C}$) for the simplicity of exposition and notation. Our construction is done at the level of $\Gamma_\delta^{(d)}$, for $\mathrm{Sym}^d\,\Gamma_\delta$ and M_δ the corresponding construction follows from it at once (e.g., in the introduction we formulated the result for M_δ). We will also restrict ourselves here to bi-Hamiltonian structures, the construction of multi-Hamiltonian structures (i.e., the case of several instead of just two Poisson brackets) also follows from it at once.

5.1. The general case

Using $\mathcal{F}$ we construct the following morphism,

$$\begin{aligned}
\mathcal{F}': B \times B &\to \mathbf{C}[x,y]\\
(b^1, b^2) &\mapsto \mathcal{F}(x, y, b^1 + \psi(x,y) b^2),
\end{aligned} \tag{5.2}$$

where we view $\mathcal{F}(x,y,b)$ as an element of $\mathcal{O}(B)[x,y]$ (i.e., for given b as an element of $\mathbf{C}[x,y]$) as before. Since $\mathcal{F}$ is a closed immersion, $\mathcal{F}'$ will also be a closed immersion for general (i.e., most) $\psi(x,y)$; this is the case treated in this paragraph, we call such a $\psi(x,y)$ simply *general*. From $\mathcal{F}'$ we construct as in (3.2) the space $\Gamma_\delta^{(d)}$ by

$$\Gamma_\delta^{(d)} = \left\{((x_1,y_1),\ldots,(x_d,y_d),b^1,b^2) \in \left(\mathbf{C}^2\right)^d \times B \times B \mid \mathcal{F}(x_i, y_i, b^1 + \psi(x_i,y_i) b^2) = 0\right\},$$

which admits two projections onto B, which we denote by π_1 and π_2. From these we construct two data of the type (3.1):

$$\begin{aligned}
\alpha_1 &= (\varphi, B \times B, \mathcal{F}', \pi_2^* \mathcal{O}(B)),\\
\alpha_2 &= (\varphi\psi, B \times B, \mathcal{F}', \pi_1^* \mathcal{O}(B)).
\end{aligned}$$

They lead to two different integrable Hamiltonian systems. Remark that their phase space is the same[5] but they have different Poisson structures, in particular they have different algebras of Casimirs. However, they bare the following relation.

[5] The divisor to be removed from $\Gamma_\delta^{(d)}$ may be different for the two Poisson structures; to have a common space we can e.g. remove their sum. We denote the divisor which we remove by D_δ.

Proposition 5.1 *The integrable Hamiltonian systems*

$$\left(\Gamma_\delta^{(d)} \setminus D_\delta, \{\cdot\,,\cdot\}_{\alpha_i}, \pi_1^*\mathcal{O}(B)\otimes\pi_2^*\mathcal{O}(B)\right) \qquad (i=1,2) \tag{5.3}$$

have many integrable vector fields in common, namely for any linear function β on $B=\mathbf{C}^d$,

$$\{\cdot\,,\pi_1^*\beta\}_{\alpha_1} = \{\cdot\,,\pi_2^*\beta\}_{\alpha_2}.$$

Proof

Recall from (3.9) that the Hamiltonian vector fields $\{\beta_i,\cdot\}_\alpha$ were determined completely by

$$M_\alpha \{\beta,x\}_\alpha = -\mathcal{N}_\alpha,$$

and a similar equation for $\{\beta,y\}_\alpha$; we have introduced here a matrix

$$(\{\beta,x\}_\alpha)_{ij} = \{\beta_i,x_j\}_\alpha\,.$$

The corresponding equation for the integrable vector fields of α_1 is given by

$$\begin{pmatrix} \frac{\partial\mathcal{F}'}{\partial\beta^1} & \frac{\partial\mathcal{F}'}{\partial\beta^2} \\ \frac{\partial}{\partial\beta^1}(\pi_2^*\gamma) & \frac{\partial}{\partial\beta^2}(\pi_2^*\gamma) \end{pmatrix} \begin{pmatrix} \{\beta^1,x\}_{\alpha_1} \\ \{\beta^2,x\}_{\alpha_1} \end{pmatrix} = -\begin{pmatrix} \frac{\partial\mathcal{F}'}{\partial y}\varphi \\ 0 \end{pmatrix}, \tag{5.4}$$

while the ones for α_2 are given by

$$\begin{pmatrix} \frac{\partial\mathcal{F}'}{\partial\beta^1} & \frac{\partial\mathcal{F}'}{\partial\beta^2} \\ \frac{\partial}{\partial\beta^1}(\pi_1^*\gamma) & \frac{\partial}{\partial\beta^2}(\pi_1^*\gamma) \end{pmatrix} \begin{pmatrix} \{\beta^1,x\}_{\alpha_2} \\ \{\beta^2,x\}_{\alpha_2} \end{pmatrix} = -\begin{pmatrix} \frac{\partial\mathcal{F}'}{\partial y}\varphi\psi \\ 0 \end{pmatrix}. \tag{5.5}$$

Our notation is the same as the one we used in (3.9), except that we denote here by β^1 (resp. β^2) the generators which come from $\mathcal{O}(B)$ using π_1 (resp. π_2). Since the elements of $\pi_2^*\mathcal{O}(B)$ (resp. $\pi_1^*\mathcal{O}(B)$) are Casimirs for $\{\cdot\,,\cdot\}_{\alpha_1}$ (resp. $\{\cdot\,,\cdot\}_{\alpha_2}$) and

$$\frac{\partial}{\partial\beta^1}(\pi_2^*\gamma) = \frac{\partial}{\partial\beta^2}(\pi_1^*\gamma) = 0,$$

(5.4) and (5.5) are equivalent to

$$\begin{aligned} \frac{\partial\mathcal{F}'}{\partial\beta^1}\{\beta^1,x\}_{\alpha_1} &= -\frac{\partial\mathcal{F}'}{\partial y}\varphi, \\ \frac{\partial\mathcal{F}'}{\partial\beta^2}\{\beta^2,x\}_{\alpha_2} &= -\frac{\partial\mathcal{F}'}{\partial y}\varphi\psi. \end{aligned} \tag{5.6}$$

Now for the last equation we have that

$$\frac{\partial\mathcal{F}'}{\partial\beta_j^2}(x_i,y_i,b^1+\psi(x_i,y_i)b^2) = \psi(x_i,y_i)\frac{\partial\mathcal{F}'}{\partial\beta_j^1}(x_i,y_i,b^1+\psi(x_i,y_i)b^2),$$

so that the last equation of (5.6) can be written as

$$\frac{\partial \mathcal{F}'}{\partial \beta^1}\{\beta^2, x\}_{\alpha_2} = -\frac{\partial \mathcal{F}'}{\partial y}\varphi,$$

and we arrive at the conclusion

$$\{\beta^1, x\}_{\alpha_1} = \{\beta^2, x\}_{\alpha_2}.$$

Since the same holds for the brackets with y, we have shown that

$$\{\beta^1, \cdot\}_{\alpha_1} = \{\beta^2, \cdot\}_{\alpha_2},$$

i.e., the integrable vector fields corresponding to linear functions of B agree. ∎

If we denote the vector field $\{\cdot\,, \beta_j^1\}_{\alpha_1} = \{\cdot\,, \beta_j^2\}_{\alpha_2}$ by X_j $(j = 1, \ldots, d)$ then we arrive at d bi-Hamiltonian hierarchies, which we depict in the following way.

$$\begin{tikzcd}
 & \beta_j^1 \arrow[dl, "2"'] \arrow[dr, "1"] & & \beta_j^2 \arrow[dl, "2"'] \arrow[dr, "1"] & \\
0 & & X_j & & 0
\end{tikzcd}$$

5.2. A special case

We next consider a case in which $\mathcal{F}'$, defined in 5.2 is not a closed immersion but still leads to bi-Hamiltonian vector fields. For a particular immersion $\mathcal{F}$ and $\psi(x, y) = x$ these were first described by us in [Van1] and later generalized in [Bue] for arbitrary $\mathcal{F}$ (still taking $\psi(x, y) = x$).

We fix some basis of $\mathbf{C}^d$ and we suppose that $\mathcal{F}$ is of the form

$$\mathcal{F}(x, y, b) = F(x, y) - (b_1 + \psi b_2 + \psi^2 b_3 + \cdots + \psi^{d-1} b_d),$$

where $b = (b_1, \ldots, b_d)$ with respect to the chosen basis. Then $\mathcal{F}'$ leads to the family of curves

$$F(x, y) - (b_1 + \psi b_2 + \cdots + \psi^d b_{d+1}),$$

leading to an affine variety $\Gamma_\delta^{(d)}$ as before. The two projection maps $\Gamma_\delta^{(d)} \to B$ are now given by

$$\begin{aligned}
\pi_1(x_1, \ldots, y_d, b_1, \ldots, b_{d+1}) &= (b_1, \ldots, b_d),\\
\pi_2(x_1, \ldots, y_d, b_1, \ldots, b_{d+1}) &= (b_2, \ldots, b_{d+1}),
\end{aligned}$$

and we have that $\beta_{i+1}^1 = \beta_i^2$, i.e., $\pi_1^* \beta_{i+1} = \pi_2^* \beta_i$, $(i = 1, \ldots, d-1)$, where $(\beta_1, \ldots, \beta_d)$ denotes the basis dual to the chosen basis. It follows that in this case Proposition 5.1 is still valid; there is now however only one bi-Hamiltonian hierarchy and it takes the following form.

$$\begin{tikzcd}
 & \beta_1^1 \arrow[dl, "2"'] \arrow[dr, "1"] & & \beta_2^1 = \beta_1^2 \arrow[dl, "2"'] \arrow[dr, "1"] & & \cdots \arrow[dl, "2"'] \arrow[dr, "1"] & & \beta_d^2 \arrow[dl, "2"'] \arrow[dr, "1"] & \\
0 & & X_1 & & X_2 & \cdots & X_d & & 0
\end{tikzcd}$$

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