INTEGRABLE REDUCTIONS OF THE DRESSING CHAIN

C. A. EVRIPIDOU, P. KASSOTAKIS, AND P. VANHAECKE

Abstract. In this paper we construct a family of integrable reductions of the dressing chain, described in its Lotka-Volterra form. For each $k, n \in \mathbb{N}$ with $n \geqslant 2k+1$ we obtain a Lotka-Volterra system $\mathrm{LV}_b(n,k)$ on $\mathbb{R}^n$ which is a deformation of the Lotka-Volterra system $\mathrm{LV}(n,k)$, which is itself an integrable reduction of the m-dimensional Bogoyavlenskij-Itoh system $\mathrm{LV}(2m+1,m)$, where $m = n-k-1$. We prove that $\mathrm{LV}_b(n,k)$ is both Liouville and non-commutative integrable, with rational first integrals which are deformations of the rational integrals of $\mathrm{LV}(n,k)$. We also construct a family of discretizations of $\mathrm{LV}_b(n,0)$, including its Kahan discretization, and we show that these discretizations are also Liouville and superintegrable.

Contents

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1. Introduction

The dressing chain is an integrable Hamiltonian system, which was constructed in [20] as a fixed point of compositions of Darboux transformations of the Schrödinger operator. It was shown in [7] that after a simple linear transformation it becomes a Lotka-Volterra system, which is a deformation of the Bogoyavlenskij-Itoh system [11, 4]. For the integrable reductions of the dressing chain which we will study here, the Lotka-Volterra formulation is the most convenient; also, we will use many results from [18, 13, 6, 7], which are all written in that formulation.

For integers n and k, satisfying $n \geqslant 2k+1$, the Hamiltonian system $\mathrm{LV}(n,k)$ has as its phase space $\mathbb{R}^n$, which we equip with its natural coordinates $x_1,\dots,x_n$. It has as Hamiltonian H the sum of these coordinates, $H := x_1+\cdots+x_n$ and as Poisson structure a quadratic Poisson structure, with brackets

$$\{x_i,x_j\} := A_{i,j}^{(n,k)}x_ix_j\ , \quad \text{where} \quad A_{i,j}^{(n,k)} = \begin{cases} +1 \text{ if } i+n>j+k\ , \\ -1 \text{ if } i+n\leqslant j+k\ , \end{cases}$$

when $1\leqslant i<j\leqslant n$. The Hamiltonian vector field $\mathcal{X}_H$ has the form

$$\dot{x}_i = \sum_{j=1}^n A_{i,j}^{(n,k)}x_ix_j\ , \qquad 1\leqslant i\leqslant n\ .$$

These systems were introduced in [6], where we also established their Liouville and non-commutative integrability (see Definition 4.1), with rational first integrals. For $n=2k+1$ one recovers the Bogoyavlenskij-Itoh system whose deformation, which is the dressing chain, was constructed in [7]. Most importantly, the systems for which $n>2k+1$ can be obtained by reduction from a Bogoyavlenskij-Itoh system $\mathrm{LV}(2m+1,m)$, with $m:=n-k-1$. The same reduction can be applied to the deformed system $\mathrm{LV}_b(2m+1,m)$, leading to a Hamiltonian system, which we will denote by $\mathrm{LV}_b(n,k)$. The Hamiltonian vector field $\mathcal{X}_H$ now has the form

$$\dot{x}_i = \sum_{j=1}^n \left(A_{i,j}^{(n,k)}x_ix_j + B_{i,j}^{(n,k)}\right)\ , \qquad 1\leqslant i\leqslant n\ .$$

where all entries $b_{i,j}$ of the skew-symmetric matrix $B^{(n,k)}$, satisfying $|i-j|\notin\{m,m+1\}$, are zero and the other entries are arbitrary parameters. Setting in this system all deformation parameters equal to zero, one recovers $\mathrm{LV}(n,k)$. A natural question, studied here, is the integrability of $\mathrm{LV}_b(n,k)$ for all n and k. For $\mathrm{LV}_b(2k+1,k)$ the answer is known [20, 7]: $\mathrm{LV}_b(2k+1,k)$ is Liouville integrable with polynomial first integrals which are deformations of the first integrals of $\mathrm{LV}(2k+1,k)$.

The main result of this paper is that $\mathrm{LV}_b(n,k)$ is on the one hand Liouville integrable, with rational integrals which are deformations of the integrals of $\mathrm{LV}(n,k)$, and is on the other hand non-commutative integrable, with such first integrals. See Theorem 3.4 for the case of $(n,0)$ and Theorem 4.9 for the case of (n,k) with $k>0$. In order to establish these results, we need to construct the deformed integrals and show that they have the desired involutivity properties; independence is in fact quite automatic and is proven by a simple deformation argument.

Surprizingly, the construction of the deformed first integrals from the undeformed ones is very simple, and is the same for all integrals of $\mathrm{LV}(n,k)$ that we constructed in [6]: from such a first integral F of $\mathrm{LV}(n,k)$ we obtain a first integral F^b of $\mathrm{LV}_b(n,k)$ by setting $F^b := e^{\mathcal{D}_b}F = F+\mathcal{D}_bF+\frac{\mathcal{D}_b^2}{2}F+\cdots$, where

$$\mathcal{D}_b = \sum_{1\leqslant i\leqslant k+1} b_{i,i+m}\frac{\partial^2}{\partial x_i\partial x_{i+m}} - \sum_{1\leqslant i\leqslant k} b_{i,i+m+1}\frac{\partial^2}{\partial x_i \partial x_{i+m+1}}\ ,$$

and where the indices of b and x are taken modulo $2m+1$.

The proof that we get in this way first integrals and that they are in involution when the undeformed integrals are in involution needs however extra work, as it does not follow directly from the definition. In the case of $\mathrm{LV}_b(n,0)$, studied in Section 3, there is only one deformation parameter $\beta = b_{1,n}$ and the action of $e^{\mathcal{D}_b}$ on the rational integrals of $\mathrm{LV}(n,0)$ which were constructed in [18] can be equivalently described as the pullback of a birational map, which we introduce. Moreover, we show that this map is a Poisson map between the deformed and undeformed systems (Proposition 3.2). This yields the integrability results for $\mathrm{LV}_b(n,0)$, since apart from the Hamiltonian, all given first integrals are rational; the fact that these rational integrals are in involution with the Hamiltonian, i.e., are first integrals, can in this case be shown by direct computation (Proposition 3.1).

When $k>0$ the above idea can also be used, but some care has to be taken because there are now $2k+1$ deformation parameters, and they can be added one by one, upon decomposing $\mathcal{D}_b = \sum_{p=1}^{2k+1}\mathcal{D}_{(p)}$, but in order to be able to view at each step the action of $e^{\mathcal{D}_{(p)}}$ on the rational integrals, as the pullback by some Poisson map, one has to add the parameters in a very specific order. The reason for this is that in this process the form of the integrals at each step is very important. With this, one gets that the deformed rational integrals of $\mathrm{LV}_b(n,k)$ are in involution (second part of Theorem 4.2). This system has $k+1$ independent polynomial integrals, which are by construction in involution, because they are restrictions to a Poisson submanifold of the involutive integrals of $\mathrm{LV}(2m+1,m)$, but they also have to be shown to be in involution with the rational first integrals. This is again done using the above Poisson maps, but since these do not produce the deformed polynomial integrals, some extra arguments which are again very much dependent on the particular structure of the integrals, are needed. In the end, this proves Theorem 4.9 which says that the deformed systems $\mathrm{LV}_b(n,k)$ are both Liouville and non-commutatively integrable.

A discrete system associated with the dressing chain was first considered in [2]. As was shown in [2], this discrete system preserves the Poisson structure as well as the integrals of the underlying continuous system. The discretization of the dressing chain was rediscovered in the context of the Painlevé equations in [17], using the Lotka-Volterra formulation of the dressing chain. We will construct in the final section of this paper a class of discretizations of the $\mathrm{LV}_b(n,0)$ system, including the Kahan discretization. We prove that the discrete maps which we obtain are birational Poisson maps and that they preserve the Hamiltonian as well as the rational first integrals of the (continuous) $\mathrm{LV}_b(n,0)$ system. These discretizations are therefore both Liouville integrable and superintegrable. It would be interesting to study the integrability properties of $\mathrm{LV}_b(n,k)$, with $n>2k+1$ and $k>0$.

The structure of the paper is as follows. We construct in Section 2 the systems $\mathrm{LV}_b(n,k)$ as (Poisson) reductions of the systems $\mathrm{LV}_b(2m+1,m)$, where $m:=n-k-1$, and we show that the inherited Poisson structure $\Pi_b^{(n,k)}$ is a deformation of the Poisson structure $\Pi^{(n,k)}$ of $\mathrm{LV}(n,k)$. In Section 3 we construct rational first integrals of $\mathrm{LV}_b(n,0)$ as deformations of the first integrals of $\mathrm{LV}(n,0)$, which were constructed in [18]. We show by using a Poisson map, which we also construct, that half of these first integrals are in involution, establishing both the Liouville and superintegrability of $\mathrm{LV}_b(n,0)$. We also give explicit solutions for this system. In Section 4 we treat the more complicated case of $k>0$, where we prove again Liouville integrability, and also non-commutative integrability, a notion which is intermediate between Liouville and superintegrability. In this case we use $2k+1$ Poisson maps, which are composed in a very specific order to obtain the results. In Section 5 we construct a family of discrete maps for $\mathrm{LV}_b(n,0)$ as compatibility conditions for a linear system, associated to the Lax operator of $\mathrm{LV}_b(n,0)$, and show their Liouville and superintegrability. We show that the Kahan discretizations of $\mathrm{LV}_b(n,0)$ is a particular case and deduce from this the Liouville and superintegrability of the Kahan discretization.

2. The Hamiltonian systems $\mathrm{LV}_b(n,k)$

In this section, we construct the polynomial Hamiltonian systems $\mathrm{LV}_b(n,k)$. Recall that n and k stand for two arbitrary integers satisfying $n\geqslant 2k+1$. We construct them as reductions of the deformed Bogoyavlenskij-Itoh systems, which we introduced in [7]; in the notation of the present paper, the latter systems are the systems $\mathrm{LV}_b(2m+1,m)$, where $m:=n-k-1$.

2.1. **The deformed Bogoyavlenskij-Itoh systems.**

We first recall the Bogoyavlenskij-Itoh systems $\mathrm{LV}(2m+1,m)$, which have first been introduced by O. Bogoyavlenskij [3, 4] and Y. Itoh [11], and their deformations $\mathrm{LV}_b(2m+1,m)$, which we constructed in [7]. In both cases, the phase space of the system is $\mathbb{R}^{2m+1}$, which is equipped with its natural coordinates $x_1, \dots, x_{2m+1}$. Since many formulas are invariant under a cyclic permutation of these coordinates, we view the indices as being taken modulo $2m+1$, i.e., we set $x_{2m+\ell+1} = x_\ell$ for all $\ell \in \mathbb{Z}$. The Poisson structure Π^m of $\mathrm{LV}(2m+1, m)$ is constructed from the skew-symmetric Toeplitz matrix[1] A^m whose first row is given by

$$(0, \underbrace{1, 1, \dots, 1}_{m}, \underbrace{-1, -1, \dots, -1}_{m}) \; .$$

It leads to a quadratic Poisson structure Π^m on $\mathbb{R}^{2m+1}$, upon defining the Poisson brackets

$$\{x_i, x_j\}^m := A^m_{i,j} x_i x_j \; , \qquad 1 \leqslant i, j \leqslant 2m+1 \; .$$

As Hamiltonian we take $H := x_1 + x_2 + \cdots + x_{2m+1}$, the sum of all coordinates. The corresponding Hamiltonian system is given by

$$\dot{x}_i = x_i \sum_{j=1}^{m} (x_{i+j} - x_{i-j}) \; , \qquad 1 \leqslant i \leqslant 2m+1 \; . \tag{2.1}$$

It is called the *Bogoyavlenskij-Itoh system* (of order m), and is denoted by $\mathrm{LV}(2m+1, m)$. Given any real skew-symmetric matrix B^m of size $2m+1$, define

$$\{x_i, x_j\}^m_b := A^m_{i,j} x_i x_j + B^m_{i,j} \; , \qquad 1 \leqslant i, j \leqslant 2m+1 \; . \tag{2.2}$$

These brackets define a Poisson structure, denoted Π^m_b, if and only if all uppertriangular entries $b_{i,j} := B^m_{i,j}$ of B^m, with $j - i \notin \{m, m+1\}$ are zero (see [7, Prop. 3]). Under this condition on B^m, we can consider the Hamiltonian system on $\mathbb{R}^{2m+1}$ with the same Hamiltonian H and Poisson structure Π^m_b. It is given by

$$\dot{x}_i = x_i \sum_{j=1}^{m} (x_{i+j} - x_{i-j}) + b_{i,i+m} - b_{i-m,i} \; , \qquad 1 \leqslant i \leqslant 2m+1 \; . \tag{2.3}$$

It is called the *deformed Bogoyavlenskij-Itoh system* (of order m) and is denoted by $\mathrm{LV}_b(2m+1, m)$. It is clear that setting all parameters $b_{i,j}$ equal to zero, one recovers $\mathrm{LV}(2m+1, m)$. A Lax equation (with spectral parameter) for (2.3) is given by

$$(X + \lambda^{-1}\Delta + \lambda M)^{\cdot} = [X + \lambda^{-1}\Delta + \lambda M, D - \lambda M^{m+1}] \; , \tag{2.4}$$

where for $1 \leqslant i, j \leqslant 2m+1$ the (i,j)-th entry of the matrices X and M, and of the diagonal matrices Δ and D, is given by

$$\begin{gathered} X_{i,j} := \delta_{i,j+m} x_i \; , \quad \Delta_{i,j} := b_{i+m,j} \delta_{i,j} \; , \quad M_{i,j} := \delta_{i+1,j} \; , \\ D_{i,j} := -\delta_{i,j} (x_i + x_{i+1} + \cdots + x_{i+m}) \; . \end{gathered}$$

[1]Later on, the matrix A^m (and similarly the matrix B^m and the Poisson structure Π^m), will have two superscripts; in that notation, A^m is written as $A^{(2m+1,m)}$, and similarly for B^m and Π^m.

It generalizes Bogoyavlenskij's Lax equation, which can be recovered from it by putting all $b_{i,j}$ equal to zero, i.e., by setting $\Delta = 0$.

2.2. **The reduced systems.** The systems $\mathrm{LV}_b(n,k)$, with $n > 2k+1$, are obtained by reduction from $\mathrm{LV}_b(2m+1,m)$, where $m := n-k-1 > k$. Consider the submanifold N_n of $\mathbb{R}^{2m+1}$, defined by $x_{n+1} = x_{n+2} = \cdots = x_{2m+1} = 0$. It is a linear space of dimension n which we identify with $\mathbb{R}^n$ and on which we take the restrictions of $x_1, x_2, \dots, x_n$ as coordinates (without changing the notation).

Proposition 2.1. *If the entries of the skew-symmetric matrix B satisfy $b_{i,j} = 0$ whenever $n+1 \leqslant i \leqslant 2m+1$, then N_n is a Poisson submanifold of $\left(\mathbb{R}^{2m+1}, \Pi_b^m\right)$.*

Proof. We need to show that all Hamiltonian vector fields $\mathcal{X}_F := \{\cdot\,, F\}_b^m$, where F is an arbitrary function on $\mathbb{R}^{2m+1}$, are tangent to N_n at all points of N_n. This is equivalent to the vanishing of $\mathcal{X}_F[x_i] = \{x_i, F\}_b^m$ at all points of N_n, for any i with $n+1 \leqslant i \leqslant 2m+1$. Since for such i, we have $b_{i,j} = 0$ for all j, and by the derivation property of the Poisson bracket,

$$\mathcal{X}_F[x_i] = \{x_i, F\}_b^m = \sum_{j=1}^{2m+1} \frac{\partial F}{\partial x_j} \left(A_{i,j}^m x_i x_j + b_{i,j}\right) = x_i \sum_{j=1}^{2m+1} \frac{\partial F}{\partial x_j} A_{i,j}^m x_j \ ,$$

which clearly vanishes on the hyperplane $x_i = 0$, and hence on N_n. $\square$

Since $N_n \simeq \mathbb{R}^n$ is a Poisson submanifold of $(\mathbb{R}^{2m+1}, \Pi_b^m)$, we can restrict Π_b^m (as given by (2.2)) to $\mathbb{R}^n$, giving a Poisson structure $\Pi_b^{(n,k)}$, with associated Poisson bracket

$$\{x_i, x_j\}_b^{(n,k)} := A_{i,j}^{(n,k)} x_i x_j + B_{i,j}^{(n,k)} \ , \qquad 1 \leqslant i, j \leqslant n \ , \tag{2.5}$$

where $A^{(n,k)}$ and $B^{(n,k)}$ denote the $n \times n$ matrices obtained from A^m and B^m by removing its last $2m+1-n$ rows and columns. Said differently, $A^{(n,k)}$ denotes the skew-symmetric $n \times n$ Toeplitz matrix whose first row is given by

$$(0, \underbrace{1, 1, \dots, 1}_{m=n-k-1}, \underbrace{-1, -1, \dots, -1}_{k}) \ .$$

Thus, all uppertriangular entries of the skew-symmetric matrix $A^{(n,k)}$ are ± 1 and $A_{i,j}^{(n,k)} = 1$ if and only if $n + i > k + j$. Also, $B^{(n,k)}$ is the skew-symmetric $n \times n$ matrix whose uppertriangular entries $b_{i,j} := B_{i,j}^{(n,k)}$ with $j - i \notin \{m, m+1\}$ are zero. So when $k > 0$, the first line of $B^{(n,k)}$ is given by

$$(\underbrace{0, 0, \dots, 0}_{m=n-k-1}, b_{1,m+1}, b_{1,m+2}, \underbrace{0, 0, \dots, 0}_{k-1}) \ ,$$

while for $k = 0$ it has the form $(0, 0, \dots, 0, b_{1,n})$. We define $\mathrm{LV}_b(n,k)$ to be the Hamiltonian system with $\Pi_b^{(n,k)}$ as Poisson bracket and $H = x_1 + x_2 + \cdots + x_n$

as Hamiltonian. Explicitly, the Hamiltonian vector field $\mathcal{X}_H$ of $\mathrm{LV}_b(n,k)$ is given by

$$\dot{x}_i = \sum_{j=1}^{n} \left(A_{i,j}^{(n,k)} x_i x_j + B_{i,j}^{(n,k)} \right) , \qquad 1 \leqslant i \leqslant n . \tag{2.6}$$

Setting $B^{(n,k)} = 0$, one recovers the Hamiltonian system $\mathrm{LV}(n,k)$, in particular its Poisson structure $\Pi^{(n,k)}$, which was constructed and studied in [6]. Therefore, the system $\mathrm{LV}_b(n,k)$ is a deformation of the system $\mathrm{LV}(n,k)$.

We show in the following proposition that the above matrices $B^{(n,k)}$ are the only ones for which the brackets given by (2.2) define a Poisson bracket (on $\mathbb{R}^n$).

Proposition 2.2. *Suppose that $B = (b_{i,j})$ is a skew-symmetric $n \times n$ matrix. Then the brackets given by*

$$\{x_i, x_j\}_b := A_{i,j}^{(n,k)} x_i x_j + b_{i,j} , \qquad 1 \leqslant i,j \leqslant n ,$$

define a Poisson bracket Π_b on $\mathbb{R}^n$ if and only if all uppertriangular entries $b_{i,j}$ of B, with $j - i \notin \{m, m+1\}$ are zero. The rank of Π_b is $2\left[\frac{n}{2}\right]$.

Proof. Let us denote by Π the Poisson structure defined by $A^{(n,k)}$, and let us denote the derived Poisson bracket by $\{\cdot\,,\cdot\}$. We know already from what precedes that if all uppertriangular entries $b_{i,j}$ of B, with $j - i \notin \{m, m+1\}$ are zero, then Π_b is the restriction of a Poisson structure to a Poisson submanifold, hence it is a Poisson structure. We therefore only need to show that if one of these entries $b_{i,j}$ with $i < j$ is non-zero, then Π_b is not a Poisson structure. Suppose first that $j - i < m$ and $i \neq 1$. Then

$$\frac{\partial}{\partial x_{i-1}} \left[\{\{x_{i-1}, x_i\}, x_j\}_B + \{\{x_i, x_j\}, x_{i-1}\}_B + \{\{x_j, x_{i-1}\}, x_i\}_B \right]$$
$$= A_{i-1,i}^{(n,k)} b_{i,j} + A_{i-1,j}^{(n,k)} b_{i,j} = 2b_{i,j} \neq 0 ,$$

so that Π_b does not satisfy the Jacobi identity. When $j - i < m$ and $i = 1$ it suffices to replace in the above computation $i - 1$ by $j + 1$ to arrive at the same conclusion. Finally, when $j - i > m + 1$ one replaces in the above computation $i - 1$ by $i + 1$ to arrive again at the same conclusion.

The rank of Π is the rank of $A^{(n,k)}$, which is equal to n when n is even and $n - 1$ when n is odd. Since Π_b is obtained by adding constants to the quadratic structure Π, its rank is at least the rank of $A^{(n,k)}$. However, the rank of Π_b is even and bounded by n, so Π_b and Π have the same rank, which is $2\left[\frac{n}{2}\right]$. $\square$

The proposition implies that from the above reduction process we get all possible deformations of $\mathrm{LV}(n,k)$ obtained by adding to $\Pi^{(n,k)}$ a constant Poisson structure.

3. The Liouville and superintegrability of $\mathrm{LV}_b(n,0)$

We construct in this section enough independent first integrals for the Hamiltonian system $\mathrm{LV}_b(n,0)$ to prove its superintegrability and then select from them enough first integrals in involution to prove its Liouville integrability. Notice that since the phase space of $\mathrm{LV}_b(n,0)$ is $\mathbb{R}^n$ and since the Poisson structure on it has rank n or $n-1$ depending on whether n is even or odd, we need to provide $n-1$ independent first integrals to prove superintegrability and $\left[\frac{n+1}{2}\right]$ independent first integrals (including the Hamiltonian) in involution to prove Liouville integrability. Throughout the section, n is fixed, and $k=0$ also, so we will drop from the notations the label $(n,0)$, except in the statements of the propositions and the theorem.

3.1. **First integrals.** We first write down the equations for the vector field $\mathcal{X}_H$ where we recall that $H = x_1 + x_2 + \cdots + x_n$ and that the Poisson structure $\Pi_b = \Pi_b^{(n,0)}$ is defined by (2.5); the matrix $B = B^{(n,0)}$ has all entries equal to zero, except for $b_{1,n} = -b_{n,1}$, which we will denote in this section by β. Also, the skew-symmetric matrix $A = A^{(n,0)}$ has all its uppertriangular entries equal to (plus!) 1. Therefore, $\mathcal{X}_H$ is given by

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 + x_3 + \cdots + x_n) + \beta\ , \\ \dot{x}_i &= x_i(-x_1 - \cdots - x_{i-1} + x_{i+1} + \cdots + x_n)\ , \quad 1 < i < n\ , \\ \dot{x}_n &= x_n(-x_1 - x_2 - \cdots - x_{n-1}) - \beta\ . \end{aligned} \tag{3.1}$$

We construct the first integrals of this system as deformations of the first integrals of $\mathrm{LV}(n,0)$, which were constructed in [18, Prop. 3.1]. We first recall the formulas for these integrals. For $1 \leqslant \ell \leqslant \left[\frac{n+1}{2}\right]$, the following functions $F_\ell = F_\ell^{(n,0)}$ are first integrals of $\mathrm{LV}(n,0)$:

$$F_\ell := \begin{cases} (x_1 + x_2 + \cdots + x_{2\ell-1})\,\dfrac{x_{2\ell+1}x_{2\ell+3}\dots x_n}{x_{2\ell}x_{2\ell+2}\dots x_{n-1}} & \text{if } n \text{ is odd,} \\[2ex] (x_1 + x_2 + \cdots + x_{2\ell})\,\dfrac{x_{2\ell+2}x_{2\ell+4}\dots x_n}{x_{2\ell+1}x_{2\ell+3}\dots x_{n-1}} & \text{if } n \text{ is even.} \end{cases} \tag{3.2}$$

More first integrals were constructed by using the anti-Poisson involution $\imath$ on $\mathbb{R}^n$, defined by

$$\imath(a_1, a_2, \dots, a_n) := (a_n, a_{n-1}, \dots, a_1)\ , \tag{3.3}$$

which leaves H is invariant, $\imath^* H := H \circ \imath = H$, so that the rational functions $G_\ell := \imath^* F_\ell$ ($\ell = 1, \dots, \left[\frac{n+1}{2}\right]$) are also first integrals of $\mathrm{LV}(n,0)$. This yields exactly $n-1$ different first integrals, because when n is even, all F_ℓ and G_ℓ are different, except for $F_{n/2} = H = G_{n/2}$, and when n is odd, all F_ℓ and G_ℓ are different, except for $F_{(n+1)/2} = H = G_{(n+1)/2}$ and $F_1 = G_1$. We recall also that the functions F_ℓ are pairwise in involution, just like the functions G_ℓ, and that all these functions are independent, which accounts for the Liouville and superintegrability of $\mathrm{LV}(n,0)$.

In order to construct from these first integrals of $\mathrm{LV}(n,0)$ first integrals of $\mathrm{LV}_b(n,0)$ we use the constant coefficient differential operator $\mathcal{D}_b$, which we define for the $n\times n$ matrix B corresponding to $\mathrm{LV}_b(n,k)$ by

$$\mathcal{D}_b := \sum_{1\leqslant i\leqslant 2m+1} b_{i,i+m}\frac{\partial^2}{\partial x_i \partial x_{i+m}}\ , \tag{3.4}$$

where we recall that $m=n-k-1$ and that the indices of B are taken modulo $2m+1$. In view of the conditions on $B=B^{(n,k)}$ (see Proposition 2.2), $\mathcal{D}_b$ can be written as

$$\mathcal{D}_b := \sum_{1\leqslant i\leqslant k+1} b_{i,i+m}\frac{\partial^2}{\partial x_i \partial x_{i+m}} - \sum_{1\leqslant i\leqslant k} b_{i,i+m+1}\frac{\partial^2}{\partial x_i \partial x_{i+m+1}}\ . \tag{3.5}$$

In the present case, $k=0$, and so the skew-symmetric matrix B has only a single non-zero entry above the diagonal, hence

$$\mathcal{D}_b = \beta\frac{\partial^2}{\partial x_1 \partial x_n}\ .$$

Notice that $\imath^*$ and $\mathcal{D}_b$ commute, $\imath^*\circ\mathcal{D}_b=\mathcal{D}_b\circ\imath^*$. As said, we use the operator $\mathcal{D}_b$ to define some first integrals of $\mathrm{LV}_b(n,0)$: we define for $1\leqslant \ell\leqslant \left[\frac{n+1}{2}\right]$ the functions $F_\ell^b=F_\ell^{(n,0),b}$ and $G_\ell^b=G_\ell^{(n,0),b}$ by

$$F_\ell^b := e^{\mathcal{D}_b}F_\ell = F_\ell + \beta\frac{\partial^2 F_\ell}{\partial x_1\partial x_n}\ , \qquad G_\ell^b := \imath^* F_\ell^b\ . \tag{3.6}$$

We have used that when the operator $\mathcal{D}_b$ is applied twice to F_ℓ, the result is zero. This follows from the fact that the variables x_1 and x_n appear linearly in F_ℓ (and hence are absent in $\mathcal{D}_bF_\ell$), as is clear from (3.2). Explicit formulas for the rational functions F_ℓ^b, with $1\leqslant \ell\leqslant\left[\frac{n-1}{2}\right]$ are given by

$$F_\ell^b = \begin{cases} \left(x_1+x_2+\cdots+x_{2\ell-1}+\frac{\beta}{x_n}\right)\dfrac{x_{2\ell+1}x_{2\ell+3}\cdots x_n}{x_{2\ell}x_{2\ell+2}\cdots x_{n-1}}\ , & \text{if } n \text{ is odd,}\\[2ex] \left(x_1+x_2+\cdots+x_{2\ell}+\frac{\beta}{x_n}\right)\dfrac{x_{2\ell+2}x_{2\ell+4}\cdots x_n}{x_{2\ell+1}x_{2\ell+3}\cdots x_{n-1}}\ , & \text{if } n \text{ is even,}\end{cases} \tag{3.7}$$

and similarly for G_ℓ^b. Also, for $\ell=\left[\frac{n+1}{2}\right]$ the above definitions (3.6) amount to $F_\ell^b=G_\ell^b=H$.

Proposition 3.1. *For $\ell=1,\dots,\left[\frac{n+1}{2}\right]$, the rational functions $F_\ell^{(n,0),b}$ and $G_\ell^{(n,0),b}$ are first integrals of $LV_b(n,0)$.*

Proof. Since $\imath$ is an anti-Poisson map which leaves the Hamiltonian H invariant, it suffices to show that the rational functions F_ℓ^b are first integrals of $\mathrm{LV}_b(n,0)$. We do this for odd n, the case of even n being completely analogous. Let $1\leqslant\ell\leqslant\left[\frac{n-1}{2}\right]$. To prove that F_ℓ^b is a first integral of (3.1) we show that its logarithmic derivative $(\log(F_\ell^b))^\cdot=\dot F_\ell^b/F_\ell^b$ is zero. Thanks to the particular form of the vector field (3.1), one easily obtains the following two formulas:

$$\begin{aligned}\left(\log\left(x_1+x_2+\cdots+x_{2\ell-1}+\frac{\beta}{x_n}\right)\right)^{\cdot} &= x_{2\ell}+x_{2\ell+1}+\cdots+x_n+\frac{\beta}{x_n}\ ,\\ \left(\log\left(\frac{x_{2\ell+1}}{x_{2\ell}}\cdots\frac{x_n}{x_{n-1}}\right)\right)^{\cdot} &= -x_{2\ell}-x_{2\ell+1}-\cdots-x_n-\frac{\beta}{x_n}\ .\end{aligned}$$

Summing them up, we find $(\log(F_\ell^b))^{\cdot}=0$, and hence that $\dot{F}_\ell^b=0$. □

3.2. **Involutivity.** We now show that the first integrals F_ℓ^b of $\mathrm{LV}_b(n,0)$ are in involution. For doing this, observe by comparing (3.2) and (3.7) that formally F_ℓ^b can be obtained from F_ℓ by replacing x_1 with $x_1+\beta/x_n$. Said differently, if we denote by $\sigma:\mathbb{R}^n\to\mathbb{R}^n$ the birational map defined for $(a_1,\dots,a_n)$ with $a_n\neq 0$ by

$$\sigma(a_1,\dots,a_n):=\left(a_1+\frac{\beta}{a_n},a_2,\dots,a_n\right)\ , \tag{3.8}$$

then $\sigma^*F_\ell=F_\ell^b$.

Proposition 3.2. *The birational map* $\sigma:\left(\mathbb{R}^n,\{\cdot\,,\cdot\}_b^{(n,0)}\right)\to\left(\mathbb{R}^n,\{\cdot\,,\cdot\}^{(n,0)}\right)$, *defined by (3.8), is a Poisson map.*

Proof. It suffices to show that $\{\sigma^*x_i,\sigma^*x_j\}_b=\sigma^*\{x_i,x_j\}$ for $1\leqslant i<j\leqslant n$. Since $\sigma^*x_i=x_i$ for $i>1$, this is obvious when $1<i<j$. We therefore only need to verify the formula for $i=1$ and $j>1$. If $1<j<n$ then

$$\{\sigma^*x_1,\sigma^*x_j\}_b=\left\{x_1+\frac{\beta}{x_n},x_j\right\}_b=\left(x_1+\frac{\beta}{x_n}\right)x_j=\sigma^*(x_1x_j)=\sigma^*\{x_1,x_j\}\ ,$$

where we have used that $\{x_n,x_j\}_b=-\{x_j,x_n\}=-x_jx_n$, with a minus sign because $j<n$. If $j=n$ then

$$\begin{aligned}\{\sigma^*x_1,\sigma^*x_n\}_b&=\left\{x_1+\frac{\beta}{x_n},x_n\right\}_b=\{x_1,x_n\}_b\\ &=x_1x_n+\beta=\sigma^*(x_1x_n)=\sigma^*\{x_1,x_n\}\ .\end{aligned}$$

□

Corollary 3.3. *The rational functions* $F_\ell^{(n,0),b}$ *defined in (3.7) are in involution with respect to the Poisson bracket* $\{\cdot\,,\cdot\}_b^{(n,0)}$. *Similarly, the rational functions* $G_\ell^{(n,0),b}$ *are in involution.*

Proof. Let $1\leqslant\ell,\ell'\leqslant\left[\frac{n-1}{2}\right]$. Then, according to Proposition 3.2,

$$\left\{F_\ell^b,F_{\ell'}^b\right\}_b=\{\sigma^*F_\ell,\sigma^*F_{\ell'}\}_b=\sigma^*\{F_\ell,F_{\ell'}\}=0\ ,$$

where we have used in the last step that the functions F_ℓ of $\mathrm{LV}(n,0)$ are in involution ([18, Prop. 3.2]). The fact that the functions G_ℓ^b are also in involution follows from the fact that $\imath$ is an anti-Poisson map of $(\mathbb{R}^n,\Pi_b)$. □

Notice that although σ is a birational Poisson isomorphism, it is *not* an isomorphism between the Hamiltonian systems $\mathrm{LV}(n,0)$ and $\mathrm{LV}_b(n,0)$ because $\sigma^*H \neq H$. In particular, (3.3) does not imply that the rational functions F_ℓ^b are in involution with the Hamiltonian H, i.e., that they are first integrals of $\mathrm{LV}_b(n,0)$; this requires a separate proof, which has been given in Proposition 3.1 above.

3.3. **Integrability.** We now prove the Liouville and superintegrability of $\mathrm{LV}_b(n,0)$. As we will see, the main result that remains to be proven is that the $n-1$ constructed first integrals, to wit the Hamiltonian, the rational functions F_ℓ^b (with $\ell = 1,\dots,\left[\frac{n-1}{2}\right]$) and the rational functions G_ℓ^b (with $\ell = 1,\dots,\left[\frac{n-1}{2}\right]$ when n is even and $\ell = 2,\dots,\left[\frac{n-1}{2}\right]$ when n is odd) are independent, i.e. have independent differentials on an open dense subset of $\mathbb{R}^n$. Since these functions are rational, it suffices to show that their differentials are independent in at least one point of $\mathbb{R}^n$.

To see this, we use the fact that the undeformed functions H, F_ℓ and G_ℓ are independent (at some point P). Since the deformed functions depend polynomially on the deformation parameter β, they will still be independent at P for β in a small interval, centered at zero. Notice that if we rescale all variables by a factor $\lambda \neq 0$ and rescale β by a factor λ^2 all these functions also get multiplied by a non-zero factor. It follows that the differentials of the deformed functions are independent at P for all values of β.

Theorem 3.4. *For any n, the Hamiltonian system $LV_b(n,0)$ is superintegrable, with first integrals the rational functions $F_\ell^{(n,0),b}$ and $G_\ell^{(n,0),b}$. Moreover, it is Liouville integrable with rational functions $F_\ell^{(n,0),b}$, where $\ell = 1,\dots,\left[\frac{n+1}{2}\right]$; also, it is Liouville integrable with rational functions $G_\ell^{(n,0),b}$, where $\ell = 1,\dots,\left[\frac{n+1}{2}\right]$.*

Proof. Recall that a superintegrable system on an n-dimensional manifold is a vector field (Hamiltonian or not), which has $n-1$ independent first integrals. As we have constructed precisely this number of independent first integrals for $\mathrm{LV}_b(n,0)$, we have proven its superintegrability. For Liouville integrability of a Hamiltonian vector field on an n-dimensional Poisson manifold of rank $2r$ we need $n-r$ independent first integrals which are in involution. Here, the rank of the Poisson structure Π_b is $2\left[\frac{n}{2}\right]$ (see Proposition 2.2) so that we need $n-\left[\frac{n}{2}\right] = \left[\frac{n+1}{2}\right]$ such integrals, which is exactly the number of independent first integrals F_ℓ^b (or G_ℓ^b) that we have, and they are in involution by Corollary 3.3. Notice that each of these sets of first integrals contains the Hamiltonian H. Notice also that when n is odd, F_1^b is a Casimir function of Π_b. □

Example 3.5. For $n = 4$ and $k = 0$ the matrices $A^{(4,0)}$ and $B^{(4,0)}$ are given by

$$A^{(4,0)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} , \quad B^{(4,0)} = \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{pmatrix} .$$

The corresponding system $\mathrm{LV}_b(4,0)$ is given by the formulas

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 + x_3 + x_4) + \beta , \\ \dot{x}_2 &= x_2(-x_1 + x_3 + x_4) , \\ \dot{x}_3 &= x_3(-x_1 - x_2 + x_4) , \\ \dot{x}_4 &= x_4(-x_1 - x_2 - x_3) - \beta , \end{aligned}$$

and besides the Hamiltonian $H = x_1 + x_2 + x_3 + x_4$ it has two more independent first rational integrals F and G, namely

$$F = \frac{(x_1 + x_2)x_4 + \beta}{x_3} \quad \text{and} \quad G = \imath^* F = \frac{(x_4 + x_3)x_1 + \beta}{x_2} ,$$

where $\imath$ is the anti-Poisson map defined in (3.3). The above three functions give the superintegrability of the system $\mathrm{LV}_b(4,0)$. The rank of the Poisson structure is 4 and each one of the pairs (H, F) and (H, G) provide the Liouville integrability of $\mathrm{LV}_b(4,0)$.

3.4. **Explicit solutions.** The Hamiltonian vector field $\mathcal{X}_H$ of $\mathrm{LV}(n,0)$ can be explicitly integrated in terms of elementary functions, as was first shown in [13]. We show that such an integration can also be done for (3.1), the Hamiltonian vector $\mathcal{X}_H$ of $\mathrm{LV}_b(n,0)$. This is most easily done by introducing some new variables, depending linearly on the coordinates x_i: for $i = 0, 1, \ldots, n$, let $u_i := x_1 + x_2 + \cdots + x_i$ and notice that $u_0 = 0$ and $u_n = H$. It is clear that $(u_1, \ldots, u_n)$ defines also linear coordinates on $\mathbb{R}^n$, so everything can be easily expressed in terms of the coordinates u_i by substituting $u_i - u_{i-1}$ for x_i $(i = 1, \ldots, n)$. As we will see, this simplifies some of the formulas (for $\mathcal{X}_H$, for example) and makes others more complex (the rational integrals, for example). For the proposition which follows, the formulas are the simplest when expressed in the u_i variables.

First, we need to express $\mathrm{LV}(n,0)$ in terms of the coordinates $u_1, \ldots, u_n$. The simplest way to do this is to first compute Π_b in terms of the u_i variables. Since for $i < j$, $\{x_i, x_j\}_b = x_i x_j$, except that $\{x_1, x_n\} = x_1 x_n + \beta$, we get

$$\begin{aligned} \{u_i, u_j\}_b &= u_i(u_j - u_i) , && \text{if } 1 \leqslant i < j < n , \\ \{u_i, u_n\}_b &= u_i(u_n - u_i) + \beta , && \text{if } 1 \leqslant i < n . \end{aligned} \tag{3.9}$$

Since $H = u_n$, we can compute $\mathcal{X}_H$ as $\{\cdot\, , u_n\}_b$, which takes in view of the above formulas the following simple, decoupled form:

$$\begin{aligned} \dot{u_i} &= u_i(H - u_i) + \beta, && i = 1, 2, \ldots, n-1 , \\ \dot{u_n} &= 0 , \end{aligned} \tag{3.10}$$

which can easily be integrated, for any initial condition. We describe the integration in a geometrical language, which will be useful when we use it in Section 5. For any point $P \in \mathbb{R}^n$, we can consider the integral curve of $\mathcal{X}_H$, starting from P, which we will denote by γ_P. Usually, the domain of an integral curve is taken to be an interval, but in the present case we will take it to be all of $\mathbb{R}$ minus a discrete subset. On the one hand, it is natural to do this because in the case of $\mathcal{X}_H$ the solutions are precisely defined on such a set. On the other hand, the systems $\mathrm{LV}_b(n,k)$ can equally be defined on a complex phase space $\mathbb{C}^n$ and then the integral curves, with complex time, are defined for all of $\mathbb{C}$, minus a discrete subset; the domain of the real integral curves which we consider is just the real part of this complex subset. Since it is convenient to express the integral curves in terms of coordinates (here the u_i coordinates) we will write, once P has been fixed, $u_i(t)$ for $u_i(\gamma_P(t))$. The u-coordinates of P will be denoted $(P_1,\dots,P_n)$, so $P_i = u_i(P)$ for $i=1,\dots,n-1$ and $P_n = u_n(P) = H(P)$.

Proposition 3.6. *Let P be any point of $\mathbb{R}^n$ and let γ_P denote the integral curve of (3.10), which is $LV_b(n,0)$, expressed in the u_i variables, starting from P. Denote by h the value of the Hamiltonian at P, i.e., $h = H(P) = u_n(P)$. Let Δ_0 be a square root of $h^2+4\beta$, which may be real or imaginary. Then, for $i = 1,2,\dots,n-1$,*

$$u_i(t) = \begin{cases} P_i\,, & \text{if } P_i^2 - P_i h - \beta = 0\,,\\[2ex] \frac{h}{2} + \frac{2P_i - h}{2+t(2P_i-h)}, & \text{if } \Delta_0^2 = h^2+4\beta = 0\,,\\[2ex] \frac{(h+\Delta_0)(h-\Delta_0-2P_i)-(h+\Delta_0-2P_i)(h-\Delta_0)e^{-t\Delta_0}}{2(h-\Delta_0-2P_i)-2(h+\Delta_0-2P_i)e^{-t\Delta_0}}\,, & \text{otherwise.}\end{cases} \tag{3.11}$$

Obviously, $u_n(t) = H$ is constant.

4. The Liouville and noncommutative integrability of $\mathrm{LV}_b(n,k)$

In this section, we generalize the results of Section 3 on the integrability of $\mathrm{LV}_b(n,0)$ to the case of $\mathrm{LV}_b(n,k)$, where $k \in \mathbb{N}$ satisfies $2k+1<n$, but is otherwise arbitrary. We do not treat here the case of $n = 2k+1$ because we have already established the Liouville integrability of $\mathrm{LV}_b(2k+1,k)$ in [7]. We show in this section that if $1 < 2k+1 < n$ then $\mathrm{LV}_b(n,k)$ is on the one hand Liouville integrable, and on the other hand is non-commutatively integrable of rank $k+1$. We start by recalling the definition of non-commutative integrability (see [16, 14]), which we specialize to $\mathbb{R}^n$.

Definition 4.1. Let Π be a Poisson structure on $\mathbb{R}^n$, with associated Poisson bracket $\{\cdot\,,\cdot\}$. Let $\mathbf{F} = (f_1,\dots,f_s)$ be an s-tuple of functions on $\mathbb{R}^n$, where $2s \geqslant n$ and set $r := n-s$. Suppose the following:

(1) The functions $f_1,\dots,f_r$ are in involution with the functions $f_1,\dots,f_s$:
$$\{f_i, f_j\} = 0, \qquad (1 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant s)\;;$$

(2) For P in a dense open subset of $\mathbb{R}^n$:

$$\mathrm{d}f_1(P) \wedge \cdots \wedge \mathrm{d}f_s(P) \neq 0 \quad \text{and} \quad \mathcal{X}_{f_1}|_P \wedge \cdots \wedge \mathcal{X}_{f_r}|_P \neq 0\ .$$

Then the triplet $(\mathbb{R}^n, \Pi, \mathbf{F})$ is called a *non-commutative integrable system* of *rank* r.

The classical case of a *Liouville integrable system* corresponds to the particular case where r is half the (maximal) rank of Π; this implies that *all* the functions $f_1, \dots, f_s$ are pairwise in involution. The case of a superintegrable system corresponds to $r = 1$; in this case, condition (1) just means that one has $n - 1$ first integrals, while the second condition in (2) is trivially satisfied: superintegrability means, as recalled in the previous section, that one has $n - 1$ independent first integrals.

In order to establish Liouville and non-commutative integrability in Section 4.3 below, we first construct a set of polynomial first integrals for $\mathrm{LV}_b(n, k)$, which are pairwise in involution, and then we construct a set of rational first integrals for $\mathrm{LV}_b(n, k)$, which are also pairwise in involution. This will be done in the two subsections which follow. Throughout the section, we suppose that $1 < 2k + 1 < n$ and that $B = B^{(n,k)}$ is a skew-symmetric $n \times n$ matrix such that (2.5) defines a Poisson structure on $\mathbb{R}^n$. Recall that this means that the uppertriangular entries $b_{i,j}$ of B with $j - i \notin \{m, m + 1\}$ are zero.

4.1. **The polynomial integrals.** Recall from Section 2.2 that the systems $\mathrm{LV}_b(n, k)$ are obtained by reduction from the systems $\mathrm{LV}_b(2m+1, m)$, where $m := n-k-1 > k$, where the last inequality comes from our assumption $n > 2k + 1$. Recall also that in order to do this reduction, one supposes that the last $2m+1-n$ rows and columns of the $(2m+1)\times(2m+1)$ matrix B are zero, so that B can be viewed as an $n \times n$ matrix by removing these zero rows and columns (see Proposition 2.1). Since $\mathrm{LV}_b(2m + 1, m)$ is Liouville integrable, with $m + 1$ independent polynomial integrals, whose formulas are recalled below, we obtain by reduction a set of first integrals of $\mathrm{LV}_b(n, k)$, which are automatically in involution with respect to the reduced Poisson structure, which is by definition the Poisson structure $\Pi_b^{(n,k)}$ of $\mathrm{LV}_b(n, k)$. One has however to be careful with the independence of the reduced integrals, for example some of these reduced integrals are zero! Moreover, since $n > 2k+1$, more integrals are needed for integrability, as we will see.

Let us first recall the formulas for the (polynomial) first integrals of $\mathrm{LV}_b(2m + 1, m)$. One method of constructing them is as coefficients of the characteristic polynomial of the Lax operator $L(\lambda) := X + \lambda^{-1}\Delta + \lambda M$, which we recalled in (2.4). It is a classical fact that the coefficients of the characteristic polynomial of a Lax operator yields first integrals for any Lax equation in which it appears [14, Sect. 12.2.5]. For $L(\lambda)$, the following expansion of its characteristic polynomial was obtained in [7, Prop. 8]:

$$\det(L(\lambda)-\mu\,\mathrm{Id}) = \lambda^{2m+1}+\frac{1}{\lambda^{2m+1}}\prod_{j=1}^{2m+1}(b_{j+m,j}-\lambda\mu)+\sum_{i=0}^{m}(\lambda\mu)^{m-i}K_i^b\ . \quad (4.1)$$

Thus, the polynomials K_i^b in this expansion are first integrals of $\mathrm{LV}_b(2m+1,m)$. Setting the deformation parameters equal to zero, one recovers the first integrals of $\mathrm{LV}(2m+1,m)$, which were first constructed by Bogoyavlenskij [4] and Itoh [12]. We construct $k+1$ first integrals of $\mathrm{LV}_b(n,k)$ by setting, for $i=0,\dots,k$,

$$K_i^{(n,k),b} := K_i^b{}_{\big|\mathbb{R}^n} := K_i^b{}_{\big|x_{n+1}=x_{n+2}=\cdots=x_{2m+1}=0}\ , \quad (4.2)$$

where the notation introduced by the latter equality is a convenient shorthand. By construction, these polynomials are first integrals of $\mathrm{LV}_b(n,k)$ and they are in involution. Also, $K_1=H$ and K_i is of degree $2i+1$ for $i=0,1,\dots,k$.

We give an alternative description of the first integrals (4.2) as deformations of the polynomial first integrals of $\mathrm{LV}(n,k)$. On the one hand, this description will be important for showing the independence of these first integrals, and on the other hand it will provide information about the structure of these integrals, which we will use to prove some of their properties (involutivity, for example).

It was shown in [7, Prop. 9] that the integrals K_i^b of $\mathrm{LV}_b(2m+1,m)$ can be obtained using the $\mathcal{D}_b$ operator (see (3.4)) by the following formula, valid for $i=0,1,\dots,m$:

$$K_i^b = e^{\mathcal{D}_b}K_i = K_i+\mathcal{D}_bK_i+\frac{1}{2!}\mathcal{D}_b^2K_i+\cdots+\frac{1}{i!}\mathcal{D}_b^iK_i\ .$$

Let us show that $\mathcal{D}_b$ commutes with restriction to $\mathbb{R}^n$. Let F be a smooth or rational function on $\mathbb{R}^{2m+1}$. In view of the conditions on B, the operator $\mathcal{D}_b$ is given by (3.5) and we see that $\mathcal{D}_b$ does not involve derivation with respect to any of the variables $x_{n+1},\ x_{n+2},\dots,\ x_{2m+1}$ (recall that $n=m+k+1$), so $\mathcal{D}_b$ commutes with restriction to the subspace $\mathbb{R}^n$ of $\mathbb{R}^{2m+1}$, which is defined by $x_{n+1}=x_{n+2}=\cdots=x_{2m+1}=0$. It follows that

$$\begin{aligned} K_i^{(n,k),b} &= K_i^b{}_{\big|\mathbb{R}^n} = \left(e^{\mathcal{D}_b}K_i\right)_{\big|\mathbb{R}^n} = e^{\mathcal{D}_b}K_i{}_{\big|\mathbb{R}^n} = e^{\mathcal{D}_b}K_i^{(n,k)} \\ &= K_i^{(n,k)}+\mathcal{D}_bK_i^{(n,k)}+\frac{1}{2!}\mathcal{D}_b^2K_i^{(n,k)}+\cdots+\frac{1}{i!}\mathcal{D}_b^iK_i^{(n,k)}\ . \end{aligned} \quad (4.3)$$

This shows that the polynomial first integrals $K_i^{(n,k),b}$ of $\mathrm{LV}_b(n,k)$ are deformations of the first integrals $K_i^{(n,k)}$ of $\mathrm{LV}(n,k)$. Notice also that (4.3) implies that $K_i^{(n,k),b}=0$ when $i>k$ since $K_i^{(n,k)}=0$ when $i>k$ (see the comments after Proposition 3.3 in [6]). That is the reason why we restricted i in (4.2) to $i=0,1,\dots,k$ rather than $i=0,1,\dots,m$.

For later use, we quickly recall from [6] a combinatorial formula for $K_i^{(n,k)}$. Let $\underline{m} = (m_1, m_2, \dots, m_{2i+1})$ be a $2i+1$-tuple of integers, satisfying $1 \leqslant m_1 < m_2 < \cdots < m_{2i+1} \leqslant n$. We view these integers as indices of the rows and columns of $A^{(n,k)}$: we denote by $A_{\underline{m}}^{(n,k)}$ the square submatrix of $A^{(n,k)}$ of size $2i+1$, corresponding to rows and columns $m_1, m_2, \dots, m_{2i+1}$ of $A^{(n,k)}$, so that

$$(A_{\underline{m}}^{(n,k)})_{s,t} = (A^{(n,k)})_{m_s,m_t} \ , \ \text{for } s,t = 1, \dots, 2i+1 \ .$$

Letting

$$\mathcal{S}_i^{(n,k)} := \left\{ \underline{m} \mid A_{\underline{m}}^{(n,k)} = A^{(2i+1,i)} \right\} \ ,$$

the first integral $K_i^{(n,k)}$ is given by

$$K_i^{(n,k)} = \sum_{\underline{m} \in \mathcal{S}_i^{(n,k)}} x_{m_1} x_{m_2} \dots x_{m_i} \dots x_{m_{2i+1}} \ . \tag{4.4}$$

One immediate consequence is that every variable x_j has degree at most one in $K_i^{(n,k)}$.

4.2. **The rational integrals.** We now construct a set of rational integrals of $\mathrm{LV}_b(n,k)$. In order to follow some of the more technical arguments in this subsection, the reader is advised to already take a look at Section 4.4 below, where explicit formulas for a few examples are given.

Recall that we assume in this section that $k > 0$ and that $n > 2k+1$. We define the rational integrals of $\mathrm{LV}_b(n,k)$ as deformations of the rational integrals of $\mathrm{LV}(n,k)$, which were first constructed in [6]. We first recall the definition of the latter integrals as pullbacks of the rational first integrals $F_\ell^{(n,0)}$ and $G_\ell^{(n,0)}$ of $\mathrm{LV}(n-2k,0)$, which we recalled in Section 3.1. Consider the polynomial map $\phi_k : \mathbb{R}^n \to \mathbb{R}^{n-2k}$, defined by

$$\phi_k(a_1, a_2, \dots, a_n) := a_1 a_2 \dots a_k (a_{k+1}, a_{k+2}, \dots, a_{n-k}) a_{n-k+1} \dots a_{n-1} a_n \ .$$

If we denote the standard coordinates on $\mathbb{R}^n$ by $x_1, \dots, x_n$, and on $\mathbb{R}^{n-2k}$ by $y_1, \dots, y_{n-2k}$, then $\phi^* y_i = x_1 x_2 \dots x_k x_{k+i} x_{n-k+1} \dots x_n$ for $i = 1, \dots, n-2k$. It was shown in [6] that for $\ell = 1, 2, \dots, \left[\frac{n+1}{2}\right] - k$, the rational functions $F_\ell^{(n,k)} := \phi_k^* F_\ell^{(n-2k,0)}$ and $G_\ell^{(n,k)} := \phi_k^* G_\ell^{(n-2k,0)}$ are first integrals of $\mathrm{LV}(n,k)$ and that the integrals $F_\ell^{(n,k)}$ are pairwise in involution with respect to $\{\cdot\,,\cdot\}^{(n,k)}$, just like the integrals $G_\ell^{(n,k)}$. Setting $s := 2\ell - 1$ when n is odd and $s := 2\ell$ when n is even, so that s and n have the same parity, one computes easily from (3.2) that

$$F_\ell^{(n,k)} = \prod_{i=1}^{k} (x_i x_{n+1-i}) \sum_{j=1}^{s} x_{j+k} \prod_{t=1}^{\frac{n-s}{2}-k} \frac{x_{s+k+2t}}{x_{s+k+2t-1}} \ . \tag{4.5}$$

When $\ell = \left[\frac{n+1}{2}\right] - k$, the last product in this expression reduces to 1 and so $F_\ell^{(n,k)}$ is actually a polynomial function. Since some of the arguments below which depend on the structure of the integrals fail for the polynomial first integrals, we will exclude these integrals in this section, and so we will throughout this section only consider the rational integrals $F_\ell^{(n,k)}$, with $\ell = 1, 2, \dots, \left[\frac{n-1}{2}\right] - k$. Notice also that every variable or its inverse appears precisely once in (4.5), with the k variables $x_1, \dots, x_k$ and the $k+1$ variables $x_{n-k}, \dots, x_n$ appearing linearly. This property will be important in what follows.

We construct the rational first integrals of $\mathrm{LV}_b(n,k)$ as deformations of the integrals $F_\ell^{(n,k)}$ by using the operators $\mathcal{D}_b$ (see (3.4) or (3.5)): for $\ell = 1, 2, \dots, \left[\frac{n-1}{2}\right] - k$, we set

$$F_\ell^{(n,k),b} := e^{\mathcal{D}_b} F_\ell^{(n,k)} = e^{\mathcal{D}_b} \left(\phi_k^* F_\ell^{(n-2k,0)} \right) ,$$

and similarly for $G_\ell^{(n,k),b}$. The present subsection is devoted to the proof of the following theorem, which says that the deformed integrals $F_\ell^{(n,k),b}$ are first integrals of $\mathrm{LV}_b(n,k)$ which are pairwise in involution.

Theorem 4.2. *For* $1 \leqslant \ell \leqslant \ell' \leqslant \left[\frac{n-1}{2}\right] - k$,

$$\left\{ F_\ell^{(n,k),b}, H \right\}_b^{(n,k)} = 0 \quad \textit{and} \quad \left\{ F_\ell^{(n,k),b}, F_{\ell'}^{(n,k),b} \right\}_b^{(n,k)} = 0 \, .$$

The same result holds for the rational functions $G_\ell^{(n,k),b}$.

For the proof of Theorem 4.2, we need some extra notation. Since throughout this subsection n and k are fixed, we will until the rest of the subsection drop (n,k) from the notation, writing F_ℓ^b for $F_\ell^{(n,k),b}$, writing B for $B^{(n,k)}$, and so on. We will need a specific ordering of the entries of the matrix B. Therefore, we label the parameters of B with single indices as follows (recall that B is skew-symmetric and that its non-zero uppertriangular entries are at positions (i,j) with $j - i = n - k$ or $j - i = n - k - 1$):

$$B = \begin{pmatrix} \dots & 0 & b_{2k+1} & -b_{2k} & 0 & \dots & 0 \\ & \dots & 0 & b_{2k-1} & -b_{2k-2} & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & b_3 & -b_2 \\ & & & & \dots & 0 & b_1 \\ & & & & & \dots & 0 \\ & & & & & \dots & \vdots \end{pmatrix}$$

Expressed in terms of a formula,

$$b_p := (-1)^{p+1} b_{k+1-\left[\frac{p}{2}\right], n-\left[\frac{p-1}{2}\right]} \, .$$

For $1 \leqslant p \leqslant 2k+1$, we denote by $B^{(p)}$ the matrix obtained by setting in B the parameters $b_{p+1}, b_{p+2}, \dots, b_{2k+1}$ equal to zero. We also set $B^{(0)}$ equal to the zero matrix. So $B^{(1)}$ contains only the parameter b_1 and $B^{(2k+1)} = B$. The corresponding Poisson structure, which can be obtained by setting in $\Pi_b = \Pi_b^{(n,k)}$ the parameters $b_{p+1}, b_{p+2}, \dots, b_{2k+1}$ equal to zero, is denoted by $\Pi_{(p)}$. In particular, $\Pi_b = \Pi_{(2k+1)}$. The associated Poisson bracket is denoted by $\{\cdot\,,\cdot\}_{(p)}$. We also associate to p the following constant coefficient differential operator

$$\mathcal{D}_{(p)} := b_p \frac{\partial^2}{\partial x_{k+1-\left[\frac{p}{2}\right]} \partial x_{n-\left[\frac{p-1}{2}\right]}} \ . \tag{4.6}$$

It is clear that $\mathcal{D}_b = \mathcal{D}_{(2k+1)} + \mathcal{D}_{(2k)} + \cdots + \mathcal{D}_{(2)} + \mathcal{D}_{(1)}$ and, since these operators commute, that

$$e^{\mathcal{D}_b} = e^{\mathcal{D}_{(2k+1)}} \circ e^{\mathcal{D}_{(2k)}} \circ \cdots \circ e^{\mathcal{D}_{(2)}} \circ e^{\mathcal{D}_{(1)}} \ .$$

We could of course consider any alternative order of the operators, but we will use the above one for some reason which will become clear later. Finally, for any smooth or rational function F on $\mathbb{R}^n$ we set $F^{(p)} := e^{\mathcal{D}_{(p)}} F^{(p-1)}$ for $1 \leqslant p \leqslant 2k+1$, and $F^{(0)} := F$. With this notation, $F_\ell^b = F_\ell^{(2k+1)}$.

A crucial fact which we will use is that the partially deformed integral $F_\ell^{(p)}$ is the pullback of $F_\ell^{(p-1)}$ by the birational Poisson map σ_p, defined for $1 \leqslant i \leqslant n$ by

$$\sigma_p^*(x_i) := \begin{cases} x_i + \frac{b_p}{x_{i+n-k-1}} \delta_{i,k+1-\frac{p-1}{2}} & \text{if } p \text{ is odd,} \\ \\ x_i + \frac{b_p}{x_{i+k-n}} \delta_{i,n+1-\frac{p}{2}} & \text{if } p \text{ is even.} \end{cases} \tag{4.7}$$

Notice that

$$\sigma_p^* \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} \right) = x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} + b_p \ , \tag{4.8}$$

independently of whether p is even or odd.

Proposition 4.3. *$F_\ell^{(p)} = \sigma_p^* F_\ell^{(p-1)}$ for $1 \leqslant p \leqslant 2k+1$ and $1 \leqslant \ell \leqslant \left[\frac{n-1}{2}\right] - k$.*

Proof. We first show that if $0 \leqslant p \leqslant 2k$ then

$$F_\ell^{(p)} = x_{n-\left[\frac{p}{2}\right]} x_{k-\left[\frac{p-1}{2}\right]} E_1 + E_2 \ , \tag{4.9}$$

where E_1 is independent of $x_{n-\left[\frac{p}{2}\right]}$ and of $x_{k-\left[\frac{p-1}{2}\right]}$, while E_2 is independent of $x_{n-\left[\frac{p}{2}\right]} = x_{n-\frac{p-1}{2}}$ when p is odd and is independent of $x_{k-\left[\frac{p-1}{2}\right]} = x_{k+1-\frac{p}{2}}$ when p is even. We do this for p odd, i.e., we show that when p is odd,

$$F_\ell^{(p)} = x_{n-\frac{p-1}{2}} x_{k-\frac{p-1}{2}} E_1 + E_2 \ ,$$

where E_1 is independent of $x_{n-\frac{p-1}{2}}$ and of $x_{k-\frac{p-1}{2}}$, while E_2 is independent of $x_{n-\frac{p-1}{2}}$. According to the explicit formula (4.5), we can write $F_\ell = F_\ell^{(0)}$

as

$$F_\ell = x_{n-\frac{p-1}{2}} x_{k-\frac{p-1}{2}} F'_\ell + F''_\ell \ ,$$

where F'_ℓ and F''_ℓ are independent of $x_{n-\frac{p-1}{2}}$ and of $x_{k-\frac{p-1}{2}}$. Then,

$$F_\ell^{(p)} = e^{\mathcal{D}_{(p)}} F_\ell^{(p-1)} = \left(e^{\mathcal{D}_{(p)}} F_\ell\right)^{(p-1)} = \left(F_\ell + \mathcal{D}_{(p)} F_\ell\right)^{(p-1)} \ ,$$

where we have used in the last step that $\mathcal{D}^2_{(p)} F_\ell = 0$, which is a consequence of the fact that F_ℓ depends linearly on $x_{k+1-\frac{p-1}{2}}$ and on $x_{n-\frac{p-1}{2}}$, which are the variables with respect to which $\mathcal{D}_{(p)}$ differentiates (see (4.6)). Since, moreover, F''_ℓ is independent of $x_{n-\frac{p-1}{2}}$,

$$F_\ell^{(p)} = \left(x_{n-\frac{p-1}{2}} x_{k-\frac{p-1}{2}} F'_\ell + F''_\ell + b_p x_{k-\frac{p-1}{2}} \frac{\partial F'_\ell}{\partial x_{k+1-\frac{p-1}{2}}}\right)^{(p-1)} \ .$$

Since $\mathcal{D}_{(1)}, \ldots, \mathcal{D}_{(p-1)}$ do not involve the variables $x_{n-\frac{p-1}{2}}$ and $x_{k-\frac{p-1}{2}}$,

$$\begin{aligned} F_\ell^{(p)} &= x_{n-\frac{p-1}{2}} x_{k-\frac{p-1}{2}} F'^{(p-1)}_\ell + \left(F''_\ell + b_p x_{k-\frac{p-1}{2}} \frac{\partial F'_\ell}{\partial x_{k+1-\frac{p-1}{2}}}\right)^{(p-1)} \\ &= x_{n-\frac{p-1}{2}} x_{k-\frac{p-1}{2}} E_1 + E_2 \ , \end{aligned}$$

where E_1 is independent of $x_{n-\frac{p-1}{2}}$ and of $x_{k-\frac{p-1}{2}}$, and E_2 is independent of $x_{n-\frac{p-1}{2}}$. This shows our claim when p is odd. The proof in case p is even is similar. We use the obtained formula (4.9) to show that $F_\ell^{(p)} = \sigma_p^* F_\ell^{(p-1)}$ for any p. According to (4.9), we can write

$$F_\ell^{(p-1)} = x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} E_1 + E_2 \ ,$$

where E_1 is independent of $x_{n-\left[\frac{p-1}{2}\right]}$ and of $x_{k+1-\left[\frac{p}{2}\right]}$, while E_2 is independent of $x_{n-\left[\frac{p-1}{2}\right]} = x_{n+1-\frac{p}{2}}$ when p is even and is independent of $x_{k+1-\left[\frac{p}{2}\right]} = x_{k+1-\frac{p-1}{2}}$ when p is odd. Therefore, on the one hand,

$$\begin{aligned} F_\ell^{(p)} &= e^{\mathcal{D}_{(p)}} F_\ell^{(p-1)} = F_\ell^{(p-1)} + \mathcal{D}_{(p)} \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} E_1 + E_2\right) \\ &= F_\ell^{(p-1)} + b_p \frac{\partial^2}{\partial x_{k+1-\left[\frac{p}{2}\right]} \partial x_{n-\left[\frac{p-1}{2}\right]}} \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} E_1 + E_2\right) \\ &= F_\ell^{(p-1)} + b_p E_1 \ , \end{aligned}$$

while on the other hand,

$$\begin{aligned} \sigma_p^* F_\ell^{(p-1)} &= \sigma_p^* \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} E_1 + E_2\right) \\ &\overset{(\star)}{=} \sigma_p^* \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]}\right) E_1 + E_2 \\ &\overset{(4.8)}{=} \left(x_{n-\left[\frac{p-1}{2}\right]} x_{k+1-\left[\frac{p}{2}\right]} + b_p\right) E_1 + E_2 = F_\ell^{(p-1)} + b_p E_1 \ , \end{aligned}$$

which shows that $F_\ell^{(p)}=\sigma_p^*F_\ell^{(p-1)}$. We have used in $(\star)$ that $\sigma_p^*(E_1)=E_1$ and that $\sigma_p^*(E_2)=E_2$, which hold because E_1 and E_2 are independent of the only variable which is not fixed by σ_p^*. $\square$

We next show that the maps σ_p are Poisson maps with respect to the appropriate Poisson structures on $\mathbb{R}^n$.

Proposition 4.4. *For $1\leqslant p\leqslant 2k+1$ the birational map*

$$\sigma_p:\left(\mathbb{R}^n,\{\cdot\,,\cdot\}_{(p)}\right)\to\left(\mathbb{R}^n,\{\cdot\,,\cdot\}_{(p-1)}\right)\ ,$$

defined by (4.7), is a Poisson map.

Proof. We give the proof in case p is even. Then (4.7) simplifies to

$$\sigma_p^*(x_i)=\begin{cases} x_i+\frac{b_p}{x_{i+k-n}} & \text{if } i=n+1-\frac{p}{2}\ ,\\ \\ x_i & \text{if } i\neq n+1-\frac{p}{2}\ .\end{cases}$$

We need to show that $\sigma_p^*\left\{x_i,x_j\right\}_{(p-1)}=\left\{\sigma_p^*(x_i),\sigma_p^*(x_j)\right\}_{(p)}$ for all $1\leqslant i<j\leqslant n$. Notice that if $(i,j)\neq\left(k+1-\frac{p}{2},n+1-\frac{p}{2}\right)$ then $\left\{x_i,x_j\right\}_{(p)}=\left\{x_i,x_j\right\}_{(p-1)}$ and otherwise $\left\{x_i,x_j\right\}_{(p)}=\left\{x_i,x_j\right\}_{(p-1)}-b_p$ (recall that p is even). It follows that, if $i,j\neq n+1-p/2$, then

$$\sigma_p^*\left\{x_i,x_j\right\}_{(p-1)}=\left\{x_i,x_j\right\}_{(p-1)}=\left\{x_i,x_j\right\}_{(p)}=\left\{\sigma_p^*(x_i),\sigma_p^*(x_j)\right\}_{(p)}\ ,$$

as was to be shown. Suppose now that $1\leqslant i=n+1-\frac{p}{2}<j$. Notice that, in this case, $\left\{x_i,x_j\right\}_{(p)}=\left\{x_i,x_j\right\}_{(p-1)}=+x_ix_j$, with a plus sign. Then

$$\sigma_p^*\left\{x_i,x_j\right\}_{(p-1)}=\sigma_p^*(x_ix_j)=\left(x_i+\frac{b_p}{x_{i+k-n}}\right)x_j\ ,$$

while

$$\left\{\sigma_p^*(x_i),\sigma_p^*(x_j)\right\}_{(p)}=\left\{x_i+\frac{b_p}{x_{i+k-n}},x_j\right\}_{(p)}=x_ix_j+\frac{b_p}{x_{i+k-n}}x_j\ ,$$

where we have used that $\left\{x_{i+k-n},x_j\right\}_{(p)}=-x_{i+k-n}x_j$, with a minus sign because $i<j$. Finally, suppose that $1\leqslant i<j=n+1-\frac{p}{2}$ and notice that

$j+k-n=k+1-\frac{p}{2}$. Then,

$$
\begin{aligned}
\left\{\sigma_p^*(x_i),\sigma_p^*(x_j)\right\}_{(p)} &= \left\{x_i, x_j+\frac{b_p}{x_{j+k-n}}\right\}_{(p)}\\
&= \begin{cases} \left\{x_i,x_j\right\}_{(p)}\ , & \text{if } i=k+1-\frac{p}{2}\ ,\\[2ex] \left\{x_i,x_j\right\}_{(p)}-\frac{b_p}{x_{j+k-n}^2}\left\{x_i,x_{j+k-n}\right\}\ , & \text{if } i\neq k+1-\frac{p}{2}\ ,\end{cases}\\
&= \begin{cases} -x_ix_j-b_p\ , & \text{if } i=k+1-\frac{p}{2}\ ,\\[2ex] -x_ix_j-\frac{b_px_i}{x_{j+k-n}}\ , & \text{if } i<k+1-\frac{p}{2}\ ,\\[2ex] x_ix_j+\frac{b_px_i}{x_{j+k-n}}\ , & \text{if } i>k+1-\frac{p}{2}\ ,\end{cases}
\end{aligned}
\tag{4.10}
$$

while

$$
\sigma_p^*\left\{x_i,x_j\right\}_{(p-1)}=\sigma_p^*(\pm x_ix_j)=\pm x_i\left(x_j+\frac{b_p}{x_{j+k-n}}\right)\ ,
$$

where the $+$ sign corresponds to the case $i>k+1-\frac{p}{2}$ and the $-$ sign to the case $i\leqslant k+1-\frac{p}{2}$. Clearly, this gives the same result as in (4.10). This shows that σ_p is a Poisson map. □

Propositions 4.3 and 4.4 imply, in that order, that for any $p=1,\dots,2k{+}1$, and for any $1\leqslant\ell\leqslant\ell'\leqslant\left[\frac{n-1}{2}\right]-k$,

$$
\left\{F_\ell^{(p)},F_{\ell'}^{(p)}\right\}_{(p)}=\left\{\sigma_p^*F_\ell^{(p-1)},\sigma_p^*F_{\ell'}^{(p-1)}\right\}_{(p)}=\sigma_p^*\left\{F_\ell^{(p-1)},F_{\ell'}^{(p-1)}\right\}_{(p-1)}
$$

and so, since the undeformed rational integrals $F_\ell=F_\ell^{(0)}$ are pairwise in involution, an easy recursion shows that their deformations $F_\ell^b=F_\ell^{(2k+1)}$ are in involution as well. This shows the second part of Theorem 4.2.

We will next show that that the deformed integrals F_ℓ^b are first integrals of $\mathrm{LV}_b(n,k)$. To do this, we first prove the following lemma.

Lemma 4.5. *Let $1\leqslant\ell\leqslant\left[\frac{n-1}{2}\right]-k$ and denote $s:=2\ell-1$ when n is odd and $s:=2\ell$ when n is even. If $i\notin\{k+1,k+2,\dots,k+s\}$ then $\left\{x_i,F_\ell^b\right\}_b=0$.*

Proof. We prove by recursion on p that $\left\{x_i,F_\ell^{(p)}\right\}_{(p)}=0$ for $p=0,1,\dots,2k+1$. For $p=0$ this amounts to showing that $\left\{x_i,F_\ell\right\}_b=0$, which was done in [6, Prop. 3.1]. Let $1\leqslant p\leqslant 2k+1$ and assume that the property is true for $p-1$. If i is such that $\sigma_p^*(x_i)=x_i$, then by Propositions 4.3 and 4.4,

$$
\left\{x_i,F_\ell^{(p)}\right\}_{(p)}=\left\{\sigma_p^*(x_i),\sigma_p^*F_\ell^{(p-1)}\right\}_{(p)}=\sigma_p^*\left\{x_i,F_\ell^{(p-1)}\right\}_{(p-1)}=0\ ,
$$

where we used the recursion hypothesis in the last step. Suppose now that $\sigma_p^*(x_i) \neq x_i$. If p is even, this means that $i = n + 1 - \frac{p}{2}$, and so

$$\begin{aligned}\left\{x_i, F_\ell^{(p)}\right\}_{(p)} &= \left\{\sigma_p^*\left(x_i - \frac{b_p}{x_{i+k-n}}\right), \sigma_p^* F_\ell^{(p-1)}\right\}_{(p)} \\ &= \sigma_p^*\left\{x_i - \frac{b_p}{x_{i+k-n}}, F_\ell^{(p-1)}\right\}_{(p-1)} = 0\ ,\end{aligned}$$

where we used again the recursion hypothesis twice: we could do so because $i + k - n \leqslant k$ so that $i + k - n \notin \{k+1, k+2, \dots, k+s\}$. If $\sigma_p^*(x_i) \neq x_i$ and p is odd, then $i = 1 + k - \frac{p-1}{2}$, so that

$$\begin{aligned}\left\{x_i, F_\ell^{(p)}\right\}_{(p)} &= \left\{\sigma_p^*\left(x_i - \frac{b_p}{x_{i+n-k-1}}\right), \sigma_p^* F_\ell^{(p-1)}\right\}_{(p)} \\ &= \sigma_p^*\left\{x_i - \frac{b_p}{x_{i+n-k-1}}, F_\ell^{(p-1)}\right\}_{(p-1)} = 0\ ,\end{aligned}$$

as before. □

Using the lemma and the fact that F_ℓ is a first integral of $\mathrm{LV}(n,k)$ (see [6]), we show that F_ℓ^b is a first integral of $\mathrm{LV}_b(n,k)$. As before, we show by recursion that $\left\{F_\ell^{(p)}, H\right\}_{(p)} = 0$, the case of $p = 0$ already being established. Suppose that $\left\{F_\ell^{(p-1)}, H\right\}_{(p-1)} = 0$ for some $p \geqslant 1$. Then

$$\begin{aligned}\left\{F_\ell^{(p)}, H\right\}_{(p)} &= \left\{\sigma_p^* F_\ell^{(p-1)}, \sigma_p^*\left(H - \frac{b_p}{x_t}\right)\right\}_{(p)} \\ &= \sigma_p^*\left\{F_\ell^{(p-1)}, H - \frac{b_s}{x_t}\right\}_{(p-1)} = 0\ ,\end{aligned}$$

since $t = k + 1 - \frac{n}{2} \leqslant k$ when p is even and $t = n - \frac{p-1}{2} \geqslant n - k$ when p is odd; in either case, $t \notin \{k+1, k+2, \dots, k+s\}$, which proves the last equality. We conclude that $\left\{F_\ell^b, H\right\}_b = 0$, which is the first statement of Theorem 4.2.

4.3. **Integrability.** We have now most ingredients to state and prove the Liouville and non-commutative integrability of $\mathrm{LV}_b(n,k)$, where we recall that $k > 0$ and $2k + 1 < n$. Since in this subsection (n,k) is fixed we will again drop (n,k) from the notation, except in the statement of the propositions and of the theorem. We have constructed in Section 4.1 a set of polynomial integrals for $\mathrm{LV}_b(n,k)$ and in Section 4.2 a set of rational integrals F_ℓ. We first show that these polynomial integrals are in involution with these rational integrals. To do this, we need a property of the polynomial integrals, which we first define.

Definition 4.6. A polynomial function K on $\mathbb{R}^n$ is said to be (n,k)-*admissible* if

(1) K is of degree at most one in each of its variables x_j;
(2) K can be written (uniquely) as $K = LK' + K''$, where K'' is independent of $x_{k+1}, \ldots, x_{n-k}$ and L is the sum of these variables, $L = x_{k+1} + x_{k+2} + \cdots + x_{n-k}$.

A key property of the polynomial Hamiltonians is that they are (n,k)-admissible:

Proposition 4.7. *For $i = 0, 1, \ldots, k$, the polynomial first integral $K_i^{(n,k),b}$ is (n,k)-admissible.*

Proof. As said, we write in the proof K_i for $K_i^{(n,k)}$ and K_i^b for $K_i^{(n,k),b}$. The fact that K_i is (n,k)-admissible follows from the following observation made in [6, Cor. 3.5(4)]: if we denote for $\underline{m} \in \mathcal{S}_i^{(n,k)}$ by $\underline{m}'$ the vector $\underline{m}$ with its middle entry m_{i+1} replaced by m'_{i+1}, then $\underline{m}' \in \mathcal{S}_i^{(n,k)}$ when $k < m'_{i+1} < n-k+1$. According to the formula (4.4) for K_i this means that when some term of K_i contains a variable x_j with $k+1 \leqslant j \leqslant n-k$, then it contains also a similar term with x_j replaced by any x_l with $k+1 \leqslant l \leqslant n-k$. Considering the sum of these substitutions yields a polynomial which is divisible by L. Therefore, K_i is (n,k)-admissible. Let us show that if for some $p \geqslant 1$, $K_i^{(p-1)}$ is (n,k)-admissible, then so is $K_i^{(p)} = e^{\mathcal{D}_{(p)}} K_i^{(p-1)}$. We can write $K_i^{(p-1)} = LK' + K''$, where K' and K'' are independent of $x_{k+1}, \ldots, x_{n-k}$, and $\mathcal{D}_{(p)}$ differentiates with respect to the variables $x_{k+1-\left[\frac{p}{2}\right]}$ and $x_{n-\left[\frac{p-1}{2}\right]}$. When $p \neq 1$ and $p \neq 2k+1$, these variables are outside the range $k+1, ..., n-k$, hence $\mathcal{D}_{(p)} K_i^{(p-1)} = L\mathcal{D}_{(p)} K' + \mathcal{D}_{(p)} K''$, with $\mathcal{D}_{(p)} K'$ and $\mathcal{D}_{(p)} K''$ independent of $x_{k+1}, \ldots, x_{n-k}$, so that $K_i^{(p)}$ is (n,k)-admissible. For $p = 1$,

$$\begin{aligned} K_i^{(1)} &= e^{\mathcal{D}_{(1)}} K_i^{(0)} = K_i^{(0)} + b_1 \frac{\partial^2}{\partial x_{k+1} \partial x_n} (LK' + K'') \\ &= K_i^{(0)} + L\mathcal{D}_{(1)} K' + \mathcal{D}_{(1)} K'' + b_1 \frac{\partial K'}{\partial x_n} , \end{aligned}$$

showing that $K_i^{(2)}$ is also (n,k)-admissible. The proof for $p = 2k+1$ is very similar, since $\mathcal{D}_{(2k+1)}$ differentiates with respect to the variables x_1 and x_{n-k}. □

We are now ready to show that every polynomial integral is in involution with every rational integral.

Proposition 4.8. *For $\ell = 1, 2, \ldots, \left[\frac{n-1}{2}\right] - k$ and $i = 0, 1, \ldots, k$,*

$$\left\{ F_\ell^{(n,k),b}, K_i^{(n,k),b} \right\}_b^{(n,k)} = 0 .$$

Proof. In view of Proposition 4.7, we can write $K_i^b = LK' + K''$ where K' and K'' are independent of $x_{k+1}, \dots, x_{n-k}$. Using Lemma 4.5 twice,

$$\left\{F_\ell^b, K_i^b\right\}_b = \left\{F_\ell^b, LK' + K''\right\}_b = \left\{F_\ell^b, LK'\right\}_b = K'\left\{F_\ell^b, L\right\}_b \ .$$

The Hamiltonian H is of course also (n,k)-admissible, $H = L + H''$, with H'' independent of the variables $x_{k+1}, \dots, x_{n-k}$. Using that F_ℓ is a first integral (Theorem 4.2) and Lemma 4.5, we can conclude that

$$\left\{F_\ell^b, K_i^b\right\}_b = K'\left\{F_\ell^b, H - H''\right\}_b = -K'\left\{F_\ell^b, H''\right\}_b = 0 \ .$$

□

Combining the results obtained in this section, we can state and prove the main theorem on the integrability of the systems $\mathrm{LV}_b(n,k)$, with $n \geqslant 2k+1$. We denote, in that order, by $H_1^{(n,k)}, H_2^{(n,k)}, \dots, H_{n-k-2}^{(n,k)}$ the following first integrals:

$$F_1^b = G_1^b, F_2^b, \dots, F_{r-1}^b, G_2^b, \dots, G_{r-1}^b, F_r^b = G_r^b, \text{ when } n-k \text{ is even,}$$
$$F_1^b, \dots, F_{r-1}^b, G_1^b, \dots, G_{r-1}^b, F_r^b = G_r^b, \text{ when } n-k \text{ is odd,}$$

where $r := \left[\frac{n-k}{2}\right]$.

Theorem 4.9. *Consider the system $LV_b(n,k)$, where $n \geqslant 2k+1$.*

(1) *When $n > 2k+1$, $LV(n,k)$ is non-commutative integrable of rank $k+1$, with first integrals*
$$H = K_0^{(n,k),b}, K_1^{(n,k),b} \dots, K_k^{(n,k),b}, H_1^{(n,k),b}, H_2^{(n,k),b}, \dots, H_{n-2k-2}^{(n,k),b} \ .$$
The first $k+1$ functions of this list have independent Hamiltonian vector fields and are in involution with every function of the complete list.

(2) *$LV(n,k)$ is Liouville integrable with first integrals*
$$H = K_0^{(n,k),b}, K_1^{(n,k),b} \dots, K_k^{(n,k),b}, H_1^{(n,k),b}, H_2^{(n,k),b}, \dots, H_{s-1}^{(n,k),b} \ ,$$
where $s := \left[\frac{n+1}{2}\right] - k$.

Proof. We first consider (1). We have already checked the first item of Definition 4.1, namely that the $k+1$ polynomials K_i are first integrals of $\mathrm{LV}(n,k)$, and are in involution with both the polynomial and rational integrals (Section 4.1 and Proposition 4.8). We need to check the second item which says that the differentials of these first integrals are independent on a dense open subset of $\mathbb{R}^n$, and similarly for the Hamiltonian vector fields associated to the polynomial integrals. To do this we use the fact that the undeformed first integrals have this property, as they define a non-commutative integrable system of rank $k+1$ (see [6, Theorem 1.1]). Since all integrals are rational functions and since the Poisson structure is polynomial, it suffices to prove that the differentials (resp. vector fields) are independent at some point. The argument is the same as the one used in Section 3.3 to derive the independence of the integrals of $\mathrm{LV}_b(n,0)$ from the independence of the integrals of $\mathrm{LV}(n,0)$: since the property is true at some point P when all parameters are zero, it is still true on a neighborhood of P for small values of the parameters; by rescaling the variables and parameters, one finds that at P the property is true for all values of the parameters. This proves (1). We now consider (2), the Liouville integrability. Since the rank of Π_b is n when n is even and $n-1$ when n is odd, we need $n/2$ independent integrals in involution when n is even and $\frac{n+1}{2}$ when n is odd. Clearly, the above list in (2) contains $k+s=\frac{n+1}{2}$ functions, which is the right number, we know that they are pairwise in involution, and by the above argument they are independent. So they define a Liouville integrable system. □

Item (1) in the theorem takes a slightly different form when $n=2k+1$. The constructed first integrals are then polynomial and they define a non-commutative integrable system of rank k, which is equivalent to saying that it is Liouville integrable, which is stated in (2), and was already proven in [7]. The reason of this drop in the rank of the non-commutative integrability when $n=2k+1$ is because, even though we have $k+1$ polynomial integrals that are in involution with all integrals, like the general case of the $\mathrm{LV}_b(n,k)$ systems, now one of these $k+1$ polynomial integrals is a Casimir and in order to establish the condition (2) of Definition 4.1 one has to exclude the Casimir from the first set of integrals.

4.4. **Examples.** For explicitness, we give below two examples, $\mathrm{LV}_b(4,1)$, which is the smallest new system with $k>0$ and $\mathrm{LV}_b(7,1)$, where one can see some non-trivial examples of the integrals which we consider.

Example 4.10. For $n=4$ and $k=1$ the matrices $A^{(4,1)}$ and $B^{(4,1)}$ are given by

$$A^{(4,1)}=\begin{pmatrix}0&1&1&-1\\-1&0&1&1\\-1&-1&0&1\\1&-1&-1&0\end{pmatrix},\quad B^{(4,1)}=\begin{pmatrix}0&0&b_3&-b_2\\0&0&0&b_1\\-b_3&0&0&0\\b_2&-b_1&0&0\end{pmatrix}.$$

The corresponding system $\mathrm{LV}_b(4,1)$ is given by

$$\begin{aligned}\dot{x}_1&=x_1(x_2+x_3-x_4)+b_3-b_2\ ,\\ \dot{x}_2&=x_2(-x_1+x_3+x_4)+b_1\ ,\\ \dot{x}_3&=x_3(-x_1-x_2+x_4)-b_3\ ,\\ \dot{x}_4&=x_4(x_1-x_2-x_3)+b_2-b_1\ .\end{aligned}$$

Besides the Hamiltonian $H=x_1+x_2+x_3+x_4$ it has an additional polynomial integral $K_1=(x_1x_4+b_2)(x_2+x_3)+b_3x_4+b_1x_1$ which is easily seen to be a $(4,1)$-admissible polynomial. The above two polynomials give the Liouville integrability of the system $\mathrm{LV}_b(4,1)$ which coincides in this case with the non-commutative integrability of rank $k+1=2$ just like in all $\mathrm{LV}_b(2k+2,k)$ systems.

Example 4.11. We now consider the case $n = 7$ with $k = 1$. The matrix $A := A^{(7,1)}$ is the skew-symmetric Toeplitz matrix with first line $(0, 1, 1, 1, 1, 1, -1)$ and $B := B^{(7,1)}$ is the skew-symmetric matrix whose only non-zero upper triangular entries are $b_{1,6} = b_3$, $b_{1,7} = -b_2$ and $b_{2,7} = b_1$. The corresponding system $\mathrm{LV}_b(7, 1)$ is given by the equations

$$\dot{x}_i = \sum_{j=1}^{7} \left(A_{i,j} x_i x_j + b_{i,j}\right), \quad \text{for} \quad i = 1, 2, \dots, 7.$$

Besides the Hamiltonian $H = x_1 + x_2 + \cdots + x_7$, the system $\mathrm{LV}_b(7, 1)$ has one more independent polynomial first integral K_1, given by

$$K_1 = (x_2 + x_3 + x_4 + x_5 + x_6)(x_1 x_7 - b_{1,7}) + b_{1,6} x_7 + b_1 x_6 \,,$$

which is a $(7, 1)$-admissible polynomial. It has also three rational first integrals given by

$$F_1 = \frac{(x_1 x_7 + b_2) x_2 x_4 x_6 + b_3 x_2 x_4 x_7 + b_1 x_1 x_4 x_6 + b_3 b_1 x_4}{x_3 x_5},$$

$$F_2 = \frac{(x_1 x_7 + b_2)(x_2 + x_3 + x_4) x_6 + b_3 (x_2 + x_3 + x_4) x_7 + b_1 (x_1 x_6 + b_3)}{x_5},$$

and $G_2 = \imath F_2$. The rank of the Poisson structure $\Pi_b^{(7,1)}$ is 6 and F_1 is a Casimir, invariant under $\imath^*$. It can be seen that the above integrals are obtained from the undeformed ones (obtained by setting the parameters equal to zero), by applying on them the operator e^{D_b} which now becomes

$$e^{D_b} = \left(I + b_3 \frac{\partial^2}{\partial x_1 \partial x_6}\right) \left(I + b_2 \frac{\partial^2}{\partial x_1 \partial x_7}\right) \left(I + b_1 \frac{\partial^2}{\partial x_2 \partial x_7}\right).$$

The system $\mathrm{LV}_b(7, 1)$ is non-commutative integrable of rank 2 with first integrals H, K_1, F_1, F_2, G_2 and is also Liouville integrable with first integrals H, K_1, F_1, F_2 or H, K_1, F_1, G_2.

5. Discretization of $\mathrm{LV}_b(n, 0)$

In this section we construct a family of discretizations of $\mathrm{LV}_b(n, 0)$. They are obtained from a discrete zero curvature condition, which is the compatibility condition of a linear system $L\Psi = \lambda\Psi, \tilde{\Psi} = N\Psi$, where L is the Lax matrix of $\mathrm{LV}(n, 0)$, which appears in (2.4). We prove that an important class of these discretizations, which includes the Kahan (also called Kahan-Hirota-Kimura) discretization of $\mathrm{LV}(n, 0)$ has the following integrability properties: it has the rational integrals of $\mathrm{LV}(n, 0)$ as invariants, and so it is both Liouville and superintegrable; also it has an invariant measure.

Throughout this section, $(n, k) = (n, 0)$ is fixed and so we will drop $(n, 0)$ from the notation for the invariants, the Poisson structure, and so on. Also, since we have in this case only one parameter $b_{1,n}$, we will denote it by β, as we did in Section 3.

5.1. **Preliminaries.** We first recall a few basic definitions and properties of discrete maps and their integrability. By a *discrete map* of $\mathbb{R}^n$ we mean an algebra homomorphism $\Phi : \mathbb{R}(x_1, x_2, \dots, x_n) \to \mathbb{R}(x_1, x_2, \dots, x_n)$, where $x_1, \dots, x_n$ are as elsewhere in this paper the natural coordinates on $\mathbb{R}^n$. Such a map is the pullback of a unique rational map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, i.e., for any rational function F, one has $\Phi(F) = \phi^*(F) = F \circ \phi$. We will also use the convenient abbreviations $\tilde{F}$ for $\Phi(F)$. Similarly for a matrix $P = (p_{i,j})$ whose entries are rational functions of $\mathbb{R}^n$, we will write $\tilde{P}$ for the matrix $(\tilde{p}_{i,j})$.

When $\mathbb{R}^n$ is equipped with a Poisson structure Π, then saying that Φ is a homomorphism of Poisson algebras is tantamount to saying that ϕ is a Poisson map; we will simply say that Φ *preserves* the Poisson structure Π. Also, on $\mathbb{R}^n$ we have a natural n-form, $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ which allows us to identify rational measures with rational n-forms and with rational functions. We will say that Φ is *measure preserving*, with preserved measure F, if it preserves the $n-$form $F\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$ in the sense that

$$F\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n = \tilde{F}\mathrm{d}\tilde{x}_1 \wedge \mathrm{d}\tilde{x}_2 \wedge \ldots \wedge \mathrm{d}\tilde{x}_n\,.$$

A rational function F is called an *invariant* of Φ if $\tilde{F} = F$. We also recall the definition of an integrable map [19].

Definition 5.1. Suppose that Φ is a discrete map of $\mathbb{R}^n$.

(1) The map Φ is *Liouville integrable* if there exist $n - r$ functionally independent invariants of Φ, which are in involution with respect to a Poisson structure Π, where r is half the rank of Π.
(2) The map Φ is *superintegrable* if it has $n-1$ functionally independent invariants and is measure preserving.

In order to simplify some of the computations below, we introduce a few more notions and notations which are related to the symmetries of the Lax matrix L of $\mathrm{LV}(2m + 1, m)$. Recall that

$$L = X + \lambda^{-1}\Delta + \lambda M \tag{5.1}$$

is the square matrix of size $2m + 1$, where

$$X_{i,j} := \delta_{i,j+m}x_i\ , \quad \Delta_{i,j} := b_{i+m,j}\delta_{i,j}\ , \quad M_{i,j} := \delta_{i+1,j}\ . \tag{5.2}$$

The entries of the above matrices are functions of $x_1, x_2, \dots, x_{2m+1}$ where all the indices are considered modulo $2m + 1$. It is a τ-circulant matrix, in the sense of the following definition:

Definition 5.2. Let $p, q \in \mathbb{N}$ and let τ be a map $\tau : \mathbb{R}^p \to \mathbb{R}^p$ of order q. Let $C = (c_{i,j})$ be an $q \times q$ matrix whose elements are functions on $\mathbb{R}^p$. Then we call C a τ-*circulant* matrix if it has the following property:

$$c_{i+1,j+1} = \tau^* c_{i,j}, \quad \text{for all} \quad i, j = 1, 2, \dots, q\,.$$

If $\underline{v}$ is the first line of C we will also denote C by $\mathcal{C}(\underline{v})$.

Taking for τ the identity map, one recovers the definition of a circulant matrix. As in the case of a circulant matrix, it is clear that a τ-circulant matrix is determined by its first line. It is easy to see that a matrix C is τ-circulant if and only if $MCM^{-1} = \tau^*(C)$, where M is the circular permutation matrix, defined in (5.2), and $\tau^*(C)$ is the matrix obtained by applying τ^* to all entries of C. As an immediate consequence, we see that the product of two τ-circulant matrices is τ-circulant; to compute the product of two such matrices one has to compute only the first line of the product, a property which we will find useful.

In order to see that the Lax operator L is τ-circulant, let us consider $\mathbb{R}^{4m+2}$ with coordinates $x_i, b_{m+i,i}$ for $i = 1, 2, \ldots, 2m+1$. The map τ from $\mathbb{R}^{4m+2}$ to itself is defined by $\tau^*(x_i) = x_{i+1}$ and $\tau^*(b_{m+i,i}) = b_{m+i+1,i+1}$ for $i = 1, 2, \ldots, 2m+1$. Since all indices are considered modulo $2m+1$, the map τ is of order $2m+1$. To see that L is τ-circulant, we show that the matrices which define L are τ-circulant, which is clear from the following formulas:

$$X = \mathcal{C}(0, 0, \ldots, 0, x_1, 0, \ldots, 0)\ , \quad \text{with } x_1 \text{ in the } (m+2)\text{-th position},$$
$$\Delta = \mathcal{C}(b_{m+1,1}, 0, \ldots, 0)\ , \quad M = \mathcal{C}(0, 1, 0, \ldots, 0)\ .$$

5.2. **Discrete maps from a linear problem.** Recall that $\mathrm{LV}_b(n, 0)$ is by definition a reduction of $\mathrm{LV}_b(2m+1, m)$, with $m := n-1$. The discrete maps of $\mathrm{LV}(n, 0)$ which we will consider below are in the same sense reductions of discrete maps of $\mathrm{LV}_b(2m+1, m)$, which we first construct. We construct a discrete map Φ by considering the compatibility conditions of the linear system

$$L\Psi = \lambda\Psi, \tilde{\Psi} = N\Psi\ , \tag{5.3}$$

where L is the Lax matrix of $\mathrm{LV}_b(2m+1, m)$, recalled above, and Ψ is an n-dimensional eigenvector of L. Recall that $\tilde{\Psi}$ is the vector Ψ with Φ applied to its entries. The $(2m+1) \times (2m+1)$ matrix N is defined as

$$N = D - \lambda K\ , \tag{5.4}$$

where $K_{i,j} := \delta_{i,j+m}$ and $D_{i,j} := \delta_{i,j} d_i$ for some functions d_i that will be determined from the compatibility condition of (5.3), which reads $\tilde{L}N = NL$. Since N is invertible, it means that $\tilde{L} = NLN^{-1}$ and therefore the coefficients of the characteristic polynomial of L, which are rational functions on $\mathbb{R}^{2m+1}$, are invariants of Φ. The above ansatz for N was taken so that NLN^{-1} equals L at the entries with constant values. Therefore, the compatibility condition $\tilde{L}N = NL$ reduces to a system of equations for the $\tilde{x}_i$ and d_i variables, which we make explicit in the following proposition:

Proposition 5.3. *The compatibility condition $\tilde{L}N = NL$ of the linear system* (5.3) *is equivalent to the following system of equations:*

$$d_{i+1} - d_i + x_{m+1+i} - \tilde{x}_i = 0\ , \ \text{ and } \quad d_{m+1+i}\tilde{x}_i - d_i x_i + b_{i,m+1+i} - b_{m+i,i} = 0\ , \tag{5.5}$$

for $i = 1, 2, \ldots, 2m+1 = 2n-1$.

Proof. Notice first that N is τ-circulant, if we extend τ^* to the variables d_i, by setting $\tau^* d_i := d_{i+1}$. The compatibility condition $\tilde{L}N = NL$ amounts therefore to the equality of the first rows of $\tilde{L}N$ and NL, which are respectively given by

$$\left(\frac{1}{\lambda}b_{m+1,1}d_1, \lambda(d_2 - \tilde{x}_1), 0, \dots, 0, d_{m+2}\tilde{x}_1 - b_{m+1,1}, -\lambda^2, 0, \dots, 0\right) ,$$
$$\left(\frac{1}{\lambda}b_{m+1,1}d_1, \lambda(d_1 - x_{m+2}), 0, \dots, 0, d_1 x_1 - b_{1,m+2}, -\lambda^2, 0, \dots, 0\right) ,$$

where the non-zero components of the above vectors appear at the positions $1, 2, m+2$ and $m+3$. From the equality of these first lines, we get

$$d_2 - d_1 + x_{m+2} - \tilde{x}_1 = 0 , \quad \text{and} \quad d_{m+2}\tilde{x}_1 - d_1 x_1 + b_{1,m+2} - b_{m+1,1} = 0 ,$$

which is (5.5) for $i = 1$. The other equations follow from it by τ-circularity. $\square$

We now reduce these equations to $\mathrm{LV}_b(n, 0)$, setting $x_i = \tilde{x}_i = 0$ for $i = n+1, n+2, \dots, 2m+1$ and $b_{m+i,i} = 0$ for $i = 2, 3, \dots, 2m+1$, where we recall that $m = n-1$ and that we denote the single parameter $b_{1,n}$ of $\mathrm{LV}(n, 0)$ as $b_{1,n} = \beta$. The system (5.5) is then transformed to the following one:

$$\begin{aligned}
\tilde{x}_i &= d_{i+1} - d_i , && i = 1, \dots, n-1, \\
\tilde{x}_n &= x_1 + d_{n+1} - d_n , && \\
x_{i+1} &= d_{n+i} - d_{n+i+1} , && i = 1, \dots, n-1, \\
d_{n+i+1}\tilde{x}_{i+1} &= d_{i+1}x_{i+1} , && i = 1, \dots, n-2, \\
d_{n+1}\tilde{x}_1 &= d_1 x_1 - \beta , && \\
d_1 \tilde{x}_n &= d_n x_n + \beta , &&
\end{aligned} \tag{5.6}$$

where the first three equation are instances of the first equation in (5.5) and the last three equations of the second one. Before solving the above system, we recall from Section 3.4 the alternative coordinates $u_1, \dots, u_n$ for $\mathbb{R}^n$, in which the system $\mathrm{LV}_b(n, 0)$ completely separates. They are defined by $u_i = \sum_{j=1}^n x_j$ for all $i = 0, 1, 2, \dots, n$. For $i = n$, u_n is just the Hamiltonian, $u_n = H = x_1 + x_2 + \ldots + x_n$.

Proposition 5.4. *For any rational function $\mathcal{R} \in \mathbb{R}(x_1, \dots, x_n)$, different from the n functions $u_i - H$, with $i = 1, \dots, n$, the reduced compatibility equations (5.6) have a unique solution for $\tilde{x}_1, \dots, \tilde{x}_n$ and for $d_2, \dots, d_{2n-1}$, with $d_1 = \mathcal{R}$. It is given by*

$$\begin{aligned}
\tilde{x}_i &= d_{i+1} - d_i , && i = 1, 2, \dots, n-1 , \\
\tilde{x}_n &= x_1 + d_{n+1} - d_n , && \\
d_i &= \frac{\mathcal{R}(\mathcal{R} + H) - \beta}{\mathcal{R} + H - u_{i-1}} , && i = 2, 3, \dots, n , \\
d_{n+i} &= \mathcal{R} + H - u_i , && i = 1, 2, \dots, n-1 .
\end{aligned} \tag{5.7}$$

Proof. We first show how the third and fourth equations in (5.7) are derived from (5.6). The last equation is obtained from the third equation in (5.6): for $i = 1, \ldots, n-1$,

$$H - u_i = \sum_{j=i+1}^{n} x_j = \sum_{j=i+1}^{n} (d_{n+j-1} - d_{n+j}) = d_{n+i} - \mathcal{R} ,$$

where we have used that, by periodicity, $d_{2n} = d_{2m+2} = d_1 = \mathcal{R}$. In order to derive the third equation in (5.7), one first uses the first three equations in (5.6) to substitue $\tilde{x}_i$ $(i = 1, \ldots, n)$ and x_i $(i = 2, 3, \ldots, n)$ in the fourth and fifth equations in (5.6), to obtain, in that order,

$$\begin{aligned} d_{i+1}d_{n+i} &= d_{i+2}d_{n+1+i} , \quad i = 1, \ldots, n-2 , \\ \mathcal{R}(x_1 + d_{n+1}) &= d_2 d_{n+1} + \beta . \end{aligned} \tag{5.8}$$

The first equation in (5.8) says that $d_{i+1}d_{n+i}$ is independent of i for $i = 1, \ldots, n-2$, while the second equation says that this constant value is equal to $\mathcal{R}(x_1 + d_{n+1}) - \beta$,

$$d_{i+1}d_{n+i} = \mathcal{R}(x_1 + d_{n+1}) - \beta = \mathcal{R}(\mathcal{R} + H) - \beta , \tag{5.9}$$

for $i = 1, \ldots, n-2$. By our assumption on $\mathcal{R}$, the $d_{n+i} = \mathcal{R} + H - u_i$ with $i = 1, \ldots, n-1$ are all different from zero, so that we can divide (5.9) by d_{n+i}. It yields the third equation in (5.7). This shows that (5.7) is the only possible solution for (5.6) with $d_1 = \mathcal{R}$. That it is indeed a solution is easily verified by substituting the formulas (5.7) in (5.6). $\square$

We now define a discrete map using the solution of (5.6) given in the previous proposition. Let $\mathcal{R}$ be a rational map, with $\mathcal{R} \neq u_i - H$ for all $i = 1, 2, \ldots, n-1$, and let $\Phi_{\mathcal{R}}$ be the discrete map $x_i \mapsto \tilde{x}_i$, defined by the formulas

$$\begin{aligned} \tilde{x}_i &= x_i \frac{\mathcal{R}(\mathcal{R} + H) - \beta}{(\mathcal{R} + H - u_i)(\mathcal{R} + H - u_{i-1})} , \quad i = 1, 2, \ldots, n-1 , \\ \tilde{x}_n &= \frac{x_n(\mathcal{R} + H) + \beta}{\mathcal{R} + x_n} , \end{aligned} \tag{5.10}$$

where we have set $u_0 = 0$. Using the first n equations in (5.6), we get $\tilde{u}_i = d_{i+1} - \mathcal{R}$ and therefore the map is given in terms of the coordinates u_i by $\tilde{u}_n = u_n$ and

$$\tilde{u}_i = \frac{\mathcal{R}u_i - \beta}{\mathcal{R} + H - u_i}, \quad i = 1, 2, \ldots, n-1 , \tag{5.11}$$

Since the map $\Phi_{\mathcal{R}}$ is by construction isospectral, it has the coefficients of the characteristic equation of the Lax matrix L as invariants. However, as we noted just after equation (4.4), we get in this way only one invariant, namely the Hamiltonian H. We show in the following proposition that $\Phi_{\mathcal{R}}$ also preserves the rational first integrals of $\mathrm{LV}_b(n, 0)$.

Proposition 5.5. *Let P be any point of $\mathbb{R}^n$ for which $Q := \phi_{\mathcal{R}}(P)$ is defined. Then Q belong to the integral curve of the continuous system $LV(n,0)$ starting at P. In particular, the discrete map $\Phi_{\mathcal{R}}$ preserves all the integrals of $LV_b(n,0)$.*

Proof. Since $\Phi_{\mathcal{R}}$ is a rational map and the integrals of $\mathrm{LV}_b(n,0)$ are rational functions, it suffices to show that for a generic P of $\mathbb{R}^n$ for which $Q := \phi_{\mathcal{R}}(P)$ is defined, Q belongs to the integral curve of the continuous system $\mathrm{LV}(n,0)$ starting at P.

We use the notation of the proof of Proposition 3.6: we denote by γ_P the integral curve of (3.10) starting from P, and we write $u_i(t) = u_i(\gamma_P(t))$. We denote by h the value of the Hamiltonian at P and by Δ_0 a square root of $h^2 + 4\beta$, which may be real or imaginary. Also, let r_0 denote the value of $\mathcal{R}$ evaluated at P and $Q = (Q_1, Q_2, \ldots, Q_n)$.

It is clear from the above that we only need to show that for each P such that Q is defined there exists a t, depending only on P, such that $Q_i = u_i(t)$ for all $i = 1, 2, \ldots, n-1$. It is also clear that we may consider our system $\mathrm{LV}_b(n,0)$ living on $\mathbb{C}^n$ and therefore the integral curves are defined on all of $\mathbb{C}$ minus a discrete set (see Section 3.4 for details and comments).

We only need to consider the case that $\Delta_0 \neq 0$. In this case the solution of $\mathrm{LV}_b(n,0)$, for

$$t = -\frac{\ln\left(\frac{h+2r_0+\Delta_0}{h+2r_0-\Delta_0}\right)}{\Delta_0},$$

gives $Q_i = u_i(t)$ for all i, as it can be seen by comparing the formulas (5.11) and the explicit solution of $\mathrm{LV}_b(n,0)$ given in Proposition 3.6. □

5.3. **Integrable discretization of $\mathbf{LV}_b(n,)$.** For a general rational function $\mathcal{R}$, the map $\Phi_{\mathcal{R}} = \phi_{\mathcal{R}}^*$ cannot be expected to have any integrability properties. We establish in this subsection a few results under the assumption that $\mathcal{R}$ is a first integral of $\mathrm{LV}(n,0)$, or under the stronger hypothesis that $\mathcal{R}$ depends on H only. We first prove that, under these conditions, $\Phi_{\mathcal{R}}$ is birational.

Proposition 5.6. *Suppose that $\mathcal{R}$ is a first integrals of $LV_b(n,0)$. Then $\phi_{\mathcal{R}}$ is a birational map, so that $\Phi_{\mathcal{R}}$ is an algebra isomorphism.*

Proof. Let $\mathcal{R}$ be as announced, so that $\tilde{\mathcal{R}} = \mathcal{R}$, in view of Proposition 5.5. Let Ψ be the involutive algebra homomorphism, defined by $\Psi(x_i) := \tilde{x}_{n+1-i}$ and $\Psi(d_i) := d_{2n+1-i}$ for $i = 1, \ldots, n$; notice that d_1 is fixed under Ψ. If we apply Ψ to the reduced compatibility equations (5.6) we get the same set of equations: the first and third equations are permuted, as well as the last ones, while the other two are unchanged. Since we know from Proposition (5.4) that given $d_1 := \mathcal{R}$ the reduced compatibility equations (5.6) have a unique solution for $\tilde{x}_1, \ldots, \tilde{x}_n$ and for $d_2, \ldots, d_{2n-1}$, in terms of $x_1, \ldots, x_n$, this means given $d_1 := \tilde{\mathcal{R}} = \mathcal{R}$, they also have a unique solution for $x_1, \ldots, x_n$

and for $d_2, \ldots, d_{2n-1}$, in terms of $\tilde{x}_1, \ldots, \tilde{x}_n$. Therefore, the map $\Phi_{\mathcal{R}}$, defined by the solutions of the system (5.6), is birational. $\square$

We now prove that $\Phi_{\mathcal{R}}$ is, under the same assumption on $\mathcal{R}$, measure preserving.

Proposition 5.7. *Suppose that $\mathcal{R}$ is a first integral of $LV_b(n,0)$. Then the discrete map $\Phi_{\mathcal{R}}$ preserves the rational n-form*

$$\Omega_b := \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n}{x_1x_2\ldots x_n + \beta x_2x_3\ldots x_{n-1}} \; .$$

Proof. We need to show that

$$\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n}{x_1x_2\ldots x_n + \beta x_2x_3\ldots x_{n-1}} = \frac{\mathrm{d}\tilde{x}_1 \wedge \mathrm{d}\tilde{x}_2 \wedge \cdots \wedge \mathrm{d}\tilde{x}_n}{\tilde{x}_1\tilde{x}_2\ldots \tilde{x}_n + \beta \tilde{x}_2\tilde{x}_3\ldots \tilde{x}_{n-1}} \; .$$

Since the coordinate change between the coordinates u_i and x_i have triangular form, and since the functions $\tilde{u}_i$ depend in the same way on the $\tilde{x}_i$, i.e., $\tilde{u}_i = \sum_{j=1}^{i} \tilde{x}_i$, we have that

$$\left|\frac{\partial(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)}{\partial(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)}\right| = \left|\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}\right| = 1 \; ,$$

where the above determinants are the Jacobian determinants of these two transformations. This implies that we need to show that

$$\frac{\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n}{x_1x_2\ldots x_n + \beta x_2x_3\ldots x_{n-1}} = \frac{\mathrm{d}\tilde{u}_1 \wedge \mathrm{d}\tilde{u}_2 \wedge \cdots \wedge \mathrm{d}\tilde{u}_n}{\tilde{x}_1\tilde{x}_2\ldots \tilde{x}_n + \beta \tilde{x}_2\tilde{x}_3\ldots \tilde{x}_{n-1}} \; .$$

We assume for the moment that $\mathcal{R}$ is any function and we denote by $\mathcal{R}_i$ the partial derivatives $\frac{\partial \mathcal{R}}{\partial u_i}$ and by $u_{i,j}$ the partial derivatives $\frac{\partial \tilde{u}_i}{\partial u_j}$. The explicit formulas for $\tilde{u}_i$ give, for any $i = 1, 2, \ldots, n-1$ and any $j \neq i, n$, that

$$u_{i,j} = \frac{\mathcal{R}_j \dot{u}_i}{(\mathcal{R} + H + u_i)^2} \; . \tag{5.12}$$

Also, $u_{i,i} = \frac{\mathcal{R}_i \dot{u}_i + \mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R}+H+u_i)^2}$ for all $i = 1, 2, \ldots, n-1$. For the partial derivatives of the $\tilde{u}_i$ with respect to u_n, as we will see, the explicit formulas are not important, we will only use the fact that $\tilde{u}_n = u_n$ and therefore $u_{n,j} = \delta_{n,j}$, which holds for any function $\mathcal{R}$. Differentiating (5.11) and rearranging, we get

$$\mathrm{d}\tilde{u}_i = \sum_{j=1}^{n} u_{i,j} \mathrm{d}u_j \; , \quad i = 1, 2, \ldots, n \; .$$

Taking the wedge product of the above n equations we obtain

$$\mathrm{d}\tilde{u}_1 \wedge \mathrm{d}\tilde{u}_2 \wedge \cdots \wedge \mathrm{d}\tilde{u}_n = \det(U)\, \mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n \; ,$$

where $U = (u_{i,j})$. The last line of U is the vector $(0, 0, \ldots, 1)$ and therefore expanding the determinant $\det(U)$ with respect to the last line we get $\det(U) = \det(V)$ where V is the minor of U obtained by removing its last row and last column. According to the formulas (5.12) the determinant of the $(n-1)\times(n-1)$ matrix V has the following form:

$$V=\frac{1}{s}\det\begin{pmatrix}\mathcal{R}_1\dot u_1+r & \mathcal{R}_2\dot u_2 & \mathcal{R}_3\dot u_3 & \cdots & \mathcal{R}_n\dot u_{n-1}\\ \mathcal{R}_1\dot u_1 & \mathcal{R}_2\dot u_2+r & \mathcal{R}_3\dot u_3 & \cdots & \mathcal{R}_n\dot u_{n-1}\\ \mathcal{R}_1\dot u_1 & \mathcal{R}_2\dot u_2 & \mathcal{R}_3\dot u_3+r & \cdots & \mathcal{R}_n\dot u_{n-1}\\ & \vdots & & \ddots & \vdots\\ \mathcal{R}_1\dot u_1 & \mathcal{R}_2\dot u_2 & \mathcal{R}_3\dot u_3 & \cdots & \mathcal{R}_n\dot u_{n-1}+r\end{pmatrix},$$

where $s=\prod_{j=1}^{n-1}(\mathcal{R}+H+u_i)^2$ and $r=\mathcal{R}H+\mathcal{R}^2-\beta$. The above matrix is written as $W+rI_{n-1}$, where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix and W has n equal lines. This means that W has only two eigenvalues λ_1 and λ_2. The first one is $\lambda_1=\sum_{j=1}^{n-1}\mathcal{R}_j\dot u_j$, which is of multiplicity 1 and the other one is $\lambda_2=0$ of multiplicity $n-2$. In the particular case where $\mathcal{R}$ is a first integral of $\mathrm{LV}_b(n,0)$, the eigenvalue λ_1 reduces to zero (since $\dot u_n=0$). This shows that, in that case,

$$\det(V)=\frac{r^{n-1}}{s}=\frac{(\mathcal{R}H+\mathcal{R}^2-\beta)^{n-1}}{\prod_{i=1}^{n-1}(\mathcal{R}+H-u_i)^2}\;.$$

Therefore, what we need to show is that

$$\prod_{i=1}^{n-1}\frac{\mathcal{R}H+\mathcal{R}^2-\beta}{(\mathcal{R}+H-u_i)^2}=\frac{\tilde x_1\tilde x_2\dots\tilde x_n+\beta\tilde x_2\tilde x_3\dots\tilde x_{n-1}}{x_1x_2\dots x_n+\beta x_2x_3\dots x_{n-1}}\;.$$

A comparison with the explicit formulas (5.10) gives that

$$\frac{x_2x_3\dots x_{n-1}}{\tilde x_2\tilde x_3\dots\tilde x_{n-1}}\frac{(\mathcal{R}H+\mathcal{R}^2-\beta)^{n-1}}{\prod_{i=1}^{n-1}(\mathcal{R}+H-u_i)^2}=\frac{\mathcal{R}H+\mathcal{R}^2-\beta}{(\mathcal{R}+H-u_1)(\mathcal{R}+H-u_{n-1})}\;.$$

To complete the proof, it remains to be shown that

$$\frac{\tilde x_1\tilde x_n+\beta}{x_1x_n+\beta}=\frac{\mathcal{R}H+\mathcal{R}^2-\beta}{(\mathcal{R}+H-u_1)(\mathcal{R}+H-u_{n-1})}\;.$$

This can be done by substituting the formulas for $\tilde x_1$ and $\tilde x_n$, given in (5.10). □

In order to preserve the Poisson structure, one needs stronger conditions on $\mathcal{R}$, as given in the following proposition:

Proposition 5.8. *Suppose that $\mathcal{R}$ is a rational function of the Hamiltonian H. Then the map $\Phi_{\mathcal{R}}$ preserves the Poisson structure Π_b.*

Proof. We give the proof using the coordinates u_i (see Section 3.4, in particular the formulas (3.9) for the Poisson structure in terms of these coordinates). According to (3.9) we need to show that $\{\tilde u_i,\tilde u_j\}_b=\tilde u_i(\tilde u_j-\tilde u_i)$ and that $\{\tilde u_\ell,u_n\}_b=\tilde u_\ell(\tilde u_n-\tilde u_\ell)+\beta$ for $1\leqslant i<j<n$ and $1\leqslant \ell<n$.

The derivatives of $\tilde{u}_i$, for any $i < n$, with respect to u_i and $u_n = H$ are

$$\begin{aligned} v_i &:= \frac{\partial \tilde{u}_i}{\partial u_i} = \frac{\mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_i)^2} , \\ v_i' &:= \frac{\partial \tilde{u}_i}{\partial u_n} = \frac{\mathcal{R}_H u_i H - \mathcal{R}_H u_i^2 + \mathcal{R}_H \beta - \mathcal{R} u_i + \beta}{(\mathcal{R} + H - u_i)^2}, \end{aligned} \tag{5.13}$$

where $\mathcal{R}_H = \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}H}$. We then have, for all i, j and ℓ as above,

$$\begin{aligned} \{\tilde{u}_i, \tilde{u}_j\}_b &= v_i v_j \{u_i, u_j\}_b + v_i v_j' \{u_i, u_n\}_b - v_i' v_j \{u_j, u_n\}_b \\ &= \frac{(u_i - u_j)(\beta - \mathcal{R} u_i)(\mathcal{R}H + \mathcal{R}^2 - \beta)}{(\mathcal{R} + H - u_i)^2 (\mathcal{R} + H - u_j)} = \tilde{u}_i(\tilde{u}_j - \tilde{u}_i) , \\ \{\tilde{u}_\ell, H\}_b &= \frac{(\mathcal{R}H + \mathcal{R}^2 - \beta)(H u_\ell - u_\ell^2 + \beta)^2}{(\mathcal{R} + H - u_\ell)^2} = \tilde{u}_\ell (H - \tilde{u}_\ell) + \beta , \end{aligned}$$

which establishes the required equalities, since $\tilde{u}_n = u_n = H$. □

The above propositions lead to the following theorem.

Theorem 5.9. *Let $\mathcal{R}$ be a rational function, depending on the Hamiltonian H only. Then the discrete map $\Phi_{\mathcal{R}}$ of $LV(n, 0)$ has the following properties:*

1. *It is birational;*
2. *It preserves the Poisson structure Π_b;*
3. *It is measure preserving: it preserves the volume form Ω_b;*
4. *It is Liouville integrable with H and the rational functions F_ℓ^b as invariants;*
5. *It is superintegrable with H and the rational functions F_ℓ^b and G_ℓ^b as invariants.*

Under the weaker hypothesis that $\mathcal{R}$ depends only on the invariants of LV$(n, 0)$, items (1), (3) and (5) still hold, but (2) and (4) may not hold.

5.4. **Kahan discretization of LV$_b(n, 0)$.** In this subsection we consider the Kahan discretization of the systems LV$_b(n, 0)$. We show that the Kahan map is of the form $\Phi_{\mathcal{R}}$, for a specific choice of the rational function $\mathcal{R}$, depending on the Hamiltonian H only, and so all integrability properties that we have seen in Theorem 5.9 hold for the Kahan map as well.

We first define the Kahan map for LV$_b(n, 0)$. Since the Kahan discretization commutes with any linear change of variables, we can do the Kahan discretization in the u_i coordinates, instead of the x_i coordinates, i.e., apply it on the vector field (3.10). Following the recipe [5], we obtain for the Kahan discretization with step size 2ϵ the following system of equations:

$$\bar{u}_i - u_i = \epsilon u_i (H - \bar{u}_i) + \epsilon \bar{u}_i (H - u_i) + 2\beta, \quad i = 1, 2, \ldots, n-1, \tag{5.14}$$

where we have used that $H = u_n$. Since H is a linear first integral of LV$_b(n, 0)$, it is an invariant for the Kahan map. The system (5.14) is diagonal with solution

$$\bar{u}_i = \frac{(1+\epsilon H)u_i + 2\epsilon\beta}{1-\epsilon H + 2\epsilon u_i}, \quad i = 1, 2, \ldots, n-1 \tag{5.15}$$

and $\bar{u}_n = u_n$. This defines the Kahan map. Comparing the formulas (5.15) and (5.11) it is clear that the Kahan map is of the form $\Phi_{\mathcal{R}}$, with

$$\mathcal{R} = -\frac{1+\epsilon H}{2\epsilon}\ . \tag{5.16}$$

Notice that $\mathcal{R}$ depends on H only. Therefore, we get by Theorem 5.9 the following results on the Kahan discretization of $\mathrm{LV}_b(n,0)$, which generalize the results on the integrability results on the Kahan discretization of $\mathrm{LV}(n,0)$, which were first established in [18]:

Theorem 5.10. *The Kahan map of LV$(n,0)$ has the following properties:*

(1) *It is birational;*
(2) *It preserves the Poisson structure Π_b;*
(3) *It is measure preserving: it preserves the volume form Ω_b;*
(4) *It is Liouville integrable with H and the rational functions F_ℓ^b as invariants;*
(5) *It is superintegrable with H and the rational functions F_ℓ^b and G_ℓ^b as invariants.*

As a byproduct of our analysis, we find that the Kahan map of $\mathrm{LV}_b(n,0)$ arises as the compatibility conditions of a linear system. It would be interesting to see if there are other examples where the Kahan map is of this form, as it links the Kahan map to isospectrality, so it may have non-trivial applications to the study of the integrability of the Kahan map of other integrable systems.

References

[1] M. Adler, P. van Moerbeke, and P. Vanhaecke. *Algebraic integrability, Painlevé geometry and Lie algebras*, volume 47 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004.

[2] V. Adler. Recutting of polygons. *Funct. Anal. Appl.*, 27(2):79–80, 1993.

[3] O. I. Bogoyavlenskij. Some constructions of integrable dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(4):737–766, 910, 1987.

[4] O. I. Bogoyavlenskij. Integrable Lotka-Volterra systems. *Regul. Chaotic Dyn.*, 13(6):543–556, 2008.

[5] E. Celledoni, R. I. McLachlan, D. I. McLaren, B. Owren, and G. R. W. Quispel. Integrability properties of Kahan's method. *Journal of Physics A: Mathematical and Theoretical*, 47(36):365202, aug 2014.

[6] P. A. Damianou, C. A. Evripidou, P. Kassotakis, and P. Vanhaecke. Integrable reductions of the Bogoyavlenskij-Itoh Lotka-Volterra systems. *J. Math. Phys.*, 58(3):032704, 17, 2017.

[7] C. A. Evripidou, P. Kassotakis, and P. Vanhaecke. Integrable deformations of the Bogoyavlenskij-Itoh Lotka-Volterra systems. *Regul. Chaotic Dyn.*, 22(6):721–739, 2017.

[8] A. Fordy and A. Hone. Discrete integrable systems and Poisson algebras from Cluster maps. *Commun. Math. Phys.*, 325:527–584, 2014.

[9] R. Hirota and K. Kimura. Discretization of the Euler top. *J. Phys. Soc. Jap.*, 69:627–630, 2000.

[10] R. Hirota and K. Kimura. Discretization of the Lagrange top. *J. Phys. Soc. Jap.*, 69:3193–3199, 2000.

[11] Y. Itoh. Integrals of a Lotka-Volterra system of odd number of variables. *Progr. Theoret. Phys.*, 78(3):507–510, 1987.

[12] Y. Itoh. A combinatorial method for the vanishing of the Poisson brackets of an integrable Lotka-Volterra system. *J. Phys. A*, 42(2):025201, 11, 2009.

[13] T. E. Kouloukas, G. R. W. Quispel, and P. Vanhaecke. Liouville integrability and superintegrability of a generalized Lotka-Volterra system and its Kahan discretization. *J. Phys. A*, 49(22):225201, 13, 2016.

[14] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke. *Poisson structures*, volume 347 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2013.

[15] A. J. Lotka. *Analytical theory of biological populations.* The Plenum Series on Demographic Methods and Population Analysis. Plenum Press, New York, 1998. Translated from the 1939 French edition and with an introduction by David P. Smith and Hélène Rossert.

[16] A. S. Miscenko and A. T. Fomenko. A generalized Liouville method for the integration of Hamiltonian systems. *Funkcional. Anal. i Prilo zen.*, 12(2):46–56, 96, 1978.

[17] M. Noumi and Y. Yamada. Affine Weyl groups, discrete dynamical systems and Painlevé equations. *Comm. Math. Phys.*, 199(2):281–295, 1998.

[18] P. H. van der Kamp, T. E. Kouloukas, G. R. W. Quispel, D. T. Tran, and P. Vanhaecke. Integrable and superintegrable systems associated with multi-sums of products. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 470(2172):20140481, 23, 2014.

[19] A. P. Veselov. Integrable maps. *Russian Mathematical Surveys*, 46(5):1–51, oct 1991.

[20] A. P. Veselov and A. B. Shabat. A dressing chain and the spectral theory of the Schrödinger operator. *Funktsional. Anal. i Prilozhen.*, 27(2):1–21, 96, 1993.

[21] V. Volterra. *Leçons sur la théorie mathématique de la lutte pour la vie.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1990. Reprint of the 1931 original.

Charalampos Evripidou, Department of Mathematics, Faculty of Science, University of Hradec Kralove, Czech Republic
E-mail address: `charalambos.evripidou@uhk.cz`

Pavlos Kassotakis, Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
E-mail address: `pavlos1978@gmail.com`

Pol Vanhaecke, Université de Poitiers, Laboratoire de Mathématiques et Applications, UMR 7348 du CNRS, Bât. H3, Boulevard Marie et Pierre Curie, Site du Futuroscope, TSA 61125, 86073 POITIERS Cedex 9
E-mail address: `pol.vanhaecke@math.univ-poitiers.fr`