Integrable systems and symmetric products of curves

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Abstract

We show how there is associated to each non-constant polynomial $F(x, y)$ an integrable system on $\mathbf{R}^{2d}$ and $\mathbf{C}^{2d}$ for each $d \geq 1$. The first integrals, which are polynomials, are not only in involution with respect to one Poisson bracket, but for a large class of compatible polynomial Poisson brackets, indexed by the family of polynomials in two variables. We show that each complex invariant manifold is isomorphic to an affine part of the d-fold symmetric product of a deformation of the algebraic curve $F(x, y) = 0$, and derive the structure of the real invariant manifolds from it. By slightly modifying our construction we obtain a large class of a.c.i. systems, in particular we obtain for any smooth curve in $\mathbf{C}^2$ an explicit representation of the holomorphic vector fields on the Jacobian of this curve, a useful tool for studying Jacobians from the point of view of algebraic geometry. We also exhibit Lax equations for the hyperelliptic case (i.e., when $F(x, y)$ is of the form $y^2 + f(x)$) and we show that in this case the invariant manifolds are affine parts of distinguished (non-linear) subvarieties of the Jacobians of the curves. As an application the geometry of the Hénon-Heiles hierarchy — a family of superimposable integrable polynomial potentials on the plane — is revealed and Lax equations for the hierarchy are given.

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1. Introduction

Finite-dimensional integrable systems first appeared in the works of Euler (1758), Lagrange (1766), Jacobi (1836), Liouville (1846) and Kowalewski (1889). They were given as systems of (non-linear) differential equations describing the motion of a mechanical system, having a sufficient number of integrals. Their investigation was based on the fact that the equations and the integrals were polynomials (in some coordinates) and led, in all cases considered, to an explicit integration of these equations in terms of (hyperelliptic) theta functions, well-known in algebraic geometry. Their work clearly showed the rich interplay between the theory of Riemann surfaces/algebraic curves (which was in that time thought of as a chapter in complex analysis) and mechanics. During the first half of the present century however, algebraic geometry was refounded and became ever more abstract, while in the theory of mechanical systems, generic smooth dynamical systems were gaining interest (as opposed to integrable ones). So both theories got separated, and integrable systems — which were at the core of this intimate relationship — faded away from the picture.

The interest in both integrable systems and their connection to algebraic geometry revived in the early seventies; many integrable systems were found as finite-dimensional solutions of certain (integrable) partial differential equations (such as the well-known Korteweg-de Vries equation) and they were again integrated in terms of theta functions. Their study led in particular to the concept of an algebraic completely integrable system (a.c.i. system): an integrable system which has a complexification for which the invariant manifolds (the smooth level sets of the integrals) are open subsets of complex algebraic tori (Abelian varieties), and the flow (run with complex time) is linear on these tori (see [AvM1], [M]). Algebraic geometry has been shown to be a useful tool for the study of a.c.i. systems and a solution to some problems about Abelian varieties was found by using an a.c.i. system.

The types of questions about Abelian varieties which have been dealt with by using a.c.i. systems concern the explicit description of certain moduli spaces of Abelian varieties (see [V2]) as well as of the Abelian varieties themselves (see [BV] and [V2]). For the former one needs an a.c.i. system in which all Abelian varieties of the moduli space to be studied appear as invariant manifolds (or covers or quotients of these), for the latter it actually suffices to have explicit equations for the holomorphic vector fields on the particular Abelian variety under consideration.

In the present paper we explicitly construct these vector fields for the Jacobian of an arbitrary (smooth) curve Γ in the plane $\mathbf{C}^2$, thereby providing the basic ingredient for giving an explicit description of the Jacobian of Γ; if this curve Γ behaves well under some deformations, then the vector fields we construct are in fact linear on all invariant manifolds of the integrable system, thereby defining an a.c.i. system, and our construction also provides us with the basic ingredient for studying the moduli space of some families of Jacobians. Apart from the construction of these systems we also construct (in the first part of the text) a (new) class of integrable systems which do not belong to the class of a.c.i. systems, but yet they have a natural complexification and their geometry is most naturally described by using algebraic geometry. This geometry will be analysed in detail and we will show how these systems can be used to explain the geometry of several known integrable systems which are not a.c.i.

The systems we construct generalize (in different directions) Mumford's explicit construction of a family of a.c.i. systems, indexed by $g > 0$, which has as its invariant manifolds all hyperelliptic Jacobians of dimension g, g being the genus of the underlying curve (see [M]). Our curves don't need to be hyperelliptic and the dimension of the invariant manifolds is in general smaller or larger that the genus of the underlying curve; note also that Mumford does not consider the corresponding Poisson or symplectic structure(s). Another generalization of Mumford's system has been discovered by Beauville (see [B]); for a Lie algebraic generalization of the Mumford system, see the forthcoming paper [PV].

The text is organized as follows.

1.1 Poisson structures on $\mathbf{R}^{2d}$

On $\mathbf{R}^{2d}$ with coordinates $(u_1,\dots,u_d,\,v_1,\dots,v_d)$ we show in Section 2.2 that there corresponds in a natural way to any non-zero polynomial $\varphi(x,y)\in\mathbf{R}[x,y]$ a Poisson bracket $\{\cdot\,,\cdot\}_d^\varphi$, which is given by

$$\begin{aligned}&\{u(\lambda),u_j\}_d^\varphi=\{v(\lambda),v_j\}_d^\varphi=0,\\&\{u(\lambda),v_j\}_d^\varphi=\{u_j,v(\lambda)\}_d^\varphi=\varphi(\lambda,v(\lambda))\left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+\ \operatorname{mod}u(\lambda),\qquad 1\le j\le d,\end{aligned}\tag{1.1}$$

where $u(\lambda)=\lambda^d+u_1\lambda^{d-1}+\cdots+u_d$ and $v(\lambda)=v_1\lambda^{d-1}+\cdots+v_d$; also $[R(\lambda)]_+$ denotes the polynomial part of a rational function $R(\lambda)$ and $f(\lambda)\operatorname{mod}g(\lambda)$ is the rest obtained when dividing $f(\lambda)$ by $g(\lambda)$. For fixed d, the map $\varphi\mapsto\{\cdot\,,\cdot\}_d^\varphi$ is clearly a linear map, showing that all our brackets are compatible; moreover this map is injective, since the Poisson structures obtained are of maximal rank except for $\varphi=0$. If $\varphi(x,y)$ is a constant, say $\varphi(x,y)=1$, then the bracket $\{\cdot\,,\cdot\}_d=\{\cdot\,,\cdot\}_d^1$ is given by the following matrix P of Poisson brackets:

$$P=\begin{pmatrix}0&U\\-U&0\end{pmatrix}\quad\text{where}\quad U=\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&u_1\\\vdots&\vdots&&\vdots&\vdots\\0&1&\cdots&u_{d-3}&u_{d-2}\\1&u_1&\cdots&u_{d-2}&u_{d-1}\end{pmatrix}.$$

Thus, (1.1) provides us with a large class of Poisson structures on $\mathbf{R}^{2d}$, which are in fact polynomial, i.e., all brackets of the coordinates u_i and v_j are polynomial.

1.2 Integrable systems on $\mathbf{R}^{2d}$

It is remarkable that there exist a lot of sets of independent functions $\{H_1,\dots,H_d\}$ which are in involution (Poisson commute) with respect to each of our Poisson structures. To describe these, let $F(x,y)$ be any polynomial in $\mathbf{R}[x,y]\setminus\mathbf{R}[x]$ and expand $F(\lambda,v(\lambda))\operatorname{mod}u(\lambda)$ as a polynomial in λ (of degree $d-1$):

$$F(\lambda,v(\lambda))\operatorname{mod}u(\lambda)=H_1\lambda^{d-1}+H_2\lambda^{d-2}+\cdots+H_d.$$

Note that $H_1,\dots,H_d$ are polynomials in u_i and v_j. The main result, established in Section 2.3, is that these polynomials are in involution with respect to all brackets $\{\cdot\,,\cdot\}_d^\varphi$ on $\mathbf{R}^{2d}$, that is

$$\{H_i,H_j\}_d^\varphi=0\quad\text{for all }1\le i,j\le d\text{ and }\varphi(x,y)\in\mathbf{R}[x,y].$$

Since $H_1,\dots,H_d$ are independent, the conclusion is that for any polynomial $F(x,y)\in\mathbf{R}[x,y]\setminus\mathbf{R}[x]$ and any $0\neq\varphi(x,y)\in\mathbf{R}[x,y]$ we have an integrable system in any dimension d (i.e., on $\mathbf{R}^{2d}$) and our construction is totally explicit.

Since everything in our construction is polynomial, these systems have a natural complexification as complex integrable systems on the Poisson manifold $\left(\mathbf{C}^{2d},\{\cdot\,,\cdot\}_d^\varphi\right)$, where the Poisson structure $\{\cdot\,,\cdot\}_d^\varphi$ is now a holomorphic one.

1.3 The geometry of the systems

The meaning of the polynomials $F(x,y)$ and $\varphi(x,y)$ and the need for considering the complexified system becomes apparent in Section 3, when we study (for generic values of c_i) the level sets $\mathcal{A}_{F,d} = \{P \in \mathbf{R}^{2d} \mid H_i(P) = c_i\}$, which are preserved by the flows of the vector fields associated to all H_i. Namely we will show in Section 3.2 that the complex invariant manifold (lying over 0)

$$\mathcal{A}_{F,d}^{\mathbf{C}} = \{(u(\lambda), v(\lambda)) \in \mathbf{C}^{2d} \mid H_{F,d}(u(\lambda), v(\lambda)) = 0\}$$

is (biholomorphic to) an affine part of the d-fold symmetric product of the plane algebraic curve $\Gamma_F \subset \mathbf{C}^2$, defined by $F(x,y) = 0$ (Γ_F is supposed generic here, i.e., smooth); a similar description of the structure of the other complex invariant manifolds (lying over $(c_1, \ldots, c_d)$) follows at once.

For different choices of $\varphi(x,y)$, all Hamiltonian vector fields $X_{H_i}^{\varphi}$ are tangent to these invariant manifolds and they are related in a quite simple way; we will call the integrable systems generated by different values of $\varphi(x,y)$ (but $F(x,y)$ being fixed) compatible integrable systems and we will compare this notion to the notion of a bi-Hamiltonian system.

The real invariant sets are the fixed points on $\mathcal{A}_{F,d}^{\mathbf{C}}$ of the complex conjugation map, which leads to a description of $\mathcal{A}_{F,d} = \mathcal{A}_{F,d}^{\mathbf{C}} \cap \mathbf{R}^{2d}$ as the set of all d-tuples in $\mathcal{A}_{F,d}^{\mathbf{C}}$, consisting only of real points and points which appear in complex conjugated pairs (see Section 3.3). We will give an explicit description of the topology of the invariant manifolds $\mathcal{A}_{F,d}$, which are in general neither tori nor cylinders; we find here a much larger class of topological types of invariant manifolds than in all other studies, the reason being that our invariant manifolds have in general nothing to do with Abelian varieties; it can be shown that a *good* compactification of the complex level manifolds $\mathcal{A}_{F,d}^{\mathbf{C}}$, i.e., a smooth compactification such that the vector fields of the system extend to them in a holomorphic way, rarely exists (see [V3] Ch. III, Sect. 3.4).

1.4 The a.c.i. systems

The construction of a.c.i. systems, motivated above, is taken up in Section 4. The polynomial $F(\lambda, v(\lambda)) \bmod u(\lambda)$ is now expanded in a different way, using (a basis for) the holomorphic differentials on the completion of the curve $F(x,y) = 0$. The main differences with the previous case are that the Hamiltonians are now rational, rather than polynomial, d is taken as the genus of the curve, and the invariant manifold (lying over 0) is now an affine part of the Jacobian of the curve; here we take for simplicity always $\varphi(x,y) = 1$.

In the special case where $F(x,y)$ is of the form $F(x,y) = y^2 + f(x)$ both constructions coincide. Then $F(x,y) = 0$ defines a hyperelliptic curve and a lot of simplifications occur. For example, the vector fields $X_{H_i}^{\varphi}$ of the integrable system can in this case (even for arbitrary φ and d) be written as Lax equations

$$X_{H_i}^{\varphi} A(\lambda) = \left[A(\lambda), [B_i(\lambda)]_+\right],$$

where

$$A(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ -\left[\frac{F(\lambda, v(\lambda))}{u(\lambda)}\right]_+ & -v(\lambda) \end{pmatrix} \quad \text{and} \quad B_i(\lambda) = \frac{\varphi(\lambda, v(\lambda))}{u(\lambda)} \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ A(\lambda);$$

see Section 4.2. Also, if $d <$ genus (Γ_F) then $\mathcal{A}_{F,d}^{\mathbf{C}}$ is interpreted as a very special non-linear subvariety of the Jacobian of Γ_F. In the special case $\varphi = 1$, $d = g$ and $f(x) = x^{2g+1} + a_{2g}x^{2g} + \cdots + a_0$ these Lax equations coincide with Mumford's description of the linear vector fields on a family of hyperelliptic Jacobians (see [M], Ch. IIIa, Sect. 3).

The algebraic geometry of the invariant manifolds of several known integrable systems, which are not a.c.i., such as the Hénon-Heiles hierarchy and its generalizations (in different aspects), the (generalized) Gaudin magnet, the discrete self-trapping timer,..., can be described completely by using the integrable systems which are introduced here. We will show in Section 4.4 in detail how this is done for the Hénon-Heiles hierarchy, which consists of a family of (superimposable) integrable potentials on the plane. For the other examples one proceeds in a completely analogous way.

2. The systems and their integrability

In this section we describe our basic construction, which gives for every polynomial $F(x,y)$ an algebra of functions which is integrable with respect to a family of compatible Poisson structures on $\mathbf{R}^{2d}$, which is parametrized by the set of all polynomials $\varphi(x,y)$ in two variables.

2.1. Notation

$\mathbf{R}^{2d}$ is throughout viewed as the space of pairs of polynomials $(u(\lambda),v(\lambda))$, with $u(\lambda)$ monic of degree d and $v(\lambda)$ of degree less than d, via

$$\begin{aligned} u(\lambda) &= \lambda^d + u_1\lambda^{d-1} + \cdots + u_{d-1}\lambda + u_d, \\ v(\lambda) &= \qquad\ \ v_1\lambda^{d-1} + \cdots + v_{d-1}\lambda + v_d, \end{aligned} \tag{2.1}$$

so the coefficients u_i and v_i serve as coordinates on $\mathbf{R}^{2d}$. Some formulas below are simplified by denoting $u_0 = 1$.

For any rational function $r(\lambda)$, we denote by $[r(\lambda)]_+$ its polynomial part and we let $[r(\lambda)]_- = r(\lambda) - [r(\lambda)]_+$. If $f(\lambda)$ is any polynomial and $g(\lambda)$ is a monic polynomial, then $f(\lambda) \bmod g(\lambda)$ denotes the polynomial of degree less than $\deg g(\lambda)$, defined by

$$f(\lambda) \bmod g(\lambda) = g(\lambda)\left[\frac{f(\lambda)}{g(\lambda)}\right]_-,$$

so $f(\lambda) = f(\lambda) \bmod g(\lambda) + h(\lambda)g(\lambda)$ for a unique polynomial $h(\lambda)$, and $f(\lambda) \bmod g(\lambda)$ is easily computed as the rest obtained by the Euclidean division algorithm.

2.2. The compatible Poisson structures $\{\cdot\,,\cdot\}_d^\varphi$

Any polynomial $\varphi(x,y)$ specifies a Poisson bracket on $\mathbf{R}^2$ by $\{y,x\} = \varphi(x,y)$, which extends to a polynomial bracket on the cartesian product $\left(\mathbf{R}^2\right)^d = \mathbf{R}^2 \times \cdots \times \mathbf{R}^2$ by taking the product bracket, i.e.,

$$\{y_i, x_j\} = \delta_{ij}\varphi(x_j, y_i), \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0; \tag{2.2}$$

here x_i and y_i are the coordinates on the i-th factor, coming from the chosen coordinates x and y on $\mathbf{C}^2$. Let Δ denote the closed subsets of $\left(\mathbf{R}^2\right)^d$ defined by

$$\Delta = \{((x_1,y_1),(x_2,y_2),\ldots,(x_d,y_d)) \mid x_i = x_j \text{ for some } i \neq j\},$$

and consider the map $\mathcal{S}\colon \left(\mathbf{R}^2\right)^d \setminus \Delta \to \mathbf{R}^{2d}$, given by

$$((x_1,y_1),(x_2,y_2),\ldots,(x_d,y_d)) \mapsto (u(\lambda),v(\lambda)) = \left(\prod_{i=1}^d(\lambda - x_i), \sum_{i=1}^d y_i \prod_{j\neq i}\frac{\lambda - x_j}{x_i - x_j}\right). \tag{2.3}$$

Note that this map is uniquely defined by $u(x_i) = 0$ and $v(x_i) = y_i$, $(i = 1,\ldots,d)$. $\mathcal{S}$ is invariant for the obvious action of the permutation group S_d on $\left(\mathbf{R}^2\right)^d$ and is a $d!\colon 1$ unramified covering map onto an open subset of $\mathbf{R}^{2d}$. Since the Poisson structure is also invariant for the action of S_d, i.e.,

$$\{f,g\}\circ\sigma = \{f\circ\sigma, g\circ\sigma\}, \qquad f,g \in C^\infty\left(\left(\mathbf{R}^2\right)^d\right),\ \sigma \in S_d,$$

a C^∞ Poisson bracket $\{\cdot\,,\cdot\}_d^\varphi$ is defined on the image of $\mathcal{S}$ by requiring that $\mathcal{S}$ is a Poisson map, i.e., that for any $f,g\in C^\infty(\mathbf{R}^{2d})$, one has $\{f,g\}_d^\varphi\circ\mathcal{S}=\{f\circ\mathcal{S},g\circ\mathcal{S}\}$. The following proposition provides us with explicit formulas for this bracket, showing in particular that it extends to a C^∞ (even polynomial) Poisson bracket on all of $\mathbf{R}^{2d}$. This bracket will also be denoted by $\{\cdot\,,\cdot\}_d^\varphi$.

Proposition 2.1 *The Poisson bracket $\{\cdot\,,\cdot\}_d^\varphi$ is given in terms of the coordinates u_i,v_i by*

$$\begin{aligned}&\{u(\lambda),u_j\}_d^\varphi=\{v(\lambda),v_j\}_d^\varphi=0,\\&\{u(\lambda),v_j\}_d^\varphi=\{u_j,v(\lambda)\}_d^\varphi=\varphi(\lambda,v(\lambda))\left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+\mod u(\lambda),\qquad 1\le j\le d,\end{aligned}\tag{2.4}$$

hence all brackets of the coordinate functions u_i and v_j are polynomials and $\{\cdot\,,\cdot\}_d^\varphi$ is defined on all of $\mathbf{R}^{2d}$. Except for the zero bracket $\{\cdot\,,\cdot\}_d^0$, all Poisson brackets $\{\cdot\,,\cdot\}_d^\varphi$ are of rank $2d$ on a dense subset of $\mathbf{R}^{2d}$ whose complement is a (possibly empty) algebraic hypersurface; moreover they are all compatible, i.e., the sum of two such Poisson brackets is again a Poisson bracket.

As a special and most important case, if x and y are canonical variables, i.e., $\varphi(x,y)=1$, then the Poisson structure $\{\cdot\,,\cdot\}_d^\varphi$, also denoted by $\{\cdot\,,\cdot\}_d$, is of maximal rank at every point of $\mathbf{R}^{2d}$, hence it defines a symplectic structure ω_d on $\mathbf{R}^{2d}$; the second equation in (2.4) reduces in this case to

$$\{u(\lambda),v_j\}_d=\{u_j,v(\lambda)\}_d=\left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+,\tag{2.5}$$

and its matrix of Poisson brackets with respect to the coordinate functions u_i and v_j, takes the form

$$P=\begin{pmatrix}0&U\\-U&0\end{pmatrix}\quad\textit{where}\quad U=\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&u_1\\\vdots&\vdots&&\vdots&\vdots\\0&1&\cdots&u_{d-3}&u_{d-2}\\1&u_1&\cdots&u_{d-2}&u_{d-1}\end{pmatrix}.$$

In terms of $\{\cdot\,,\cdot\}_d$, the Poisson structure $\{\cdot\,,\cdot\}_d^\varphi$ is given by

$$\begin{aligned}\{u(\lambda),f\}_d^\varphi&=\varphi(\lambda,v(\lambda))\,\{u(\lambda),f\}_d\mod u(\lambda),\\\{v(\lambda),f\}_d^\varphi&=\varphi(\lambda,v(\lambda))\,\{v(\lambda),f\}_d\mod u(\lambda).\end{aligned}\tag{2.6}$$

Proof

Clearly $\{u(\lambda),u(\mu)\}_d^\varphi=0$. If $1\le j\le d$, then

$$\begin{aligned}\{u_j,v(\lambda)\}_d^\varphi&=(-1)^j\left\{\sum_{i_1<i_2<\cdots<i_j}x_{i_1}x_{i_2}\cdots x_{i_j}\,,\sum_{l=1}^d y_l\prod_{k\neq l}\frac{\lambda-x_k}{x_l-x_k}\right\}_d^\varphi\\&=(-1)^{j-1}\sum_{i_1<i_2<\cdots<i_j}\sum_{t=1}^j x_{i_1}x_{i_2}\cdots\widehat{x_{i_t}}\cdots x_{i_j}\varphi(x_{i_t},y_{i_t})\prod_{k\neq i_t}\frac{\lambda-x_k}{x_{i_t}-x_k}\\&=(-1)^{j-1}\sum_{l\notin\{i_1<i_2<\cdots<i_{j-1}\}}x_{i_1}x_{i_2}\cdots x_{i_{j-1}}\varphi(x_l,y_l)\prod_{k\neq l}\frac{\lambda-x_k}{x_l-x_k}\\&=(-1)^{j-1}\sum_{l=1}^d\varphi(x_l,y_l)\prod_{k\neq l}\frac{\lambda-x_k}{x_l-x_k}\,(-1)^{j-1}\sum_{m=0}^{j-1}x_l^m u_{j-m-1}\\&=\sum_{l=1}^d\sum_{m=0}^{j-1}x_l^m u_{j-m-1}\varphi(x_l,y_l)\prod_{k\neq l}\frac{\lambda-x_k}{x_l-x_k}.\end{aligned}$$

Since $y_l = v(x_l)$ this shows that $\{u_j, v(\lambda)\}_d^\varphi$ is the (unique) polynomial in λ of degree less than d, which takes at $\lambda = x_l$ the value $\sum_{m=0}^{j-1} x_l^m u_{j-m-1}\varphi(x_l, v(x_l))$, for $l = 1, \ldots, d$. As the x_l are the roots of $u(\lambda)$, the same is true for $\sum_{m=0}^{j-1} \lambda^m u_{j-m-1}\varphi(\lambda, v(\lambda)) \bmod u(\lambda)$, and we find

$$\begin{aligned} \{u_j, v(\lambda)\}_d^\varphi &= \sum_{m=0}^{j-1} \lambda^m u_{j-m-1}\varphi(\lambda, v(\lambda)) \bmod u(\lambda) \\ &= \varphi(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \bmod u(\lambda), \end{aligned}$$

which proves the second equality in (2.4).

Since $\varphi(x, y)$ is a polynomial, it follows from the fact that $u(\lambda)$ is monic and formulas (2.4) that all brackets $\{u_i, v_j\}_d^\varphi$ are polynomial, hence extend to a Poisson bracket on $\mathbf{R}^{2d}$, also denoted by $\{\cdot\,,\cdot\}_d^\varphi$. Compatibility of the brackets derives from the formula $\{\cdot\,,\cdot\}_d^\varphi + \{\cdot\,,\cdot\}_d^\psi = \{\cdot\,,\cdot\}_d^{\varphi+\psi}$, an easy consequence of (2.4).

For $\varphi = 1$ one obtains (2.5), because the degree of $\left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+$ is less than d for any $j = 1, \ldots, d$, which also leads at once to the matrix representation of $\{\cdot\,,\cdot\}_d$ — since its determinant equals $(-1)^d$, it is of rank $2d$ everywhere. Note also that $\{\cdot\,,\cdot\}_d$ is not compatible with the standard symplectic structure $\sum du_i \wedge dv_i$ on $\mathbf{R}^{2d}$.

To see where the rank of the Poisson structure $\{\cdot\,,\cdot\}_d^\varphi$ fails to be maximal, we need to investigate the determinant of the matrix of Poisson brackets $\{u_i, v_j\}_d^\varphi$. By some elementary properties of determinants one finds that for any values $x_1, \ldots, x_d$,

$$\det\left(\{u_i, v(x_j)\}_d^\varphi\right)_{1\le i,j\le d} = \det\left(\{u_i, v_j\}_d^\varphi\right)_{1\le i,j\le d} \prod_{k<l}(x_k - x_l). \tag{2.7}$$

Choosing $x_1, \ldots, x_d$ to be the roots of $u(\lambda)$ (which may be complex), we get from (2.4)

$$\begin{aligned} \det\left(\{u_i, v(x_j)\}_d^\varphi\right)_{1\le i,j\le d} &= \det\left(\varphi(x_j, v(x_j)) \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_{+|\lambda=x_j}\right)_{1\le i,j\le d} \\ &= \det\left(\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_{+|\lambda=x_j}\right)_{1\le i,j\le d} \prod_{m=1}^{d} \varphi(x_m, v(x_m)) \\ &= \det\left(\{u_i, v(x_j)\}_d\right)_{1\le i,j\le d} \prod_{m=1}^{d} \varphi(x_m, v(x_m)) \\ &\overset{(i)}{=} (-1)^{[d/2]} \prod_{k<l}(x_k - x_l) \prod_{m=1}^{d} \varphi(x_m, v(x_m)), \end{aligned}$$

where in *(i)* we used (2.7) for $\varphi = 1$. It follows that (even if $u(\lambda)$ has multiple roots)

$$\det\left(\{u_i, v_j\}_d^\varphi\right)_{1\le i,j\le d} = (-1)^{[d/2]} \prod_{m=1}^{d} \varphi(x_m, v(x_m)),$$

on all of $\mathbf{R}^{2d}$, hence the Poisson structure is of lower rank on the locus $\prod_{j=1}^d \varphi(x_j, v(x_j)) = 0$, which for given φ is easy written as the equation of an algebraic hypersurface in $\mathbf{R}^{2d}$.

Finally, (2.6) follows immediately from the Leibniz property of Poisson brackets. ∎

Amplification 2.2

The condition that $\varphi(x,y)$ is a polynomial is not essential: if $\varphi(x,y)$ is any smooth function, then all the above formulas remain valid, yielding yet more examples of compatible Poisson structures. In this more general case, for $f(\lambda)$ any smooth funtion and $g(\lambda)$ a monic polynomial as before, $f(\lambda) \bmod g(\lambda)$ denotes the unique[1] polynomial of degree less than $\deg g(\lambda)$ which takes at the roots x_i of $g(\lambda)$ the value $f(x_i)$. The Poisson brackets $\{u_i, v_j\}_d^\varphi$ are no longer polynomial and can't be computed by the Euclidean division algorithm.

Of interest is also the case in which $\varphi(x,y)$ is rational. Then all brackets $\{u_i, v_j\}_d^\varphi$ are rational functions of the coordinates u_i and v_j. Obviously, if $\varphi(x,y)$ has poles on $\mathbf{R}^2$, the bracket $\{\cdot\,,\cdot\}_d^\varphi$ will also have poles on $\mathbf{R}^{2d}$, and is in this case only a Poisson bracket on a dense subset of $\mathbf{R}^{2d}$.

Amplification 2.3

If φ depends only on x and has degree at most d, then $\{\cdot\,,\cdot\}_d^\varphi$ is a modified Lie-Poisson structure. Explicitly, for $\varphi = x^n$, $0 \le n < d$ the Poisson matrix P_n, which contains the brackets of the coordinates $u_1, \dots, u_d, v_1, \dots, v_d$ (in that order), is given by

$$P_n = \begin{pmatrix} 0 & 0 & U_n & 0 \\ 0 & 0 & 0 & -U_n' \\ -U_n & 0 & 0 & 0 \\ 0 & U_n' & 0 & 0 \end{pmatrix}$$

where

$$U_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & u_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & u_{d-n-3} & u_{d-n-2} \\ 1 & u_1 & \cdots & u_{d-n-2} & u_{d-n-1} \end{pmatrix} \quad \text{and} \quad U_n' = \begin{pmatrix} u_{d-n+1} & u_{d-n+2} & \cdots & u_{d-1} & u_d \\ u_{d-n+2} & u_{d-n+3} & \cdots & u_d & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ u_{d-1} & u_d & \cdots & 0 & 0 \\ u_d & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For $n = d$ one finds a Lie-Poisson bracket which is given by

$$P_d = \begin{pmatrix} 0 & -U_d' \\ U_d' & 0 \end{pmatrix}, \qquad \text{where} \qquad U_d' = \begin{pmatrix} u_1 & u_2 & \cdots & u_{d-1} & u_d \\ u_2 & u_3 & \cdots & u_d & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ u_{d-1} & u_d & \cdots & 0 & 0 \\ u_d & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For $\varphi = \sum_{i=0}^d c_i x^i$ the corresponding Poisson matrix is given by $\sum_{i=0}^d c_i P_i$. Note that these are the only $\varphi(x,y)$ for which $\{\cdot\,,\cdot\}_d^\varphi$ is a modified Lie-Poisson structure and that $\varphi(x,y) = cx^d$, $(c \in \mathbf{R})$ is the only one which gives a Lie-Poisson structure.

2.3. Polynomials in involution for $\{\cdot\,,\cdot\}_d^\varphi$

We will now show how an arbitrary polynomial $F(x,y)$ leads to a natural set of d polynomials on $\mathbf{R}^{2d}$ which have the remarkable property to be in involution with respect to *all* our Poisson

[1] If $g(\lambda)$ has multiple roots, then $f(\lambda) \bmod g(\lambda)$ is not unique; since in this paper $g(\lambda) = u(\lambda)$ depends on the coordinates u_i, it is (as a function on $\mathbf{R}^{2d}$) uniquely defined on a dense subset of $\mathbf{R}^{2d}$, hence its extension to $\mathbf{R}^{2d}$ is also unique.

structures $\{\cdot,\cdot\}_d^\varphi$. These functions are functional independent, (except in the special case when $F(x,y)$ is independent of y, $F(x,y) \in \mathbf{R}[x]$,) hence they define an integrable system on $\mathbf{R}^{2d}$ for any structure $\{\cdot,\cdot\}_d^\varphi$. We stress that the integrable systems obtained for different choices of $\varphi(x,y)$ and $F(x,y)$ (and obviously d) are all different, in particular their Hamiltonian vector fields are completely different; our situation should in particular not be confused with the notion of a bi- (or multi-) Hamiltonian system (see Section 3.4 below).

Let $F(x,y) \in \mathbf{R}[x,y] \setminus \mathbf{R}[x]$ and let us view $\mathbf{R}^d$ as the space of polynomials (say in λ) of degree less than d. Then there is a natural map $\hat{H}_{F,d}$ from $\left(\mathbf{R}^2\right)^d \setminus \Delta$ to $\mathbf{R}^d$, which assigns to a d-tuple $((x_1,y_1),\dots,(x_d,y_d))$ the unique polynomial in $\mathbf{R}[\lambda]$ of degree less than d, which takes for $\lambda = x_i$ the value $F(x_i,y_i)$ (for $i=1,\dots,d$). Since $\hat{H}_{F,d}$ is invariant under the action of S_d, a map $H_{F,d}$ such that $\hat{H}_{F,d} = H_{F,d} \circ \mathcal{S}$, is defined on the image of $\mathcal{S}$: $H_{F,d}$ is given by

$$H_{F,d}(u(\lambda),v(\lambda)) = F(\lambda,v(\lambda)) \bmod u(\lambda). \tag{2.8}$$

The d components of the map $H_{F,d}$ define d functions on $\mathbf{R}^{2d}$, which will be simply denoted by $H_1,\dots,H_d$ (omitting the dependence on F and d in the notation), i.e., $H_{F,d}(u(\lambda),v(\lambda)) = H_1\lambda^{d-1} + H_2\lambda^{d-2} + \cdots + H_d$. As $u(\lambda)$ is a *monic* polynomial, these functions H_i are polynomial in our coordinates on $\mathbf{R}^{2d}$ hence are defined on all of $\mathbf{R}^{2d}$. The main result of this section is the following.

Theorem 2.4 *The coefficients $H_1,\dots,H_d$ of $H(\lambda) = F(\lambda,v(\lambda)) \bmod u(\lambda)$ define for any non-zero polynomial $\varphi(x,y)$ a completely integrable system on the Poisson manifold $\left(\mathbf{R}^{2d},\{\cdot,\cdot\}_d^\varphi\right)$ with polynomial first integrals (constants of motion). More precisely, $\{H_1,\dots,H_d\}$ forms a set of d functional independent polynomials on $\mathbf{R}^{2d}$, which are in involution for all brackets $\{\cdot,\cdot\}_d^\varphi$.*

Before proving this theorem we prove a key lemma and write down explicit equations for the Hamiltonian vector fields $X_{H_i}^\varphi = \{\cdot,H_i\}_d^\varphi$, which — by the above theorem — commute as differential operators, in view of the formula (see [AM]) $\left[X_{H_i}^\varphi, X_{H_j}^\varphi\right] = X_{\{H_j,H_i\}_d^\varphi}^\varphi$.

Lemma 2.5 *Let $p(\lambda)$, $q(\lambda)$ and $r(\lambda)$ be polynomials, with $\deg q(\lambda) \geq \deg r(\lambda)$ and let $i \in \mathbf{N}$.*

$$\begin{aligned} &(1) \qquad r(\lambda)\left[\lambda^{-i}q(\lambda)\right]_+ \bmod q(\lambda) = r(\lambda)\left[\lambda^{-i}q(\lambda)\right]_+ - q(\lambda)\left[\lambda^{-i}r(\lambda)\right]_+, \\ &(2) \qquad \sum_{l=1}^{\deg q}\mu^{l-1}p(\lambda)\left[\lambda^{-l}q(\lambda)\right]_+ \bmod q(\lambda) = \sum_{l=1}^{\deg q}\lambda^{l-1}p(\mu)\left[\mu^{-l}q(\mu)\right]_+ \bmod q(\mu). \end{aligned} \tag{2.9}$$

Proof

For the proof of *(1)* note that if $\deg r(\lambda) \leq \deg q(\lambda)$ then the right hand side of

$$r(\lambda)\left[\lambda^{-i}q(\lambda)\right]_+ - q(\lambda)\left[\lambda^{-i}r(\lambda)\right]_+ = -r(\lambda)\left[\lambda^{-i}q(\lambda)\right]_- + q(\lambda)\left[\lambda^{-i}r(\lambda)\right]_-$$

is of degree less than $\deg q(\lambda)$, hence also the left hand side. To show *(2)* we may assume that $\deg p(\lambda) < \deg q(\lambda)$ because the equality depends only on $p(\lambda) \bmod q(\lambda)$. Then

$$\begin{aligned} \sum_{l=1}^{\deg q}\lambda^{l-1}p(\mu)\left[\mu^{-l}q(\mu)\right]_+ \bmod q(\mu) &\overset{(i)}{=} \sum_{l=1}^{\deg q}\lambda^{l-1}\left(p(\mu)\left[\mu^{-l}q(\mu)\right]_+ - q(\mu)\left[\mu^{-l}p(\mu)\right]_+\right) \\ &\overset{(ii)}{=} \sum_{l=1}^{\deg q}\mu^{l-1}\left(p(\lambda)\left[\lambda^{-l}q(\lambda)\right]_+ - q(\lambda)\left[\lambda^{-l}p(\lambda)\right]_+\right) \\ &= \sum_{l=1}^{\deg q}\mu^{l-1}p(\lambda)\left[\lambda^{-l}q(\lambda)\right]_+ \bmod q(\lambda). \end{aligned}$$

In *(i)* we applied part *(1)* of this lemma; the exchange property in *(ii)* is proven at once by expanding the polynomials or by induction on $\deg q(\lambda)$. ∎

Proposition 2.6 *The coefficients H_i of $F(\lambda, v(\lambda)) \bmod u(\lambda)$ determine d independent polynomial vector fields $X^{\varphi}_{H_i}$ on $\mathbf{R}^{2d}$, which are explicitly given by*

$$\begin{aligned} X^{\varphi}_{H_i} u(\lambda) &= \varphi(\lambda, v(\lambda)) \frac{\partial F}{\partial y}(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ \bmod u(\lambda), \\ X^{\varphi}_{H_i} v(\lambda) &= \varphi(\lambda, v(\lambda)) \left[\frac{F(\lambda, v(\lambda))}{u(\lambda)}\right]_+ \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ \bmod u(\lambda). \end{aligned} \tag{2.10}$$

Moreover, the following remarkable identities hold for all $1 \le i, j \le d$:

$$\{u_i, H_j\}^{\varphi}_d = \{u_j, H_i\}^{\varphi}_d \text{ and } \{v_i, H_j\}^{\varphi}_d = \{v_j, H_i\}^{\varphi}_d. \tag{2.11}$$

Proof

Writing X_{H_i} as a shorthand for $X^1_{H_i}$, we first compute $X_{H_i} u(\lambda) = \{u(\lambda), H_i\}_d$, which we obtain as the coefficient of μ^{d-i} in $\{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d$.

$$\begin{aligned} \{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d &= \sum_{j=1}^{d} \{u(\lambda), v_j\}_d \frac{\partial H_{F,d}}{\partial v_j}(u(\mu), v(\mu)) \\ &= \sum_{j=1}^{d} \left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \frac{\partial H_{F,d}}{\partial v_j}(u(\mu), v(\mu)) \\ &= \sum_{j=1}^{d} \sum_{k=0}^{j-1} u_k \lambda^{j-k-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \mu^{d-j} \bmod u(\mu) \\ &= \sum_{l=1}^{d} \sum_{j=l}^{d} u_{j-l} \lambda^{l-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \mu^{d-j} \bmod u(\mu) \\ &= \sum_{l=1}^{d} \lambda^{l-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \left[\frac{u(\mu)}{\mu^l}\right]_+ \bmod u(\mu) \\ &= \sum_{l=1}^{d} \mu^{l-1} \frac{\partial F}{\partial y}(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^l}\right]_+ \bmod u(\lambda), \end{aligned}$$

where we used the exchange property (2.9) in the last step. Since H_i is the coefficient of μ^{d-i} in $H(\lambda)$ this leads to equation (2.10) for $X_{H_i} u(\lambda)$ in case $\varphi(x, y) = 1$. In a similar way $X_{H_i} v(\lambda)$ is found, the computation of $\frac{\partial}{\partial u_j} H_{F,d}(u(\mu), v(\mu))$ is however more involved: let $1 \leq j \leq d$ then

$$\begin{aligned}
\frac{\partial}{\partial u_j}\left(F(\mu, v(\mu)) \bmod u(\mu)\right) &= \frac{\partial}{\partial u_j}\left(u(\mu)\left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_-\right) \\
&= -\frac{\partial}{\partial u_j}\left(u(\mu)\left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+\right) \\
&= -u(\mu)\left(\frac{\mu^{d-j}}{u(\mu)}\left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+ - \left[\frac{\mu^{d-j}}{u(\mu)}\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+\right) \\
&\overset{(i)}{=} -u(\mu)\left[\frac{\mu^{d-j}}{u(\mu)}\left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+\right]_- \\
&= -\mu^{d-j}\left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+ \bmod u(\mu).
\end{aligned}$$

In *(i)* we used that if $R = R(\mu)$ and $P = P(\mu)$ are rational functions, with $[R]_+ = 0$, then

$$R\,[P]_+ - [RP]_+ = R\,[P]_+ - \left[R\,[P]_+\right]_+ = \left[R\,[P]_+\right]_- .$$

Granted this, we obtain as above

$$\begin{aligned}
\{v(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d &= \sum_{j=1}^{d} \mu^{d-j}\left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \left[\frac{F(\mu, v(\mu))}{u(\mu)}\right]_+ \bmod u(\mu) \\
&= \sum_{l=1}^{d} \mu^{l-1}\left[\frac{u(\lambda)}{\lambda^{l}}\right]_+ \left[\frac{F(\lambda, v(\lambda))}{u(\lambda)}\right]_+ \bmod u(\lambda),
\end{aligned}$$

which leads to the expression (2.10) for $X_{H_i} v(\lambda)$ in case $\varphi(x, y) = 1$. Having obtained the formulas (2.10) for $X_{H_i} u(\lambda)$ and $X_{H_i} v(\lambda)$, the formulas for $X^{\varphi}_{H_i} u(\lambda)$ and $X^{\varphi}_{H_i} v(\lambda)$, follow at once upon using (2.6).

Finally, the exchange property (2.9) implies that λ and μ are everywhere interchangeable in the above computations so we get $\{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}^{\varphi}_d = \{u(\mu), H_{F,d}(u(\lambda), v(\lambda))\}^{\varphi}_d$, which is tantamount to the identity $\{u_i, H_j\}^{\varphi}_d = \{u_j, H_i\}^{\varphi}_d$. The second formula in (2.11) follows in the same way.

∎

Proof of Theorem 2.4

We first prove that $\{H_i, H_{F,d}(u(\lambda), v(\lambda))\}^{\varphi}_d = 0$ for $1 \leq i \leq d$. To make the proof more transparent, we use the following abbreviations:

$$F_y = \frac{\partial F}{\partial y}(\lambda, v(\lambda)), \quad F_{(u)} = \frac{F(\lambda, v(\lambda))}{u(\lambda)} \quad \text{and} \quad U_i = \frac{\varphi(\lambda, v(\lambda))}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ ,$$

so that (2.10) is rewritten as $X^{\varphi}_{H_i}u(\lambda)=u(\lambda)\left[U_iF_y\right]_-$ and $X^{\varphi}_{H_i}v(\lambda)=u(\lambda)\left[U_i\left[F_{(u)}\right]_+\right]_-$. Then

$$
\begin{aligned}
\{H_{F,d}(u(\lambda),v(\lambda)),H_i\}^{\varphi}_d &= X^{\varphi}_{H_i}\left(u(\lambda)\left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_-\right)\\
&= X^{\varphi}_{H_i}u(\lambda)\left[F_{(u)}\right]_- + u(\lambda)\left[\frac{X^{\varphi}_{H_i}F(\lambda,v(\lambda))}{u(\lambda)}\right]_- - u(\lambda)\left[\frac{F_{(u)}X^{\varphi}_{H_i}u(\lambda)}{u(\lambda)}\right]_-\\
&= u(\lambda)\left(\left[U_iF_y\right]_-\left[F_{(u)}\right]_- + \left[F_y\left[U_i\left[F_{(u)}\right]_+\right]_-\right]_- - \left[F_{(u)}\left[U_iF_y\right]_-\right]_-\right)\\
&= u(\lambda)\left[\left[U_iF_y\right]_-\left[F_{(u)}\right]_- + F_y\left[U_i\left[F_{(u)}\right]_+\right]_- - F_{(u)}\left[U_iF_y\right]_-\right]_-\\
&\overset{(i)}{=} u(\lambda)\left[-\left[U_iF_y\right]_-\left[F_{(u)}\right]_+ + F_yU_i\left[F_{(u)}\right]_+\right]_-\\
&= u(\lambda)\left[\left[U_iF_y\right]_+\left[F_{(u)}\right]_+\right]_-\\
&= 0.
\end{aligned}
$$

In *(i)* we used the fact that F_y is a polynomial, i.e., $[F_y]_-=0$.

We now show that the d coefficients of $H_{F,d}(u(\lambda),v(\lambda))=F(\lambda,v(\lambda)) \bmod u(\lambda)$ are functional independent. Clearly the last d coefficients $\tilde H_1,\dots,\tilde H_d$ of $F(\lambda,v(\lambda))$ are independent because v_i appears only in $\tilde H_1,\dots,\tilde H_i$ (it *does* appear since $F(x,y)\notin \mathbf{R}[x]$). Reducing $F(\lambda,v(\lambda))$ modulo $u(\lambda)$ amounts to substracting from $\tilde H_i$ polynomials of lower degree in the variables v_j, so it cannot make these functions dependent and the independence of $\{H_1,\dots,H_d\}$ follows. ∎

Amplification 2.7

If $F(x,y)$ and $F'(x,y)$ differ only by a polynomial which is independent of y and is of degree less than d in x, then clearly the integrable systems on $\mathbf{R}^{2d}$ which are associated to F and F' are the same; in this sense, for $\varphi(x,y)$ fixed, a system is associated to a coset

$$\tilde F(x,y)=\left\{F(x,y)+\sum_{i=0}^{d-1}c_ix^i \mid c_i\in\mathbf{R}\right\}.$$

If a (differentiable) deformation family M of classes $\tilde F(x,y)$ is given (rather than a single class) then our construction is easy adapted to give (for each non-zero $\varphi(x,y)\in\mathbf{R}[x,y]$) a d-dimensional integrable system on a Poisson manifold, which is the product of the deformation manifold M and $\mathbf{R}^{2d}$. Namely let the brackets (2.2) on $\left(\mathbf{R}^2\right)^d$ be extended trivially to $\left(\mathbf{R}^2\right)^d\times M$, i.e., if π_M denotes the projection map $\left(\mathbf{R}^2\right)^d\to M$ then the annihilator of the Poisson bracket is chosen as $\{f\circ\pi_M \mid f\in C^\infty(M)\}$. Also the map $\mathcal{S}$ given by (2.3) is extended to the map

$$\mathcal{S}\times \mathrm{Id}_M\colon \left(\left(\mathbf{R}^2\right)^d\setminus\Delta\right)\times M\to\mathbf{R}^{2d}\times M,$$

which is the identity map Id_M on the second component. This Poisson structure and these maps are invariant for the action of S_d (on the first component) so that we obtain, as before, a Poisson structure $\{\cdot\,,\cdot\}^{\varphi}_{d,M}$ on the image of $\mathcal{S}\times\mathrm{Id}_M\subset\mathbf{R}^{2d}\times M$, which extends to all of $\mathbf{R}^{2d}\times M$, because all brackets are polynomial. The commuting vector fields $\{\cdot,H_i\}^{\varphi}_{d,M}$ are tangent to the (linear)

Poisson submanifolds $\{\tilde{F}\} \times \mathbf{R}^{2d}$, $(\tilde{F} \in M)$, to which $\{\cdot\,,\cdot\}_{d,M}^{\varphi}$ restricts as $\{\cdot\,,\cdot\}_d^{\varphi}$. Therefore, these commuting vector fields restrict to these submanifolds, giving the vector fields $\{\cdot, H_i\}_d^{\varphi}$ (as given by (2.10)) of the integrable system associated to $\tilde{F}$ (i.e., to F).

Amplification 2.8

In all the above definitions, $\mathbf{R}$ can be replaced by $\mathbf{C}$; our construction then associates to each complex polynomial in two variables, a maximal set of holomorphic functions (polynomials), defined on $\mathbf{C}^{2d}$, which are in involution with respect to a holomorphic Poisson bracket, itself determined by an arbitrary non-zero polynomial in two variables.

3. The geometry of the invariant manifolds

The integrable systems introduced in Section 2 provide us (for each $d \geq 1$ and $F(x,y) \in \mathbf{R}[x,y] \setminus \mathbf{R}[x]$) with a surjective map defined by $H_{F,d}(u(\lambda),v(\lambda)) = F(\lambda,v(\lambda)) \bmod u(\lambda)$. The fibers of $H_{F,d}$ are preserved by the flows of the d vector fields $X^{\varphi}_{H_i}$ which correspond via $\{\cdot\,,\cdot\}^{\varphi}_d$ to the components of this map — note that this map (hence also its fibers) is independent of the choice of $\varphi(x,y)$. By Sard's Theorem, the generic fiber of this map is smooth. These smooth fibers are called the *invariant manifolds* of the system; they are Lagrangian submanifolds of $(\mathbf{R}^{2d},\{\cdot\,,\cdot\}^{\varphi}_d)$, i.e., the restriction of $\{\cdot\,,\cdot\}^{\varphi}_d$ to these d-dimensional submanifolds vanishes. In this section we investigate the geometry of these invariant manifolds and discuss the role of $\varphi(x,y)$.

3.1. The invariant manifolds $\mathcal{A}_{F,d}$ and $\mathcal{A}^{\mathbf{C}}_{F,d}$

Since $H_{F,d}(u(\lambda),v(\lambda))$ is defined as $F(\lambda,v(\lambda)) \bmod u(\lambda)$, the fiber over $h(\lambda) \in \mathbf{R}_{d-1}[\lambda]$ is the same as the fiber over 0 for $H_{F',d}$, where $F'(x,y) = F(x,y) - h(x)$. Therefore we may restrict ourselves to the fiber lying over 0, denoted by $\mathcal{A}_{F,d}$; thus, by definition, $\mathcal{A}_{F,d}$ is given by

$$\mathcal{A}_{F,d} = \left\{ (u(\lambda),v(\lambda)) \in \mathbf{R}^{2d} \mid \left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_{-} = 0 \right\}. \tag{3.1}$$

Sard's Theorem implies that this fiber is smooth if $F(x,y)$ is generic. Clearly if $F(x,y)$ is generic then the *complex* algebraic curve $\Gamma_F \subset \mathbf{C}^2$, defined by $F(x,y) = 0$, is smooth. We show in the following proposition how smoothness of the curve and of the fiber are related.

Proposition 3.1 *If the algebraic curve $\Gamma_F \subset \mathbf{C}^2$ defined by $F(x,y) = 0$ is smooth, then the fiber $\mathcal{A}_{F,d} \subset \mathbf{R}^{2d}$ is also smooth.*

Proof

$\mathcal{A}_{F,d}$ will be smooth if and only if $H_{F,d}$ is submersive at each point of $\mathcal{A}_{F,d}$, i.e., iff

$$\operatorname{rank}\left(\frac{\partial H_i}{\partial u_1},\ldots,\frac{\partial H_i}{\partial u_d},\frac{\partial H_i}{\partial v_1},\ldots,\frac{\partial H_i}{\partial v_d}\right)_{1\leq i\leq d} = d \text{ along } \mathcal{A}_{F,d}.$$

From the proof of Theorem 2.4 and the definition (3.1) of $\mathcal{A}_{F,d}$, the jth and $(d+j)$th columns of this matrix are respectively given by

$$\lambda^{d-j}\frac{F(\lambda,v(\lambda))}{u(\lambda)} \bmod u(\lambda) \quad \text{and} \quad \lambda^{d-j}\frac{\partial F}{\partial y}(\lambda,v(\lambda)) \bmod u(\lambda).$$

It is therefore sufficient to show that if Γ_F is smooth then the dimension of the linear space

$$\left(R_1(\lambda)\frac{F(\lambda,v(\lambda))}{u(\lambda)} + R_2(\lambda)\frac{\partial F}{\partial y}(\lambda,v(\lambda))\right) \bmod u(\lambda),\ \deg R_i(\lambda) < d, \tag{3.2}$$

equals d. Let $\lambda_1,\ldots,\lambda_r$ be the distinct roots of $u(\lambda)$, λ_i having multiplicity s_i. We claim that

$$\frac{F(\lambda_i,v(\lambda_i))}{u(\lambda_i)} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y}(\lambda_i,v(\lambda_i)) = 0 \tag{3.3}$$

cannot hold simultaneously if Γ_F is smooth. Otherwise $(x_i, y_i) = (\lambda_i, v(\lambda_i))$ would be a singular point of Γ_F: if (3.3) holds then clearly $\frac{\partial F}{\partial y}(x_i, y_i) = 0$, but also $F(x_i, y_i) = \frac{\partial F}{\partial x}(x_i, y_i) = 0$ because in this case $F(x, y_i)$ has a double root at $x = x_i$.

The dimension of (3.2) is now investigated by using the fact that for any polynomial $p(\lambda)$, the value of $p(\lambda) \bmod u(\lambda)$ at λ_i is just $p(\lambda_i)$, and the values of the first $s_i - 1$ derivatives of $p(\lambda) \bmod u(\lambda)$ at λ_i are given by the values of the corresponding derivatives of $p(\lambda)$ at λ_i (s_i is the multiplicity of λ_i in $p(\lambda)$). Let us suppose that the different roots of $u(\lambda)$ are ordered such that $\lambda_1, \ldots, \lambda_t$ are also roots of $\frac{\partial F}{\partial y}(\lambda, v(\lambda))$, while $\lambda_{t+1}, \ldots, \lambda_r$ are not. As a first restriction, let $R_1(\lambda)$ (resp. $R_2(\lambda)$) be such that its first $s_i - 1$ derivatives vanish at λ_i for $t+1 \le i \le r$ (resp. $1 \le i \le t$). As a further restriction it is (by the first restriction and as (3.3) cannot happen) now easy to see that $R_1(\lambda)$ (resp. $R_2(\lambda)$) can be determined such that the polynomial given by (3.2) and the first $s_i - 1$ derivatives of (3.2) take any given values at λ_i for $1 \le i \le t$ (resp. $t+1 \le i \le r$). These d conditions are independent, hence the dimension of (3.2) equals d and $\mathcal{A}_{F,d}$ is smooth. ∎

We aim at a more precise description of the structure of the invariant manifolds $\mathcal{A}_{F,d}$, which will be useful for describing their topological structure. If the fixed point set of the complex conjugation map $\tau\colon \mathbf{C}^{2d} \to \mathbf{C}^{2d}\colon z \mapsto \bar{z}$ is denoted as $\mathrm{Fix}(\tau)$, then clearly $\mathcal{A}_{F,d}$ is given by

$$\mathcal{A}_{F,d} = \mathrm{Fix}(\tau) \cap \mathcal{A}^{\mathbf{C}}_{F,d}, \text{ where } \mathcal{A}^{\mathbf{C}}_{F,d} = \left\{ (u(\lambda), v(\lambda)) \in \mathbf{C}^{2d} \mid \left[\frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_- = 0 \right\}. \tag{3.4}$$

Note that $\mathcal{A}^{\mathbf{C}}_{F,d}$ is the complex invariant manifold lying over 0 of the integrable system on $\mathbf{C}^{2d}$ associated to F (see Amplification 2.7). The following proposition is the complex analog of Proposition 3.1.

Proposition 3.2 *The curve $\Gamma_F \subset \mathbf{C}^2$ is smooth if and only if the fiber $\mathcal{A}^{\mathbf{C}}_{F,d} \subset \mathbf{C}^{2d}$ is smooth.*

Proof

If Γ_F has a singular point $P_1 = (x_1, y_1)$, choose for $i = 2, \ldots, d$ a point $P_i = (x_i, y_i)$ on Γ_F and define $(u(\lambda), v(\lambda)) = \mathcal{S}((x_1, y_1), \ldots, (x_d, y_d)) \in \mathcal{A}^{\mathbf{C}}_{F,d}$. All polynomials given by (3.2) vanish for $\lambda = x_1$, hence they span a linear space of dimension less than d. Thus $H_{F,d}$ is not submersive at $(u(\lambda), v(\lambda))$ and $\mathcal{A}_{F,d}$ is singular at this point. This shows the if part of the proposition; the only if part is proven verbatim as in the real case (Proposition 3.1). ∎

It will be seen that a clear understanding of the structure of the complex manifolds $\mathcal{A}^{\mathbf{C}}_{F,d}$ (for Γ_F smooth), leads also to a precise description of the real manifolds $\mathcal{A}_{F,d}$.

3.2. The structure of the complex invariant manifolds $\mathcal{A}^{\mathbf{C}}_{F,d}$

We will show that $\mathcal{A}^{\mathbf{C}}_{F,d}$ is an affine part of the d-fold symmetric product $\mathrm{Sym}^d\Gamma_F$ of $\Gamma_F \subset \mathbf{C}^2$; our proof is a generalization of Mumford's construction in [M] Ch. IIIa, Sect. 1, which is specific for hyperelliptic curves. Recall (e.g. from [Gu]) that $\mathrm{Sym}^d\Gamma_F$ is defined as the orbit space of the obvious action of the permutation group S_d on the cartesian product $\Gamma_F^d = \Gamma_F \times \cdots \times \Gamma_F$ (d factors), i.e.,

$$\mathrm{Sym}^d\Gamma_F = \Gamma_F^d / S_d.$$

$\mathrm{Sym}^d\Gamma_F$ inherits its structure as a complex algebraic variety from the algebraic structure of Γ_F. Moreover the smoothnes of Γ_F implies smoothnes of $\mathrm{Sym}^d\Gamma_F$: namely each point $P = \langle P_1^{m_1}, \ldots, P_r^{m_r} \rangle \in \mathrm{Sym}^d\Gamma$ (with all P_i different; m_i is the multiplicity of P_i in P) has a neighborhood which is isomorphic to a neighborhood of $(\langle P_1^{m_1} \rangle, \ldots, \langle P_r^{m_r} \rangle)$ in $\mathrm{Sym}^{m_1}\Gamma_F \times \cdots \times \mathrm{Sym}^{m_r}\Gamma_F$, and a point $\langle P_i^{m_i} \rangle$ on the diagonal of $\mathrm{Sym}^{m_i}\Gamma_F$ has coordinates given by the m_i elementary symmetric functions of the m_i coordinate functions on $\Gamma_F^{m_i}$.

Theorem 3.3 *If the algebraic curve Γ_F in $\mathbf{C}^2$, defined by $F(x,y)=0$ is smooth, then $\mathcal{A}^{\mathbf{C}}_{F,d}$ is biholomorphic to the (Zariski) open subset of* $\mathrm{Sym}^d\Gamma_F$*, obtained by removing from it the divisor*

$$\mathcal{D}_{F,d}=\left\{\langle P_1,\dots,P_d\rangle \mid \exists i,j\colon 1\leq i<j\leq d, \begin{pmatrix} x(P_i)=x(P_j) \text{ with } P_i\neq P_j, \text{ or} \\ P_i=P_j \text{ is a ramification point of } x \end{pmatrix}\right\}.$$

Proof

- *Construction of the map* $\phi_{F,d}\colon \mathcal{A}^{\mathbf{C}}_{F,d}\to \mathrm{Sym}^d\Gamma_F\setminus\mathcal{D}_{F,d}$

Given a point $(u(\lambda),v(\lambda))\in\mathcal{A}^{\mathbf{C}}_{F,d}$, a point in $\mathrm{Sym}^d\Gamma_F$ is associated to it as follows: for every root λ_i of $u(\lambda)$ one has $F(\lambda_i,v(\lambda_i))=0$, because $\left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_-=0$, so each root λ_i of $u(\lambda)$ determines a point $(\lambda_i,v(\lambda_i))$ on Γ_F. Thus there corresponds to $(u(\lambda),v(\lambda))\in\mathcal{A}^{\mathbf{C}}_{F,d}$ an unordered set of d points $\langle P_1,\dots,P_d\rangle\in\mathrm{Sym}^d\Gamma_F$, where P_i is defined by $(x(P_i),y(P_i))=(\lambda_i,v(\lambda_i))$. Clearly, if $x(P_i)=x(P_j)$ then $P_i=P_j$; therefore, to show that $\langle P_1,\dots,P_d\rangle$ never belongs to $\mathcal{D}_{F,d}$ we only need to prove that $P_i=P_j$ cannot occur for $i\neq j$ if P_i is a ramification point for x, i.e., if $y(P_i)$ is a multiple root of $F(x(P_i),y)$ (as a polynomial in y). As $P_i=P_j$ $(i\neq j)$ implies that $u(\lambda)$ has a multiple root $x(P_i)$, in such a case $F(x,y(P_i))$ would have a multiple root $x=x(P_i)$, again because $\left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_-=0$. If moreover P_i is a ramification point of x then also $\frac{\partial F}{\partial y}(x(P_i),y(P_i))=0$ and it follows that $(x(P_i),y(P_i))$ is a singular point of Γ_F, a contradiction.

- $\mathcal{D}_{F,d}$ *is a divisor on* $\mathrm{Sym}^d\Gamma_F$

This means that $\mathcal{D}_{F,d}$ is given locally as the zero locus of a holomorphic function. If $\langle P_1,\dots,P_g\rangle\in\mathcal{D}_{F,d}$ let the set of indices $\{1,\dots,d\}$ be decomposed as $S_1\cup\dots\cup S_n$, such that all points P_i where i runs through one of the subsets S_j have the same x-coordinate, which is disjoint from the x-coordinates of the points which correspond to the other subsets. For each P_i $(i=1,\dots,d)$ let x_i denote the lifting of x to a small neighborhood of $\langle P_1,\dots,P_d\rangle$ (corresponding to the factor P_i). Then a local defining equation of $\mathcal{D}_{F,d}$ is given by

$$\prod_{i=1}^{n}\prod_{\substack{j,k\in S_i\\ j<k}}(x_j-x_k)=0.$$

- $\phi_{F,d}$ *is a biholomorphism*

We first construct the inverse of $\phi_{F,d}$, which is closely related to the map $\mathcal{S}$, as given by (2.3). Let $\langle P_1,\dots,P_d\rangle\in\mathrm{Sym}^d\Gamma_F\setminus\mathcal{D}_{F,d}$. Clearly $u(\lambda)$ is taken as

$$u(\lambda)=\prod_{i=1}^{d}(\lambda-x(P_i)). \tag{3.5}$$

If all $x(P_i)$ are different then $v(\lambda)$ is uniquely determined as the polynomial of degree $d-1$ whose value at $\lambda=x(P_i)$ is $y(P_i)$, i.e., $v(\lambda)$ is given by

$$v(\lambda)=\sum_{l=1}^{d}y_l\prod_{k\neq l}\frac{\lambda-x_k}{x_l-x_k} \tag{3.6}$$

and is holomorphic there. If two values coincide, say $x(P_1) = x(P_2)$, then $P_1 = P_2$ is not a ramification point (since the point does not belong to $\mathcal{D}_{F,d}$), hence the equation $F(x,y) = 0$ can be solved uniquely as $y = f(x)$ in a neighborhood of $P_1 = P_2$. For P_1' and P_2' in this neighborhood, substitute

$$f(x(P_i')) = f(x(P_1)) + (x(P_i') - x(P_1))\frac{df}{dx}(x(P_1)) + \mathcal{O}\left(x(P_1') - x(P_1)\right)^2, \qquad (i = 1,2)$$

for y_1 and y_2 in (3.6), to obtain that $v(\lambda)$ has no poles as $P_1', P_2' \to P_1$, hence extends to a holomorphic function on the larger subset where at most two points coincide. Since the complement of this larger subset in $\mathrm{Sym}^d\Gamma_F \setminus \mathcal{D}_{F,d}$ is of codimension at least two, $v(\lambda)$ extends to a holomorphic function on $\mathrm{Sym}^d\Gamma_F \setminus \mathcal{D}_{F,d}$. It also follows that this holomorphic function is the inverse of $\phi_{F,d}$ on all of $\mathrm{Sym}^d\Gamma_F \setminus \mathcal{D}_{F,d}$: if the point P_i has multiplicity s_i, then the first $s_i - 1$ derivatives of $v(\lambda)$ at $x(P_i)$ coincide with those of $f(\lambda)$ at $x(P_i)$, hence $F(\lambda, y(P_i))$ has a root of multiplicity s_i at $\lambda = x(P_i)$. Finally, the inverse of a holomorphic bijection between complex manifolds is always holomorphic (see [GH]), hence $\phi_{F,d}$ is a biholomorphism. ∎

Amplification 3.4

Mumford shows that in the hyperelliptic case the biholomorphism is actually an isomorphism (a biregular map). His argument goes over verbatim to the general case, showing that as an algebraic variety $\mathcal{A}_{F,d}^{\mathbf{C}}$ is isomorphic to an affine part of $\mathrm{Sym}^d\Gamma_F$.

3.3. The structure of the real invariant manifolds $\mathcal{A}_{F,d}$

Since $\mathcal{A}_{F,d}$ is given as $\mathcal{A}_{F,d}^{\mathbf{C}} \cap \mathrm{Fix}(\tau)$, it consists of those polynomials $(u(\lambda), v(\lambda)) \in \mathcal{A}_{F,d}$ whose coefficients are all real. We figure out what this means for the corresponding point in $\mathrm{Sym}^d\Gamma_F$.

Proposition 3.5 *Under the biholomorphism $\phi_{F,d}$, the real invariant manifolds $\mathcal{A}_{F,d}$ correspond to the set of all unordered d-tuples of points $\langle P_1, \ldots, P_d\rangle$ on Γ_F, consisting only of real points $P_i \in \mathbf{R}^2 \cap \Gamma_F$ and complex conjugated pairs $P_i = \bar{P}_j$, each ramification point (of x) occurring at most once, and $x(P_i) = x(P_j)$ only if $P_i = P_j$. Moreover its manifold structure derives from the structure of the d-fold symmetric product of Γ_F.*

Proof

$u(\lambda)$ is real if and only if its roots consist of real roots and roots which occur in complex conjugate pairs. Obviously, if $v(\lambda)$ is real, then at each root x_i of $u(\lambda)$, with multiplicity s_i, $v(\lambda)$ and the first $s_i - 1$ derivatives of $v(\lambda)$ take complex conjugate values when evaluated at complex conjugate points (in particular, real values at real points). It is checked that this is also a sufficient condition for $v(\lambda)$ to be real. Since $v(x_i) = y_i$, this means that the real polynomials $(u(\lambda), v(\lambda))$ on $\mathcal{A}_{F,d}^{\mathbf{C}}$ correspond to those points $\langle P_1, \ldots, P_d\rangle$ in $\mathrm{Sym}^d\Gamma_F$ consisting of real points $P_i = (x(P_i), y(P_i)) \in \mathbf{R}^2$ and complex conjugated pairs $P_j = (x(P_j), y(P_j)) = \left(\overline{x(P_k)}, \overline{y(P_k)}\right) = \bar{P}_k$, but not belonging to $\mathcal{D}_{F,d}$, i.e., the multiplicity of each ramification point (of x) is at most one, and $x(P_i) = x(P_j)$ only if $P_i = P_j$. ∎

Proposition 3.5 can be used to obtain a precise description of the topology of the real invariant manifolds $\mathcal{A}_{F,d}$, as we show now for $d = 2$ (for $d = 1$, $\mathcal{A}_{F,d}$ is just $\Gamma_F \cap \mathbf{R}^2$, the real part of Γ_F).

For a fixed F such that Γ_F is smooth, let the connected components of $\Gamma_F \cap \mathbf{R}^2$ (if any) be denoted by $\Gamma_1, \ldots, \Gamma_s$ and define for $1 \leq i, j, k \leq s$, $i < j$,

$$\Gamma_{00} = \{\langle P, \bar{P}\rangle \mid P \in \Gamma_F,\ x(P) \notin \mathbf{R}\},$$
$$\Gamma_{ij} = \{(P_1, P_2) \in \Gamma_i \times \Gamma_j \mid x(P_1) = x(P_2) \Rightarrow P_1 = P_2\},$$
$$\Gamma_{kk} = \left\{\langle P_1, P_2\rangle \in \frac{\Gamma_k \times \Gamma_k}{S_2} \mid x(P_1) = x(P_2) \Rightarrow (P_1 = P_2 \text{ and is not a ramification point of } x)\right\}.$$

Then the union of Γ_{00} with all the sets Γ_{ij} and Γ_{kk} is easy identified with $\phi_{F,2}(\mathcal{A}_{F,2})$, the surface to be described. Note that the only paths in it which are not contained in $\mathbf{R}^2$, are in Γ_{00}, and Γ_{00} connects exactly the surfaces Γ_{kk}. Therefore, if $i \neq j$ then Γ_{ij} is not connected to any other Γ_{mn}, Γ_{kk}, nor to Γ_{00}.

Therefore we first concentrate on such a subset Γ_{ij}, say on Γ_{12}. If the intervals $x(\Gamma_1)$ and $x(\Gamma_2)$ are disjoint, then $\Gamma_{12} = \Gamma_1 \times \Gamma_2$, so Γ_{12} is either homeomorphic to a torus, a cylinder or a disc, depending on whether the components Γ_1 and Γ_2 are closed or open. If $x(\Gamma_1)$ and $x(\Gamma_2)$ have a value x_0 in common, then one finds again these surfaces, but with a number of punctures (holes), equal to

$$\prod_{i=1}^{2} \#\{Q \in \Gamma_i \mid x(Q) = x_0\}.$$

If $x(\Gamma_1)$ and $x(\Gamma_2)$ have an interval in common, Γ_{12} may even disconnect in different pieces. The structure of these pieces is easy determined from a picture of the real part of the curve. Namely, on a square representing $\Gamma_1 \times \Gamma_2$, the divisor $\{(P_1, P_2) \in \Gamma_i \times \Gamma_j \mid x(P_1) = x(P_2)\}$ is drawn by counting points on the vertical lines $x =$ constant, the only care one needs to take is that if Γ_1 (or Γ_2) is closed, then an origin should be marked on it, and if one passes this origin, one needs to pass over the corresponding edge of the rectangle. The following table shows some examples (all possibilities for which Γ_1 and Γ_2 are closed, and x is 2:1 when restricted to Γ_1 and Γ_2).

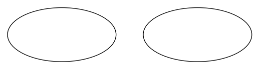
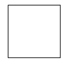
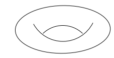
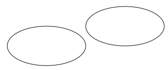
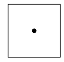
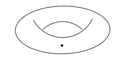
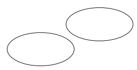
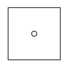
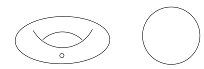
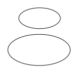
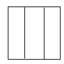
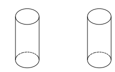
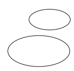
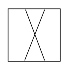
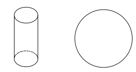
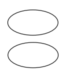
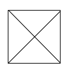
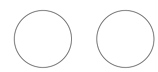
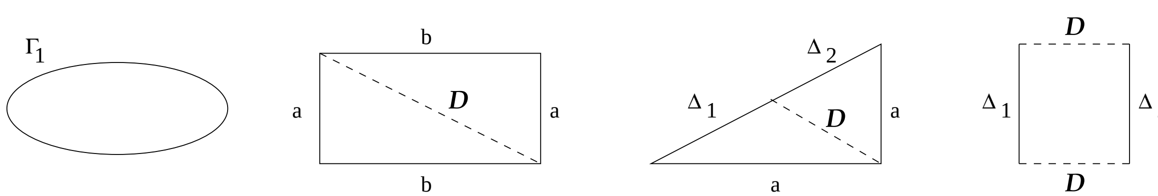
Γ_1 and Γ_2	Divisor	Component Γ_{12}	Picture
		torus	
		torus minus point	
		(torus minus disc) + disc	
		two cylinders	
		cylinder + disc	
		two discs	

Table 1

In the same way Γ_{kk} is investigated by drawing the divisor

$$\left\{(P_1,P_2)\in\Gamma_i\times\Gamma_i \mid x(P_1)=x(P_2) \text{ and } \left(\begin{array}{l}P_1\neq P_2 \text{ or,}\\ P_1=P_2 \text{ is a ramification point of } x\end{array}\right)\right\}.$$

on a rectangle representing $\Gamma_i\times\Gamma_i$. Each triangle which is cut off from the rectangle by its main diagonal then represents $\frac{\Gamma_i\times\Gamma_i}{S_2}$ and Γ_{ii} is the complement of the divisor in the triangle. For example, consider a component Γ_1 as in Figure 1.a below. Then Figure 1.b shows a torus with a circle on it (the anti-diagonal of the rectangle), which is the divisor D to be removed. The resulting piece Γ_{11} is drawn in Figure 1.c and is redrawn in a simpler way in Figure 1.d.



(a) (b) (c) (d)

Figure 1

For every Γ_i, such a piece is found and will be glued to Γ_{00} precisely along the part of its boundary which comes from the diagonal in the rectangle (the solid lines in Figure 1.d).

In order to explain how Γ_{00} is described, we recall the classical picture of a (smooth, complete) algebraic curve $\bar{\Gamma}$. An equation $F(x,y)=0$ of such a curve defines an $m{:}1$ ramified covering map to $\mathbf{P}^1$ by $(x,y)\mapsto x$, when m is the degree of $F(x,y)$ in y. This may be visualized by drawing concentric spheres (called *sheets*), on which there are marked some non-intersecting intervals (called *cuts*, every cut is equally present on all sheets). The topology is such that if you are walking on a sheet i and pass a cut j (from a fixed side) then you move to a sheet $p_j(i)$, each p_j being a permutation of $\{1,\dots,m\}$. It is clear that the cuts and their corresponding permutations determine the topology of the curve completely. Since each cut connects two ramification points (of x), these cuts may, for a real curve, be taken on the real axis and orthogonal to it.

Γ_{00} is now given as follows. Consider the described picture for the smooth completion $\bar{\Gamma}_F$ of Γ_F. Clearly the conjugation map interchanges the upper and lower hemispheres and is fixed on the equator(s) $\{P\in\bar{\Gamma}_F \mid x(P)\in\mathbf{R}\cup\infty\}$. It follows that the open upper (lower) hemispheres give precisely Γ_{00}. A convenient way to represent them is by drawing a disc for each upper hemisphere and labelling the different parts of the boundary which correspond to the horizontal and vertical cuts. A moment's thought reveals that the different sheets are to be connected along those lines which correspond to the vertical cuts, while the pieces Γ_{kk} are to be connected to the corresponding horizontal cuts. This gives a topological model of $\Gamma_{00}\cup\bigcup_{k=1}^{s}\Gamma_{kk}$ as a disc with holes. The following example may highlight the different steps.

Example 3.6
We consider a hyperelliptic curve $F(x,y)=y^2-f(x)=0$, where f is an irreducible monic polynomial of degree seven with five real roots. The curve Γ_F has genus three and its graph and related representation as a cover of $\mathbf{P}^1$ are given by

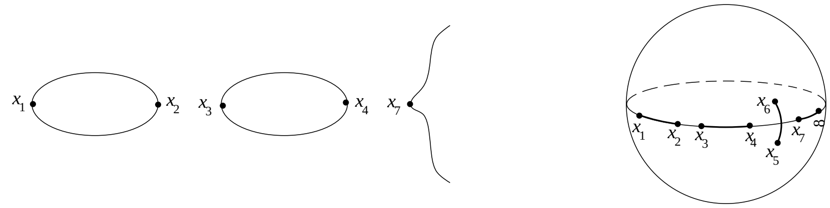


Figure 2

where the imaginary ramification points (of x) are not seen from the graph. For Γ_{00} we get two upper hemispheres

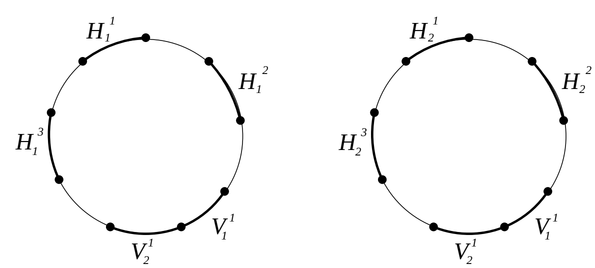


Figure 3

which become one disc after gluing the vertical cut V_1^1, V_2^1. We also get two subsets Γ_{11} and Γ_{22} given as in Figure 1.d by

Figure 4

and one disconnected piece Γ_{33} (since $\infty \in \Gamma_3$)

Figure 5

Now, glue Figures 3, 4 and 5 according to their labels H_j^i to find a disc with two holes. Since the other components of $\mathcal{A}_{F,2}$ are direct products of the real components, we find that

$$\mathcal{A}_{F,2} \simeq \text{ one torus } + \text{ two cylinders } + \text{ one disc with two holes.}$$

Example 3.7

It is shown in the same way that, if $F(x,y)$ is of the form

$$F(x,y) = y^2 + \prod_{i=1}^{n}(x-\alpha_i) \prod_{j=1}^{m} \left(x^2+\beta_j^2\right),$$

with $\alpha_i, \beta_j \in \mathbf{R}$ (all α_i being different, as well as all β_i^2), then

$$\begin{array}{llll} \mathcal{A}_{F,2} \simeq \binom{(n-1)/2}{2} \text{ tori} + \frac{n-1}{2} \text{ cylinders} + & \text{one disc with } g-1 \text{ holes} & & \text{if } n \text{ is odd,} \\ \mathcal{A}_{F,2} \simeq \quad \binom{n}{2} \quad \text{ tori} + & \text{one disc with } g \text{ holes} & & \text{if } n \text{ is even,} \end{array}$$

where g is the genus $\left[\frac{n+1}{2}\right] + m - 1$ of the curve $F(x,y) = 0$.

3.4. The significance of the Poisson structures $\{\cdot\,,\cdot\}_d^{\varphi}$

As we have shown, the choice of $F(x,y)$ determines the invariant manifolds on which the flows of our systems evolve. In particular, for different choices of $\varphi(x,y)$ the Hamiltonian vector fields will all be tangent to the same invariant manifolds. They have this property in common with bi-Hamiltonian integrable systems. Therefore we propose the following definition.

Definition 3.8 On a manifold M let there be given two (different) compatible Poisson structures $\{\cdot\,,\cdot\}_1$ and $\{\cdot\,,\cdot\}_2$ and an algebra $\mathcal{A}$ of functions on M. If $\mathcal{A}$ makes both $(M, \{\cdot\,,\cdot\}_1, \mathcal{A})$ and $(M, \{\cdot\,,\cdot\}_2, \mathcal{A})$ into an integrable system then we call these systems *compatible integrable systems*; in the particular case that both integrable systems have a vector field in common, i.e., if there exist $f_1, f_2 \in \mathcal{A}$ for which $\{\cdot\,, f_1\}_1 = \{\cdot\,, f_2\}_2$ then they are collectively called a *bi-Hamiltonian* integrable system.

Clearly we have defined in the present paper a lot of compatible integrable systems, which are not bi-Hamiltonian. The dependence of the vector fields on $\varphi(x,y)$ is explicitly given by the following proposition.

Proposition 3.9 *Let H_i $(i=1,\dots,d)$ denote the functions on $\mathbf{R}^{2d}$, defined in (2.8), and let X_i^1 and X_i^φ denote their Hamiltonian vector fields with respect to $\{\cdot,\cdot\}_d$ and $\{\cdot,\cdot\}_d^\varphi$. Then the transfer matrix $\mathcal{T}_1^\varphi$, which is defined by*

$$(X^\varphi_{H_1},\dots,X^\varphi_{H_d}) = (X^1_{H_1},\dots,X^1_{H_d})\ \mathcal{T}_1^\varphi,$$

is given by

$$\mathcal{T}_1^\varphi = \varphi(M, v(M)), \quad \textit{where} \quad M = \begin{pmatrix} -u_1 & -u_2 & -u_3 & \dots & -u_d \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{3.7}$$

The general transfer matrices $\mathcal{T}_{\varphi_1}^{\varphi_2}$ are immediately computed from (3.7) upon using the cocycle identities

$$\mathcal{T}_{\varphi_1}^{\varphi_2}\mathcal{T}_{\varphi_2}^{\varphi_1} = 1 \qquad \textit{and} \qquad \mathcal{T}_{\varphi_1}^{\varphi_2}\mathcal{T}_{\varphi_2}^{\varphi_3}\mathcal{T}_{\varphi_3}^{\varphi_1} = 1.$$

Proof

We first write down the vector field X_{H_i} at a generic point $(u(\lambda),v(\lambda)) \in \mathcal{A}_{F,d}^{\mathbf{C}}$; the genericity condition taken here is that for $\phi_{F,d}(u(\lambda),v(\lambda)) = \langle (x_1,y_1),\dots,(x_d,y_d)\rangle$ all x_i are different and none of the points (x_i,y_i) is a ramification point of x. Varying the point $(u(\lambda),v(\lambda))$ in a small neighborhood, each x_i gives a local coordinate on a neighborhood $U_i \subset \Gamma_F$ of (x_i,y_i) as well as a local coordinate on a neighborhood $U \subset \mathcal{A}_{F,d}^{\mathbf{C}}$ of $\langle (x_1,y_1),\dots,(x_d,y_d)\rangle$. Since on the one hand the derivative of $u(\lambda) = \prod_{k=1}^d (\lambda - x_k)$ at $\lambda = x_j$ is $X_{H_i}u(x_j) = -\prod_{l\neq j}(x_j - x_l)\, X_{H_i}x_j$, while at the other hand, direct substitution in (2.10) gives

$$X_{H_i}u(x_j) = \frac{\partial F}{\partial y}(x_j,y_j)\sum_{k=0}^{i-1} u_k x_j^{i-k-1},$$

we find that

$$X_{H_i}x_j = -\prod_{l\neq j}(x_j - x_l)^{-1}\frac{\partial F}{\partial y}(x_j,y_j)\sigma_{i-1}(\hat{x}_j), \tag{3.8}$$

where $\sigma_{i-1}(\hat{x}_j)$ is the $(i-1)$th symmetric function in $x_1,\dots,x_d$, evaluated at $x_j = 0$. Using (2.6) it follows that

$$X_H^\varphi = \Delta^\varphi X_H \qquad \text{with} \qquad \Delta^\varphi = \operatorname{diag}(\varphi(x_1,y_1),\dots,\varphi(x_d,y_d));$$

X_H^φ denotes the matrix with entries $(X_H^\varphi)_{ij} = X^\varphi_{H_j}x_i$ and $X_H = X_H^1$. Therefore $\mathcal{T}_1^\varphi$ is given by

$$\begin{aligned} T_1^\varphi &= (X_H)^{-1}\,\Delta^\varphi X_H \\ &\overset{(i)}{=} V\Delta^\varphi V^{-1} \\ &= V\varphi(\Delta^x, v(\Delta^x))V^{-1} \\ &= \varphi\left(V\Delta^x V^{-1}, v(V\Delta^x V^{-1})\right) \\ &= \varphi(M, v(M)), \end{aligned}$$

where $M = V\Delta^x V^{-1}$ is easily checked to have the form announced in (3.7). Step (i) requires some extra work (one uses (3.8)); also we have introduced the notation V for the Vandermonde matrix

$$V = \begin{pmatrix} x_1^{d-1} & x_2^{d-1} & \dots & x_d^{d-1} \\ x_1^{d-2} & x_2^{d-2} & \dots & x_d^{d-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

∎

Remark 3.10

In the special case where $d = 2$ and $\varphi(x, y) = x$ one obtains what Caboz et al. call a (ρ, s) bi-Hamiltonian structure (see [CGR]). Our definition of compatible integrable systems and Proposition 3.9 generalize and clarify this concept.

4. Algebraic completely integrable systems

We arrive now to what was our original motivation, namely the explicit construction of algebraic completely integrable systems (a.c.i. systems) for general Jacobians. Recall from [AvM1] that an a.c.i. system is defined as an integrable system on $\mathbf{R}^N$ which behaves well under complexification. At first this means that both the Poisson bracket and the functions in involution are supposed to be polynomial, so that the generic invariant manifold has a natural complexification as a (complex) algebraic variety. Second, these varieties are required to complete into Abelian varieties (that is, complex algebraic tori), on which the standard coordinates on $\mathbf{C}^N$ provide meromorphic functions. Third, the divisors to be adjoined to the complex invariant manifolds are assumed to be minimal in the sense that on each of its components at least one of these meromorphic functions has a pole. Finally the complex flow of the vector fields which define the integrable system is assumed to be linear. If all these conditions are satisfied with the first integrals (constants of motion) being rational instead of polynomial, then we will speak of an a.c.i. system *with rational first integrals.*

Recall also from [GH] that a complex (algebraic) torus is associated to any (complex, smooth, complete) algebraic curve $\bar{\Gamma}$ of genus g as follows: choose a base $\{\omega_1, \dots, \omega_g\}$ for the holomorphic differentials and a symplectic base $\{A_1, \dots, A_g, B_1, \dots, B_g\}$ for $H_1(\bar{\Gamma}, \mathbf{Z})$, symplectic meaning here that $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$. If we denote $\vec{\omega} = (\omega_1, \dots, \omega_g)$ then the lattice

$$\Lambda_{\bar{\Gamma}} = \operatorname{span} \left\{ \int_{A_i} \vec{\omega}\,, \int_{B_i} \vec{\omega} \mid 1 \leq i \leq g \right\}$$

is of rank $2g$ and a complex algebraic torus (which can be shown to be independent of the choices made) is defined by $\operatorname{Jac}(\bar{\Gamma}) = \mathbf{C}^g / \Lambda_{\bar{\Gamma}}$, the so-called *Jacobian* of $\bar{\Gamma}$. Fixing any base point $P_0 \in \bar{\Gamma}$ there is for each $d \in \mathbf{N}$ a well-defined holomorphic map $A_d \colon \operatorname{Sym}^d \bar{\Gamma} \to \operatorname{Jac}(\bar{\Gamma})$ defined by

$$A_d(\langle P_1, \dots, P_d \rangle) = \sum_{i=1}^{d} \int_{P_0}^{P_i} \vec{\omega} \bmod \Lambda_{\bar{\Gamma}},$$

classically known as *Abel's map*; Abel's Theorem says that $A_d(\langle P_1, \dots, P_d \rangle) = A_d(\langle Q_1, \dots, Q_d \rangle)$ if and only if there exists a meromorphic function on $\bar{\Gamma}$ with zeros at the points $P_1, \dots, P_d$ and poles at $Q_1, \dots, Q_d$. Moreover A_d is surjective for $d \geq g$.

In particular, writing ω_i locally as $\omega_i(P) = \Omega_i(P) dz_i$, the torus $\operatorname{Jac}(\bar{\Gamma})$ is obtained from $\operatorname{Sym}^g \bar{\Gamma}$ by blowing down those $\langle P_1, \dots, P_g \rangle$ for which $\det \Omega_i(P_j) = 0$.

4.1. The general case

We will now modify our construction to obtain for a large class of plane curves an a.c.i. system for which each complex invariant manifold is an affine part of the Jacobian of a deformation of this curve. This new construction will coincide with the previous one in one case (called the hyperelliptic case), considered in the next paragraph.

We start with a smooth plane curve $\Gamma_F \subset \mathbf{C}^2$, defined by $F(x,y) = 0$. Let g denote the genus of the smooth completion $\bar{\Gamma}_F$ of Γ_F. Each holomorphic differential ω on $\bar{\Gamma}_F$ can be written as

$$\omega = \frac{R(x,y)}{\frac{\partial F}{\partial y}(x,y)} dx,$$

for some polynomial $R(x,y)$, hence the choice of a basis for the space of differentials leads to g polynomials $R_i(x,y)$. Having fixed such a basis, we define for any $c = (c_1, \ldots, c_g) \in \mathbf{C}^g$ a polynomial F_c with corresponding curve Γ_{F_c} by

$$F_c(x,y) = F(x,y) + \sum_{i=1}^{g} c_j R_i(x,y).$$

The following property will be assumed on the curve Γ_F:

Assumption. For generic values of c, a basis for the space of holomorphic differentials on Γ_{F_c} is given by

$$\frac{R_i(x,y)}{\frac{\partial F_c}{\partial y}(x,y)}dx \qquad (i = 1, \ldots, g).$$

We stress that the assumption, which is easily checked for any concrete curve at hand, is obviously valid for hyperelliptic, trigonal, say n-gonal curves. As H. Knörrer pointed out the assumption needs not be satisfied when the curve behaves badly at infinity (for his counter-example, see [V3] Ch. VI, Sect. 4).

Theorem 4.1 *Let Γ_F be a any smooth algebraic curve of genus g in $\mathbf{C}^2$. Then the g rational functions $H_i = H_i(u_j, v_j)$, defined by*

$$F(\lambda, v(\lambda)) - \sum_{i=1}^{g} H_i R_i(\lambda, v(\lambda)) = 0 \bmod u(\lambda) \tag{4.1}$$

define g linear vector fields on the level manifold over 0, which is an affine part of the Jacobian of $\bar{\Gamma}_F$. If the above assumption is valid for Γ_F, these Hamiltonians H_i define an a.c.i. system with rational first integrals on $\left(\mathbf{R}^{2d}, \{\cdot\,,\cdot\}_d\right)$, whose invariant manifolds are affine parts of the Jacobians of the family of curves

$$\Gamma_{F_c} \,:\, F(x,y) - \sum_{i=1}^{g} c_i R_i(x,y) = 0, \qquad c = (c_1, \ldots, c_g) \in \mathbf{C}^g.$$

Proof

Let $R(x,y) = (R_1(x,y), \ldots, R_g(x,y))$ and equip $\mathbf{R}^g$ with its standard inner product, denoted by $\langle\cdot\,,\cdot\rangle$. Similar to the map $\hat{H}_{F,d}$: $\left(\mathbf{R}^2\right)^d \setminus \Delta \to \mathbf{R}^d$, introduced in Section 2, we now define a map

$$\hat{H}_F \colon \left(\mathbf{R}^2\right)^g \setminus \Delta' \to \mathbf{R}^g,$$

(the subscript g is omitted in the notation since it is implicit in F: $g = \text{genus}(\bar{\Gamma}_F)$), by requiring that the polynomial

$$\langle \hat{H}_F((x_1, y_1) \ldots, (x_g, y_g)), R(x,y)\rangle$$

has for $(x,y) = (x_i, y_i)$ the value $F(x_i, y_i)$, $(i = 1, \ldots, g)$. Note that if $R(x,y) = (x^{g-1}, \ldots, x, 1)$ this corresponds to our earlier definition of $\hat{H}_{F,d}$ (for $d = g$). As in this earlier case, $\hat{H}_F$ is invariant under the action of S_g, leading to a map H_F (with components H_i) given by the explicit formula (4.1). This map is not polynomial: indeed, Δ' is given by $\det R_i(x_j, y_j) = 0$, and $\hat{H}_F$ has its poles there. This has an important consequence for the complex invariant manifolds: let $c = (c_1, \dots, c_g)$ be generic, then we know from Theorem 3.3 that the set of all $(u(\lambda), v(\lambda))$ for which

$$F(\lambda, v(\lambda)) - \sum_{i=1}^{g} c_i R_i(\lambda, v(\lambda)) = 0 \bmod u(\lambda),$$

is biholomorphic to an affine part of $\mathrm{Sym}^g \Gamma_{F_c}$; to obtain the invariant manifold from it we need to remove the intersection with Δ', that is, those g-tuples of points on Γ_{F_c} for which $\det R_i(x_j, y_j) = 0$. By the assumption this corresponds exactly to the divisor which is blown down in the symmetric product $\mathrm{Sym}^g \bar{\Gamma}_{F_c}$ in order to obtain the Jacobian of $\bar{\Gamma}_{F_c}$. It follows that the complex invariant manifolds are affine parts of Abelian varieties. Without the assumption the result is only proven for $c = 0$.

We now show that the (complex) flows of the vector fields are linear (on the Abelian varieties), showing in particular that they commute. By construction, we have at a generic point,

$$F(x_j, y_j) = \sum_{i=1}^{g} \hat{H}_i R_i(x_j, y_j),$$

where $\hat{H}_i$ are the components of $\hat{H}_F$. Taking the bracket $\{\cdot, \cdot\}_g$ with x_k and recalling that $\{y_i, x_j\}_g = \delta_{ij}$, we have

$$\frac{\partial F}{\partial y}(x_j, y_j)\delta_{jk} = -\sum_{i=1}^{g} R_i(x_j, y_j) X_{\hat{H}_i} x_k + \sum_{i=1}^{g} \hat{H}_i \frac{\partial R_i}{\partial y}(x_j, y_j)\delta_{jk}.$$

Restricted to the invariant manifolds $\hat{H}_i = c_i$ we have

$$\sum_{i=1}^{g} R_i(x_j, y_j) X_{\hat{H}_i} x_k = -\frac{\partial F_c}{\partial y}(x_j, y_j)\delta_{jk},$$

$$\sum_{i=1}^{g} \frac{R_i(x_j, y_j)}{\frac{\partial F_c}{\partial y}(x_j, y_j)} X_{\hat{H}_i} x_k = -\delta_{jk},$$

which is easily rewritten as

$$\sum_{i=1}^{g} \frac{R_j(x_i, y_i)}{\frac{\partial F_c}{\partial y}(x_i, y_i)} X_{\hat{H}_k} x_i = -\delta_{jk}.$$

If Γ_F satisfies the assumption, then we have on the left exactly a basis for the holomorphic differentials on Γ_{F_c}, so we find that the vector fields $X_{\hat{H}_i}$ (hence also the vector fields X_{H_i}) linearize under the Abel map, that is, their flow is linear on the Jacobian of Γ_{F_c}. Without the assumption we still have linearity of the flow on the Jacobian of Γ_F (the invariant manifold over 0).

To conclude the proof of our statement that we have an a.c.i. system if Γ_F satisfies the condition, we need to show that along each of the components of the divisor which is missing from $\mathrm{Jac}(\bar{\Gamma}_{F_c})$ at least one of the functions u_i, v_i has a pole. Note that u_i and v_i are given as symmetric functions on Γ_{F_c} hence giving indeed meromorphic functions on $\mathrm{Jac}(\bar{\Gamma}_{F_c})$. The missing components are the images under the Abel map of the divisors $\bar{\mathcal{D}}_{F,g}$ and $\bar{\mathcal{E}}_{F,g}$, where $\bar{\mathcal{D}}_{F,g}$ is the closure of $\mathcal{D}_{F,g}$

in $\mathrm{Sym}^d\bar\Gamma_F$ and $\bar{\mathcal{E}}_{F,d}$ is a divisor whose irreducible components $\bar{\mathcal{E}}_{F,d}(\infty_i)$ correspond to the points ∞_i in $\bar\Gamma_F\setminus\Gamma_F$, namely

$$\bar{\mathcal{E}}_{F,d}(\infty_k)=\left\{\langle\infty_k,P_2,\dots,P_d\rangle\mid P_k\in\bar\Gamma_F \text{ for } 2\le k\le d\right\}.$$

As for the former, the functions v_i obviously all have a pole along it, for the latter, on a component $\bar{\mathcal{E}}_{F,g}(\infty_k)$ where $x(\infty_k)$ is infinite, all u_i have a pole, and if $y(\infty_k)$ is infinite, then at least v_1 has a pole along it. ∎

4.2. The hyperelliptic case

Our definitions of $H_{F,d}$ and H_F agree in one case, namely the case that F is of the form $F(x,y)=y^2+f(x)$ for some polynomial $f(x)$. In order to see that both definitions coincide, just note that a basis for the space of holomorphic differentials on $\bar\Gamma_F$ is given by

$$\left\{\frac{x^{g-1}dx}{y},\dots,\frac{xdx}{y},\frac{dx}{y}\right\},$$

leading to the polynomials $R_i(x,y)=x^{g-i}\,(i=1,\dots,g)$.

In the present paragraph we will investigate this case, but without putting any conditions on d or φ. Thus, these systems are only a.c.i. if $d=g$ and $\varphi=1$. Since $F(x,y)=0$ defines a hyperelliptic curve, we call this case the *hyperelliptic case.* Our first result is that equations (2.10) can be written as Lax equations, i.e., they can be written as a commutator in some Lie algebra (see e.g. [Gr]), as given by the following theorem.

Theorem 4.2 *The differential equations describing the vector fields for the hyperelliptic case are written in the Lax form (with spectral parameter λ)*

$$X^{\varphi}_{H_i}A(\lambda)=\left[A(\lambda),[B_i(\lambda)]_+\right],$$

where

$$A(\lambda)=\begin{pmatrix} v(\lambda) & u(\lambda)\\ w(\lambda) & -v(\lambda)\end{pmatrix},\ B_i(\lambda)=\frac{\varphi(\lambda,v(\lambda))}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+A(\lambda)\ \text{and}\ w(\lambda)=-\left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_+. \tag{4.2}$$

The spectral curve $\det(A(\lambda)-\mu\,\mathrm{Id})=0$, preserved by the flow of the vector fields $X^{\varphi}_{H_i}$, is given by $\mu^2+f(\lambda)=H_{F,d}(u(\lambda),v(\lambda))$.

Proof

If we define the polynomial $w(\lambda)$ as stated above, then equations (2.10) are easy rewritten as

$$\begin{aligned} X^{\varphi}_{H_i}u(\lambda)&=2\varphi(\lambda,v(\lambda))v(\lambda)\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+-2u(\lambda)\left[\varphi(\lambda,v(\lambda))\frac{v(\lambda)}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right]_+,\\ X^{\varphi}_{H_i}v(\lambda)&=-\varphi(\lambda,v(\lambda))w(\lambda)\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_++u(\lambda)\left[\varphi(\lambda,v(\lambda))\frac{w(\lambda)}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right]_+. \end{aligned} \tag{4.3}$$

upon using

$$\frac{\partial F}{\partial y}(\lambda,v(\lambda))=2v(\lambda).$$

From (4.3) let us also compute $X^{\varphi}_{H_i}w(\lambda)$ and observe that the explicit dependence on F disappears completely!

$$\begin{aligned}
X^{\varphi}_{H_i}w(\lambda) &= -X^{\varphi}_{H_i}\left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\right]_+ \\
&= -2\left[\frac{v(\lambda)}{u(\lambda)}X^{\varphi}_{H_i}v(\lambda)\right]_+ + \left[\frac{F(\lambda,v(\lambda))}{u(\lambda)}\frac{X^{\varphi}_{H_i}u(\lambda)}{u(\lambda)}\right]_+ \\
&= 2\left[v(\lambda)\left[\frac{w(\lambda)\varphi(\lambda,v(\lambda))}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right]_- - w(\lambda)\frac{X^{\varphi}_{H_i}u(\lambda)}{u(\lambda)}\right]_+ \\
&= -2v(\lambda)\left[\varphi(\lambda,v(\lambda))\frac{w(\lambda)}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right]_+ + 2w(\lambda)\left[\varphi(\lambda,v(\lambda))\frac{v(\lambda)}{u(\lambda)}\left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right]_+ .
\end{aligned}$$

This leads at once to the above Lax equations. The associated spectral curve is computed as follows:

$$\begin{aligned}
\det(A(\lambda)-\mu\,\mathrm{Id}) &= \mu^2 - v^2(\lambda) + u(\lambda)\left[\frac{f(\lambda)+v^2(\lambda)}{u(\lambda)}\right]_+ \\
&= \mu^2 + f(\lambda) - u(\lambda)\left[\frac{f(\lambda)+v^2(\lambda)}{u(\lambda)}\right]_- \\
&= \mu^2 + f(\lambda) - H_{F,d}(u(\lambda),v(\lambda)).
\end{aligned}$$

∎

For example, if we restrict ourselves to $d = 1$ (i.e., one degree of freedom), then $u(\lambda) = \lambda + u_1$, $v(\lambda) = v_1$ and

$$\begin{aligned}
H_{F,1}(u_1,v_1) &= \left(v^2(\lambda)+f(\lambda)\right) \bmod u(\lambda) \\
&= \left(v_1^2+f(\lambda)\right) \bmod (\lambda+u_1) \\
&= v_1^2 + f(-u_1),
\end{aligned}$$

and $\{\cdot,\cdot\}^1_1$ is the standard bracket on $\mathbf{R}^2$, so we find that for $\varphi = 1$ the hyperelliptic case in one degree of freedom corresponds exactly to the case of polynomial potentials on the line.

Amplification 4.3

In the special case that $\varphi = 1$ and $d = g = \mathrm{genus}(\Gamma_F)$ one obtains the so-called odd or even master systems, according to whether the degree of $f(x)$ is odd or even. The odd master system was introduced by Mumford in [M] and his construction was adapted by us in [V1] to obtain the even master system: in this special case we may rewrite the matrix $[B_i(\lambda)]_+$ as

$$[B_i(\lambda)]_+ = \left[\frac{A(\lambda)}{\lambda^{g-i+1}}\right]_+ + \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix},$$

where $b_i = -u_i$ or $b_i = -u_i\lambda + 2u_1u_i - u_{i+1}$, according to whether the degree of $f(x)$ is odd or even[2], showing that these systems coincide indeed with the master systems in [V1]. Both our a.c.i. and non-a.c.i. systems can be seen as generalizations — in several different directions — of Mumford's odd master system.

[2] In the latter case we killed the coefficient of x^{2g+1} in $f(x)$, precisely as we did in [V1].

4.3. $\mathcal{A}_{F,d}^{\mathbf{C}}$ as strata of hyperelliptic Jacobians

The complex invariant manifolds $\mathcal{A}_{F,d}^{\mathbf{C}}$ behave well with respect to the Abel map in the hyperelliptic case, as is shown in the following proposition. Recall that in the case of a hyperelliptic curve $\Gamma\colon y^2+f(x)=0$ the completion $\bar\Gamma$ of Γ is obtained by adding to Γ one or two points, depending on whether the degree of $f(x)$ is odd or even; these points will be denoted by ∞, resp. ∞_1 and ∞_2. Note that if $F(x,y)=y^2+f(x)$ then Γ_F is smooth (or, equivalently, $\mathcal{A}_{F,d}^{\mathbf{C}}$ is smooth) if and only if $f(x)$ has no multiple roots.

Proposition 4.4 *In the hyperelliptic case $F(x,y)=y^2+f(x)$, the complex invariant manifold $\mathcal{A}_{F,d}^{\mathbf{C}}$ is for $d\le g$ biholomorphic to a (smooth) affine part of a distinguished d-dimensional subvariety W_d of $\operatorname{Jac}(\bar\Gamma_F)$, namely*

$$\begin{aligned}
\mathcal{A}_{F,d}^{\mathbf{C}} &\cong W_d\setminus W_{d-1} && \deg f(x)\ \textit{odd},\\
\mathcal{A}_{F,d}^{\mathbf{C}} &\cong W_d\setminus (W_{d-1}\cup(\vec e+W_{d-1})) && \deg f(x)\ \textit{even},
\end{aligned}$$

where $\vec e\in\operatorname{Jac}(\bar\Gamma_F)$ is given by $\vec e=A_1(\infty_1)-A_1(\infty_2)=\int_{\infty_2}^{\infty_1}\vec\omega \bmod \Lambda_{\bar\Gamma_F}$. Also

$$\begin{aligned}
W_g &= \operatorname{Jac}(\bar\Gamma_F),\\
W_{g-1} &= \textit{theta divisor } \Theta\subset\operatorname{Jac}(\bar\Gamma_F),\\
&\ \ \vdots\\
W_1 &= \textit{curve } \bar\Gamma_F \textit{ embedded in } \operatorname{Jac}(\bar\Gamma_F),\\
W_0 &= \textit{origin of } \operatorname{Jac}(\bar\Gamma_F).
\end{aligned}$$

Proof

We prove the proposition only for the case in which $\deg f(x)$ is odd. We choose ∞ as the base point for the Abel map and define W_k for $k=1,\dots,g$ as $W_k=A_k(\operatorname{Sym}^k\bar\Gamma_F)$. By a theorem due to Jacobi $W_g=\operatorname{Jac}(\bar\Gamma_F)$ and by Riemann's Theorem, W_{g-1} is (a translate of) the Riemann theta divisor (see [GH]). Clearly for each $k\le g$, W_{k-1} is a divisor in W_k and, by another theorem of Riemann, $W_k\setminus W_{k-1}$ is smooth. We claim that

$$A_d(\operatorname{Sym}^d\Gamma_F\setminus\mathcal{D}_{F,d})=W_d\setminus W_{d-1},$$

more precisely A_d realizes a holomorphic bijection between these smooth varieties. Namely,

$$\begin{aligned}
&\langle P_1,\dots,P_d\rangle\in\operatorname{Sym}^d\Gamma_F\setminus\mathcal{D}_{F,d}\\
&\qquad \text{iff } \forall i\ P_i\neq\infty \text{ and } \exists i\neq j: x(P_i)=x(P_j)\Rightarrow\begin{pmatrix} P_i=P_j \text{ and}\\ P_i \text{ is not a ramification point of } x\end{pmatrix}\\
&\qquad \text{iff } A_d(P_1,\dots,P_d)\notin W_d\setminus W_{d-1},
\end{aligned}$$

where we used Abel's Theorem in the last step. It follows that $\operatorname{Sym}^d\Gamma_F\setminus\mathcal{D}_{F,d}$ and $W_d\setminus W_{d-1}$ are biholomorphic, hence by Proposition 3.3, $A_d\circ\phi_{F,d}$ is a biholomorphism and the manifolds $\mathcal{A}_{F,d}^{\mathbf{C}}$ and $W_d\setminus W_{d-1}$ are biholomorphic. ∎

4.4. The Hénon-Heiles hierarchy

It was found by Ramani (see [DGR]) that the integrable Hénon-Heiles potential $V_3 = 8q_2^3 + 4q_1^2 q_2$ is part of a hierarchy of integrable potentials

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} \binom{n-k}{k} q_1^{2k} q_2^{n-2k}.$$

Namely, the energy $E_n = \left(p_1^2 + p_2^2\right)/2 + V_n$ has a first integral, given by

$$G_n = -q_2 p_1^2 + q_1 p_1 p_2 + q_1^2 V_{n-1},$$

as is checked immediately by direct computation. These potentials have moreover the special property that they can be superimposed freely in the sense that any linear combination of them gives an integrable potential. The case $n = 3$ was studied in [AvM4] and the case $n = 4$ in [V1] (it was called the quartic potential there). In fact, in [V1] we constructed a map which relates this quartic potential to the two-dimensional even master system. This map will prove useful to understand the geometry of the whole Hénon-Heiles hierarchy. Namely define a map $T\colon \mathbf{C}^4 \to \mathbf{C}^4$ by

$$T(q_1, q_2, p_1, p_2) = (\lambda^2 - 2q_2\lambda - q_1^2,\ -2p_2\lambda - 2q_1 p_1),$$

which is invariant for the action of $\mathbf{Z}_2$ on each complex invariant manifold

$$\mathcal{A}_{eg,n}^{\mathbf{C}} = \left\{P \in \mathbf{C}^4 \mid E_n(P) = e,\ G_n(P) = g\right\},$$

the action being given by $(q_1, q_2, p_1 p_2) \mapsto (-q_1, q_2, -p_1, p_2)$. It is fixed point free on $\mathcal{A}_{eg,n}^{\mathbf{C}}$ if $g \neq 0$.

Proposition 4.5 *The map $T\colon \mathbf{C}^4 \to \mathbf{C}^4$ given by*

$$T(q_1, q_2, p_1, p_2) = (\lambda^2 - 2q_2\lambda - q_1^2,\ -2p_2\lambda - 2q_1 p_1),$$

restricts to an unramified $2{:}1$ covering map on each invariant manifold $\mathcal{A}_{eg,n}^{\mathbf{C}}$ (with $g \neq 0$) and this restriction is onto $\mathcal{A}_{F,2}^{\mathbf{C}}$, where F is given by

$$F(x, y) = y^2 + 8\left(x^{n+2} - ex^2 - gx\right),$$

and Γ_F has genus $\left[\frac{n+1}{2}\right]$. Therefore, if n is odd (resp. even) then $\mathcal{A}_{eg,n}^{\mathbf{C}}$ is an unramified cover of the complement of one (resp. two) curve(s), isomorphic to $\bar{\Gamma}_F$, in the W_2 stratum of $\operatorname{Jac}(\bar{\Gamma}_F)$. The restriction $\tilde{T}$ of T to $\mathcal{A}_{eg,n}^{\mathbf{C}}$ maps also the vector fields X_{E_n} and X_{G_n} to (a multiple of) X_{H_1} and X_{H_2}, and leads to the Lax equations $X_{E_n} A(\lambda) = \frac{1}{2}[A(\lambda), B_n(\lambda)]$, for the Hénon-Heiles hierarchy, where

$$A(\lambda) = \begin{pmatrix} -2p_2\lambda - 2q_1 p_1 & \lambda^2 - 2q_2\lambda - q_1^2 \\ 4p_1^2 - 8\sum_{i=0}^{n-1} V_i \lambda^{n-i} & 2p_2\lambda + 2q_1 p_1 \end{pmatrix}, \quad B(\lambda) = \begin{pmatrix} 0 & 1 \\ -8\sum_{i=1}^{n-1} V_j' \lambda^{n-i-1} & 0 \end{pmatrix}$$

and $V_j' = \frac{\partial V_j}{\partial q_2}(q_1, q_2)$.

Proof

Let us fix values e, g and denote by $\tilde{T}$ the restriction of T to $\mathcal{A}^{\mathbf{C}}_{eg,n}$. We show that $\tilde{T}$ maps $\mathcal{A}^{\mathbf{C}}_{eg,n}$ in $\mathcal{A}^{\mathbf{C}}_{F,2}$, when $F(x,y)$ is defined as $F(x,y) = y^2 + 8(x^{n+2} - e_n x^2 - g_n x)$. To show this, let $(q_1, q_2, p_1, p_2) \in \mathcal{A}^{\mathbf{C}}_{eg,n}$ and let $u(\lambda) = \lambda^2 - 2q_2\lambda - q_1^2$ and $v(\lambda) = -2p_2\lambda - 2q_1p_1$. Then the equality

$$\frac{F(\lambda, v(\lambda))}{u(\lambda)} = 8\sum_{i=0}^{n-1} V_i\lambda^{n-i} - 4p_1^2, \tag{4.4}$$

follows immediately from

$$\begin{aligned}\left(\sum_{i=0}^{n-1} V_i\lambda^{n-i}\right)(\lambda^2 - 2q_2\lambda - q_1^2) &= \sum_{i=-2}^{n-3} V_{i+2}\lambda^{n-i} - 2q_2\sum_{i=-1}^{n-2} V_{i+1}\lambda^{n-i} - q_1^2\sum_{i=-1}^{n-1} V_i\lambda^{n-i}\\ &= \lambda^{n+2} + \sum_{i=-1}^{n-2}(V_{i+2} - 2q_2V_{i+1} - q_1^2V_i)\lambda^{n-i} - V_n\lambda^2 - q_1^2V_{n-1}\lambda\\ &= \lambda^{n+2} - V_n\lambda^2 - q_1^2V_{n-1}\lambda,\end{aligned}$$

where we used in the last step the recursion formula

$$V_{i+2} = 2q_2V_{i+1} + q_1^2V_i \tag{4.5}$$

for the potentials V_i (valid for $i \geq -1$; $V_{-1} = 0$). It follows that $\tilde{T}$ maps $\mathcal{A}^{\mathbf{C}}_{eg,n}$ indeed in $\mathcal{A}^{\mathbf{C}}_{F,2}$. Clearly $\tilde{T}$ is surjective and unramified.

To obtain a Lax pair, let $\varphi(x,y) = 1$ and compute the entries in $[B_i(\lambda)]_+$ as given by (4.2). The only non-trivial element in $B_i(\lambda)$ is $\left[\frac{w(\lambda)}{u(\lambda)}\right]_+$, where $w(\lambda) = 4p_1^2 - 8\sum_{i=0}^{n} V_i\lambda^{n-i}$, as follows from the definition of $w(\lambda)$ in (4.2) and (4.4). As in the preceding calculation we get

$$(\lambda^2 - 2q_2\lambda - q_1^2)\sum_{j=1}^{n-1} V_j'\lambda^{n-j-1} = 2u(\lambda)\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} V_i\lambda^{n-i} + \text{ (polynomial of degree} \leq 1),$$

by using the relation

$$V_{i+2} - 2q_2V_{i+1}' - q_1^2V_i' = 2V_{i+1},$$

which is the derivative of (4.5) with respect to q_2. From this representation it is seen at once that $\tilde{T}_*X_{E_n} = \frac{1}{2}X_{H_1}$. Similarly one shows that $\tilde{T}_*X_{G_n}$ is a multiple of X_{H_2}. ∎

Using the results of Section 3.3, the topology of the real invariant manifolds as well as the bifurcations of the Hénon-Heiles hierarchy can be determined, in analogy with [Ga], where this is done for the case $n = 3$ (the Hénon-Heiles potential).

Amplification 4.6

The map T can be seen as a morphism from the Hénon-Heiles potential V_n to the odd or even master system, but a (compatible) Poisson structure, *different* from the one considered here, has to be taken for the odd or even master system (see [V3] Ch. VII, Sect. 3).

Amplification 4.7

As we learned from V. Kuznetsov, the Hénon-Heiles hierarchy has a higher dimensional generalization, which consists of a family of potentials on $\mathbf{R}^d$, defined by a recursion relation which generalizes (4.5), namely let B and $A_1, \ldots, A_{d-1}$ be arbitrary parameters, the A_i being all different. Then the potentials are defined by

$$V_{i+2}^{(d)} = 2(q_d - B)V_{i+1}^{(d)} + \sum_{k=1}^{d-1} \sum_{j=0}^{i} (-1)^j q_k^2 V_{i-j}^{(d)} A_k^j;$$

the Hénon-Heiles hierarchy discussed above corresponds then to the case $d = 2$, $A_1 = B = 0$. Using the results obtained in [EEKL], it is easy to construct the generalization of our map T and to generalize Proposition 4.5, i.e., to prove that for the nth member V_n^d of the hierarchy ($n \geq 3$), the complex invariant manifolds are $2^{d-1}{:}\,1$ unramified covers of (an affine part of) the W_d stratum of the hyperelliptic Jacobian $\mathrm{Jac}(\bar{\Gamma}_F)$, where

$$F(x,y) = y^2 + \pi(x)\left(16x^{n-2}(x+B)^2 + 8e_n + \sum_{i=1}^{d-1} f_i \frac{\pi(x)}{x+A_i}\right), \qquad \pi(x) = \prod_{i=1}^{d-1}(x+A_i),$$

which defines a hyperelliptic curve of genus $[\frac{n-3}{2}]+d$. It leads also in a natural way to Lax equations for this hierarchy.

Acknowledgments. I wish to thank M. Adler, M. Audin, P. Bueken, L. Haine, H. Knörrer, V. Kutznetsov and P. van Moerbeke for valuable and stimulating discussions which contributed to this paper. The support of the Max-Planck-Institut für Mathematik is also greatly acknowledged.

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