# INTEGRABLE DEFORMATIONS OF THE BOGOYAVLENSKIJ-ITOH LOTKA-VOLTERRA SYSTEMS

C. A. EVRIPIDOU, P. KASSOTAKIS AND P. VANHAECKE

Abstract. We construct a family of integrable deformations of the Bogoyavlenskij-Itoh systems and construct a Lax operator with spectral parameter for it. Our approach is based on the construction of a family of compatible Poisson structures for the undeformed system, whose Casimirs are shown to yield a generating function for the integrals in involution of the deformed systems. We show how these deformations are related to the Veselov-Shabat systems.

## Contents

## 1. Introduction

The Bogoyavlenskij-Itoh system was introduced by Bogoyavlenskij in [2] in his study of the integrability of Lotka-Volterra systems. Recall that the most general form of Lotka-Volterra equations in dimension $n$ is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} A_{i,j} x_i x_j, \quad i = 1, 2, \ldots, n . \tag{1.1}$$

For the case of the Bogoyavlenskij-Itoh system, $n = 2k + 1$ is odd, there are no linear terms ($\varepsilon_i = 0$ for all $i$) and the matrix $A$ is skew-symmetric with entries

$$A_{i,j} = \left\{ \begin{array}{ll} 1 & i+1 \leqslant j \leqslant \min\{i+k, n\} , \\ -1 & \min\{i+k, n\} \leqslant j \leqslant n ; \end{array} \right.$$

see the matrix (2.2) below. It is a Hamiltonian system, with Poisson structure defined by

$$\{x_i, x_j\} = A_{i,j} x_i x_j \,, \tag{1.2}$$

and linear Hamiltonian $H = x_1 + x_2 + \cdots + x_{2k+1}$. Bogoyavlenskij provides for this system the following Lax equation (with spectral parameter $\lambda$)

$$(X + \lambda M)^{\cdot} = [X + \lambda M, B - \lambda M^{k+1}] \tag{1.3}$$

where for $i, j \in \{1, 2, \ldots, 2k+1\}$ the $(i,j)$-th entry of the matrices $X$, $M$ and $B$ is respectively given by

$$X_{i,j} := \delta_{i,j+k} x_i \,, \quad M_{i,j} := \delta_{i+1,j} \,, \quad B_{i,j} := -\delta_{i,j}(x_i + x_{i+1} + \cdots + x_{i+k}) \,.$$

The characteristic polynomial of $X + \lambda M$ leads to $k + 1$ independent constants of motion, among which are both the Hamiltonian and the Casimir

$$C := \prod_{i=1}^{2k+1} x_i = x_1 x_2 \ldots x_{2k+1} \tag{1.4}$$

of the Poisson structure (1.2). Itoh gives in [6] a combinatorial description of these integrals and proves in [7] by a combinatorial argument that they are in involution, thereby proving the Liouville integrability of the Bogoyavlenskij-Itoh system. Some integrable reductions of the Bogoyavlenskij-Itoh system were recently constructed and studied by us in [5].

In this paper we construct and study a certain type of integrable deformations of the Bogoyavlenskij-Itoh systems. Our approach is based on (compatible) deformations of the Poisson structure (1.2) and of its Casimir (1.4). We show that the only constant Poisson structures

$$\{x_i, x_j\}_b := b_{i,j} \,, \qquad 1 \leqslant i, j \leqslant 2k + 1 \,,$$

which are compatible with (1.2) are the ones for which $b_{i,j} = 0$ for all $i$ and $j$ such that $|i - j| \notin \{k, k+1\}$. If one writes the corresponding Hamiltonian vector field, with $H$ as Hamiltonian, one finds

$$\dot{x}_i = \sum_{j=1}^{2k+1} A_{i,j} x_i x_j + c_i \,, \quad i = 1, 2, \ldots, 2k + 1 \,, \tag{1.5}$$

where $c_i = b_{i,i+k} - b_{i-k,i}$. Notice that the $c_i$ sum up to zero, but this is the only relation between these constants. Up to a minor change of variables (see Section 6) this system coincides with the so-called *Veselov-Shabat system*, which was constructed in [13] as fixed point of compositions of Darboux transformations of the Schrödinger operator. Notice that this system is also known as (one of) the *Noumi-Yamada* system(s) (see [11]), but it seems that Noumi and Yamada were unaware of the Veselov-Shabat system. Also, neither in the Noumi-Yamada paper [11] nor in the Veselov-Shabat paper [13] are the systems that they consider put in relation with the Bogoyavlenskij-Itoh systems, of which they are deformations.

We prove the Liouville integrability of (1.5) by using the deformed Casimir, i.e., the Casimir of the deformed Poisson structure $\{\cdot,\cdot\} + \{\cdot,\cdot\}_b$. A combinatorial description of the deformed Casimir is given in Proposition 4.2. From it, one gets immediately a Casimir for the Poisson pencil $\{\cdot,\cdot\} + \lambda\{\cdot,\cdot\}_b$ and so, using the Lenard-Magri scheme, a family of polynomials in involution, which we show to be independent, thereby proving the Liouville integrability of the deformed Bogoyavlenskij-Itoh systems. We also provide a Lax equation with spectral parameter for these systems, which is a deformation of the Lax equation (2.6). Notice that our $(2k+1) \times (2k+1)$ Lax operator is different from the $2 \times 2$ Lax operator which was constructed by Veselov and Shabat; our Lax equation has the advantage that the phase variables appear in it linearly, which makes it easier to extract the phase variables (in order to know for example their time evolution) from the Lax operator.

It can be shown that the deformed Bogoyavlenskij-Itoh systems admit a natural discretization, which is constructed by using the deformed Lax operator which we constructed in this paper. This discretization will be worked out and studied in a future publication.

The structure of the paper is as follows. We recall the main facts about the Bogoyavlenskij-Itoh systems in Section 2. The combinatorial constructions that we will use, in the style of Itoh's combinatorial constructions of the (undeformed) first integrals, are prepared in Section 3. We proceed in Section 4 with the construction of the deformed Poisson strutures and deformed Casimirs. Section 5 is devoted to the Lax equation and the Liouville integrability of the deformed Bogoyavlenskij-Itoh systems; in particular we match the combinatorial description of the first integrals with the coefficients of our Lax equation. The upshot is that *as integrable systems* they are deformation of the Bogoyavlenskij-Itoh systems. In the last section we show how these deformed systems are related to the ones constructed by Veselov and Shabat in [13].

## 2. The Bogoyavlenskij-Itoh systems and their integrability

We recall in this section the basic results on the Bogoyavlenskij-Itoh systems, which were obtained in [2, 3, 6, 7]; the notation is a slight simplification of the one used in [5], where more general systems are considered. Fix $k \in \mathbb{N}^*$ and denote $\mathcal{I} := \{1, 2, \ldots, 2k+1\}$. We consider on $\mathbb{R}^{2k+1}$ (or on $\mathbb{C}^{2k+1}$) with linear coordinates $x_1, \ldots, x_{2k+1}$ the homogeneous quadratic Poisson structure defined by

$$\{x_i, x_j\} := A_{i,j} x_i x_j, \qquad i, j \in \mathcal{I}, \qquad (2.1)$$

where $A = (A_{i,j})$ is the constant skew-symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & 1 & \cdots & 1 & 1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \cdots & 1 & 1 & 1 & \ddots & -1 & -1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ 1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & -1 & -1 & -1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & 1 & 0 \end{pmatrix} , \tag{2.2}$$

with $-1$ appearing $k$ times on the first row (and hence on every row). When $k$ is not clear from the context, we write $A^{(k)}$ for $A$ and $\{\cdot, \cdot\}^{(k)}$ for $\{\cdot, \cdot\}$. The rank of $A$ is constant and equals $2k$; a Casimir for $\{\cdot, \cdot\}$ is given by the polynomial function

$$C := \prod_{i \in \mathcal{I}} x_i = x_1 x_2 \ldots x_{2k+1} .$$

As Hamiltonian function, one takes $H := \sum_{i \in \mathcal{I}} x_i = x_1 + x_2 + \cdots + x_{2k+1}$, the sum of all coordinates. If we set $x_{2k+\ell+1} = x_\ell$ for all $\ell \in \mathbb{Z}$, then the Hamiltonian vector field $\mathcal{X}_H := \{\cdot, H\}$ is given by

$$\dot{x}_i = x_i \sum_{j=1}^k (x_{i+j} - x_{i-j}) , \qquad i \in \mathcal{I} . \tag{2.3}$$

The automorphism of $\mathbb{R}^{2k+1}$ of order $2k+1$, given by

$$(x_1, x_2, \ldots, x_{2k}, x_{2k+1}) \mapsto (x_2, x_3, \ldots, x_{2k+1}, x_1) , \tag{2.4}$$

is a Poisson map and it preserves the Hamiltonian, hence it is an automorphism of the system. The following Lax equation (with spectral parameter $\lambda$) was provided by Bogoyavlenskij in [2]:

$$(X + \lambda M)\dot{} = [X + \lambda M, B - \lambda M^{k+1}] \tag{2.5}$$

where for $i, j \in \mathcal{I}$ the $(i,j)$-th entry of the matrices $X$, $M$ and $B$ is respectively given by

$$X_{i,j} := \delta_{i,j+k} x_i , \quad M_{i,j} := \delta_{i+1,j} , \quad B_{i,j} := b_i \delta_{i,j} := -\delta_{i,j}(x_i + x_{i+1} + \cdots + x_{i+k}) . \tag{2.6}$$

The characteristic polynomial of $X + \lambda M$ has the form

$$\det(X + \lambda M - \mu \operatorname{Id}) = \lambda^{2k+1} - \mu^{2k+1} + \sum_{\ell=0}^k K_\ell \lambda^{k-\ell} \mu^{k-\ell} , \tag{2.7}$$

where, by homogeneity, each $K_\ell$ is a homogeneous polynomial (in $x_1, \ldots, x_{2k+1}$) of degree $2\ell + 1$. One has $K_0 = H$, the Hamiltonian, and $K_k = C$, the above Casimir. Being a coefficient of the characteristic polynomial of the Lax operator $X + \lambda M$, each one of the $K_\ell$ is a first integral of (2.3).

In order to give an explicit formula for $K_\ell$, we need some more notation, which will also be needed in the rest of the paper. With $k$ fixed as above, let $\ell$ be an integer with $0 \leqslant \ell \leqslant k$. We will consider besides $A = A^{(k)}$ also $A^{(\ell)}$. Let $\underline{m} = (m_1, m_2, \ldots, m_{2\ell+1})$ be a $(2\ell+1)$-tuple of integers, satisfying $1 \leqslant m_1 < m_2 < \cdots < m_{2\ell+1} \leqslant 2k+1$. We view them as indices of the rows and columns of $A$: we denote by $A'_{\underline{m}}$ the square submatrix of $A$ of size $2\ell+1$, corresponding to rows and columns $m_1, m_2, \ldots, m_{2\ell+1}$ of $A$, so that

$$(A'_{\underline{m}})_{i,j} = A_{m_i, m_j} , \quad \text{for } i, j = 1, \ldots, 2\ell+1 . \tag{2.8}$$

Let

$$\mathcal{S}_\ell := \left\{ \underline{m} \mid A'_{\underline{m}} = A^{(\ell)} \right\} . \tag{2.9}$$

With this notation, the polynomials $K_\ell$ which appear in the characteristic polynomial (2.7) can be written as

$$K_\ell = \sum_{\underline{m} \in \mathcal{S}_\ell} x_{m_1} x_{m_2} \ldots x_{m_\ell} \ldots x_{m_{2\ell+1}} . \tag{2.10}$$

For example, $\mathcal{S}_0 = \{1, 2, \ldots, 2k+1\}$ and $\mathcal{S}_k = \{(1, 2, \ldots, 2k+1)\}$, so that $K_0 = H$ and $K_k = C$, as above. Moreover, Itoh shows by a beautiful combinatorial argument that the polynomials $K_\ell$ are in involution, $\{K_\ell, K_m\} = 0$ for $0 \leqslant \ell < m \leqslant k$. Since these $k+1$ polynomials are moreover functionally independent, and since the rank of the Poisson structure $\{\cdot, \cdot\}$ is $2k$, the triplet $(\mathbb{R}^{2k+1}, \{\cdot, \cdot\}, (K_0, K_1, \ldots, K_k))$ is a Liouville integrable system.

## 3. The sets $\mathcal{S}_\ell$

We establish in this section some combinatorial properties of the sets $\mathcal{S}_\ell$ which will be used in the subsequent sections. We recall that $k$ is a fixed integer and that $\mathcal{I}$ stands for $\{1, 2, \ldots, 2k+1\}$. We also denote for $s \in \mathcal{I}$ by $\mathcal{I}^{(s)}$ the set of all strictly ordered $s$-tuplets $\underline{m} = (m_1, m_2, \ldots, m_s)$, with entries in $\mathcal{I}$. We will often use set theory notation for such elements, for example we write $i \in \underline{m}$ and $\{i, j\} \subset \underline{m}$, with the obvious meanings. If $\underline{m}$ and $\underline{n}$ are two vectors with elements in $\mathcal{I}$, satisfying $\underline{m} \cap \underline{n} = \emptyset$, we write $\underline{m} \oplus \underline{n}$ for the vector which contains the elements of $\underline{m}$ and $\underline{n}$ (in increasing order); also, if $\underline{n} \subset \underline{m}$, we write $\underline{m} \ominus \underline{n}$ for the vector with elements in $\underline{m}$ and not in $\underline{n}$. By a slight abuse of notation, we will denote for $r \in \mathbb{Z}$ by $r$ mod $2k+1$ the unique element of $\mathcal{I}$ which is congruent to $r$ modulo $2k+1$. For $0 \leqslant \ell \leqslant k$ the set $\mathcal{S}_\ell$ is defined by

$$\mathcal{S}_\ell := \left\{ \underline{m} \in \mathcal{I}^{(2\ell+1)} \mid A_{m_i, m_j} = A^{(\ell)}_{i,j} , \text{ for } i, j = 1, \ldots, 2\ell+1 \right\} .$$

We recall from [5] the following characterization of the elements of $\mathcal{S}_\ell$:

**Proposition 3.1.** *Let $\underline{m} = (m_1, \ldots, m_{2\ell+1})$ be an element of $\mathcal{I}^{(2\ell+1)}$. Then $\underline{m} \in \mathcal{S}_\ell$ if and only if the following conditions are satisfied:*

(1) $m_{\ell+i} < m_i + k + 1 \leqslant m_{\ell+i+1}$ *for* $i = 1, \dots, \ell;$

(2) $m_{2\ell+1} < m_{\ell+1} + k + 1.$

There are for any $s \in \mathcal{I}$ two natural permutations of $\mathcal{I}^{(s)}$: first, there is an involution $\sigma_s$, given by

$$\sigma_s(m_1, m_2, \dots, m_s) := (2k + 2 - m_s, 2k + 2 - m_{s-1}, \dots, 2k + 2 - m_1).$$

If we write $\underline{n} := \sigma_s(\underline{m})$, then $n_i = 2k + 2 - m_{s+1-i}$, for $i = 1, 2, \dots, s$. Next, there is a cyclic permutation of order $2k + 1$,

$$\tau_s(m_1, m_2, \dots, m_s) := \begin{cases} (m_1 + 1, m_2 + 1, \dots, m_s + 1) & \text{when } m_s < 2k + 1 \text{ ;} \\ (1, m_1 + 1, m_2 + 1, \dots, m_{s-1} + 1) & \text{when } m_s = 2k + 1 \text{ .} \end{cases}$$
(3.1)

Said differently, $\tau_s$ simply adds 1 to all entries of the vector $\underline{m}$, but the result needs to be slightly reordered when one of the entries of $\underline{m}$ gets bigger than $2k + 1$. We show in following lemma that $\sigma_{2\ell+1}$ and $\tau_{2\ell+1}$ both restrict to a permutation of $\mathcal{S}_\ell$.

**Lemma 3.2.** *If $\underline{m} \in \mathcal{S}_\ell$ then $\sigma_{2\ell+1}(\underline{m}) \in \mathcal{S}_\ell$ and $\tau_{2\ell+1}(\underline{m}) \in \mathcal{S}_\ell$.*

*Proof.* Suppose that $\underline{m} \in \mathcal{S}_\ell$ and let $\underline{n} := \sigma_{2\ell+1}(\underline{m})$. According to Proposition 3.1 we need to show that

$$1 \leqslant i \leqslant \ell + 1 \implies n_{\ell+i} < n_i + k + 1 \text{ and } 1 \leqslant i \leqslant \ell \implies n_i + k + 1 \leqslant n_{\ell+i+1} \text{ . (3.2)}$$

In terms of $\underline{m}$ these two inequalities become $m_{2\ell+2-i} < m_{\ell+2-i} + k + 1$ and $m_{\ell+1-i} + k + 1 \leqslant m_{2\ell+2-i}$. Setting $j := \ell + 2 - i$ in the first inequality and $j := \ell + 1 - i$ in the second inequality, (3.2) becomes

$$1 \leqslant j \leqslant \ell + 1 \implies m_{\ell+j} < m_j + k + 1 \text{ and } 1 \leqslant j \leqslant \ell \implies m_j + k + 1 \leqslant m_{\ell+j+1} \text{ ,}$$

which are exactly the conditions (1) and (2) in Proposition 3.1 which express that $\underline{m} \in \mathcal{S}_\ell$. This shows that $\sigma_{2\ell+1}(\underline{m}) \in \mathcal{S}_\ell$. Next, let $\underline{n} := \tau_{2\ell+1}(\underline{m})$. Again, we need to verify (3.2): when $m_{2\ell+1} < 2k + 1$ this is completely obvious, so let us assume that $m_{2\ell+1} = 2k + 1$. We have that $n_1 = 1$ and $n_i = m_{i-1} + 1$ for $i = 2, 3, \dots, 2\ell + 1$. We need to prove (3.2) for this vector $\underline{n}$. Again, for $i \geqslant 2$ this is completely obvious, so we only need to check that $m_\ell < k + 1 \leqslant m_{\ell+1}$. Both follow from the characterizations of $\underline{m}$ in Proposition 3.1: the first one from the second inequality in (1), with $i = \ell$, and the second one from (2), with $m_{2\ell+1} = 2k+1$. $\square$

We decompose $\mathcal{S}_\ell$ in two subsets $\mathcal{S}_{\ell,+}$ and $\mathcal{S}_{\ell,-}$, where

$$\mathcal{S}_{\ell,+} := \{\underline{m} \in \mathcal{S}_\ell \mid 1 \in \underline{m}\} \text{ , } \quad \mathcal{S}_{\ell,-} := \{\underline{m} \in \mathcal{S}_\ell \mid 1 \notin \underline{m}\} \tag{3.3}$$

and we define the maps

$$\begin{aligned} \phi_1 &: \{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \leqslant k + 1\} &\rightarrow& \mathcal{S}_\ell, & \underline{m} &\mapsto& \underline{m} \oplus (1, k + 2) \text{ ,} \\ \phi_2 &: \{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \geqslant k + 2\} &\rightarrow& \mathcal{S}_\ell, & \underline{m} &\mapsto& \underline{m} \oplus (1, k + 1) \text{ .} \end{aligned} \tag{3.4}$$

**Lemma 3.3.** *The formulas (3.4) define injective maps with values in $\mathcal{S}_{\ell,+}$. More precisely, they are part of the following commutative diagram, where all (restriced) maps are bijections.*

$$
\begin{array}{ccc}
\{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \leqslant k+1\} & \xrightarrow{\ \phi_1\ } & \{\underline{n} \in \mathcal{S}_{\ell,+} \mid k+2 \in \underline{n}\} \\
\Big\downarrow{\scriptstyle\tau_{\ell-1}\circ\sigma_{\ell-1}} & & \Big\downarrow{\scriptstyle\tau_\ell\circ\sigma_\ell} \\
\{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \geqslant k+2\} & \xrightarrow{\ \phi_2\ } & \{\underline{n} \in \mathcal{S}_{\ell,+} \mid k+1 \in \underline{n}\}
\end{array}
$$

*Proof.* We first show that $\phi_1$ is well-defined. Let $\ell \geqslant 1$ and suppose that $\underline{m} \in \mathcal{S}_{\ell-1,-}$ with $m_\ell \leqslant k+1$. We need to show that $\underline{m} \cap (1, k+2) = \emptyset$ and that $\underline{n} := \phi_1(\underline{m}) = \underline{m} \oplus (1, k+2)$ belongs to $\mathcal{S}_\ell$. According to Proposition 3.1,

$$m_{\ell-1+i} < m_i + k + 1 \leqslant m_{\ell+i} \quad \text{for} \quad i = 1, \ldots, \ell-1 \,; \tag{3.5}$$

$$m_{2\ell-1} < m_\ell + k + 1 \,, \tag{3.6}$$

and, by assumption,

$$m_1 > 1 \quad \text{and} \quad m_\ell \leqslant k + 1 \,. \tag{3.7}$$

According to (3.5) with $s = 1$ and (3.7), $m_{\ell+1} > k + 2 > m_\ell$, which shows that $\underline{m} \cap (1, k+2) = \emptyset$. Since $m_{\ell+1} > k + 2$ and $m_\ell \leqslant k + 1$,

$$
n_j = 
\begin{cases}
1 & j = 1 \,, \\
m_{j-1} & j = 2, \ldots, \ell+1 \,, \\
k+2 & j = \ell+2 \,, \\
m_{j-2} & j = \ell+3, \ldots, 2\ell+1 \,.
\end{cases}
\tag{3.8}
$$

We need to check that $\underline{n} \in \mathcal{S}_\ell$, i.e., that

$$n_{\ell+i} < n_i + k + 1 \leqslant n_{\ell+i+1} \quad \text{for} \quad i = 1, \ldots, \ell \,; \tag{3.9}$$

$$n_{2\ell+1} < n_{\ell+1} + k + 1 \,. \tag{3.10}$$

To do this, we use (3.8) to write the latter inequalities in terms of the entries of $\underline{m}$: for $i = 1$, $i = 2$ and $i = 3, \ldots, \ell$, (3.9) reads respectively

$$m_\ell < k + 2 \leqslant k + 2 \,,$$

$$k + 2 < m_1 + k + 1 \leqslant m_{\ell+1} \,,$$

$$m_{\ell+i-2} < m_{i-1} + k + 1 \leqslant m_{\ell+i-1} \,.$$

The first line follows from $m_\ell \leqslant k + 1$ (see (3.7)), the other two from $m_1 > 1$ and (3.5). Finally, (3.10), written in terms of $\underline{m}$ is precisely (3.6). This shows that $\underline{n} \in \mathcal{S}_\ell$, hence that $\phi_1$ takes values in $\mathcal{S}_{\ell,+}$, more precisely in $\{\underline{n} \in \mathcal{S}_{\ell,+} \mid (1, k+2) \in \underline{n}\}$. Obviously, $\phi_1$ is injective. It is proven similarly that $\phi_2$ takes values in $\mathcal{S}_{\ell,+}$ and is injective; let us just point out that the formula for $\underline{n} := \phi_2(\underline{m})$ (as in (3.8)) is now given by

$$
n_j = 
\begin{cases}
1 & j = 1 \,, \\
m_{j-1} & j = 2, \ldots, \ell \,, \\
k+1 & j = \ell+1 \,, \\
m_{j-2} & j = \ell+2, \ldots, 2\ell+1 \,.
\end{cases}
\tag{3.11}
$$

In order to show that the (restricted) horizontal maps $\phi_1$ and $\phi_2$ in the diagram are bijections, is suffices to construct their inverse maps, which are defined as restrictions of the following two maps:

$$\begin{array}{rcccl}
\phi_1^{-1} & : & \{\underline{m} \in \mathcal{S}_{\ell,+} \mid k+2 \in \underline{m}\} & \to & \mathcal{S}_\ell, \quad \underline{m} \mapsto \underline{m} \ominus (1, k+2) , \\
\phi_2^{-1} & : & \{\underline{m} \in \mathcal{S}_{\ell,+} \mid k+1 \in \underline{m}\} & \to & \mathcal{S}_\ell, \quad \underline{m} \mapsto \underline{m} \ominus (1, k+1) .
\end{array}$$

Clearly, these maps are well-defined. We need to show that if $\underline{n} \in \mathcal{S}_\ell$ with $(1, k+2) \subset \underline{n}$ then $\underline{m} := \underline{n} \ominus (1, k+2)$ satisfies $m_\ell \leqslant k+1$. Since $\underline{n} \in \mathcal{S}_\ell$ it satisfies (3.9), which yields with $i = 1$ and $n_1 = 1$ that $n_{\ell+1} < k+2 \leqslant n_{\ell+2}$ so that $n_{\ell+2} = k+2$ and $m_\ell = n_{\ell+1} \leqslant k+1$. Similarly, if $\underline{n} \in \mathcal{S}_\ell$ with $(1, k+1) \subset \underline{n}$ then $\underline{m} := \underline{n} \ominus (1, k+1)$ satisfies $m_\ell > k+1$. This shows that the horizontal arrows in the diagram are bijections.

Let us show that the vertical arrows of the diagram are also bijections. In view of Lemma 3.3, $\tau_{\ell-1} \circ \sigma_{\ell-1}$ is a permutation (involution) of $\mathcal{S}_{\ell-1}$; explicitly, it is given for $\underline{m} \in \mathcal{S}_{\ell-1,-}$ by

$$\tau_{\ell-1} \circ \sigma_{\ell-1}(\underline{m}) = (2k+3 - m_{2\ell-1}, 2k+3 - m_{2\ell-2}, \ldots, 2k+3 - m_1) . \quad (3.12)$$

The formula shows that $1 \notin \tau_{\ell-1} \circ \sigma_{\ell-1}(\underline{m})$ so that $\tau_{\ell-1} \circ \sigma_{\ell-1}$ restricts to a bijection of $\mathcal{S}_{\ell-1,-}$, and that the $\ell$-th component of $\tau_{\ell-1} \circ \sigma_{\ell-1}(\underline{m})$ is given by $2k+3 - m_\ell$, showing that $\tau_{\ell-1} \circ \sigma_{\ell-1}$ restricts further to a bijection between the subsets $\{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \leqslant k+1\}$ and $\{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \geqslant k+2\}$ of $\mathcal{S}_{\ell,-}$. Similarly, $\tau_\ell \circ \sigma_\ell$ is an involution of $\mathcal{S}_\ell$, which is given, for $\underline{n} \in \mathcal{S}_{\ell,+}$ by

$$\tau_\ell \circ \sigma_\ell(\underline{n}) = (1, 2k+3 - n_{2\ell+1}, 2k+3 - n_{2i}, \ldots, 2k+3 - n_3, 2k+3 - n_2) . \quad (3.13)$$

Also, it restricts to a bijection between the subsets $\{\underline{n} \in \mathcal{S}_{\ell,+} \mid (1, k+2) \subset \underline{n}\}$ and $\{\underline{n} \in \mathcal{S}_{\ell,+} \mid (1, k+1) \subset \underline{n}\}$ of $\mathcal{S}_{\ell,+}$: if $(1, k+2) \subset \underline{n}$ (resp. $(1, k+1) \subset \underline{n}$), then $n_{\ell+2} = k+2$ (resp. $n_{\ell+1} = k+1$), so that $\tau_\ell(\sigma_\ell(\underline{n})) \ni k+1$ (resp. $\tau_\ell(\sigma_i(\underline{n})) \ni k+2$). The commutativity of the diagram follows at once from the explicit formulas (3.8) and (3.11) – (3.13). □

For an element $\underline{m} \in \mathcal{I}^{(s)}$, we denote by $\underline{m}'$ the vector whose elements are those elements of $\mathcal{I}$ which are absent from $\underline{m}$ (again these elements will always be put in the increasing order). In formula, $\underline{m}' = \mathcal{I} \ominus \underline{m}$. We also denote, for $0 \leqslant \ell \leqslant k$,

$$\mathcal{S}_\ell' := \{\underline{m}' \mid \underline{m} \in \mathcal{S}_\ell\} .$$

Clearly, if $\underline{m}' \in \mathcal{S}_\ell'$ then $\underline{m}'$ has $2(k-\ell)$ entries. We give in the following proposition a characterization of the elements of $\mathcal{S}_\ell'$; it will be used in the construction of integrals in the following sections.

**Proposition 3.4.** *Let $0 \leqslant \ell \leqslant k$ and let $\underline{m}' = (r_1, r_2, \ldots, r_{k-\ell}, s_1, s_2, \ldots, s_{k-\ell}) \in \mathcal{I}^{(2k-2\ell)}$. Then $\underline{m}' \in \mathcal{S}_\ell'$ if and only if*

$$s_j - r_j \in \{k, k+1\} , \quad for\ j = 1, \ldots, k - \ell .$$

*Proof.* The proof goes by induction on $\ell$, starting from $\ell = k$, downwards to $\ell = 0$. When $\ell = k$, the equivalence is trivially satisfied, because $\mathcal{S}_k$ has a single element $\underline{m} = (1, 2, \ldots, 2k+1)$, whose complement $\underline{m}'$ has no entries. Suppose now that the above equivalence is true for some $\ell$ with $0 < \ell \leqslant k$ and for all $\underline{m}' \in \mathcal{I}^{(2k-2\ell)}$. Let $\underline{m}' = (r_1, r_2, \ldots, r_{k-\ell+1}, s_1, s_2, \ldots, r_{k-\ell+1})$. We prove that

$$\underline{m}' \in \mathcal{S}'_{\ell-1} \quad \text{if and only if} \quad s_j - r_j \in \{k, k+1\} , \quad \text{for } j = 1, \ldots, k-\ell+1 . \quad (3.14)$$

Let us assume first that $\underline{m}' \in \mathcal{S}'_{\ell-1}$, i.e., that $\underline{m} \in \mathcal{S}_{\ell-1}$. According to Lemma 3.2, $\tau_{2\ell-1}(\underline{m}) \in \mathcal{S}_{\ell-1}$ and it is easy to see that if $\underline{m}'$ satisfies the right hand side of (3.14) then so does $\tau_{2k-2\ell+2}(\underline{m}') = \tau_{2\ell-1}(\underline{m})'$. We may therefore assume that $1 \notin \underline{m}$, but also that $m_\ell \leqslant k+1$: indeed, if $1 \notin \underline{m}$ and $m_\ell > k+1$ then it suffices to replace $\underline{m}$ by $\tau^k_{2\ell-1}(\underline{m})$ (note that $m_{\ell-1} \leqslant k$, as follows easily by taking $s = \ell-1$ in item (1) of Proposition 3.1). According to Lemma 3.3, $\underline{m} \oplus (1, k+2) \in \mathcal{S}_\ell$. By the recurrence hypothesis, we can write $(\underline{m} \oplus (1, k+2))' = (\bar{r}_2, \ldots, \bar{r}_{k-\ell+1}, \bar{s}_2, \ldots, \bar{s}_{k-\ell+1})$, satisfying $s_j - r_j \in \{k, k+1\}$ for $j = 2, \ldots, k-\ell+1$; also, $\bar{r}_2 > 1$ and $\bar{r}_{k-\ell+1} < k+2 < \bar{s}_2$. Then

$$(1, \bar{r}_2, \ldots, \bar{r}_{k-\ell+1}, k+2, \bar{s}_2, \ldots, \bar{s}_{k-\ell+1}) = \underline{m}' = (r_1, r_2, \ldots, r_{k-\ell}, s_1, s_2, \ldots, s_{k-\ell}) ,$$

so that $r_j = \bar{r}_j$ and $s_j = \bar{s}_j$ for $j = 2, 3, \ldots, k_\ell + 1$. As a consequence, the entries of $\underline{m}'$ satisfy the right hand side of (3.14).

We now assume that the entries of $\underline{m}'$ satisfy the right hand side of (3.14). Again, using $\tau_{2\ell-1}$ we may assume that $r_1 = 1$ and that $s_1 = k+2$. Thus, $1 \notin \underline{m}$ and $k+2 \notin \underline{m}$. Then

$$(\underline{m} \oplus (1, k+2))' = (r_2, r_3, \ldots, r_{k-\ell+1}, s_2, s_3, \ldots, s_{k-\ell+1})$$

with $s_j - r_j \in \{k, k+1\}$ for $j = 2, \ldots, k-\ell+1$. By the recursion hypothesis, $\underline{m} \oplus (1, k+2) \in \mathcal{S}_\ell$; more precisely, $\underline{m} \oplus (1, k+2) \in \{\underline{n} \in \mathcal{S}_{\ell,+} \mid k+2 \in \underline{n}\}$. By Lemma 3.3, $\underline{m} \in \mathcal{S}_{\ell-1}$, and so $\underline{m}' \in \mathcal{S}'_{\ell-1}$, as was to be shown. $\qquad \square$

## 4. The deformed Poisson structure and its basic Casimir

In this section, we introduce a class of deformations of the Poisson structure $\{\cdot, \cdot\}$, defined by (2.1) and construct a Casimir for it. In the following proposition, we determine all constant Poisson structures on $\mathbb{R}^{2k+1}$ which are compatible with $\{\cdot, \cdot\}$.

**Proposition 4.1.** *Let $(b_{i,j})_{1 \leqslant i,j \leqslant 2k+1}$ be an arbitrary skew-symmetric matrix with entries in $\mathbb{R}$ and consider the corresponding constant Poisson structure on $\mathbb{R}^{2k+1}$, defined by $\{x_i, x_j\}_b := b_{i,j}$ for $1 \leqslant i, j \leqslant 2k+1$. Then $\{\cdot, \cdot\}_b$ is compatible with the quadratic Poisson structure $\{\cdot, \cdot\}$ if and only if $b_{i,j} = 0$ for all $i$ and $j$ such that $|i - j| \notin \{k, k+1\}$. In particular, the constant Poisson structures which are compatible with $\{\cdot, \cdot\}$ form a vector space of dimension $2k + 1$.*

*Proof.* Recall (e.g. from [8, Section 3.3.2]) that two Poisson structures are said to be *compatible* when their sum is a Poisson structure, i.e., when it satisfies the Jacobi identity. Denote

$$\{\cdot,\cdot\}_b^{(k)} := \{\cdot,\cdot\} + \{\cdot,\cdot\}_b \ . \tag{4.1}$$

Then $\{\cdot,\cdot\}_b^{(k)}$ satisfies the Jacobi identity if and only if

$$\left\{\{x_i, x_j\}_b^{(k)}, x_m\right\}_b^{(k)} + \circlearrowright (i,j,m) = 0 \ ,$$

for all triplets of distinct indices $i, j, m \in \mathcal{I}$, where $\circlearrowright (i,j,m)$ has the obvious meaning. Since $\{\cdot,\cdot\}_b$ is a constant Poisson bracket this condition amounts to $\{\{x_i, x_j\}, x_m\}_b + \circlearrowright (i,j,m) = 0$ , that is,

$$A_{i,j} \{x_i x_j, x_m\}_b + \circlearrowright (i,j,m) = A_{i,j} x_i b_{j,m} + A_{i,j} x_j b_{i,m} + \circlearrowright (i,j,m) =$$

$$A_{i,j} x_i b_{j,m} + A_{m,i} x_i b_{m,j} + \circlearrowright (i,j,m) = x_i b_{j,m} \left(A_{i,j} + A_{i,m}\right) + \circlearrowright (i,j,m) = 0 \ ,$$

which is in turn equivalent with

$$b_{j,m} \left(A_{i,j} + A_{i,m}\right) = 0 \tag{4.2}$$

for all triplets of distinct indices $i, j, m \in \mathcal{I}$. It follows that $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_b$ are compatible if and only if $b_{j,m} = 0$ for all $j, m \in \mathcal{I}$ such that $A_{i,j} + A_{i,m} \neq 0$ for some $i \in \mathcal{I}$, with $i \neq j$ and $i \neq m$.

Suppose that $j$ and $m$ satisfy $|j - m| \in \{k, k+1\}$; without loss of generality, we may assume that $m = i + k$. From the structure of the $i$-th line of the matrix $A$ (see (2.2) we see that $A_{i,j} + A_{i,m} = 0$ whenever $i \neq j$ and $i \neq m$. Therefore, there are no constraints on the entries $b_{j,m}$, when $|j - m| \in \{k, k+1\}$. We show that the other entries $b_{j,m}$ all have to be zero. By permuting $j$ and $m$ if necessary, we may suppose that $j < m$. If $m - j < k$, then picking $i = j - 1$ (or $i = m + 1$ in case $j = 1$) we have $A_{i,j} + A_{i,m} \neq 0$; one arrives at the same result when $m - j > k + 1$ by picking $i = j + 1$. In both cases, it follows that the corresponding coefficient $b_{j,m}$ must be zero. $\qquad\square$

Notice that the conditions on $b$ which are given in the above proposition are void when $k = 1$.

The rank of the Poisson structure $\{\cdot,\cdot\}$ is $2k$, as follows from the fact that the rank of $A$ is $2k$ (see e.g. [8, Example 8.14]). Thus, the Poisson matrix of $\{\cdot,\cdot\}$ has a non-vanishing $(2k \times 2k)$-minor; the latter is a homogeneous polynomial $P_{4k}(x)$ of degree $4k$ since the entries of the Poisson matrix are homogeneous and quadratic. The corresponding $(2k \times 2k)$-minor of the Poisson matrix of $\{\cdot,\cdot\}_b^{(k)}$ is a non-homogeneous polynomial of degree $4k$, whose homogeneous part of top degree is $P_{4k}(x)$, hence this minor is also non-zero (at a generic point of $\mathbb{R}^{2k+1}$), so that the rank of $\{\cdot,\cdot\}_b^{(k)}$ is $2k$.

We show that $\{\cdot,\cdot\}_b^{(k)}$ has a polynomial Casimir of degree $2k+1$ whose leading term is the Casimir $C = x_1 x_2 \ldots x_{2k+1}$ of $\{\cdot,\cdot\}$. To do this, we need some more notation. First, in order to eliminate many signs in the formulas that follow, we define new constants $b'_{i,j}$, which are just the constants $b_{i,j}$, possibly up to a sign: we define, $b'_{i,j} := -b_{i,j} = b_{j,i}$ when $|i - j| = k + 1$ and $b'_{i,j} := b_{i,j}$ otherwise. Also, for $\underline{m} \in \mathcal{S}_\ell$, we denote by $x_{\underline{m}}$ the product $x_{\underline{m}} = x_{m_1} x_{m_2} \ldots x_{m_{2\ell+1}}$ and by $b'_{\underline{m}}$ the product

$$b'_{\underline{m}} := b'_{r_1,s_1} b'_{r_2,s_2} \ldots b'_{r_{k-\ell},s_{k-\ell}} \, ,$$

where $\underline{m}' = (r_1, \ldots, r_{k-\ell}, s_1, \ldots, s_{k-\ell})$ is the ordered list of all indices of $\mathcal{I}$ that are absent in $\underline{m}$ (see Proposition 3.4).

**Proposition 4.2.** *The polynomial*

$$K_k^b := \sum_{\ell=0}^{k} \sum_{\underline{m} \in \mathcal{S}_\ell} b'_{\underline{m}} x_{\underline{m}} \tag{4.3}$$

*is a Casimir of* $\{\cdot,\cdot\}_b^{(k)} = \{\cdot,\cdot\} + \{\cdot,\cdot\}_b$.

*Proof.* In order to simplify the notation in the proof, we will write $K_k^b$ simply as $C$. We need to prove that $\{x_j, C\}_b^{(k)} = 0$ for all $j \in \mathcal{I}$. To do this, it is sufficient to prove that $\{x_1, C\}_b^{(k)} = 0$. Indeed, consider the cyclic permutation $\tau : \mathcal{I} \to \mathcal{I}$ which sends $j$ to $j+1$, for $j < 2k+1$ and sends $2k+1$ to $1$ (it is $\tau_1$, as defined in (3.1)). The induced permutation on the variables $x_i$ is denoted by $\tau^*$, so that $\tau^* x_1 = x_2$, for example. As we already recalled, $\tau^*$ is a Poisson automorphism of $\{\cdot,\cdot\}$; it is also a Poisson isomorphism between $\{\cdot,\cdot\}_b$ and $\{\cdot,\cdot\}_{\tau^* b}$, where $\tau^* b$ is defined by $(\tau^* b)_{i,j} := b_{i+1,j+1}$. Indeed,

$$\{\tau^* x_i, \tau^* x_j\}_b = \{x_{i+1}, x_{j+1}\}_b = b_{i+1,j+1} = \{x_i, x_j\}_{\tau^* b} = \tau^* \{x_i, x_j\}_{\tau^* b} \, .$$

It follows that $\tau^*$ is a Poisson automorphism between $\{\cdot,\cdot\}_b^{(k)}$ and $\{\cdot,\cdot\}_{\tau^* b}^{(k)}$. According to Lemma 3.2, $\tau^* C = C$, so that

$$\{x_2, C\}_b^{(k)} = \{\tau^* x_1, \tau^* C\}_b^{(k)} = \tau^* \{x_1, C\}_{\tau^* b}^{(k)} \, ,$$

and we can conclude from $\{x_1, C\}_b^{(k)} = 0$ for all $b$ that $\{x_2, C\}_b^{(k)} = 0$ for all $b$, and similarly that $\{x_j, C\}_b^{(k)} = 0$ for all $j \in \mathcal{I}$ and for all $b$.

In order to show that $\{x_1, C\}_b^{(k)} = 0$, we decompose $C$ as $C = \sum_{\ell=0}^{k}(C_{\ell,+} + C_{\ell,-})$, where

$$C_{\ell,+} := \sum_{\underline{m} \in \mathcal{S}_{\ell,+}} b'_{\underline{m}} x_{\underline{m}} \, , \quad \text{and} \quad C_{\ell,-} := \sum_{\underline{m} \in \mathcal{S}_{\ell,-}} b'_{\underline{m}} x_{\underline{m}} \, .$$

With this notation and in view of the definition (4.1) of $\{\cdot,\cdot\}_b^{(k)}$, proving that $\{x_1, C\}_b^{(k)} = 0$ amounts to proving that

$$\sum_{\ell=0}^{k} \left( \{x_1, C_{\ell,+}\} + \{x_1, C_{\ell,+}\}_b + \{x_1, C_{\ell,-}\} + \{x_1, C_{\ell,-}\}_b \right) = 0 \, .$$

We will do this by showing that

$$\{x_1, C_{\ell,+}\} = \{x_1, C_{\ell,+}\}_b + \{x_1, C_{i-1,-}\} = \{x_1, C_{\ell,-}\}_b = 0 , \qquad (4.4)$$

for all $\ell \geqslant 0$ (with the convention that $C_{-1,-} = 0$). We start with the first term. Let $\underline{m} \in \mathcal{S}_\ell$ with $\underline{m} = (m_1, m_2, \ldots, m_{2\ell+1})$. Recall that the sets $\mathcal{S}_\ell$ are defined such that $\underline{m} \in \mathcal{S}_\ell$ if and only if $A'_{\underline{m}} = A^{(\ell)}$ (see (2.8)). If $\underline{m} \in \mathcal{S}_{\ell,+}$ then $m_1 = 1$ and

$$\left\{x_1, x_{\underline{m}}\right\}^{(k)} = x_1 x_{\underline{m}} \left(\sum_{j=1}^{2\ell+1} A_{m_1,m_j}^{(k)}\right) = x_1 x_{\underline{m}} \left(\sum_{j=2}^{2\ell+1} A_{1,j}^{(\ell)}\right) = 0 .$$

This shows that $\{x_1, C_{\ell,+}\} = 0$. We next prove the last equality in (4.4). From the definition of $\{\cdot, \cdot\}_b$ it follows that for any function $F$

$$\{x_1, F\}_b = b_{1,k+1}\frac{\partial F}{\partial x_{k+1}} + b_{1,k+2}\frac{\partial F}{\partial x_{k+2}} . \qquad (4.5)$$

Therefore

$$\{x_1, C_{\ell,-}\}_b = \sum_{\underline{m}} \left\{x_1, b'_{\underline{m}}x_{\underline{m}}\right\}_b ,$$

where the sum can be restricted to those $\underline{m} \in \mathcal{S}_{\ell,-}$ which contain $k+1$ or $k+2$. Since $1 \notin \underline{m}$, (3.14) implies that $k+1$ and $k+2$ cannot both belong to $\underline{m}$. Also, the inequalities in Proposition 3.1 imply that the vector $\psi(\underline{m})$ whose effect is to replace $k+1$ in $\underline{m}$ by $k+2$ leads to a new element of $\mathcal{S}_{\ell,-}$ containing $k+2$ (and vice-versa). Therefore

$$
\begin{aligned}
\{x_1, C_{\ell,-}\}_b &= \sum_{\substack{\underline{m}\in\mathcal{S}_{\ell,-}\\k+1\in\underline{m}\not\ni k+2}} \left\{x_1, b'_{\underline{m}}x_{\underline{m}}\right\}_b + \sum_{\substack{\underline{m}\in\mathcal{S}_{\ell,-}\\k+1\notin\underline{m}\ni k+2}} \left\{x_1, b'_{\underline{m}}x_{\underline{m}}\right\}_b \\
&= \sum_{\substack{\underline{m}\in\mathcal{S}_{\ell,-}\\k+1\in\underline{m}\not\ni k+2}} \left(\left\{x_1, b'_{\underline{m}}x_{\underline{m}}\right\}_b + \left\{x_1, b'_{\psi(\underline{m})}x_{\psi(\underline{m})}\right\}_b\right) \\
&= \sum_{\substack{\underline{m}\in\mathcal{S}_{\ell,-}\\k+1\in\underline{m}\not\ni k+2}} \left(\frac{b_{1,k+1}b'_{\underline{m}}x_{\underline{m}}}{x_{k+1}} + \frac{b_{1,k+2}b'_{\psi(\underline{m})}x_{\psi(\underline{m})}}{x_{k+2}}\right) = 0 ,
\end{aligned}
$$

since $x_{\underline{m}}/x_{k+1} = x_{\psi(\underline{m})}/x_{k+2}$ and $b_{1,k+1}b'_{\underline{m}} = b'_{1,k+1}b'_{\underline{m}} = b'_{1,k+2}b'_{\psi(\underline{m})} = -b_{1,k+2}b'_{\psi(\underline{m})}$ for all $\underline{m} \in \mathcal{S}_{\ell,-}$ with $k+1 \in \underline{m}$ and $k+2 \notin \underline{m}$.

It remains to be shown that $\{x_1, C_{\ell,+}\}_b + \{x_1, C_{i-1,-}\} = 0$. Using (4.5), we find

$$\{x_1, C_{\ell,+}\}_b = \sum_{\substack{\underline{n}\in\mathcal{S}_{\ell,+}\\k+1\in\underline{n}}} b_{1,k+1}b'_{\underline{n}}\frac{x_{\underline{n}}}{x_{k+1}} + \sum_{\substack{\underline{n}\in\mathcal{S}_{\ell,+}\\k+2\in\underline{n}}} b_{1,k+2}b'_{\underline{n}}\frac{x_{\underline{n}}}{x_{k+2}} . \qquad (4.6)$$

For $\underline{m} \in \mathcal{S}_{\ell-1,-}$, if $m_\ell \leqslant k+1$ then $\left\{x_1, x_{\underline{m}}\right\}^{(k)} = x_1 x_{\underline{m}}\left(\sum_{j=1}^{2\ell-1} A_{1,m_j}\right) = x_1 x_{\underline{m}}$, and similarly if $m_\ell \geqslant k+2$ then $\left\{x_1, x_{\underline{m}}\right\}^{(k)} = -x_1 x_{\underline{m}}$. Therefore,

$$\{x_1, C_{\ell-1,-}\}^{(k)} = -\sum_{\substack{\underline{m}\in\mathcal{S}_{\ell-1,-}\\m_\ell>k+1}} b'_{\underline{m}}x_1 x_{\underline{m}} + \sum_{\substack{\underline{m}\in\mathcal{S}_{\ell-1,-}\\m_\ell\leqslant k+1}} b'_{\underline{m}}x_1 x_{\underline{m}} . \qquad (4.7)$$

Up to a minus sign, the first sum in (4.6) and the first sum in (4.7) are equal, and similarly for the second sums. We show this for the first sum, using the bijection $\phi_2$ from Lemma 3.3 (for the second sum, one uses $\phi_1$):

$$\phi_2 \quad : \quad \{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \geqslant k+2\} \quad \to \quad \{\underline{n} \in \mathcal{S}_{\ell,+} \mid k+1 \in \underline{n}\}$$
$$\underline{m} \qquad\qquad \mapsto \qquad \underline{m} \oplus (1, k+1) \ .$$

For $\underline{n} := \phi_2(\underline{m}) = \underline{m} \oplus (1, k+1)$, with $\underline{m} \in \{\underline{m} \in \mathcal{S}_{\ell-1,-} \mid m_\ell \geqslant k+2\}$, we have $x_{\underline{n}} = x_{\underline{m}} x_1 x_{k+1}$ and $b'_{\underline{n}} = b'_{\underline{m}}/b'_{1,k+1} = b'_{\underline{m}}/b_{1,k+1}$, so that

$$\sum_{\substack{\underline{n} \in \mathcal{S}_{\ell,+} \\ k+1 \in \underline{n}}} b_{1,k+1} b'_{\underline{n}} \frac{x_{\underline{n}}}{x_{k+1}} = \sum_{\substack{\underline{m} \in \mathcal{S}_{\ell-1,-} \\ m_\ell > k+1}} b'_{\underline{m}} x_1 x_{\underline{m}} \ ,$$

as was to shown.                                                                                     $\square$

## 5. LAX EQUATION, FIRST INTEGRALS AND INTEGRABILITY

We now introduce a family of deformations of the Bogoyavlenskij-Itoh Lotka-Volterra systems, which we will show to be Liouville integrable. Let $c_1, c_2, \ldots, c_{2k+1}$ be arbitrary constants, satisfying $\sum_{i=1}^{2k+1} c_i = 0$. The *deformed Bogoyavlenskij-Itoh Lotka-Volterra system* is the system defined by the following differential equations:

$$\dot{x}_i = \sum_{j=1}^{2k+1} A_{i,j} x_i x_j + c_i \ , \quad i = 1, 2, \ldots, 2k+1 \ , \tag{5.1}$$

where the constants $A_{i,j}$ are given by (2.2). We first show that (5.1) is Hamiltonian with respect to some of the Poisson structures $\{\cdot, \cdot\}_b^{(k)}$, introduced in Section 4, with $H = \sum_{i=1}^{2k+1} x_i$ as Hamiltonian function. Before doing this, notice that if (5.1) admits $H$ as a Hamiltonian function, then $H$ is a constant of motion of (5.1), and so $\sum_{i=1}^{2k+1} c_i = 0$, which explains why we have imposed the latter condition on the deformation constants $c_i$. Let $b$ be as in Proposition 4.1, i.e., $b$ is skew-symmetric and $b_{i,j} = 0$ for all $i$ and $j$ such that $|i - j| \notin \{k, k+1\}$. Then the Hamiltonian vector field $\{\cdot, H\}_b^{(k)}$ is given by

$$\dot{x}_i = \sum_{j=1}^{2k+1} A_{i,j} x_i x_j + b_{i,i+k} - b_{i-k,i} \ , \quad i = 1, 2, \ldots, 2k+1 \ , \tag{5.2}$$

where, as above, all indices are taken modulo $2k + 1$ and with values in $\mathfrak{I} = \{1, 2, \ldots, 2k+1\}$. Comparing (5.1) and (5.2), we need to show that the following linear system admits a solution:

$$b_{i,i+k} - b_{i-k,i} = c_i \ , \quad i \in \mathfrak{I} \ . \tag{5.3}$$

Let $b_{1-k,1} := c$ be arbitray. Then the solution to the equations (5.3) can be obtained by solving them for the following values of $i$ (in that order and, as always, modulo $2k+1$): $i = 1, k+1, 2k+1, \ldots$, yielding unique values for $b_{1,1+k}, b_{1+k,1+2k}, b_{1+2k,1+3k} = b_{1+2k,k}, \ldots$. The final formula is given by

$$b_{i,i+k} = c_1 + c_{1+k} + c_{1+2k} + \cdots + c_{i-k} + c_i + c \ , \quad \text{for} \quad i \in \mathfrak{I} \ . \tag{5.4}$$

indeed, it follows from the latter formula at once that $b_{i,i+k} - b_{i,i-k} = c_i$ for all $i \in \mathcal{I}$ and that

$$b_{1-k,1} = c_1 + c_{1+k} + c_{1+2k} + \cdots + c_2 + c_{2+k} + c = \sum_{i=1}^{2k+1} c_i + c = c \,,$$

since $\sum_{i=1}^{2k+1} c_i = 0$. We have thereby proven the following proposition:

**Proposition 5.1.** *The deformed Bogoyavlenskij-Itoh Lotka-Volterra system*

$$\dot{x}_i = \sum_{j=1}^{2k+1} A_{i,j} x_i x_j + c_i \,, \quad i = 1, 2, \ldots, 2k+1 \,,$$

*with $\sum_{i=1}^{2k+1} c_i = 0$ is a Hamiltonian system with respect to a (non-unique) Poisson structure $\{\cdot,\cdot\}_b^{(k)}$, the non-zero entries of b being given by (5.4) and with $H = \sum_{i=1}^{2k+1} x_i$ as Hamiltonian function.*

In the following proposition we give a Lax equation (with spectral parameter) for the deformed Bogoyavlenskij-Itoh Lotka-Volterra system (5.1). Bogoyavlenskij's Lax equation (2.5) is obtained from it by putting all $b_{i,j}$ equal to zero.

**Proposition 5.2.** *The system (5.1) can be written as the following Lax equation (with spectral parameter $\lambda$)*

$$(X + \lambda^{-1}\Delta + \lambda M)^{\cdot} = [X + \lambda^{-1}\Delta + \lambda M, B - \lambda M^{k+1}] \tag{5.5}$$

*where for $i, j \in \mathcal{I}$ the $(i,j)$-th entry of the matrices $X$, $M$ and $B$ is respectively given by*

$$X_{i,j} := \delta_{i,j+k} x_i \,, \quad \Delta_{i,j} := b_{i+k,j} \delta_{i,j} \,, \quad M_{i,j} := \delta_{i+1,j} \,, \tag{5.6}$$

$$B_{i,j} := b_i \delta_{i,j} := -\delta_{i,j}(x_i + x_{i+1} + \cdots + x_{i+k}) \,. \tag{5.7}$$

*Proof.* To check that (2.5) is equivalent to (5.2) it is sufficient to check that (5.2) is equivalent with $\dot{X} = [X, B] - [\Delta, M^{k+1}]$ and (since $M$ is constant) that $[M, B] - [X, M^{k+1}] = 0$; indeed, $[M, M^{k+1}] = 0$ and $[\Delta, B] = 0$ because $\Delta$ and $B$ are diagonal matrices. For the second equality, one finds at once from (5.6) that

$$([M, B] - [X, M^{k+1}])_{i,j} = \delta_{i+1,j}(b_j - b_i - x_i + x_{j+k}) = 0 \,.$$

Since $B$ and $\Delta$ are diagonal matrices, $[X, B]_{i,j} = X_{i,j}(b_j - b_i)$ and $-[\Delta, M^{k+1}]_{i,j} = (b_{i,i+k} - b_{i-k,i})\delta_{i,j+k}$, with non-zero entries only when $j = i - k$; for these entries, one has from the Lax equation

$$\dot{x}_i = \dot{X}_{i,i-k} = [X, B]_{i,i-k} - [\Delta, M^{k+1}]_{i,i-k} = x_i(b_{i-k} - b_i) + b_{i,i+k} - b_{i-k,i} \,,$$

which is the right hand side of (5.2). $\qquad\square$

Each coefficient of the characteristic polynomial of the Lax operator

$$L^b(\lambda) := X + \lambda^{-1}\Delta + \lambda M \tag{5.8}$$

gives a first integral for the deformed Bogoyavlenskij-Itoh Lotka-Volterra system. We know already one constant of motion: the Casimir $K_k^b$, introduced in Proposition 4.2. Indeed, $\dot{K}_k^b = \{K_k^b, H\}_b^{(k)} = 0$. We will show in the Proposition 5.4 that $K_k^b$ appears as a coefficient of the characteristic polynomial of the Lax operator $L^b(\lambda)$ (namely the coefficient of $\lambda^0$). But first we prove a basic lemma on permutations, which will be used in the Proposition 5.4 and will clarify its proof.

**Lemma 5.3.** *For a permutation $\sigma$ of $\mathcal{I}$, the following two conditions are equivalent:*

  (i) $\sigma(i) \in \{i, i+k, i+k+1\}$ *for all* $i \in \mathcal{I}$;
  (ii) $\sigma$ *is equal to one of the following:*
     (1) *The $(2k+1)$-cycle $(1, 1+k, 1+2k, \ldots, 2, k+2)$;*
     (2) *The inverse of the $(2k+1)$-cycle $(1, 1+k, 1+2k, \ldots, 2, k+2)$;*
     (3) *A product of $k - \ell$ transpositions $(r_j, s_j)$ with disjoint support, and satisfying $s_j - r_j \in \{k, k+1\}$ for $j = 1, \ldots, k - \ell$.*

*Proof.* The implication $(ii) \implies (i)$ is clear. We prove the other implication. Suppose that $\sigma(i) \in \{i, i+k, i+k+1\}$ for all $i \in \mathcal{I}$ and that $\sigma$ is not as stated in (3) above. Then there exist $j \in \mathcal{I}$ such that $\sigma^2(j) \neq j$. Since $\sigma(j) \in \{i, i+k, i+k+1\}$, we have either $\sigma(j) = j+k$ or $\sigma(j) = j+k+1$. By replacing $\sigma$ with $\sigma^{-1}$, if needed, we may suppose that $\sigma(j) = j+k$. Since $\sigma^2(j) \neq j$ $(= j+2k+1)$, we must have $\sigma^2(j) = \sigma(j+k) = j+2k$. By recursion, $\sigma^s(j) = j+sk$ for $s = 0, 1, 2, \ldots$. Since $k$ and $2k+1$ are relatively prime,

$$\sigma = (j,\ j+k,\ j+2k,\ \ldots,\ j+1,\ j+1+k) = (1,\ 1+k,\ 1+2k,\ \ldots,\ 2,\ 2+k).$$

This proves that $\sigma$ is of the form (1). $\qquad\square$

**Proposition 5.4.** *The determinant of the Lax operator $L^b(\lambda)$ is given by*

$$\det L^b(\lambda) = \lambda^{2k+1} + \frac{1}{\lambda^{2k+1}} \prod_{j=1}^{2k+1} b_{j+k,j} + K_k^b, \tag{5.9}$$

*where $K_k^b$ is the Casimir (4.3) of $\{\cdot, \cdot\}_b^{(k)}$.*

*Proof.* Let $\Lambda := L^b(\lambda)E$, where $E$ denotes the $(2k+1) \times (2k+1)$-matrix with entries $E_{j,j+k} = 1$ for all $j \in \mathcal{I}$ and all other entries equal to zero. Thus, $\Lambda$ is just the matrix $L^b(\lambda)$ with its last $k$ columns placed first. The nonzero entries of $\Lambda$ are the following ($j \in \mathcal{I}$):

  (1) $\Lambda_{i,i} = x_i$;
  (2) $\Lambda_{i,i+k} = \frac{b_{i+k,i}}{\lambda}$;
  (3) $\Lambda_{i,i+k+1} = \lambda$.

Also, $\det \Lambda = \det L^b(\lambda)$ since $\det E = 1$. We compute this determinant by using the Leibniz formula:

$$\det(\Lambda) = \sum_{\sigma \in S_{\mathcal{J}}} \mathrm{sgn}(\sigma) \prod_{i=1}^{2k+1} \Lambda_{i,\sigma(i)} \ ,$$

where $S_{\mathcal{J}}$ is the permutation group of $\mathcal{J}$ and $\mathrm{sgn}(\sigma)$ stands for the sign of a permutation $\sigma$. Because of items (1) – (3) above, $\prod_i \Lambda_{i,\sigma(i)} \neq 0$ if and only if $\sigma(i) \in \{i, i+k, i+k+1\}$ for every $i \in \mathcal{J}$. According to Lemma 5.3, there are three possibilities for a permutation $\sigma$ satisfying these conditions. For the first one, $\sigma = (1, 1+k, 1+2k, \dots, 2, k+2)$. Since $\mathrm{sgn}(\sigma) = 1$, it leads to the following contribution in $\det \Lambda$:

$$\prod_{i=1}^{2k+1} \Lambda_{i,i+k} = \frac{1}{\lambda^{2k+1}} \prod_{j \in \mathcal{J}} b_{j+k,j} \ .$$

For the second one, $\sigma^{-1} = (1, 1+k, 1+2k, \dots, 2, k+2)$, leading to $\lambda^{2k+1}$ since again $\mathrm{sgn}(\sigma) = 1$ and since $\Lambda_{i,i+k+1} = \lambda$ for all $i$. For the third and final one, $\sigma$ is the product of transpositions,

$$\sigma = (r_1, \ s_1)(r_2, \ s_2) \cdots (r_{k-\ell}, \ s_{k-\ell}) \ ,$$

with $s_j - r_j \in \{k, k+1\}$ for $j = 1, \dots, k - \ell$. Let $\underline{m} \in \mathcal{S}_\ell$ be the element defined by $\underline{m}' = (r_1, r_2, \dots, r_{k-\ell}, s_1, s_2, \dots, s_{k-\ell}) \in \mathcal{S}_\ell$ (see Proposition 3.4). The corresponding term in $\det(\Lambda)$ is[1]

$$\mathrm{sgn}(\sigma) \prod_{j=1}^{k-\ell} (\Lambda_{r_j,s_j} \Lambda_{s_j,r_j}) \prod_{j=1}^{2\ell+1} x_{m_j} = \prod_{j=1}^{k-\ell} b'_{r_j,s_j} x_{\underline{m}} = b'_{\underline{m}} x_{\underline{m}} \ .$$

Summing up all these contributions, we get in view of the definition (4.3) of $K_k^b$ precisely (5.9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We can obtain from the above formula (5.9) for the determinant of the Lax operator $L^b(\lambda)$ easily a formula for its characteristic polynomial $\det(L^b(\lambda) - \mu \, \mathrm{Id})$ by the following trick: since the $j$-th diagonal entry of $L^b(\lambda)$ is $b_{j+k,j}/\lambda$ and since the parameters $b_{i,j}$ appear nowhere else in the Lax matrix, it suffices to replace in all formulas $b_{j+k,j}/\lambda$ by $b_{j+k,j}/\lambda - \mu$. This can be done by replacing every $b_{j+k,j}$ by $b_{j+k,j} - \lambda\mu$. Notice that when the constants $b_{j+k,j}$ are viewed as a vector $b$ (with $2k+1$ entries), then this substitution amounts to replacing $b$ with $b - \lambda\mu\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)$. This proves part of the following proposition:

**Proposition 5.5.** *The characteristic polynomial of the Lax operator $L^b(\lambda)$ is given by*

$$\chi(L^b(\lambda), \mu) = \det(L^b(\lambda) - \mu \, \mathrm{Id}) = \lambda^{2k+1} + \frac{1}{\lambda^{2k+1}} \prod_{j=1}^{2k+1} (b_{j+k,j} - \lambda\mu) + K_k^{b-\lambda\mu\mathbf{1}} \ .$$

$$(5.10)$$

---

[1]Something has to be said about the signs, which disappear thanks to the primes!

The coefficients $K_\ell^b$ in the expansion

$$K_k^{b-\lambda\mu\mathbf{1}} = \sum_{\ell=0}^{k} (\lambda\mu)^{k-\ell} K_\ell^b \,, \qquad (5.11)$$

are first integrals of the deformed Bogoyavlenskij-Itoh Lotka-Volterra system (5.2). Also, each $K_\ell^b$ is of total degree $2\ell+1$ (as a polynomial in $x_1,\ldots,x_{2k+1}$), with leading term $K_\ell$.

*Proof.* Formula (5.10) was proven above. In order to prove the other statements, we first point out that if we attribute the weight 2 to every $b_{j+k,j}$ and the weight 1 to all variables $x_i$, to $\lambda$ and to $\mu$ then every entry of the matrix $L^b(\lambda)-\mu\,\mathrm{Id}$ is homogeneous of degree 1, and so the characteristic polynomial (5.10) is weight homogeneous of degree $2k+1$ with respect to these weights. It follows that $K_k^{b-\lambda\mu\mathbf{1}}$ can indeed be expanded as in (5.11) and that every coefficient $K_\ell^b$ is weight homogeneous of degree $2\ell+1$ (depending polynomially on the variables $x_i$ and the constants $b_{j+k,j}$). Since these polynomials are (possibly up to a constant) coefficients of the characteristic polynomial of the Lax operator, they are constants of motion. Setting $b=0$ in (5.10) and in (5.11) we have

$$\chi(L(\lambda),\mu) = \lambda^{2k+1} - \mu^{2k+1} + K_k^{-\lambda\mu\mathbf{1}} = \lambda^{2k+1} - \mu^{2k+1} + \sum_{\ell=0}^{k} (\lambda\mu)^{k-\ell} K_\ell^0 \,,$$

which, compared with (2.7), to wit,

$$\chi(L(\lambda),\mu) = \lambda^{2k+1} - \mu^{2k+1} + \sum_{\ell=0}^{k} (\lambda\mu)^{k-\ell} K_\ell \,,$$

shows that $K_\ell$ is obtained from $K_\ell^b$ by putting all constants in $b$ equal to zero; since the constants in $b$ are of weight 2 and both $K_\ell$ and $K_\ell^b$ are weight homogeneous of degree $2\ell+1$, it follows that $K_\ell^b$ is of total degree $2\ell+1$ (as a polynomial in $x_1,\ldots,x_{2k+1}$), with leading term $K_\ell$, as asserted. $\square$

We have constructed $k+1$ polynomial first integrals $K_0^b, K_1^b, \ldots, K_k^b$ for the deformed Bogoyavlenskij-Itoh Lotka-Volterra system (5.1), which is a Hamiltonian system on the Poisson manifold $(\mathbb{R}^{2k+1}, \{\cdot,\cdot\}_b^{(k)})$. We now prove the Liouville integrability of this system.

**Theorem 5.6.** *The deformed Bogoyavlenskij-Itoh Lotka-Volterra system (5.1) is Liouville integrable, with independent first integrals $K_0^b, K_1^b, \ldots, K_k^b$ which are pairwise in involution with respect to $\{\cdot,\cdot\}_b^{(k)}$.*

*Proof.* According to Proposition 5.5, the $k+1$ polynomials $K_\ell^b$, defined by

$$K_k^{b-\nu\mathbf{1}} = \sum_{\ell=0}^{k} \nu^{k-\ell} K_\ell^b \qquad (5.12)$$

are first integrals of (5.1). By the same proposition, the leading term of $K_\ell^b$ (as a polynomial in $x_1, \ldots, x_{2k+1}$) is $K_\ell$; since the polynomials $K_0, K_1, \ldots, K_k$ are (functionally) independent, the latter fact implies that the same is true for the deformed polynomials $K_0^b, K_1^b, \ldots, K_k^b$. It remains to be shown that the latter polynomials are in involution with respect to $\{\cdot, \cdot\}_b^{(k)}$. To do this, we use the Lenard-Magri scheme in the context of Poisson pencils (see [10]). Recall from Proposition 4.2 that $K_k^b$ is a Casimir for $\{\cdot, \cdot\}_b^{(k)}$. It follows that for any $\nu$, $K_k^{b-\nu\mathbf{1}}$ is a Casimir of $\{\cdot, \cdot\}_{b-\nu\mathbf{1}}^{(k)} = \{\cdot, \cdot\}_b^{(k)} - \nu\{\cdot, \cdot\}_{\mathbf{1}}$. Said differently, $K_k^{b-\nu\mathbf{1}}$ is a Casimir for the Poisson pencil $\{\cdot, \cdot\}_b^{(k)} - \nu\{\cdot, \cdot\}_{\mathbf{1}}$. Its expansion (5.12) implies according to the Lenard-Magri scheme the following facts:

(1) The polynomials $K_\ell^b$ are in involution with respect to $\{\cdot, \cdot\}_b^{(k)}$;
(2) They are also in involution with respect to $\{\cdot, \cdot\}_{\mathbf{1}}$;
(3) Each of the integrable vector fields is bi-Hamiltonian: $\{\cdot, K_\ell^b\}_b^{(k)} = \{\cdot, K_{\ell+1}^b\}_{\mathbf{1}}$;
(4) $K_k^b$ is a Casimir of $\{\cdot, \cdot\}_b^{(k)}$ (as we already know) and $K_0^b$, which is just $H$, is a Casimir of $\{\cdot, \cdot\}_{\mathbf{1}}$ (which is quite obvious).

In particular, the first integrals are in involution with respect to $\{\cdot, \cdot\}_b^{(k)}$, as was to be shown. $\qquad\square$

*Remark* 5.7. The proof shows that the functions $K_\ell^b, \ell = 0, 1, \ldots, k$ are also in involution with respect to the bracket $\{\cdot, \cdot\}_{\mathbf{1}}$.

*Remark* 5.8. For $b = 0$, we get that the functions $K_\ell, \ell = 0, 1, \ldots, k$ are in involution with respect to the brackets $\{\cdot, \cdot\}^{(k)}$, which provides an alternative proof to Itoh's combinatorial proof of this fact.

## 6. Relation with the Veselov-Shabat system

In this section we will give a recursive formula for constructing the first integrals $K_\ell^b$ which shows that the combinatorial argument of Itoh can be used for their definition as in the case of the polynomials $K_\ell^{\mathbf{0}}$, the first integrals of the Bogoyavlenskij-Itoh system (see [5]). We will also show that the system (5.2) and the Veselov-Shabat system constructed in [13] are isomorphic by constructing a Poisson isomorphism between $(\mathbb{R}^{2k+1}, \{\cdot, \cdot\}_b^{(k)})$ and $(\mathbb{R}^{2k+1}, \{\cdot, \cdot\}_v)$ where $\{\cdot, \cdot\}_v$ is the Poisson structure constructed in [13] as part of the Hamiltonian formalism of the Veselov-Shabat system.

The Poisson structure $\{\cdot, \cdot\}_v$ is defined by the formulas $\{g_i, g_j\}_v = (-1)^{j-i+1} g_i g_j$ if $j > i + 1$ and $\{g_i, g_{i+1}\}_v = g_i g_{i+1} + \beta_{i+1}$ and the Veselov-Shabat system is the Hamiltonian system with Poisson structure $\{\cdot, \cdot\}_v$ and Hamiltonian the sum of the variables $\sum_{j=1}^n g_j$. We define a permutation $\rho$ of $\mathfrak{I}$ by setting $\rho(\ell) := \rho_\ell = (\ell - 1)k + 1$ for all $\ell \in \mathbb{Z}$. Then, one defines the Poisson map $g : (\mathbb{R}^{2k+1}, \{\cdot, \cdot\}_b^{(k)}) \to$

$(\mathbb{R}^{2k+1}, \{\cdot,\cdot\}_v)$ by

$$(x_1, x_2, \ldots, x_{2k+1}) \mapsto (g_1, g_2, \ldots, g_{2k+1}) := (x_{\rho_1}, x_{\rho_2}, \ldots, x_{\rho_{2k+1}})$$

which is easily checked to be a Poisson isomorphism. The parameters $b_{i,i+k}$ of the first Poisson structure are related to the parameters $\beta_i$ of the second Poisson structure by $\beta_{i+1} = b_{\rho_i, \rho_i+k}$ for all $i = 1, 2, \ldots, 2k + 1$. Since the Hamiltonian $\sum_{j=1}^n x_j$ of the system (5.2) is mapped under $g$ to the Hamiltonian $\sum_{j=1}^n g_j$ of the Veselov-Shabat system, the two systems are isomorphic.

Following [13] we define new variables $f_i$ which are related to the $g_i$ by the formulas $g_i = f_i + f_{i+1}$ or equivalently by $f_i = \frac{1}{2}\sum_{j=1}^{2k+1}(-1)^{j+1}g_{i+j-1}$, $i = 1, 2, \ldots, 2k+1$. Then the system of Veselov and Shabat in the $f_i$ variables reads

$$\dot{f}_i + \dot{f}_{i+1} = f_{i+1}^2 - f_i^2 + \beta_{i+1} - \beta_i, \quad i = 1, 2, \ldots, 2k + 1 \tag{6.1}$$

and the matrix equation $\dot{L}_i = L_i \cdot U_{i+1} - U_i \cdot L_i$ where

$$L_i = \begin{pmatrix} f_i & 1 \\ f_i^2 + \beta_i - \lambda & f_i \end{pmatrix}, \qquad U_i = \begin{pmatrix} 0 & 1 \\ \dot{f}_i + f_i^2 + \beta_i - \lambda & 0 \end{pmatrix}$$

gives the $i$-th equation of the system (6.1). Notice that $U_{2k+2} = U_1$ and therefore if we write $L = \prod_{i=1}^n L_i$ we have that

$$\begin{aligned}
\dot{L} &= \sum_{i=1}^{2k+1} L_1 L_2 \cdots \dot{L}_i \cdots L_{2k+1} \\
&= \sum_{i=1}^{2k+1} L_1 L_2 \cdots L_i U_{i+1} \cdots L_{2k+1} - \sum_{i=1}^{2k+1} L_1 L_2 \cdots U_i L_i \cdots L_{2k+1} \\
&= L U_{2k+2} - U_1 L = [L, U_1]
\end{aligned}$$

a Lax equation for the system (6.1). The simple form of the matrices $L_i$ allows us to compute easily the trace of $L$ and as it turns out the trace is equal to the Casimir $K_k^{b-\nu}$. If $R_i$ is the matrix $R_i = (\beta_i - \lambda)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then one can show that the following formulas on the matrices $L_i, R_i$ and $U_i$ hold

$$\prod_{i=j}^{\ell}(L_i - R_i) = \prod_{i=j}^{\ell-1} g_i \begin{pmatrix} f_\ell & 1 \\ f_j f_\ell & f_j \end{pmatrix}, \quad R_i R_j = \mathbf{0},$$

$$\begin{pmatrix} f_\ell & 1 \\ f_j f_\ell & f_j \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_{2k+1} & 1 \\ f_{\ell+2} f_{2k+1} & f_{\ell+2} \end{pmatrix} = \begin{pmatrix} f_{2k+1} & 1 \\ f_j f_{2k+1} & f_j \end{pmatrix}.$$

As a consequence we get the following formula for the trace of $L$

$$\operatorname{tr}(L) = \prod_{i=1}^{2k+1} L_i = \prod_{i=1}^{2k+1}(L_i - R_i + R_i) = \prod_{i=1}^{2k+1}\left(1 + (\beta_{i+1} - \lambda)\frac{\partial^2}{\partial g_i \partial g_{i+1}}\right)\prod_{j=1}^{2k+1} g_i. \tag{6.2}$$

The trace of $L$, written in the variables $x_i$, becomes

$$\operatorname{tr}(L) = \prod_{i=1}^{2k+1} (1 + (b_{\rho_i,\rho_i+k} - \lambda)\frac{\partial^2}{\partial x_{\rho_i}\partial x_{\rho_{i+1}}}) \prod_{j=1}^{2k+1} x_{\rho_i}$$

$$= \prod_{i=1}^{2k+1} (1 + (b_{i,i+k} - \lambda)\frac{\partial^2}{\partial x_i\partial x_{i+k}}) \prod_{j=1}^{2k+1} x_i.$$

The coefficients of the thace of $L$ (viewed as a polynomial in $\lambda$) are first integrals of (5.2). We will show that the previous formula for the trace of $L$ is exactly the polynomial $K_k^{b-\lambda}$ and we will also give a recursive formula for explicitly producing the polynomials $K_j^b$. To do this, we first define the following constant coefficient linear operators

$$d_i = \frac{\partial}{\partial x_i}, \quad D_{i,j} = d_i d_j, \quad D = \sum_{i=1}^{2k+1} D_{i,i+k}$$

and also the

$$D_{i,j}^b = b_{i,j} d_i d_j, \, D_b = \sum_{i=1}^{2k+1} D_{i,i+k}^b.$$

Since they have constant coefficients, all these operators commute with each other (at least when applied on polynomials, which is the case which we consider here). In the following lemma we show that the first integrals $K_i$ of (5.2) can be defined recursively using the operator $D$.

**Lemma 6.1.** *The polynomials $K_i$, defined by (2.7) (which are the same thing as $K_i = K_i^{\mathbf{0}}$) satisfy the following recursive equation*

$$DK_i = (k - i + 1)K_{i-1}.$$

*Proof.* Each element $\underline{m}' \in \mathcal{S}_i'$ (and therefore each element $\underline{m} \in \mathcal{S}_i$) is in one to one correspondence with the $i'$-tubles consisting of elements $b_{\ell,\ell+k}, \ell = 1, 2, \ldots, n$ which do not have any common indices (e.g. the choice $b_{1,5}, b_{5,9}, \ldots$ is not allowed). This is evident from Proposition 3.4. Therefore there is a bijection $\Phi : \mathcal{S}_i \to \mathcal{B}_{i'}$ where

$$\mathcal{B}_{i'} = \left\{ \{b_{r_1,r_1+k}, b_{r_2,r_2+k}, \ldots, b_{r_{i'},r_{i'}+k}\} : \#\{r_1, r_2, \ldots, r_{i'}, r_1 + k, r_2 + k, \ldots, r_{i'} + k\} = 2i' \right\}.$$

We can easily see that the map $\Phi$ is the complement, meaning that

$$\Phi^{-1}\left(\{b_{r_1,r_1+k}, b_{r_2,r_2+k}, \ldots, b_{r_{i'},r_{i'}+k}\}\right) = \underline{m} \iff$$

$$\underline{m}' = (r_1, r_2, \ldots, r_{i'}, r_1 + k, r_2 + k, \ldots, r_{i'} + k).$$

Let $\underline{m} = (m_1, m_2, \ldots, m_{2i+1}) \in \mathcal{S}_i$ such that $\Phi(\underline{m}) = \{b_{r_1,r_1+k}, b_{r_2,r_2+k}, \ldots, b_{r_{i'},r_{i'}+k}\}$. Then $x_{\underline{m}} = \left(\prod_{\ell=1}^{i'} D_{r_\ell,r_\ell+k}\right) K_k$ and, using the combinatorial formulas (4.3) we get

$$K_i = \sum_{\{b_{r_1,r_1+k}, b_{r_2,r_2+k}, \ldots, b_{r_{i'},r_{i'}+k}\} \in \mathcal{B}_i} \left(\prod_{\ell=1}^{i'} D_{r_\ell,r_\ell+k}\right) K_k.$$

Because the degree of the monomial $K_k$, viewed as a polynomial in one variable $x_i$, is one, it follows that $d_j^2 K_k = 0$ for any $j \in \mathcal{I}$. The multinomial formula

$$D^{i'} = \left( \sum_{j=1}^{n} D_{j,j+k} \right)^{i'} = \sum_{m_1+m_2+\ldots+m_k=i'} \binom{i'}{m_1, m_2, \ldots, m_\ell} \prod_{j=1}^{\ell} D_{j,j+k}^{m_j}$$

gives that

$$D^{i'} K_k = \sum_{\left\{ b_{r_1,r_1+k}, b_{r_2,r_2+k}, \ldots, b_{r_{i'},r_{i'}+k} \right\} \in \mathcal{B}_i} \binom{i'}{1,1,\ldots,1} \left( \prod_{j=1}^{i'} D_{r_j,r_j+k} \right) K_k = i'! K_i$$

and we deduce that

$$D^{i'} K_k = i'! K_i \tag{6.3}$$

Therefore,

$$K_{i-1} = \frac{1}{(i'+1)!} D^{i'+1} K_k = \frac{i'!}{(i'+1)!} DK_i = \frac{1}{i'+1} DK_i.$$

$\square$

We prove now a similar formula for the polynomials $K_i^b$ which shows that the formula (6.2) of Veselov and Shabat for the trace of the Lax operator $L$ coincides with the polynomial $K_k^{b-\lambda}$ (item (1) of the next proposition) which is the characteristic polynomial of the Lax operator we constructed in (5.5).

**Proposition 6.2.**    (1) *For each $i = 0, 1, 2, \ldots, k$*

$$K_i^b = K_i + D_b K_i + \frac{1}{2!} D_b^2 K_i + \ldots + \frac{1}{i!} D_b^i K_i$$

(2) *For each $i = 0, 1, 2, \ldots, k-1$*

$$K_i^b = \frac{1}{k-i} DK_{i+1}^b.$$

*Proof.* First of all, note that (1) holds for $i = k$. This can be seen from the proof of Lemma 6.1 and the definition of the polynomial $K_k^b$; in the polynomial $K_k^b$, for each variable missing from $x_{\underline{m}}$ there is one and only one constant $b_{.,.}$ associated to it (see also Proposition 3.4).

We also notice that the operator $D_b$ satisfies the following formula

$$D_{b+\lambda \mathbf{1}} = D_b + \lambda D \Rightarrow D_{b+\lambda \mathbf{1}}^l = \sum_{m=0}^{l} \binom{l}{m} \lambda^m D_b^{l-m} D^m.$$

Therefore for item (1) we have

$$K_k^{b+\lambda \mathbf{1}} = \sum_{l=0}^{k} \frac{1}{l!} D_{b+\lambda \mathbf{1}}^l K_k = \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=0}^{l} \binom{l}{m} \lambda^m D_b^{l-m} D^m K_k.$$

Since $K_i^b$ is the coefficient of $\lambda^{k-i}$ in the expansion of $K_k^{b+\lambda \mathbf{1}}$ it follows that

$$K_i^b = \sum_{l=0}^{k} \frac{1}{l!} \binom{l}{k-i} D_b^{l-k+i} D^{k-i} K_k = \sum_{l=k-i}^{k} \frac{1}{l!} \binom{l}{k-i} D_b^{l-k+i} D^{k-i} K_k.$$

In view of the formula (6.3), the previous line can be re-written as

$$K_i^b = \sum_{l=k-i}^{k} \frac{1}{l!} \binom{l}{k-i} (k-i)! D_b^{l-k+i} K_i$$

which is exactly item (1).

Item (2) can be deduced by combining item (1) and Lemma 6.1. $\qquad\square$

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Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

*E-mail address*: `cevrip02@ucy.ac.cy, pavlos1978@gmail.com`

Pol Vanhaecke, Laboratoire de Mathématiques, UMR 7348 du CNRS, Université de Poitiers, 86962 Futuroscope Chasseneuil Cedex, France

*E-mail address*: `pol.vanhaecke@math.univ-poitiers.fr`