INTERMEDIATE TODA SYSTEMS

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Abstract. We construct a large family of Hamiltonian systems which interpolate between the classical Kostant-Toda lattice and the full Kostant-Toda lattice and we discuss their integrability. There is one such system for every nilpotent ideal $\mathcal{I}$ in a Borel subalgebra $\mathfrak{b}_+$ of an arbitrary simple Lie algebra $\mathfrak{g}$. The classical Kostant-Toda lattice corresponds to the case of $\mathcal{I} = [\mathfrak{n}_+, \mathfrak{n}_+]$, where $\mathfrak{n}_+$ is the unipotent ideal of $\mathfrak{b}_+$, while the full Kostant-Toda lattice corresponds to $\mathcal{I} = \{0\}$. We mainly focus on the case $\mathcal{I} = [[\mathfrak{n}_+, \mathfrak{n}_+], \mathfrak{n}_+]$. In this case, using the theory of root systems of simple Lie algebras, we compute the rank of the underlying Poisson manifolds and we construct a set of (rational) functions in involution, large enough to ensure Liouville integrability. These functions are restrictions of well-chosen integrals of the full Kostant-Toda lattice, except for the case of the Lie algebras of type C and D where a different function (Casimir) is needed. The latter fact, and other ones listed in the paper, point at the Liouville integrability of all the systems we construct, but also at the non-triviality of obtaining the result in full generality.

Dedicated to Valery V. Kozlov on the occasion of his 65^{th} birthday.

1. Introduction

This paper has overlap with [6] which appeared in the Proceedings of the fifth International Workshop on Group Analysis of Differential Equations and Integrable Systems in Protaras, Cyprus. In [6] we announced a list of results without proofs and we promised that the proofs will appear in a future publication. We chose to present it in this special issue of RCD in honor of Valery Kozlov on the occasion of this 65th birthday. The choice is quite appropriate. In fact we discovered these systems in an effort to find Lax pairs in the context of real and complex Lie algebras of the so called Kozlov-Treshchev potentials which were classified in [12]. This paper is therefore an expanded version of [6] with the inclusion of complete proofs of all results announced in [6]. The main purpose of this paper is to define the new systems, give their connection with simple Lie algebras and some of their basic properties. We will not attempt to prove the Liouville integrability of these systems in full generality.

We begin with a short overview of the various Toda type systems relevant to our construction. The classical Toda lattice is the mechanical system with Hamiltonian function

$$H(q_1, \dots, q_N, p_1, \dots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$

It describes a system of N particles on a line, connected by exponential springs. The differential equations which govern this lattice can be transformed via a change of variables due to Flaschka [10] to a Lax equation $\dot{L} = [L_+, L]$, where L is the Jacobi matrix

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$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}, \tag{1}$$

and L_+ is the skew-symmetric part of L in the Lie algebra decomposition lower triangular plus skew-symmetric. Lax equations define isospectral deformations; though the entries of L vary over time, the eigenvalues of L remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Trace} L^i$ are constants of motion. Moreover, they are in involution with respect to a Poisson structure, associated to the above Lie algebra decomposition.

There is a generalization due to Deift, Li, Nanda and Tomei [5] who showed that the system remains integrable when L is replaced by a full symmetric $N \times N$ matrix. The resulting system is called the full symmetric Toda lattice. The functions $H_i := \frac{1}{i}\operatorname{Trace} L^i$ are still in involution but they are not enough to ensure integrability. It was shown in [5] that there are additional integrals, called chop integrals, which are rational functions of the entries of L. They are constructed as follows. For $k = 0, \ldots, [\frac{(N-1)}{2}]$, denote by $(L - \lambda\,\mathrm{Id}_N)_k$ the result of removing the first k rows and the last k columns from $L - \lambda\,\mathrm{Id}_N$, and let

$$\det(L - \lambda\,\mathrm{Id}_N)_k = E_{0k}\lambda^{N-2k} + \cdots + E_{N-2k,k}\ . \tag{2}$$

Set

$$\frac{\det\,(L - \lambda\,\mathrm{Id}_N)_k}{E_{0k}} = \lambda^{N-2k} + I_{1k}\lambda^{N-2k-1} + \cdots + I_{N-2k,k}\ . \tag{3}$$

The functions I_{rk}, where $r = 1, \ldots, N-2k$ and $k = 0, \ldots, [\frac{N-1}{2}]$, are independent constants of motion, they are in involution and sufficient to account for the integrability of the full Toda lattice.

The classical Toda lattice was generalized in another direction. One can define a Toda type system for each simple Lie algebra. The finite, non–periodic Toda lattice corresponds to a root system of type A_ℓ. This generalization is due to Bogoyavlensky [3]. These systems were studied extensively in [11] where the solution of the system was connected intimately with the representation theory of simple Lie groups. See also Olshanetsky-Perelomov [13] and Adler-van Moerbeke [1]. We will call these systems the Bogoyavlensky-Toda lattices. They can be described as follows.

Let $\mathfrak{g}$ be any simple Lie algebra, equipped with its Killing form $\langle\cdot\,|\,\cdot\rangle$. One chooses a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a basis Π of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. The corresponding set of positive roots is denoted by Δ^+. To each positive root α one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in $\mathfrak{g}$ which generate a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha\in\Delta^+} \cup (H_\alpha)_{\alpha\in\Pi}$ is basis of $\mathfrak{g}$, called a root basis. To these data one associates the Lax equation $\dot{L} = [L_+, L]$, where L and L_+ are defined as follows:

$$\begin{aligned} L &= \sum_{i=1}^{\ell} b_i H_{\alpha_i} + \sum_{i=1}^{\ell} a_i (X_{\alpha_i} + X_{-\alpha_i}), \\ L_+ &= \sum_{i=1}^{\ell} a_i (X_{\alpha_i} - X_{-\alpha_i}). \end{aligned}$$

The affine space M of all elements L of $\mathfrak{g}$ of the above form is the phase space of the Bogoyavlensky-Toda lattice, associated to $\mathfrak{g}$. The functions which yield the integrability of the system are the Ad-invariant functions on $\mathfrak{g}$, restricted to M.

Let D be the diagonal $N \times N$ matrix with entries $d_i := \prod_{j=1}^{i-1} a_j$. In [11] Kostant conjugates the matrix L, given by (1), by the matrix D to obtain a matrix of the form

$$X = \begin{pmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ c_1 & b_2 & 1 & \ddots & & \vdots \\ 0 & c_2 & b_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & c_{N-1} & b_N \end{pmatrix}. \tag{4}$$

The Lax equation takes the form

$$\dot{X} = [X_+, X],$$

where X_+ is the strictly lower triangular part of X, according to the Lie algebra decomposition strictly lower plus upper triangular. This form is convenient in applying Lie theoretic techniques to describe the system. Note that the diagonal elements correspond to the Cartan subalgebra while the subdiagonal elements correspond to the set Π of simple roots. The full-Kostant Toda lattice is obtained by replacing Π with Δ^+, in the sense that one fills the lower triangular part of X in (4) with additional variables. It leads on the affine space of all such matrices to the Lax equation

$$\dot{X} = [X_+, X], \tag{5}$$

where X_+ is again the projection to the strictly lower part of X.

Generalizing the above procedure, we can introduce the following Lax pair (L_Φ, B_Φ), where Φ is any subset of Δ^+ containing Π. Thus, we have :

$$L_\Phi = \sum_{\alpha \in \Pi} b_\alpha H_\alpha + \sum_{\alpha \in \Phi} a_\alpha (X_\alpha + X_{-\alpha})$$

$$B_\Phi = \sum_{\alpha \in \Phi} a_\alpha (X_\alpha - X_{-\alpha})$$

In order to have consistency in the Lax equation, the Lax matrix being symmetric, the bracket $[B_\Phi, L_\Phi]$ should give an element of the form $\sum_{\alpha \in \Phi} c_\alpha H_\alpha + \sum_{\alpha \in \Phi} d_\alpha (X_\alpha + X_{-\alpha})$. In this case, we will say that Φ is adapted. A straightforward computation leads to the following result :

Proposition 1.1. *The set Φ is adapted if and only if it satisfies to the following property:*

$$\forall \alpha, \beta \in \Phi, \alpha - \beta \text{ or } \beta - \alpha \in \Phi \cup \{0\}$$

Recall that $\alpha - \beta = 0$ means that $\alpha - \beta$ is not a root.

Thus, for each Φ which is adapted we obtain a corresponding Hamiltonian system. Note that the special case $\Phi = \Pi$ corresponds to the classical Toda lattice while the case $\Phi = \Delta^+$ corresponds to the full symmetric Toda of [5]. There is an analogous construction for systems of the Kostant form which is the focus of this paper. With the exception of the following example we will switch to the Kostant-type systems. Even though the chain of systems under the full symmetric Toda and the chain of systems under the full Kostant-Toda are not isomorphic, the techniques used are quite similar.

Example 1.2. We consider a Lie algebra of type B_2. The set of positive roots $\Delta^+ = \{\alpha, \beta, \alpha + \beta, \beta + 2\alpha\}$ which corresponds to the full symmetric Toda lattice with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & a_4 & 0 \\ a_1 & b_2 & a_2 & 0 & -a_4 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ a_4 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & -a_4 & -a_3 & -a_1 & -b_1 \end{pmatrix}.$$

This system is completely integrable with integrals $h_2 = \frac{1}{2}\mathrm{Tr}L^2$ which is the Hamiltonian, $h_4 = \frac{1}{2}\mathrm{Tr}L^4$ and a rational integral which is obtained by the method of chopping as in [5].

Taking $\Phi = \{\alpha, \beta, \alpha+\beta\}$ we obtain another integrable system with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ 0 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & 0 & -a_3 & -a_1 & -b_1 \end{pmatrix} .$$

The matrix B is defined as above, i.e. the skew-symmetric part of L. Again there is rational integral given by

$$I_{11} = \frac{a_1 a_2 - a_3 b_2}{a_3} .$$

Defining the Poisson bracket by $\{a_1, a_2\} = a_3$, $\{a_i, b_i\} = -a_i \quad i = 1, 2$ and $\{a_1, b_2\} = a_1$ we verify easily that h_2 plays the role of the Hamiltonian and I_{11} is a Casimir. The set $\{h_2, h_4, I_{11}\}$ is an independent set of functions in involution.

2. Intermediate Toda lattices

We have defined some Hamiltonian systems associated to a subset Φ consisting of positive roots (which we call adapted). The associated matrix is symmetric. As in the case of classical and full Toda there is also an analogous system defined by a Lax matrix which is lower triangular (the Kostant-Toda lattices). In this paper we restrict our attention to this version of the systems. In this section we show that these Hamiltonian systems are associated to a nilpotent ideal of a Borel subalgebra of a semi-simple Lie algebra $\mathfrak{g}$. Since for particular (extreme) choices of the ideal one finds the classical Kostant-Toda lattice or the full Kostant-Toda lattice, associated to $\mathfrak{g}$, we call these Hamiltonian systems *Intermediate Toda lattices.*

2.1. **The phase space $M_{\mathcal{I}}$.** Throughout this section, $\mathfrak{g}$ is an arbitrary complex semi-simple Lie algebra, whose rank we denote by ℓ. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The choice of Π amounts to the choice of a Borel subalgebra $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. It also leads to a Borel subalgebra $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$, corresponding to the negative roots. We fix an element ε in $\mathfrak{n}_+$, satisfying $\langle \varepsilon \mid [\mathfrak{n}_-, \mathfrak{n}_-]\rangle = 0$, where $\langle \cdot \mid \cdot \rangle$ stands for the Killing form of $\mathfrak{g}$. One usually picks for ε a principal nilpotent element of $\mathfrak{n}_+$. For example, for $\mathfrak{g} = \mathfrak{sl}_N(\mathbf{C})$, viewed as the Lie algebra of traceless $N \times N$ matrices, one can take for $\mathfrak{h}$ and for $\mathfrak{b}_+$ the subalgebras of diagonal, respectively upper triangular matrices and for ε one can choose

$$\varepsilon := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} .$$

Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$. The quotient map $\mathfrak{b}_+ \to \mathfrak{b}_+/\mathcal{I}$ will be denoted by $P_{\mathcal{I}}$. Using the isomorphism $\mathfrak{b}_+^* \simeq \mathfrak{b}_-$ induced by the Killing form, we can think of the orthogonal $\mathcal{I}^\perp$ of $\mathcal{I}$ in $\mathfrak{b}_+^*$ as a vector subspace of $\mathfrak{b}_-$. We consider the affine space $M_{\mathcal{I}} := \varepsilon + \mathcal{I}^\perp$. Explicitly,

$$M_{\mathcal{I}} = \{X + \varepsilon \mid X \in \mathfrak{b}_- \text{ and } \langle X \mid \mathcal{I}\rangle = 0\} .$$

When $\mathcal{I} = \{0\}$, $M_{\mathcal{I}} = \mathfrak{b}_- + \varepsilon$, which is the phase space of the full Kostant-Toda lattice. On the other extreme, taking $\mathcal{I} = [\mathfrak{n}_+, \mathfrak{n}_+]$ the manifold $M_{\mathcal{I}}$ is the phase space of the classical Kostant-Toda lattice. We therefore call $M_{\mathcal{I}}$ the intermediate Kostant-Toda phase space. Notice that if $\mathcal{I} \subset \mathcal{J}$ then $M_{\mathcal{J}} \subset M_{\mathcal{I}}$.

2.2. **Hamiltonian structure.** We show that $M_{\mathcal{I}}$ has a natural Poisson structure. To do this, we prove that $M_{\mathcal{I}}$ is a Poisson submanifold of $\mathfrak{g}$, equipped with a Poisson structure $\{\cdot\,,\cdot\}$ whose construction[1] we first recall, using the theory of R-matrices (see [2, Chapter 4.4] for the general theory of R-matrices). Write $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+ := \mathfrak{b}_+$ and $\mathfrak{g}_- := \mathfrak{n}_-$. For $X \in \mathfrak{g}$, its projection in $\mathfrak{g}_\pm$ is denoted by $X_\pm$. The endomorphism $R : \mathfrak{g} \to \mathfrak{g}$, defined for all $X \in \mathfrak{g}$ by $R(X) := X_+ - X_-$ is an R-matrix, which means that the bracket on $\mathfrak{g}$, defined by

$$[X, Y]_R := \frac{1}{2}([R(X), Y] + [X, R(Y)]) = [X_+, Y_+] - [X_-, Y_-],$$

for all $X, Y \in \mathfrak{g}$, is a (new) Lie bracket on $\mathfrak{g}$. The Lie-Poisson bracket on $\mathfrak{g}$ which corresponds to $[\cdot\,,\cdot]_R$, and which we denote simply by $\{\cdot\,,\cdot\}$ (since it is the only Poisson bracket on $\mathfrak{g}$ which we will use) is given by

$$\{F, G\}(X) = \langle X \mid [(\nabla_X F)_+, (\nabla_X G)_+]\rangle - \langle X \mid [(\nabla_X F)_-, (\nabla_X G)_-]\rangle\,, \tag{6}$$

for every pair of functions F, G on $\mathfrak{g}$ and for all $X \in \mathfrak{g}$. In this formula, the gradient $\nabla_X F$ of F at X is the element of $\mathfrak{g}$, defined by

$$\langle \nabla_X F \mid Y\rangle = \langle \mathrm{d}_X F, Y\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(X + tY). \tag{7}$$

Proposition 2.1. *Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$.*

(1) *The affine space $M_{\mathcal{I}}$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot\,,\cdot\})$;*
(2) *Equipped with the induced Poisson structure, $M_{\mathcal{I}}$ is isomorphic to $(\mathfrak{b}_+/\mathcal{I})^*$, equipped with the canonical Lie-Poisson bracket;*
(3) *A function F on $M_{\mathcal{I}}$ is a Casimir function if and only if $(\nabla_X \tilde{F})_+ \in \mathcal{I}$ for all $X \in M_{\mathcal{I}}$, where $\tilde{F}$ is an arbitrary extension of F to $\mathfrak{g}$.*

Proof. First notice that the second term in the right hand side of (6) vanishes at every point X of $M_{\mathcal{I}} \subset \varepsilon + \mathfrak{b}_-$, since $\langle \varepsilon \mid [\mathfrak{n}_-, \mathfrak{n}_-]\rangle = 0$ and $\langle \mathfrak{b}_- \mid \mathfrak{n}_-\rangle = 0$. Let F be a function on $\mathfrak{g}$ which vanishes on $M_{\mathcal{I}}$. Since $\mathfrak{n}_-$ is isotropic with respect to $\langle \cdot \mid \cdot\rangle$, it follows from (7) that $\left\langle (\nabla_X F)_+ \mid Y \right\rangle = 0$ for all $Y \in \mathfrak{b}_-$ satisfying $\langle Y \mid \mathcal{I}\rangle = 0$. Therefore $(\nabla_X F)_+ \in \mathcal{I}$, so that the first term in (6) also vanishes at all points of $M_{\mathcal{I}}$ for such a function F and for all functions G on $\mathfrak{g}$. It follows that all Hamiltonian vector fields which correspond to functions which vanish on $M_{\mathcal{I}}$, also vanish on $M_{\mathcal{I}}$, so that $M_{\mathcal{I}}$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot\,,\cdot\})$, and its Poisson bracket, still denoted by $\{\cdot\,,\cdot\}$, is given, for $F, G \in \mathcal{F}(M_{\mathcal{I}})$ at $X \in M_{\mathcal{I}}$, by

$$\{F, G\}(X) = \left\langle X \mid \left[P_{\mathcal{I}}(\nabla_X \tilde{F})_+, P_{\mathcal{I}}(\nabla_X \tilde{G})_+\right]\right\rangle, \tag{8}$$

where $\tilde{F}$ (resp. $\tilde{G}$) is an arbitrary element of $\mathcal{F}(\mathfrak{g})$, whose restriction to $M_{\mathcal{I}}$ is F (resp. G). Notice that the projections $P_{\mathcal{I}}$ in (8) are optional since $\langle X \mid \mathcal{I}\rangle = 0$ for all $X \in M_{\mathcal{I}}$. From (8), as it is written, it is clear that the map $M_{\mathcal{I}} \to (\mathfrak{b}_+/\mathcal{I})^*$, which sends $X + \varepsilon$ to $\langle X \mid \cdot\rangle$, viewed as a linear form on $\mathfrak{b}_+/\mathcal{I}$, is an isomorphism of Poisson manifolds, where $M_{\mathcal{I}}$ is equipped with the bracket (8), and $(\mathfrak{b}_+/\mathcal{I})^*$ is equipped with the canonical Lie-Poisson brackets. We also read off from (8) that a function F on $M_{\mathcal{I}}$ whose extension $\tilde{F}$ to $\mathfrak{g}$ satisfies $(\nabla_X \tilde{F})_+ \in \mathcal{I}$ for all $X \in M_{\mathcal{I}}$ is a Casimir function. In fact, this condition characterizes Casimir functions since the center of $\mathfrak{b}_+/\mathcal{I}$ is trivial, a consequence of the fact that $\mathcal{I} \subset \mathfrak{n}_+$ (see subsection 2.3 below). □

For a function H on $M_{\mathcal{I}}$, we denote its Hamiltonian vector field by $\mathcal{X}_H$; our sign convention is that $\mathcal{X}_H := \{\cdot\,, H\}$, so that $\mathcal{X}_H[F] = \{F, H\}$ for all $F \in \mathcal{F}(M)$. The Hamiltonian of the intermediate Kostant-Toda lattice is the polynomial function on $M_{\mathcal{I}}$, given by

$$H := \frac{1}{2}\langle X \mid X\rangle\,, \tag{9}$$

[1] See the appendix of [7] for an alternative construction, using symplectic reduction to the cotangent bundle $T^*\mathbf{G}$, where $\mathbf{G}$ is any Lie group integrating $\mathfrak{g}$.

so that the vector field of the intermediate Kostant-Toda lattice is given by the Lax equation (on $M_{\mathcal{I}}$)

$$\dot{X} = [X_+, X]\,. \tag{10}$$

2.3. **Nilpotent ideals in $\mathfrak{b}^+$.** We now summarize some facts about nilpotent ideals in $\mathfrak{b}^+$ (see [4]).

If $\mathcal{I}$ is a nilpotent ideal of $\mathfrak{b}_+$, then $\mathcal{I}$ is contained in $\mathfrak{n}_+$. For example, $\mathfrak{n}_+$ itself is a nilpotent ideal of $\mathfrak{b}_+$. For $\alpha \in \Delta^+$, let $X_{\mathfrak{a}}$ denote an arbitrary root vector, corresponding to α, i.e., $[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha$, for all $H \in \mathfrak{h}$. Consider a subset Φ of Δ^+, which has the property that if $\alpha \in \Phi$ then every root of the form $\alpha + \beta$, with $\beta \in \Pi^+$, belongs to Φ; we call such a set Φ an admissible set of roots. For such α and β, the Jacobi identity implies that $[X_\alpha, X_\beta]$ is a multiple of $X_{\alpha+\beta}$. It follows that the (vector space) span of $\{X_\alpha \mid \alpha \in \Phi\}$ is a nilpotent ideal of $\mathfrak{b}_+$. Most importantly, every nilpotent ideal of $\mathfrak{b}_+$ is of this form, for a certain admissible set of roots Φ. Thus, the nilpotent ideals of a given Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ are parametrized by the family of all subsets Φ of Δ^+, which have the property that if $\alpha \in \Phi$ then every root of the form $\alpha + \beta$, with $\beta \in \Pi^+$, belongs to Φ.

To give an idea of the number of intermediate Toda systems, we list the cardinality of nilpotent ideals for each complex simple Lie algebra. This number is given table 2.3. In the table

$$\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$$

is the Catalan number.

Lie Algebra	Number of Positive Roots	Number of Ideals
A_n	$\binom{n+1}{2}$	C_{n+1}
B_n, C_n	n^2	$\binom{2n}{n}$
D_n	$n^2 - n$	$(n+1)\mathcal{C}_n - nC_{n-1}$
G_2	6	8
F_4	24	105
E_6	36	832
E_7	63	4160
E_8	120	25080

It is interesting that there is a uniform formula for counting nilpotent ideals of $\mathfrak{b}^+$. It is given by

$$\frac{1}{|W|}\prod_{i=1}^{\ell}(h + m_i + 1) = \prod_{i=1}^{\ell}\frac{(h + m_i + 1)}{m_i + 1}$$

where W is the Weyl group, h is the Coxeter number and m_i are the exponents.

2.4. **Height k Kostant-Toda lattices.** We will in the sequel of this paper mainly study the case when $\mathcal{I}$ is an ideal of height 2, a notion which we introduce in this paragraph. Every positive root $\alpha \in \Delta^+$ can be written as a linear combination of the simple roots, $\alpha = \sum_{i=1}^{\ell} n_i\alpha_i$, where all n_i are non-negative integers. The integer $ht(\alpha) := \sum_{i=1}^{\ell} n_i$ is called the *height* of α. For $k \in \mathcal{N}$, let Φ_k denote the set of all roots of height larger than k. It is clear that Φ_k is an admissible set of roots. We denote the corresponding ideal of $\mathfrak{b}_+$ by $\mathcal{I}_k$ and we call it a height k ideal. An alternative description of $\mathcal{I}_k$ is as $\mathrm{ad}^k_{\mathfrak{n}_+}\mathfrak{n}_+$. For $k = 1$, $\mathcal{I}_1 = [\mathfrak{n}_+, \mathfrak{n}_+]$ is the ideal which leads to the classical Toda lattice. We will consider in the sequel mainly $\mathcal{I}_2 = [\mathfrak{n}_+, [\mathfrak{n}_+, \mathfrak{n}_+]]$ and the corresponding affine space $M_{\mathcal{I}_2}$.

Example 2.2. Consider a Lie algebra of type C_4. Take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4\}$.

The Lax matrix is

$$L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_3 & b_4 & -a_4 & -1 & 0 & 0 \\ 0 & 0 & 0 & c_3 & -b_3 & -a_3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & -b_2 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -c_1 & -b_1 & -a_1 \end{pmatrix}.$$

The function

$$a_1 - a_2 + a_3 - a_4 + \frac{2b_1b_2c_3 + b_1c_2b_4 + b_3b_4c_1}{c_1c_3}$$

is a Casimir. We need five functions to establish integrability. Since $det(L - \lambda I)$ is an even polynomial of the form

$$\lambda^8 + \sum_{i=0}^{3} f_i \lambda^{2i}$$

we obtain four polynomial integrals. Using an one-chop we obtain a characteristic polynomial of the form $A\lambda^2 + B$. The function $f_5 = B/A$ is the fifth integral.

3. Computation of the rank

In this section, we compute the index of the Lie algebra $\mathfrak{b}_+/\mathcal{I}_2$, when $\mathfrak{b}_+$ is a Borel subalgebra of a simple Lie algebra of type A_ℓ, B_ℓ or C_ℓ. It yields the rank of the corresponding intermediate Kostant-Toda phase space (see paragraph 2.4). We first recall a few basic facts about stable linear forms, the index of a Lie algebra and the relation to the rank of the corresponding Lie-Poisson structure.

3.1. **Stable linear forms.** Let $\mathfrak{a}$ be any complex algebraic Lie algebra, $\mathfrak{a}^*$ its dual vector space. The *stabilizer* of a linear form $\varphi \in \mathfrak{a}^*$ is given by

$$\mathfrak{a}^\varphi := \{x \in \mathfrak{a} \mid \mathrm{ad}_x^* \varphi = 0\} = \{x \in \mathfrak{a} \mid \forall y \in \mathfrak{a},\ \langle \varphi, [x, y]\rangle = 0\}.$$

The integer $\min\{\dim \mathfrak{a}^\varphi \mid \varphi \in \mathfrak{a}^*\}$ is called the *index* of $\mathfrak{a}$ and is denoted by $\mathrm{ind}(\mathfrak{a})$. Since the symplectic leaves of the canonical Lie-Poisson structure on $\mathfrak{a}^*$ are the coadjoint orbits, the codimension of the symplectic leaf through φ is the dimension of $\mathfrak{a}^\varphi$. It follows that the index of $\mathfrak{a}$ is the codimension of a symplectic leaf of maximal dimension, i.e., the rank of the canonical Lie-Poisson structure on $\mathfrak{a}^*$ is given by $\dim \mathfrak{a} - \mathrm{ind}(\mathfrak{a})$; notice that since the latter rank is always even, the index of $\mathfrak{a}$ and the dimension of $\mathfrak{a}$ have the same parity. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *regular* if $\dim \mathfrak{a}^\varphi = \mathrm{ind}(\mathfrak{a})$; thus, we can use regular linear forms to compute the index of $\mathfrak{a}$, and hence the rank of the canonical Lie-Poisson structure on $\mathfrak{a}^*$.

We will use the following proposition to compute the index of $\mathfrak{b}_+/\mathcal{I}_2$.

Proposition 3.1. *Let $\mathfrak{a}$ be a subalgebra of a semi-simple complex Lie algebra $\mathfrak{g}$. Suppose that φ is a linear form on $\mathfrak{a}$, such that $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semi-simple elements. Then φ is regular, so that the index of $\mathfrak{a}$ is given by* $\dim \mathfrak{a}^\varphi$.

Proof. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *stable* if there exists a neighborhood U of φ in $\mathfrak{a}^*$ such that for every $\psi \in U$, the stabilizer $\mathfrak{a}^\psi$ is conjugate to $\mathfrak{a}^\varphi$, with respect to the adjoint group of $\mathfrak{a}$. According to [9], every stable linear form is regular. According to ([8], [9, thm 1.7, cor. 1.8]), φ is stable if and only if $[\mathfrak{a}, \mathfrak{a}^\varphi] \cap \mathfrak{a}^\varphi = \{0\}$. The latter equality holds when $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semi-simple elements (see [9, Lemma 2.6]). Thus, φ is stable, hence regular. □

3.2. **Computation of the index.** In this paragraph we compute the index of $\mathfrak{b}/\mathcal{I}$ under the following assumption on (the root system of) $\mathfrak{g}$:

(H) *The roots of height 2 of* $\mathfrak{g}$ *are given by* $\{\alpha_k+\alpha_{k+1} \mid 1 \leqslant k \leqslant \ell-1\}$

For classical Lie algebras, the basis Π can be ordered such that this assumption occurs when $\mathfrak{g}$ is of type A_ℓ, B_ℓ or C_ℓ. Let $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta^+}(\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha})$ be the decomposition of $\mathfrak{g}$ according to the adjoint action of $\mathfrak{h}$. To each positive root α corresponds a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of elements of $\mathfrak{g}$, where $X_\alpha\in\mathfrak{g}_\alpha, X_{-\alpha}\in\mathfrak{g}_{-\alpha}, H_\alpha\in\mathfrak{h}$ and $(X_\alpha, X_{-\alpha}, H_\alpha)$ generates a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. Let us recall shortly how such a triple can be constructed. Let h_α be the unique element in $\mathfrak{h}$ such that $\langle\alpha, H\rangle=\langle h_\alpha \mid H\rangle$ for all $H\in\mathfrak{h}$. Define a scalar product on the real vector space $\mathfrak{h}^*_{\mathbf{R}}$ by

$$\langle\alpha\,|\,\beta\rangle := \langle h_\alpha\,|\,h_\beta\rangle = \langle\beta, h_\alpha\rangle = \langle\alpha, h_\beta\rangle\,,$$

for all $\alpha,\beta\in\Delta$. We set

$$H_\alpha := \frac{2}{\langle\alpha\,|\,\alpha\rangle}h_\alpha.$$

It is clear that $\langle\alpha, H_\alpha\rangle=2$. Choose $X_\alpha\in\mathfrak{g}_\alpha, X_{-\alpha}\in\mathfrak{g}_{-\alpha}$ such that $\langle X_\alpha\,|\,X_{-\alpha}\rangle=\frac{2}{\langle\alpha\,|\,\alpha\rangle}$. Then, $(X_\alpha, X_{-\alpha}, H_\alpha)$ is the required triple. Moreover,

$$\left[X_{\pm\alpha_k}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right]=\epsilon_k^{\pm}X_{\mp\alpha_{k+1}}, \qquad \left[X_{\pm\alpha_{k+1}}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right]=\eta_k^{\pm}X_{\mp\alpha_k},$$

where each of the integers $\epsilon_k^{\pm}$ and $\eta_k^{\pm}$ is equal to 1 or to -1, depending on $\mathfrak{g}$.

For all $\alpha,\beta\in\Pi$, let $C_{\alpha\beta}:=\langle\beta, H_\alpha\rangle=2\dfrac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}$. The $\ell\times\ell$-matrix $C:=(C_{ij}, 1\leqslant i,j\leqslant\ell)$, where $C_{ij}:=C_{\alpha_i\alpha_j}$, is invertible. It is called the *Cartan matrix* of $\mathfrak{g}$.

Proposition 3.2. *Consider the linear form* φ *on* $\mathfrak{b}_+$*, defined for* $Z\in\mathfrak{b}_+$ *by* $\langle\varphi, Z\rangle:=\langle X\,|\,Z\rangle$*, where* X *is defined by*

$$X := \delta_\ell X_{-\alpha_\ell}+\sum_{i=1}^{\ell-1}X_{-\alpha_i-\alpha_{i+1}}, \tag{11}$$

with $\delta_\ell:=1$ *if* ℓ *is odd and* $\delta_\ell:=0$ *otherwise. Denote by* $\bar\varphi$ *the induced linear form on* $\mathfrak{b}_+/\mathcal{I}_2$.

(1) $\bar\varphi$ *is a regular linear form on* $\mathfrak{b}_+/\mathcal{I}_2$*;*
(2) $\dim(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi}=1-\delta_\ell$*;*
(3) *The index of* $\mathfrak{b}_+/\mathcal{I}_2$ *is 1 if the rank* ℓ *of* $\mathfrak{g}$ *is even and is 0 otherwise.*

Proof. The proof of (3) follows at once from (1) and (2). We prove (1) and (2) at the same time by determining $(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi}$. Notice that φ vanishes on $\mathcal{I}_2$, so that φ induces indeed a linear form $\bar\varphi$ on $\mathfrak{b}_+/\mathcal{I}_2$, as asserted. We compute its stabilizer

$$\begin{aligned}(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi} &= \{\bar Y\in\mathfrak{b}_+/\mathcal{I}_2 \mid \langle X\,|\,[Y,Z]\rangle=0,\ \text{for all } Z\in\mathfrak{b}_+\}\\ &= \{\bar Y\in\mathfrak{b}_+/\mathcal{I}_2 \mid [X,Y]\in\mathfrak{n}_+\}.\end{aligned}$$

Let $\bar Y\in\mathfrak{b}_+/\mathcal{I}_2$. In view of the assumption (H), any representative $Y\in\mathfrak{b}_+$ of $\bar Y$ can be written as

$$Y=\sum_{i=1}^{\ell}a_iH_{\alpha_i}+\sum_{i=1}^{\ell}b_iX_{\alpha_i}+\sum_{i=1}^{\ell-1}c_iX_{\alpha_i+\alpha_{i+1}}.$$

By a direct computation,

$$\begin{aligned}[X,Y] &= \sum_{i=1}^{\ell} a_i C_{i1}\delta_\ell X_{-\alpha_1} - b_1\delta_\ell H_{\alpha_1} + \epsilon_1^- c_1 \delta_\ell X_{\alpha_2}\\ &\quad + \sum_{i=1}^{\ell-1}\sum_{k=1}^{\ell} a_k (C_{ki} + C_{k,i+1}) X_{-\alpha_i-\alpha_{i+1}}\\ &\quad - \sum_{i=1}^{\ell-1} (b_{i+1}\eta_i^+ X_{-\alpha_i} + b_i \epsilon_i^+ X_{-\alpha_{i+1}}) - \sum_{k=1}^{\ell-1} c_k (H_{\alpha_k} + H_{\alpha_{k+1}}).\end{aligned}$$

It follows that $[X,Y]\in\mathfrak{n}_+$ if and only if the coefficients satisfy the following system of linear equations:

$$\begin{aligned}&c_1 = \cdots = c_{\ell-1} = 0,\\ &b_{\ell-1} = b_1\delta_\ell = 0,\\ &\eta_k^+ b_{k+1} + \epsilon_{k-1}^+ b_{k-1} = 0 \text{ for } k = 2,\dots,\ell-1,\\ &\sum_{i=1}^{\ell} a_i (C_{ik} + C_{i,k+1}) = 0 \text{ for } k = 1,\dots,\ell-1,\\ &\delta_\ell \sum_{i=1}^{\ell} a_i C_{i1} - \eta_1^+ b_2 = 0.\end{aligned}$$

Since all ϵ_k^+ and all η_k^+ are different from zero, the first three equations imply that all b_k and all c_k are equal to zero (recall that $\delta_\ell := 1$ if ℓ is odd and $\delta_\ell := 0$ otherwise). This shows that if $[X,Y]\in\mathfrak{n}_+$ then $Y\in\mathfrak{h}$; in particular $(\mathfrak{b}_+/\mathcal{I}_2)^\varphi \subset \mathfrak{h}$. Since $\mathfrak{h}$ is Abelian, we may conclude from Proposition 3.1 that φ is regular, which is the content of (2). We continue solving the above equations in order to determine the dimension of $(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi}$, which by the above amounts to compute the dimension of the solution space of the linear system

$$\delta_\ell \sum_{k=1}^{\ell} a_k C_{k1} = 0, \qquad \sum_{k=1}^{\ell} a_k (C_{ki} + C_{k,i+1}) = 0, \text{ for } i = 1,\dots,\ell-1, \tag{12}$$

where we recall that the C_{ij} are the entries of the Cartan matrix of $\mathfrak{g}$. When ℓ is odd (so that $\delta_\ell = 1$) this system is clearly equivalent to

$$\sum_{k=1}^{\ell} a_k C_{ki} = 0, \text{ for } i = 1,\dots,\ell, \tag{13}$$

which means that $(a_1,\dots,a_\ell)$ belongs to the null space of the Cartan matrix. Since the Cartan matrix is non-degenerate, only the trivial solution remains. In that case $\dim(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi} = 0$. When ℓ is even, the homogeneous linear systems (12) and (13) differ by one equation and the latter has no non-trivial solutions, so the solution space of (12) is at most one-dimensional. Since $\dim \mathfrak{b}_+/\mathcal{I}_2 = 3\ell - 1$ is odd, then index of $\mathfrak{b}_+/\mathcal{I}_2$ is odd, hence it is equal to 1. This proves (3). □

4. Invariants

We now examine the integrability of the intermediate Kostant-Toda lattice on $M_{\mathcal{I}_2} \subset \mathfrak{g}$, for any semi-simple Lie algebra $\mathfrak{g}$ of type A_ℓ, B_ℓ or C_ℓ. We also make some remarks for the case of D_ℓ. More specifically we show that the number of functions needed for integrability is correct. However, we will not deal with the independence of these functions which is a more complicated issue. Recall that integrability means that the Hamiltonian is part of a family of

s independent functions in involution, where s is related to the dimension and the rank of the Poisson manifold $M_{\mathcal{I}_2}$ by the formula

$$\dim M_{\mathcal{I}_2} = \frac{1}{2}\mathrm{Rk}\, M_{\mathcal{I}_2} + s.$$

Since $\dim M_{\mathcal{I}_2} = 3\ell - 1$ and since the corank of $M_{\mathcal{I}_2}$ is 1 if ℓ is even and 0 otherwise (see item (3) in proposition 3.2), we need $s = [3\ell/2]$ such functions. According to the Adler Kostant Symes Theorem, the ℓ basic Ad-invariant polynomials provide already ℓ independent functions in involution. Thus, one needs $[\ell/2]$ extra ones. As we will see, they can be constructed by restricting certain chop-type integrals, except for the case of C_ℓ where another integral (Casimir) is needed. We first recall from [5] the construction of the chop integrals on $M := \varepsilon + \mathfrak{b}_-$ in the case of $\mathfrak{g} = \mathfrak{sl}_N(\mathbf{C})$ and explain why they are in involution. Since $M_{\mathcal{I}_2}$ is a Poisson manifold of M, their restrictions to $M_{\mathcal{I}_2}$ are still in involution (but they may become trivial or dependent).

We consider $\mathfrak{g} = \mathfrak{sl}_N(\mathbf{C})$ with the standard choice of $\mathfrak{h}$ and Π (see paragraph 2.1). Let k be an integer, $0 \leqslant k \leqslant [\frac{N-1}{2}]$. For any matrix X, we denote by X_k the matrix obtained by removing the first k rows and last k columns from X. We denote by $\mathbf{G}_k$ the subgroup of $\mathbf{G} := \mathbf{GL}_N(\mathbf{C})$, consisting of all $N \times N$ invertible matrices of the form

$$g = \begin{pmatrix} \Delta & A & B \\ 0 & D & C \\ 0 & 0 & \Delta' \end{pmatrix}, \tag{14}$$

where Δ and Δ' are arbitrary upper triangular matrices of size $k \times k$ and A, B, C, D are arbitrary[2]. The Lie algebra of $\mathbf{G}_k$ is denoted by $\mathfrak{g}_k$. A first fundamental, non-trivial observation, due to [5], is that for all $g \in \mathbf{G}_k$, decomposed as in (14),

$$\det\left(gXg^{-1}\right)_k = \frac{\det \Delta'}{\det \Delta} \det X_k. \tag{15}$$

This leads to (rational) $\mathbf{G}_k$-invariant functions on $\mathfrak{g}$ (and hence on M), constructed as follows. For $X \in \mathfrak{g}$ and for an arbitrary scalar λ, consider the so-called k-chop polynomial of X, defined by

$$Q_k(X, \lambda) := \det(X - \lambda\, \mathrm{Id}_N)_k.$$

In view of (15), the coefficients of Q_k (as a polynomial in λ) define polynomial functions on $\mathfrak{g}$, which transform under the action of $g \in \mathbf{G}_k$, with the same factor $\det \Delta' / \det \Delta$. Writing

$$Q_k(X, \lambda) = \sum_{i=0}^{N-2k} E_{i,k}(X) \lambda^{N-2k-i},$$

each of the rational functions $E_{i,k}/E_{j,k}$ is $\mathbf{G}_k$-invariant. By restriction to M, this yields $\mathbf{G}_k$-invariant elements of $\mathcal{F}(M)$. They are called k-chop integrals because they are integrals (constants of motion) for the full Kostant-Toda lattice. Notice that the constants of motion $H_i := \frac{1}{i}\,\mathrm{Trace}\, X^i$ are 0-chop integrals and that the Toda Hamiltonian is expressible in terms of them as $H = \frac{1}{2}(H_1^2 - 2H_2)$.

We show that all chop integrals are in involution. To do this, let F be a k-chop integral and let $\tilde{F}$ denote its extension to a $\mathbf{G}_k$-invariant rational function on $\mathfrak{g}$. Similarly, let G be a l-chop integral, with $\mathbf{G}_\ell$-invariant extension $\tilde{G}$. We may suppose that $k \leqslant \ell$. Infinitesimally, the fact that $\tilde{F}$ is $\mathbf{G}_k$ invariant yields that

$$\left\langle X \mid \left[\nabla_X \tilde{F}, Y\right]\right\rangle = 0, \tag{16}$$

for all $X \in \mathfrak{g}$ and for all $Y \in \mathfrak{g}_k$. Since $\mathfrak{b}_+ \subset \mathfrak{g}_k$, it follows that

$$\left\langle X \mid \left[(\nabla_X \tilde{F})_+, \nabla_X \tilde{G}\right]\right\rangle = 0 = \left\langle X \mid \left[\nabla_X \tilde{F}, (\nabla_X \tilde{G})_+\right]\right\rangle$$

[2]With the understanding that, since X is supposed invertible, Δ, Δ' and D are invertible.

so that (6) can be rewritten, for $X \in M$, as

$$\{F, G\}(X) = -\left\langle X \,|\, \left[\nabla_X \tilde{F}, \nabla_X \tilde{G}\right]\right\rangle. \tag{17}$$

We claim that $\nabla_X \tilde{G} \in \mathfrak{g}_\ell$. This follows from the construction of the function $\tilde{G} \in \mathcal{F}(\mathfrak{g})$: the rational function $\tilde{G}(X)$ depends only on X_ℓ, the ℓ-chop of X, while if an element Z of $\mathfrak{g}$ satisfies $\langle \mathfrak{g}_\ell \,|\, Z\rangle = 0$, then Z_ℓ is the zero matrix. Thus, $\nabla_X \tilde{G} \in \mathfrak{g}_\ell \subset \mathfrak{g}_k$, so that (16) implies that the right hand side of (17) is zero, for all $X \in M$. It follows that F and G Poisson commute.

Notice that in the case of the height 2 intermediate Kostant-Toda lattice all k-chops with $k > 1$ vanish and that only a few 1-chops survive. In what follows we consider separately the cases of various classical Lie algebras.

4.1. **The case of A_ℓ.** We first consider $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbf{C})$, the Lie algebra of traceless matrices of size $N = \ell + 1$, taking for $\mathfrak{h}$, Π and ε the standard choices, as before. A general element of $\mathcal{M}_{\mathcal{I}_2}$ is then of the form

$$X = \begin{pmatrix} a_1 & 1 & 0 & \dots & \dots & 0 \\ b_1 & a_2 & 1 & \ddots & & \vdots \\ c_1 & b_2 & a_3 & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & c_{\ell-1} & b_\ell & a_{\ell+1} \end{pmatrix},$$

with $\sum_{i=1}^{\ell+1} a_i = 0$. The 1-chop matrix of X is given by

$$(X - \lambda\,\mathrm{Id}_{\ell+1})_1 = \begin{pmatrix} b_1 & a_2^\lambda & 1 & 0 & \dots & 0 \\ c_1 & b_2 & a_3^\lambda & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & c_3 & b_4 & \ddots & 1 \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix},$$

where a_i^λ is a shorthand for $a_i - \lambda$. We also use the matrix $X(\lambda, \alpha)$, defined by

$$X(\lambda, \alpha) = \begin{pmatrix} b_1 & a_2^\lambda & \alpha_{13} & \dots & \dots & \alpha_{1\ell} \\ c_1 & b_2 & a_3^\lambda & \alpha_{24} & & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_3 & b_4 & \ddots & \alpha_{\ell-2,\ell} \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix}.$$

We define functions $\beta_k(\alpha)$ by setting $\det X(\lambda, \alpha) = \sum_{k=0}^{d} \beta_k(\alpha)\lambda^k$.

Lemma 4.1.

(1) *The polynomials* $\det(X - \lambda\,\mathrm{Id}_{\ell+1})_1$ *and* $\det X(\lambda, \alpha)$ *have degree* $d := [\frac{\ell}{2}]$ *in* λ*;*
(2) *When ℓ is even (resp. odd), the coefficients β_{d-1} and β_d (resp. β_d) are independent of the variables α.*

Proof. To each variable which occurs in the determinant of $X(\lambda, \alpha)$, we associate a weight according to the following rule: each c_i is of weight -1, each b_i is of weight 0, each a_i^λ is of weight 1, each α_{kj} is of weight $k-j \geqslant 2$. Using the usual expansion formula we decompose $\det X(\lambda,\alpha)$ as a sum of terms of weight 0, each of them composed of ℓ factors.

Suppose that a term of this determinant contains at least $[\frac{\ell}{2}]+1$ factors a_i^λ. Then there are at most $[\frac{\ell-1}{2}]$ other factors in this term and their global weight should be at most $-[\frac{\ell}{2}]-1$, which is impossible since every variable has weight at least -1. It follows that the degree of $\det X(\lambda,\alpha)$, and hence the degree of $\det(X-\lambda\,\mathrm{Id}_{\ell+1})_1$ is at most $d:=[\frac{\ell}{2}]$.

The coefficient β_d defined initem (2) above is a sum of terms of the form $\gamma_1.\gamma_2\dots\gamma_{\ell-d}$, of total weight $-d$. If ℓ is even then $\ell-d=d$, so that each of the γ_i has weight -1, i.e., is (proportional to) one of the variables c_j. If ℓ is odd then $\ell-d=d+1$, so that one of the γ_i has weight 0 (is proportional to a b_j) and all other ones have weight -1 (are proportional to one of the variables c_j). In particular, the coefficient β_d is in both cases non-zero and it is independent of the variables α. This shows *(1)* and part of *(2)*.

Suppose now that ℓ is even, so that $\ell=2d$. Every term in β_{d-1} is of the form $\gamma_1'.\gamma_2'\dots\gamma_{d+1}'$ and is of total weight $1-d$. Then there are two possibilites. Either two of the γ_i' have weight 0 (are proportional to a b_j) and all other ones have weight -1 (are proportional to one of the variables c_j) or one of the γ_i' has weight 1 (is proportional to one of the variables a_j) and all other ones have weight -1 (are proportional to a c_j). In particular, the coefficient β_{d-1} is independent of the variables α. Notice that if ℓ is odd (and at least equal to 3) then β_{d-1} does depend on the variables α. $\square$

If follows from the lemma that we have precisely the correct number of functions in involution which are necessary for Liouville integrability in the case of A_ℓ. Elaborating on the weights of the variables, defined in the proof of item (2) of the lemma, it can be shown that these functions are independent, thereby proving Liouville integrability.

4.2. **The case of B_ℓ.** A Lie algebra of type B_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N:=2\ell+1$, satisfying $XJ+JX^t=0$, where J is the matrix of size $2\ell+1$, all of whose entries are zero, except for the entries on the anti-diagonal, which are all equal to one. Clearly, X satisfies $XJ+JX^t=0$ if and if X is skew-symmetric with respect to its anti-diagonal. It follows for such X that $\det(X-\lambda\,\mathrm{Id}_{\ell+1})=(-1)^N\det(X+\lambda\,\mathrm{Id}_{\ell+1})$, so that the characteristic polynomial is an odd polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1J+JX_1^t=0$, so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Π^+ the matrices $E_{i,i+1}-E_{2\ell-i,2\ell-i+1}$, for $i=1,\dots,\ell$. If one finally chooses ε to be the matrix $\sum_{i=1}^{\ell}(E_{i,i+1}-E_{2\ell-i,2\ell-i+1})$, then the height 2 phase space is given by all matrices of the form

$$\begin{pmatrix}
a_1 & 1 & & & & & & & & \\
b_1 & \ddots & \ddots & & & & & & & \\
c_1 & \ddots & \ddots & 1 & & & & & & \\
 & \ddots & b_{\ell-1} & a_\ell & 1 & & & & & \\
 & & c_{\ell-1} & b_\ell & 0 & -1 & & & & \\
 & & & 0 & -b_\ell & -a_\ell & \ddots & & & \\
 & & & & -c_{\ell-1} & -b_{\ell-1} & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & -c_1 & -b_1 & -a_1 &
\end{pmatrix}$$

In this case $N=2\ell+1$, the 1-chop polynomial is even, so the 1-chop polynomial is degree ℓ when ℓ is even and of degree $\ell-1$ when ℓ is odd. This yields $\frac{\ell}{2}$ integrals when ℓ is even and $\frac{\ell-1}{2}$ when ℓ is odd. Therefore the number of integrals in involution is again precisely the number needed for Liouville integrability.

4.3. **The case of C_ℓ.** A Lie algebra of type C_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N := 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} .$$

It follows for such X that $\det(X - \lambda \operatorname{Id}_{\ell+1}) = (-1)^{2\ell} \det(X + \lambda \operatorname{Id}_{\ell+1})$, so that the characteristic polynomial is an even polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1 J + J X_1^t = 0$, so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Π^+ the matrices $E_{i,i+1} - E_{2\ell-1-i,2\ell-i}$, for $i = 1, \dots, \ell$. The height 2 phase space for C_ℓ is given by all matrices of the form

$$\begin{pmatrix}
a_1 & 1 & & & & & & & \\
b_1 & a_2 & \ddots & & & & & & \\
c_1 & b_2 & \ddots & 1 & & & & & \\
 & \ddots & \ddots & a_\ell & 1 & & & & \\
 & & c_{\ell-1} & b_\ell & -a_\ell & -1 & & & \\
 & & & c_{\ell-1} & -b_{\ell-1} & \ddots & \ddots & & \\
 & & & & -c_{\ell-2} & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & & -c_1 & -b_1 & -a_1
\end{pmatrix} .$$

In this case $N = 2\ell$, the 1-chop polynomial is even, so we get $\frac{\ell}{2} - 1$ integrals from the 1-chop when ℓ is even and $\frac{\ell-2}{2}$ integrals when ℓ is odd. Therefore, when ℓ is odd, we have the correct number of integrals in involution for Liouville integrability. When ℓ is even, we need an extra function, but then there exists a Casimir function f which does not arise from the method of chopping, and it does the job. We construct f as follows:

$$f := A + \frac{B}{C} ,$$

where

$$A := \sum_{i=1}^{\frac{\ell}{2}} (a_{2i-1} - a_{2i}) , \quad B := \sum_{i,j} d_{ij} m_{ij} \quad \text{and} \quad C := \prod_{i=1}^{\ell-1} c_{2i-1} .$$

The terms $d_{ij} m_{ij}$ of B are defined as follows: we associate variables $b_1, b_2, \dots, b_\ell$ to the simple roots $\alpha_1, \alpha_2, \dots, \alpha_\ell$ and associate variables $c_1, c_2, \dots, c_{\ell-1}$ to the height 2 roots $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots, \alpha_{\ell-1} + \alpha_\ell$. Take simple roots α_i, α_j (with corresponding variables b_i, b_j) such that i is odd and j is even. The remaining variables correspond to the height two roots $\alpha_k + \alpha_{k+1}$ where $k \neq i, i-1$ and $k \neq j, j-1$. The term m_{ij} is a product of b_i, b_j and $\frac{l-1}{2}$ c variables. The coefficient d_{ij} is 2 if m_{ij} includes the term $c_{\ell-1}$ (corresponding to the root $\alpha_{\ell-1} + \alpha_\ell$), and is equal to 1 otherwise. The proof that this formula produces a Casimir is a straightforward (but long) calculation which we omit.

Example 4.2. For $\ell = 6$, the Casimir f is explicitly given by

$$f = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \frac{b_5 b_6 c_1 c_3 + 2 b_1 b_4 c_2 c_5 + b_3 b_6 c_1 c_4 + 2 b_1 b_2 c_3 c_5 + 2 b_3 b_4 c_1 c_5 + b_1 b_6 c_2 c_4}{c_1 c_3 c_5} .$$

4.4. **The case of D_ℓ.** A Lie algebra of type D_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} .$$

As in the case of C_ℓ the characteristic polynomial is an even polynomial. On the other hand the 1-chop polynomial is odd so the degree of this polynomial is $\ell - 1$ when ℓ is even. But when ℓ is odd the degree of the 1-chop polynomial is again ℓ. This gives $\frac{\ell}{2} - 1$ integrals when ℓ is even and $\frac{\ell-1}{2}$ integrals in involution when ℓ is odd. In the even case we need an extra function i.e. a Casimir but at this point we do not have an explicit formula for it. There is no stable form in this case, but we can produce a form which gives a lower bound for the rank and this lower bound is good enough, once we have the Casimir. However, if we restrict (in the even case) the system to a generic leaf of the symplectic foliation, we have enough integrals in involution required by Liouville integrability. Clearly, an explicit description of the symplectic foliation is tantamount to an explicit description of the Casimir.

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