Liouville integrability and superintegrability of a generalized Lotka-Volterra system and its Kahan discretization

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ABSTRACT. We prove the Liouville and superintegrability of a generalized Lotka-Volterra system and its Kahan discretization.

Contents

1. Introduction

The Kac-van Moerbeke system is a prime example of an integrable system, described by the differential equations

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}), \qquad (i = 1, \ldots, n), \tag{1.1}$$

where $x_0 = x_{n+1} = 0$. It was first introduced and studied, together with some of its generalizations, by Lotka to model oscillating chemical reactions and by Volterra to describe population evolution in a hierarchical system of competing species (see

[**12, 18**]). By now, many generalizations of (1.1) have been introduced and studied, often from the point of (Liouville or algebraic) integrability [**2, 8, 9**] or Lie theory [**2, 5**], but also in relation with other integrable systems [**7, 13, 17**]. In our recent study [**17**], a natural generalization of (1.1) came up in the study of a class of multi-sums of products: we considered the system

$$\dot{x}_i = x_i \left(\sum_{j>i} x_j - \sum_{j<i} x_j \right) , \qquad (i = 1, \dots, n) , \tag{1.2}$$

we showed its Liouville and superintegrability and we used it to show the Liouville and superintegrability (or non-commutative integrability) of the Hamiltonian system defined by the above-mentionned class of functions. The system (1.2) has a Hamiltonian structure, described by the Hamiltonian function and Poisson structure, which are respectively given by

$$H = \sum_{i=1}^n x_i , \qquad \{x_i, x_j\} = x_i x_j , \quad (i < j) . \tag{1.3}$$

We consider in the present paper the case of a general linear Hamiltonian

$$H = \sum_{i=1}^n a_i x_i , \tag{1.4}$$

with the Poisson structure still given by (1.3). The differential equations which describe this Hamiltonian system are given by

$$\dot{x}_i = x_i \left(\sum_{j>i} a_j x_j - \sum_{j<i} a_j x_j \right) , \qquad (i = 1, \dots, n) . \tag{1.5}$$

When all the parameters a_i are different from zero, a trivial rescaling (which preserves the Poisson structure) leads us back to (1.3), so the novelty of our study is mainly concerned with the case where at least one (but not all!) of the parameters a_i is zero, though all results below are also valid in case all the parameters a_i are different from zero. By explicitly exhibiting a set of $[(n+1)/2]$ involutive (Poisson commuting) rational functions, which are shown to be functionally independent, we show that (1.4) is Liouville integrable (Theorems 2.3 and 2.4). We also exhibit $n-1$ functionally independent first integrals, thereby showing that (1.5) is superintegrable (Theorem 2.5). Finally, we construct for any initial conditions explicit solutions of (1.5) (Proposition 2.6).

In Section 3 we study the Kahan discretization (see [**4, 10, 14, 15**] and the references therein) of (1.5), which we explicitly describe (Proposition 3.1). We also show that the map defined by the Kahan discretization is a Poisson map (Proposition 3.2). Upon comparing the latter map with the solutions to the continuous system (1.5), we prove that the Kahan map is a time advance map for this Hamiltonian system, and we derive from it that the discrete system is both Liouville and

superintegrable, with the same first integrals as the continuous system (Proposition 3.4 and Corollary 3.3).

We finish the paper with some comments and perspectives for future work (Section 4).

2. A generalized Lotka-Volterra system

Let n be an arbitrary positive integer. We consider on $\mathbb{R}^n$ the generalized Lotka-Volterra system

$$\dot{x}_i = x_i \sum_{j=1}^{n} A_{ij} x_j , \qquad (i = 1, \ldots, n) , \tag{2.1}$$

where A is the square matrix

$$A = \begin{pmatrix} 0 & a_2 & a_3 & \ldots & a_n \\ -a_1 & 0 & a_3 & \ldots & a_n \\ -a_1 & -a_2 & 0 & \ldots & a_n \\ \vdots & \vdots & \vdots & \ddots & \\ -a_1 & -a_2 & -a_3 & \ldots & 0 \end{pmatrix} , \tag{2.2}$$

and $(a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$. Like most Lotka-Volterra system, it has a linear function as Hamiltonian, to wit $H := a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$; the corresponding (quadratic) Poisson structure is defined by the brackets $\{x_i, x_j\} := x_i x_j$, for $1 \leqslant i < j \leqslant n$. The following elementary lemma, which will play a key rôle in the proof of Theorem 2.3 below, shows that rescaling the parameters a_i by non-zero constants leads to isomorphic Hamiltonian systems.

LEMMA 2.1. *Let $c_1, \ldots, c_n$ be arbitrary non-zero real constants. Then the linear change of coordinates $x_i \mapsto x_i/c_i$ transforms the generalized Lotka-Volterra system with parameters $a_1, \ldots, a_n$ into the generalized Lotka-Volterra system with parameters $a_1 c_1, \ldots, a_n c_n$.*

PROOF. Let $y_i := x_i/c_i$. Then $\{y_i, y_j\} = \{x_i, x_j\}/(c_i c_j) = \frac{x_i x_j}{c_i c_j} = y_i y_j$, for any $i < j$, which shows that the change of variables preserves the Poisson structure. Clearly, in terms of the new variables, the Hamiltonian reads $H = a_1 c_1 y_1 + \cdots + a_n c_n y_n$, which is the Hamiltonian of the generalized Lotka-Volterra system with constants $a_i c_i$. $\qquad \square$

As an application of the lemma, we have that when the parameters a_i are all non-zero, we can rescale them all to 1, and (2.1) becomes (1.2) (which is system (3.5) in [**17**]). In this case, the matrix A is skew-symmetric and so (2.1) is a genuine Lotka-Volterra system, whose Liouville and superintegrability have extensively been studied in [**17**]. When some of the parameters a_i are zero, we get new (non-isomorphic) systems. As we will show in this section, all these systems are Liouville and superintegrable.

For the study of the general case, it is convenient to introduce the functions $v_i := a_1 x_1 + \cdots + a_i x_i$, for $i = 1, \ldots, n$; we also set $v_0 := 0$. In terms of these functions, $H = v_n$ and the system (2.1) can equivalently be written as

$$\dot{x}_i = x_i(H - v_i - v_{i-1}), \qquad (i = 1, \ldots, n). \tag{2.3}$$

For $i < j$, one has $\{v_i, x_j\} = v_i x_j$, and so the Poisson brackets of the functions v_i are given by

$$\{v_i, v_j\} = v_i(v_j - v_i), \qquad (i < j). \tag{2.4}$$

In particular, remembering that $H = v_n$,

$$\dot{v}_i = \{v_i, H\} = v_i(H - v_i), \tag{2.5}$$

for $i = 1, \ldots, n$. If $a_1 a_2 \ldots a_n \neq 0$, the functions v_i define new coordinates on $\mathbb{R}^n$, since then $x_k = (v_k - v_{k-1})/a_k$ for $k = 1, \ldots, n$; moreover, the system (2.1) totally decouples in terms of these coordinates since it takes the simple form $\dot{v}_i = v_i(H - v_i)$, for $i = 1, \ldots, n$. However, the functions v_i do not define coordinates when at least one of the a_k is zero, because if $a_k = 0$ then $v_k = v_{k-1}$.

With a view to proving Liouville integrability, we define for $k = 1, \ldots, \left[\frac{n}{2}\right]$ the functions

$$J_k := \frac{x_1 x_3 \ldots x_{2k-1}}{x_2 x_4 \ldots x_{2k}}, \tag{2.6}$$

and for $k = 1, \ldots, \left[\frac{n+1}{2}\right]$ the functions

$$F_k := \begin{cases} v_{2k-1} \dfrac{x_{2k+1} x_{2k+3} \ldots x_n}{x_{2k} x_{2k+2} \ldots x_{n-1}} & \text{if } n \text{ is odd,} \\[4mm] v_{2k} \dfrac{x_{2k+2} x_{2k+4} \ldots x_n}{x_{2k+1} x_{2k+3} \ldots x_{n-1}} & \text{if } n \text{ is even.} \end{cases} \tag{2.7}$$

Notice that $F_{[(n+1)/2]} = v_n = H$, the Hamiltonian (1.4). For odd n, we also introduce the function

$$C := \frac{x_1 x_3 \ldots x_n}{x_2 x_4 \ldots x_{n-1}}.$$

PROPOSITION 2.2. *For any $k, l \in \{1, \ldots, \left[\frac{n}{2}\right]\}$,*

$$\{J_k, J_l\} = \{F_k, F_l\} = \{F_k, H\} = 0. \tag{2.8}$$

Moreover, when n is odd, C is a Casimir function of the Poisson bracket $\{\cdot, \cdot\}$.

PROOF. First, we notice that for any $k = 1, \ldots, \left[\frac{n}{2}\right]$

$$x_i \frac{\partial J_k}{\partial x_i} = \begin{cases} (-1)^{i+1} J_k & \text{for } 1 \leqslant i \leqslant 2k, \\ 0 & \text{for } 2k < i \leqslant n. \end{cases} \tag{2.9}$$

It follows that, for $k < l \in \{1, \ldots, \left[\frac{n}{2}\right]\}$, we have

$$
\begin{aligned}
\{J_k, J_l\} &= \sum_{1 \leqslant i < j \leqslant n} x_i x_j \left(\frac{\partial J_k}{\partial x_i} \frac{\partial J_l}{\partial x_j} - \frac{\partial J_k}{\partial x_j} \frac{\partial J_l}{\partial x_i} \right) \\
&= \sum_{1 \leqslant i < j \leqslant 2k} [(-1)^{i+1} J_k (-1)^{j+1} J_l - (-1)^{j+1} J_k (-1)^{i+1} J_l] \\
&\quad + \sum_{1 \leqslant i \leqslant 2k < j \leqslant 2l} (-1)^{i+1} J_k (-1)^{j+1} J_l = 0 \ .
\end{aligned}
$$

This shows the first equality of (2.8). We show the two other equalities of (2.8) for even n. To do this, it suffices to show that $\{F_k, F_l\} = 0$ for $1 \leqslant k < l \leqslant n/2$ since $F_{n/2} = H$. We set $F_k = v_{2k} I_k$, i.e., we define I_k by

$$
I_k := \frac{x_{2k+2} x_{2k+4} \cdots x_n}{x_{2k+1} x_{2k+3} \cdots x_{n-1}} \ .
$$

As in (2.9), we have that

$$
x_i \frac{\partial I_k}{\partial x_i} = \begin{cases} 0 & \text{for } 1 \leqslant i \leqslant 2k \ , \\ (-1)^i I_k & \text{for } 2k < i \leqslant n \ , \end{cases} \tag{2.10}
$$

from which it follows, as above, that $\{I_k, I_l\} = 0$ and that $\{I_k, J_l\} = 0$ for all $k, l \in \{1, \ldots, n/2\}$. Also, for any $j \in \{1, \ldots, n\}$

$$
\{I_k, x_j\} = \sum_{i=1}^{n} \frac{\partial I_k}{\partial x_i} \{x_i, x_j\} = \left(\sum_{1 \leqslant i < j} x_i \frac{\partial I_k}{\partial x_i} - \sum_{j < i \leqslant n} x_i \frac{\partial I_k}{\partial x_i} \right) x_j
$$

and using (2.10) we derive that

$$
\{I_k, x_j\} = \begin{cases} 0 & \text{for } j \leqslant 2k \ , \\ -I_k x_j & \text{for } 2k < j \ , \end{cases} \quad \text{and } \{I_k, v_j\} = \begin{cases} 0 & \text{for } j \leqslant 2k \ , \\ -I_k(v_j - v_{2k}) & \text{for } 2k < j \ . \end{cases} \tag{2.11}
$$

It follows from (2.4) and (2.11) that, for any $k < l \leqslant n/2$,

$$
\begin{aligned}
\{F_k, F_l\} &= \{v_{2k} I_k, v_{2l} I_l\} = v_{2k} I_l \{I_k, v_{2l}\} + v_{2l} I_k \{v_{2k}, I_l\} + I_k I_l \{v_{2k}, v_{2l}\} \\
&= -v_{2k} I_k I_l (v_{2l} - v_{2k}) + 0 + v_{2k} I_k I_l (v_{2l} - v_{2k}) = 0 \ .
\end{aligned}
$$

This shows the second half of (2.8) for n even; for n odd, the proof is very similar (in this case, $H = F_{(n+1)/2}$ and one proves as above that $\{F_k, F_l\} = 0$ for $1 \leqslant k < l \leqslant (n+1)/2$). Finally we show that C is a Casimir function (when n is odd). For $j = 1, \ldots, n$,

$$
\begin{aligned}
\{C, x_j\} &= \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} \{x_i, x_j\} = \left(\sum_{1 \leqslant i < j} x_i \frac{\partial C}{\partial x_i} - \sum_{j < i \leqslant n} x_i \frac{\partial C}{\partial x_i} \right) x_j \\
&= \sum_{1 \leqslant i < j} (-1)^{i+1} C x_j - \sum_{j < i \leqslant n} (-1)^{i+1} C x_j = 0 \ ,
\end{aligned}
$$

which shows our claim. $\qquad \square$

THEOREM 2.3. *Suppose that n is even. Let ℓ denote the smallest integer such that $a_{\ell+1} \neq 0$ (in particular, $\ell = 0$ when $a_1 \neq 0$) and let $\lambda := \left[\frac{\ell}{2}\right]$. The $\frac{n}{2}$ functions $J_1, J_2, \ldots, J_\lambda, H, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1}$ are pairwise in involution and functionally independent, hence they define a Liouville integrable system on $(\mathbb{R}^n, \{\cdot, \cdot\})$.*

PROOF. We know already from Proposition 2.2 that the functions J_k are pairwise in involution, and also the functions F_l (recall that $F_{n/2} = H$). We show that $\{J_k, F_l\} = 0$ for $k = 1, \ldots, \lambda$ and $l = \lambda + 1, \ldots, \frac{n}{2}$. To do this, we use the following analog of (2.11), which is easily obtained from (2.9):

$$\{J_k, v_j\} = \left\{ \begin{array}{ll} J_k v_j & \text{for } j \leqslant 2k\ , \\ J_k v_{2k} & \text{for } 2k < j\ . \end{array} \right.$$

It follows that, for the above values of k, l, which satisfy $k \leqslant \lambda < l$, one has $\{J_k, v_{2l}\} = J_k v_{2k} = 0$ (the last equality follows from $2k \leqslant 2\lambda \leqslant \ell$ and $v_i = 0$ for $i \leqslant \ell$), and so

$$\{J_k, F_l\} = \{J_k, v_{2l} I_l\} = v_{2k}\{J_k, I_l\} + I_l\{J_k, v_{2l}\} = 0\ ;$$

in the last step we also used that the functions I_i and J_j are in involution (see the proof of Proposition 2.2). This shows that the $\frac{n}{2}$ functions

$$J_1, J_2, \ldots, J_\lambda, H, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1} \tag{2.12}$$

are pairwise in involution.

We now show that these functions are functionally independent. We first do this when all a_i are zero, except for $a_{\ell+1}$ which we may suppose to be equal to 1; then $v_i = x_{\ell+1} = H$ for $i > \ell$ and $v_i = 0$ for $i \leqslant \ell$. The Jacobian matrix of the above functions (2.12) with respect to $x_1, \ldots, x_n$ (in that order) is easily seen to have the following block form:

$$Jac = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \star & B \end{pmatrix}\ ,$$

where A has size $\lambda \times \ell$ and B has size $(\frac{n}{2} - \lambda - 1) \times (n - \ell - 1)$. We show that this matrix has full rank $n/2$ (which is equal to the number of rows of Jac). To do this, it is sufficient to show that A has full rank λ and that B has full rank $n/2 - \lambda - 1$ (the value of the column vector $\star$ is irrelevant). Consider the square submatrix A' of A consisting only of its even-numbered columns. For $k < l$ we have $A'_{kl} = A_{k,2l} = \partial J_k / \partial x_{2l} = 0$, since J_k only depends on $x_1, \ldots, x_{2k}$. It follows that A' is a lower triangular matrix. Moreover, $A'_{kk} = A_{k,2k} = \partial J_k / \partial x_{2k} \neq 0$, hence A' is non-singular. This shows that $\operatorname{rank}(A) = \operatorname{rank}(A') = \lambda$. Similarly, we extract from B a square submatrix B' by selecting from B its even-numbered (respectively odd-numbered) columns when ℓ is even (respectively odd). For $k > l$ we have $B'_{kl} = \partial F_{\lambda+k} / \partial x_{2\lambda+1+2l} = \partial(v_{2\lambda+2k} I_{\lambda+k}) / \partial x_{2\lambda+1+2l} = \partial(x_{\ell+1} I_{\lambda+k}) / \partial x_{2\lambda+1+2l} = x_{\ell+1} \partial I_{\lambda+k} / \partial x_{2\lambda+1+2l} = 0$, since $I_{\lambda+k}$ is independent of $x_1, \ldots, x_{2\lambda+2k}$. However, $B'_{kk} = x_{\ell+1} \partial I_{\lambda+k} / \partial x_{2\lambda+1+2k} \neq 0$, because $I_{\lambda+k}$ does depend on $x_{2\lambda+1+2k}$. This shows that B' is a non-singular upper triangular matrix, hence $\operatorname{rank}(B) = \operatorname{rank}(B') = n/2 - \lambda - 1$. We have thereby shown that if $H = x_{\ell+1}$, then the $n/2$ functions in (2.12) are functionally independent; since the rank of the Poisson structure $\{\cdot, \cdot\}$ is n, we have shown Liouville integrability in this case.

We now consider the general case, where several of the a_i may be non-zero. We may still suppose that $a_{\ell+1} = 1$; as above, $a_1 = \cdots = a_\ell = 0$. Let us view $a_{\ell+2}, \ldots, a_n$ as arbitrary parameters and consider the matrix

$$Jac' := \begin{pmatrix} A' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \star & B' \end{pmatrix} ,$$

where A' and B' are square matrices which are constructed as in the previous paragraph. It depends polynomially on the parameters $a_{\ell+2}, \ldots, a_n$ and we have shown that the determinant of Jac' is non-zero when we set all the parameters $a_{\ell+2}, \ldots, a_n$ equal to zero. By continuity, the determinant remains non-zero when the parameters $a_{\ell+2}, \ldots, a_n$ are sufficiently close to zero, which proves that the $n/2$ functions in (2.12) are functionally independent for such values of the parameters. In view of Lemma 2.1, any non-zero rescaling of the parameters leads to isomorphic systems, so for any values of $a_{\ell+2}, \ldots, a_n$, the functions in (2.12) are functionally independent. This shows Liouville integrability for any values of the parameters $a_1, \ldots, a_n$. $\square$

When n is odd, the rank of the Poisson structure $\{\cdot, \cdot\}$ is $n-1$, so for Liouville integrability we need $(n + 1)/2$ functionally independent functions in involution. Recall from Proposition 2.2 that in this case C is a Casimir function. The Liouville integrability is in this case given by the following theorem, whose proof is omitted because it is very similar to the proof of Theorem 2.3.

THEOREM 2.4. *Suppose that n is odd. As before, let ℓ denote the smallest integer such that $a_{\ell+1} \neq 0$ and let $\lambda := \left[\frac{\ell}{2}\right]$. The $\frac{n+1}{2}$ functions $J_1, J_2, \ldots, J_\lambda, H, F_{\lambda+2},$ $F_{\lambda+3}, \ldots, F_{\frac{n-1}{2}}, C$ are pairwise in involution and functionally independent, hence define a Liouville integrable system on $(\mathbb{R}^n, \{\cdot, \cdot\})$.*

We show in the following theorem that the Hamiltonian vector field defined by H is also superintegrable.

THEOREM 2.5. *The Hamiltonian system (1.5) has $n - 1$ functionally independent first integrals, hence is superintegrable.*

PROOF. We denote, as before, by ℓ the smallest integer such that $a_{\ell+1} \neq 0$ (in particular, $\ell = 0$ when $a_1 \neq 0$). Suppose first that $a_{\ell+1}$ is the only a_i which is different from zero; by a simple rescaling, we may assume $a_{\ell+1} = 1$, so that $H = x_{\ell+1}$. Then the equations of motion (1.5) take the following simple form:

$$\dot{x}_i = \begin{cases} x_i H & i \leqslant \ell, \\ 0 & i = \ell + 1, \\ -x_i H & i > \ell + 1. \end{cases} \tag{2.13}$$

When $\ell = 0$, a complete set of $n - 1$ independent first integrals of (2.13) is given by $H = x_1$ and x_i/x_2, $(i = 3, \ldots, n)$. When $\ell \neq 0$, we can take besides the Hamiltonian $H = x_{\ell+1}$ the functions x_i/x_1, $(i = 2, \ldots, \ell)$ and $x_1 x_i$, $(i = \ell + 2, \ldots, n)$.

In the general case, we partition the set $\{1, 2, \ldots, n\}$ into three subsets (A or C may be empty):

$$
\begin{aligned}
A &:= \{1, 2, \ldots, \ell\}\,, \\
B &:= \{i \mid a_i \neq 0\}\,, \\
C &:= \{i \mid i > \ell + 1 \text{ and } a_i = 0\}\,.
\end{aligned}
$$

Since we have treated the case $\#B = 1$, we may henceforth assume that $\#B \geqslant 2$. Notice that each function v_i (and in particular H) depends only on the variables x_i with $i \in B$. It follows that the differential equations (2.3),

$$
\dot{x}_i = x_i(H - v_i - v_{i-1})\,, \qquad (i \in B)\,,
$$

involve only the variables x_j with $j \in B$, so they form a subsystem which is the same as the original system, but now of dimension $m := \#B$, and with all parameters a_i, $i \in B$ different from zero. As explained above (see Lemma 2.1 and the remarks which follow its proof) this subsystem is by a simple rescaling isomorphic to the system (1.2), for which we know from [**17**] that it is superintegrable, with $m - 1$ first integrals which we denote here by $G_1, \ldots, G_{m-1}$. We do not need here the precise formulas for these functions, but only the fact that they depend only on the variables x_j with $j \in B$; this obvious fact implies that the functions $G_1, \ldots, G_{m-1}$ are first integrals of the full system (1.5) as well. Consider, for $i \in A \cup C$ the following rational function:

$$
K_i := \begin{cases}
\dfrac{(H - a_{\ell+1} x_{\ell+1}) x_i}{x_{\ell+1}}\,, & i \in A\,, \\[3mm]
\dfrac{(H - a_{\ell+1} x_{\ell+1}) v_i^2}{x_i x_{\ell+1}}\,, & i \in C\,.
\end{cases}
$$

Notice that $H - a_{\ell+1} x_{\ell+1}$ is different from zero, because $\#B \geqslant 2$. For $i \in A$, we have that

$$
\begin{aligned}
(\ln K_i)^{\cdot} &= (\ln(H - a_{\ell+1} x_{\ell+1}))^{\cdot} + (\ln(x_i/x_{\ell+1}))^{\cdot} \\
&= -\frac{a_{\ell+1} \dot{x}_{\ell+1}}{H - a_{\ell+1} x_{\ell+1}} + a_{\ell+1} x_{\ell+1} = 0\,.
\end{aligned}
$$

Indeed, $\dot{x}_{\ell+1} = x_{\ell+1}(H - v_{\ell+1} - v_\ell) = x_{\ell+1}(H - a_{\ell+1} x_{\ell+1})$. Similarly, for $i \in C$, we have from (2.3) and (2.5) that

$$
\begin{aligned}
(\ln K_i)^{\cdot} &= (\ln(H - a_{\ell+1} x_{\ell+1}))^{\cdot} + 2(\ln v_i)^{\cdot} - (\ln(x_i x_{\ell+1}))^{\cdot} \\
&= -a_{\ell+1} x_{\ell+1} + 2(H - v_i) - (H - 2v_i) - (H - a_{\ell+1} x_{\ell+1}) = 0\,.
\end{aligned}
$$

This shows that the $n - 1$ functions $G_1, \ldots, G_{m-1}$ and K_i, $i \in A \cup C$, are first integrals of (1.5). Recall that the functionally independent functions $G_1, \ldots, G_{m-1}$ depend on x_i with $i \in B$ only and notice that for $i \in A \cup C$ the variable x_i appears

only in K_i. It follows that these $n - 1$ first integrals of (1.5) are functionally independent, hence (1.5) is superintegrable. $\square$

Finally, we compute the solution $x(t)$ of (2.1) which corresponds to any given initial condition $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$. We also introduce the derived functions $v_i(t) = a_1 x_1(t) + \cdots + a_i x_i(t)$, for $i = 1, \ldots, n$. We denote by h_0 the value of the Hamiltonian H at the initial condition $x^{(0)}$ and we denote $v_i^{(0)} := v_i(0)$. It follows from (2.3) and (2.5) that we need to solve

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = x_i(t)(h_0 - v_i(t) - v_{i-1}(t)) , \qquad (i = 1, \ldots, n) , \qquad (2.14)$$

where

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}(t) = v_i(t)(h_0 - v_i(t)) , \qquad (i = 1, \ldots, n) . \qquad (2.15)$$

When $v_i^{(0)} = 0$, the latter equation has $v_i(t) = 0$ as its unique solution; otherwise (2.15) is easily integrated by a separation of variables, giving

$$v_i(t) = \frac{1}{\frac{1}{h_0} + C_i e^{-h_0 t}} , \quad \text{or} \quad v_i(t) = \frac{1}{t + C_i'} , \qquad (2.16)$$

depending on whether $h_0 \neq 0$ or $h_0 = 0$. The integrating constants C_i and C_i' are computed from $v_i(0) = v_i^{(0)}$, which leads to

$$C_i = \frac{1}{v_i^{(0)}} - \frac{1}{h_0} , \quad \text{and} \quad C_i' = \frac{1}{v_i^{(0)}} .$$

The functions $v_i(t)$ in (2.16) have very simple primitives, to wit

$$\int v_i(t)\mathrm{d}t = \ln\left(\frac{e^{h_0 t}}{h_0} + C_i\right) , \qquad \text{or} \qquad \int v_i(t)\mathrm{d}t = \ln(t + C_i') . \qquad (2.17)$$

Substituted in (2.14), which we write now as $\frac{\mathrm{d}\ln x_i}{\mathrm{d}t}(t) = h_0 - v_i(t) - v_{i-1}(t)$, we obtain by integration and by using the primitives (2.17) (or $\int v_i(t)\mathrm{d}t = \text{constant in}$ case $v_i^{(0)} = 0$) and the initial condition $x_i(0) = x_i^{(0)}$, the following result:

PROPOSITION 2.6. *The solution $x(t)$ of (2.1) which corresponds to the initial condition $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ is given by*

$$x_i(t) = x_i^{(0)} \frac{(1 - f(t)h_0)(1 + f(t)h_0)}{\left(1 - f(t)h_0 + 2f(t)v_{i-1}^{(0)}\right)\left(1 - f(t)h_0 + 2f(t)v_i^{(0)}\right)} , \quad (i = 1, \ldots, n) ,$$

(2.18)

where $f(t) = \frac{e^{h_0 t} - 1}{(e^{h_0 t} + 1)h_0} = \frac{1}{h_0} \tanh(\frac{h_0 t}{2})$ when h_0 (the value of H at $x^{(0)}$) is different from zero and $f(t) = t/2$ otherwise. Also, $v_i^{(0)} = a_1 x_1^{(0)} + \cdots + a_i x_i^{(0)}$.

Notice that when $h_0 \neq 0$, (2.18) can be rewritten as

$$x_i(t) = \frac{x_i^{(0)} e^{th_0} h_0^2}{\left(h_0 + (e^{th_0} - 1)v_{i-1}^{(0)}\right)\left(h_0 + (e^{th_0} - 1)v_i^{(0)}\right)} , \quad (i = 1, \ldots, n) .$$

REMARK 2.7. When several of the parameters a_i in the Hamiltonian function H are equal to zero, so that H is independent of the corresponding variables x_i, the vector field (1.5) is a Hamiltonian vector field with respect to a family of compatible Poisson structures, always with the same Hamiltonian H. Indeed, suppose that $a_i = a_j = 0$, with $i < j$. Then, in the computation of the vector field $\dot{x}_k = \{x_k, H\}$, $k = 1, \ldots, n$, the Poisson brackets $\{x_i, x_j\} = -\{x_j, x_i\}$ are not used, so we may replace $\{x_i, x_j\} = -\{x_j, x_i\}$ by an arbitrary function f_{ij} of $x_1, \ldots, x_n$ without any effect on the vector field. However, in order for the new bracket to be a Poisson bracket, it has to satisfy the Jacobi identity, which puts several restrictions on the function f_{ij}. One way to satisfy this restriction is to take $f_{ij} := a_{ij} x_i x_j$, where a_{ij} is an arbitrary constant. In fact, replacing $\{x_i, x_j\} = x_i x_j$ by $\{x_i, x_j\} = a_{ij} x_i x_j$ for all $i < j$ for which $a_i = a_j = 0$, the new brackets will still be of the general form $\{x_i, x_j\} = b_{ij} x_i x_j$, known in the literature as *diagonal brackets*; such brackets are known to automatically satisfy the Jacobi identity [**11**, Example 8.14] so they are Poisson brackets. Clearly, any linear combination of these diagonal Poisson brackets is again a diagonal Poisson bracket, hence all these brackets are compatible. The upshot is that when $k \geqslant 2$ parameters are equal to zero, then (1.5) has a multi-Hamiltonian structure: it is Hamiltonian with respect to a $\binom{k}{2}$-dimensional family of Poisson brackets.

3. The Kahan discretization

In this section we consider the Kahan discretization of the system (2.1). Let us recall quickly the construction of the Kahan discretization of a quadratic vector field $\dot{x}_i = Q_i(x)$ (see e.g. [**4**]). Let $\Phi_i(y, z)$ denote the symmetric bilinear form which is associated to the quadratic form Q_i and let ϵ denote a positive parameter, which should be thought of as being small. Then the *Kahan discretization with step size* ϵ is the map[1] $x_i \mapsto \tilde{x}_i$, implicitly defined by

$$\tilde{x}_i - x_i = \epsilon \Phi_i(x, \tilde{x}) . \tag{3.1}$$

We refer to this map as the *Kahan map* (associated to $\dot{x}_i = Q_i(x)$). It is well known that the Kahan map preserves the linear integrals of the initial continuous system (quadratic vector field). So, in our case of the generalized Lotka-Volterra system, its Hamiltonian function $H = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ is an invariant of the Kahan map. As we are going to show in this section the Kahan map (of this system) preserves the Poisson structure as well; we will also see in the next section that all constants of motion, in particular the ones that appear in Theorems 2.3, 2.4 and 2.5, are also invariants of the Kahan map.

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[1] When the map which is defined by the discretization is iterated, one often writes it as $x_i^{(m)} \mapsto x_i^{(m+1)}$.

We begin with a lemma which provides an explicit formula for the Kahan discretization of the generalized Lotka-Volterra system.

PROPOSITION 3.1. *The Kahan discretization with step size 2ϵ of the system (2.1) is the rational map* $\mathcal{K} : (x_1, \ldots, x_n) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_n)$, *given by*

$$\tilde{x}_i = x_i \frac{(1 - \epsilon H)(1 + \epsilon H)}{(1 - \epsilon H + 2\epsilon v_{i-1})(1 - \epsilon H + 2\epsilon v_i)}, \quad (i = 1, \ldots, n). \qquad (3.2)$$

PROOF. Let us write $\tilde{v}_j = a_1 \tilde{x}_1 + \cdots + a_j \tilde{x}_j$, in analogy with the functions v_j. According to (3.1), the Kahan discretization of (2.3) (which is equivalent to (2.1)) is given by

$$\tilde{x}_i - x_i = \epsilon x_i(H - \tilde{v}_i - \tilde{v}_{i-1}) + \epsilon \tilde{x}_i(H - v_i - v_{i-1}), \qquad (i = 1, \ldots, n), \qquad (3.3)$$

where we have used that H is invariant ($\tilde{H} = H$). Summing up these equations, multiplied by a_i, for $i = 1, \ldots, j$, we get

$$\tilde{v}_j - v_j = \epsilon(v_j H + \tilde{v}_j H - \delta_j), \qquad (3.4)$$

where δ_j is given by

$$\delta_j := \sum_{i=1}^{j} a_i x_i(\tilde{v}_i + \tilde{v}_{i-1}) + \sum_{i=1}^{j} a_i \tilde{x}_i(v_i + v_{i-1}) = 2v_j \tilde{v}_j.$$

The last equality can be proven by an easy recursion on j: on the one hand, $\delta_1 = a_1 x_1 \tilde{v}_1 + a_1 \tilde{x}_1 v_1 = 2v_1 \tilde{v}_1$, while on the other hand

$$\begin{aligned}
\delta_{j+1} - \delta_j &= a_{j+1} x_{j+1}(\tilde{v}_{j+1} + \tilde{v}_j) + a_{j+1} \tilde{x}_{j+1}(v_{j+1} + v_j) \\
&= 2a_{j+1} x_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} v_j + 2a_{j+1}^2 x_{j+1} \tilde{x}_{j+1},
\end{aligned}$$

and so

$$\begin{aligned}
\delta_{j+1} &= 2a_{j+1} x_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} v_j + 2a_{j+1}^2 x_{j+1} \tilde{x}_{j+1} + 2v_j \tilde{v}_j \\
&= 2(v_j + a_{j+1} x_{j+1})(\tilde{v}_j + a_{j+1} \tilde{x}_{j+1}) = 2v_{j+1} \tilde{v}_{j+1}.
\end{aligned}$$

Solving (3.4) (with $\delta_j = 2v_j \tilde{v}_j$) linearly for $\tilde{v}_j$ we get

$$\tilde{v}_j = v_j \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon v_j}. \qquad (3.5)$$

Substituting this into (3.3) leads to

$$\tilde{x}_i - x_i = \epsilon x_i \left(H - v_i \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon v_i} - v_{i-1} \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon v_{i-1}} \right) + \epsilon \tilde{x}_i(H - v_i - v_{i-1}),$$

which can be solved linearly for $\tilde{x}_i$. It yields the formula (3.2). $\qquad \square$

PROPOSITION 3.2. *The Kahan map* $\mathcal{K}$, *given by (3.2), is a Poisson map with respect to the Poisson bracket* $\{\cdot, \cdot\}$.

PROOF. Recall that the Poisson bracket $\{\cdot, \cdot\}$ is given by $\{x_i, x_j\} = x_i x_j$, for $1 \leqslant i < j \leqslant n$. Therefore, we need to show that $\{\tilde{x}_i, \tilde{x}_j\} = \tilde{x}_i \tilde{x}_j$, for $1 \leqslant i < j \leqslant n$. We set, for $k = 1, \ldots, n$,

$$A_k = x_k(1 - \epsilon H)(1 + \epsilon H), \ B_k = (1 - \epsilon H + 2\epsilon v_{k-1})(1 - \epsilon H + 2\epsilon v_k) ,$$

so that $\tilde{x}_k = A_k/B_k$. Then

$$\{\tilde{x}_i, \tilde{x}_j\} = \frac{A_i A_j \{B_i, B_j\} - A_i B_j \{B_i, A_j\} - B_i A_j \{A_i, B_j\} + B_i B_j \{A_i, A_j\}}{B_i^2 B_j^2} .$$

The Poisson brackets in the right-hand side of this equation can be computed using besides (2.4) the following formulas:

$$\{x_i, H\} = x_i(H - v_i - v_{i-1}) , \qquad \{x_i, v_j\} = \begin{cases} x_i(v_j - v_i - v_{i-1}) & \text{for } i \leqslant j , \\ -x_i v_j & \text{for } i > j . \end{cases}$$

After some computation, it leads to

$$\begin{aligned} \{\tilde{x}_i, \tilde{x}_j\} &= \frac{(1 - \epsilon^2 H^2)^2 x_i x_j}{(1 - \epsilon H + 2\epsilon v_{i-1})(1 - \epsilon H + 2\epsilon v_i)(1 - \epsilon H + 2\epsilon v_{j-1})(1 - \epsilon H + 2\epsilon v_j)} \\ &= \tilde{x}_i \tilde{x}_j , \end{aligned}$$

as was to be shown.

$\square$

An easy comparison of the solution (2.18) to the continuous system and the Kahan map (3.2) shows that the Kahan map is a time advance map for the continuous system, hence preserves all integral curves of the continuous system and so all constants of motion of the continuous system are invariants for the Kahan map. Precisely, let $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ be any point of $\mathbb{R}^n$ and let $\epsilon \in \mathbb{R}$ be small but positive. As above, the value of H at $x^{(0)}$ is denoted by h_0. Let t_ϵ denote the unique solution to the equation $f(t_\epsilon) = \epsilon$, where $f(t)$ is the function given in Proposition 2.6. With these notations, (2.18) and (3.2) imply that $x_i(t_\epsilon) = \tilde{x}_i^{(0)}$. It leads, in view of Theorems 2.3 and 2.4, to the following corollary:

COROLLARY 3.3. *The Kahan discretization (3.2) is Liouville integrable, with invariants given in Theorem 2.3 (resp. Theorem 2.4) when n is even (resp. when n is odd). It is also superintegrable, with invariants given in Theorem 2.5.*

Let us denote the k-th iterate of the Kahan map (3.2) starting from the initial condition $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ by $x^{(k)}$. Then the relation between the solutions to the continuous system and the Kahan map can be written as $x_i(t_\epsilon) = x_i^{(1)}$. Now notice that t_ϵ depends only on $x^{(0)}$ through h_0; this implies that the restriction of $\mathcal{K}$ to the integral curve through $x^{(0)}$ is the time t_ϵ flow of the continuous system (restricted to the integral curve through $x^{(0)}$). Thus, $x^{(2)}$ is obtained from $x^{(1)}$ by the time t_ϵ flow, and hence from $x^{(0)}$ by the time $2t_\epsilon$ flow, $x^{(2)} = x(2t_\epsilon)$; more generally, $x^{(m)}$ is obtained from $x^{(0)}$ by the time mt_ϵ flow, $x^{(m)} = x(mt_\epsilon)$. It leads to the following proposition.

PROPOSITION 3.4. *The solution of the discrete system*

$$\tilde{x}_i = x_i \frac{(1 - \epsilon H)(1 + \epsilon H)}{(1 - \epsilon H + 2\epsilon v_{i-1})(1 - \epsilon H + 2\epsilon v_i)} \, , \quad (i = 1, \ldots, n) \tag{3.6}$$

with $H = \sum a_i x_i$ and initial condition $x^{(0)}$ is given by

$$x_i^{(m)} = x_i^{(0)} \frac{\left(\frac{1+\epsilon h_0}{1-\epsilon h_0}\right)^m h_0^2}{\left(h_0 + v_{i-1}^{(0)}\left(\left(\frac{1+\epsilon h_0}{1-\epsilon h_0}\right)^m - 1\right)\right)\left(h_0 + v_i^{(0)}\left(\left(\frac{1+\epsilon h_0}{1-\epsilon h_0}\right)^m - 1\right)\right)} \, . \tag{3.7}$$

when h_0 (the value of H at $x^{(0)}$) is different from zero. When $h_0 = 0$,

$$x_i^{(m)} = x_i^{(0)} \frac{1}{\left(1 + 2m\epsilon v_{i-1}^{(0)}\right)\left(1 + 2m\epsilon v_i^{(0)}\right)} \, . \tag{3.8}$$

PROOF. In view of Proposition 2.6,

$$\begin{aligned} x_i^{(m)} &= x_i(mt_\epsilon) \\ &= x_i^{(0)} \frac{(1 - f(mt_\epsilon)h_0)(1 + f(mt_\epsilon)h_0)}{\left(1 - f(mt_\epsilon)h_0 + 2f(mt_\epsilon)v_{i-1}^{(0)}\right)\left(1 - f(mt_\epsilon)h_0 + 2f(mt_\epsilon)v_i^{(0)}\right)} \, . \end{aligned} \tag{3.9}$$

When $h_0 \neq 0$, it follows easily from $f(t) = \frac{e^{th_0}-1}{(e^{th_0}+1)h_0}$ and $f(t_\epsilon) = \epsilon$ that $e^{t_\epsilon h_0} = \frac{1+h_0\epsilon}{1-h_0\epsilon}$. In turn, we can compute $f(mt_\epsilon)$ from it, namely

$$f(mt_\epsilon) = \frac{1}{h_0} \frac{e^{mt_\epsilon h_0} - 1}{e^{mt_\epsilon h_0} + 1} = \frac{1}{h_0} \frac{\left(\frac{1+h_0\epsilon}{1-h_0\epsilon}\right)^m - 1}{\left(\frac{1+h_0\epsilon}{1-h_0\epsilon}\right)^m + 1} \, . \tag{3.10}$$

It now suffices to substitute (3.10) in (3.9) and to simplify the resulting expression to obtain (3.7). When $h_0 = 0$, we have that $f(mt_\epsilon) = m\epsilon$, since $f(t) = t/2$. Substituted in (3.9) (with $h_0 = 0$), we get at once (3.8). $\square$

4. Conclusion

We presented a new class of generalized Lotka-Volterra systems which are, together with their Kahan discretizations, Liouville integrable and superintegrable, and we provided their explicit solutions. Since linear Hamiltonians are always preserved under Kahan discretization and since the Poisson structure that we used is quadratic, it is natural to ask which quadratic Poisson structures on $\mathbb{R}^n$ are preserved by the Kahan discretization of every Hamiltonian vector field with linear Hamiltonian; in view of what we have shown, the Poisson structure defined by defined by the brackets $\{x_i, x_j\} := x_i x_j$, for $1 \leqslant i < j \leqslant n$, belongs to this class. The Hamiltonian systems which are defined by them would then be good candidates for being Liouville integrable and/or superintegrable. In view of the recent developments in discretization of polynomial vector fields by polarization ([**3**]), similar questions can also be considered for higher degree polynomial Hamiltonian vector fields.

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