

# A PDE for non-intersecting Brownian motions and applications

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## Abstract

Consider  $N = n_1 + n_2 + \dots + n_p$  non-intersecting Brownian motions on the real line, starting from the origin at  $t = 0$ , with  $n_i$  particles forced to reach  $p$  distinct target points  $\beta_i$  at time  $t = 1$ , with  $\beta_1 < \beta_2 < \dots < \beta_p$ . This can be viewed as a diffusion process in a sector of  $\mathbb{R}^N$ . This work shows that the transition probability, that is the probability for the particles to pass through windows  $\tilde{E}_k$  at times  $t_k$ , satisfies, in a new set of variables, a non-linear PDE which can be expressed as a near-Wronskian; that is a determinant of a matrix of size  $p + 1$ , with each row being a derivative of the previous, except for the last column. It is an interesting open question to understand those equations from a more probabilistic point of view.

As an application of these equations, let the number of particles forced to the extreme points  $\beta_1$  and  $\beta_p$  tend to infinity; keep the number of particles forced to intermediate points fixed (inliers), but let the target points themselves go to infinity according to a proper scale. A new critical process appears at the point of bifurcation, where the bulk of the particles forced to  $-\sqrt{n}$  depart from those going to  $\sqrt{n}$ . These statistical fluctuations near that point of bifurcation are specified by a kernel, which is a rational perturbation of the Pearcey kernel. This work also shows that such equations are an essential tool in obtaining certain asymptotic results. Finally, the paper contains a conjecture.

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## 1 Introduction

Consider  $N$  non-intersecting Brownian motions  $x_1(t) < x_2(t) < \dots < x_N(t)$  on  $\mathbb{R}$  (Dyson's Brownian motions), all starting at source points  $\gamma_1 < \gamma_2 < \dots < \gamma_N$  at time  $t = 0$  and forced to target points  $\delta_1 < \delta_2 < \dots < \delta_N$  at  $t = 1$ . According to the Karlin-McGregor formula [17], the probability that the  $N$  particles pass through the subsets  $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_m \subset \mathbb{R}$  respectively at times  $0 < t_1 < t_2 < \dots < t_m < 1$  is given by (setting  $t_0 := 0$  and  $t_{m+1} := 1$ ),

$$\mathbb{P} \left( \bigcap_{k=1}^m \left\{ \text{all } x_i(t_k) \in \tilde{E}_k \right\} \mid \begin{array}{l} x_j(0) = \gamma_j, \quad x_j(1) = \delta_j, \\ \text{for } j = 1, \dots, N \end{array} \right)$$

$$= \frac{1}{Z_N} \int_{\tilde{E}_1^N} \prod_{i=1}^N du_i^{(1)} \int_{\tilde{E}_2^N} \prod_{i=1}^N du_i^{(2)} \dots \int_{\tilde{E}_m^N} \prod_{i=1}^N du_i^{(m)} \det(p(t_1 - t_0; \gamma_i, u_j^{(1)}))_{1 \leq i, j \leq N}$$

$$\times \det(p(t_2 - t_1; u_i^{(1)}, u_j^{(2)}))_{1 \leq i, j \leq N} \dots \det(p(t_{m+1} - t_m; u_i^{(m)}, \delta_j))_{1 \leq i, j \leq N} \quad (1.1)$$

where  $p(t, x, y)$  denotes the standard Brownian transition probability,

$$p(t, x, y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(y-x)^2}{t}}. \quad (1.2)$$

There has been a great deal of interest in non-intersecting Brownian motions and especially in some critical infinite-dimensional diffusions arising when the number of particles  $N \rightarrow \infty$ . This in turn has been motivated by random matrix theory and Dyson's observation [14] that letting the entries of GUE matrices run according to independent Ornstein-Uhlenbeck processes leads to such non-intersecting Brownian motions for the random eigenvalues of the matrix.

When some source points and some target points coincide, the formula (1.1) for the probability must be adapted by taking appropriate limits; see [17, 16, 10, 7]. In this paper, we consider the situation where the source points all coincide with 0, while some target points may coincide. Consider thus  $N = n_1 + n_2 + \dots + n_p$  non-intersecting Brownian motions starting from the origin at  $t = 0$ , with  $n_i$  particles forced to reach  $p$  distinct target points  $\beta_i$  at time  $t = 1$ , with  $\beta_1 < \beta_2 < \dots < \beta_p$  in  $\mathbb{R}$ ; see Figure 1.

Given positive integers  $n = (n_1, \dots, n_p)$ , given  $m$  subsets  $\tilde{E}_1, \dots, \tilde{E}_m \subset \mathbb{R}$  and times  $t_0 = 0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$ , this paper deals with the *probability*<sup>5</sup>  $\mathbb{P}_n^{(\beta)}(t, \tilde{\mathbb{E}})$ , as in (1.3) below (i.e., the probability for the particles to pass through the windows  $\tilde{E}_k$  at times  $t_k$ ); as is well-known, (see [20, 10, 19, 7]),  $\mathbb{P}_n^{(\beta)}(t, \tilde{\mathbb{E}})$  can also be viewed as the probability for the eigenvalues of a chain of  $m$  coupled Hermitian random matrices, after some change of variables:

$$\begin{aligned} \mathbb{P}_n^{(\beta)}(t, \tilde{\mathbb{E}}) &:= \mathbb{P} \left( \bigcap_{k=1}^m \left\{ \text{all } x_i(t_k) \in \tilde{E}_k \right\} \middle| \begin{array}{l} \text{all } x_i(0) = 0; \\ n_j \text{ paths end up at } \beta_j \text{ at } t = 1, \\ \text{for } 1 \leq j \leq p \end{array} \right) \\ &= \frac{1}{\tilde{Z}_n} \int_{\text{spec}(M_k) \in E_k} e^{-\frac{1}{2} \text{tr} \left( \sum_{k=1}^m M_k^2 - 2 \sum_{k=1}^{m-1} c_k M_k M_{k+1} - 2AM_m \right)} \prod_{k=1}^m dM_k \\ &=: \mathbb{P}_n^A(c, \mathbb{E}). \end{aligned} \quad (1.3)$$

The change of variables is given by the following formulae<sup>6</sup>,

$$A := \text{diag}(\overbrace{b_1, \dots, b_1}^{n_1}, \overbrace{b_2, \dots, b_2}^{n_2}, \dots, \overbrace{b_p, \dots, b_p}^{n_p}), \text{ with } b_\ell = \sqrt{\frac{2(t_m - t_{m-1})}{(1-t_m)(1-t_{m-1})}} \beta_\ell$$

$$E_k := \tilde{E}_k \sqrt{\frac{2(t_{k+1} - t_{k-1})}{(t_k - t_{k-1})(t_{k+1} - t_k)}}, \quad c_k^2 := \frac{(t_{k+2} - t_{k+1})(t_k - t_{k-1})}{(t_{k+2} - t_k)(t_{k+1} - t_{k-1})}, \quad (1.4)$$

for  $\ell = 1, \dots, p$  and  $k = 1, \dots, m$ . It is quite natural to impose a *linear constraint on the rescaled target points*  $\beta_1, \dots, \beta_p$ , namely

$$\sum_{\ell=1}^p \kappa_\ell \beta_\ell = 0, \text{ with } \sum_{\ell=1}^p \kappa_\ell = 1, \text{ set } \kappa_0 := -1. \quad (1.5)$$

<sup>5</sup> $\tilde{\mathbb{E}} = \tilde{E}_1 \times \dots \times \tilde{E}_m$ .

<sup>6</sup>For  $m = 1$ , the matrix integral above becomes a one-matrix integral with external potential. The change of variables below becomes:  $b_\ell = \sqrt{\frac{2t}{1-t}} \beta_\ell$ ,  $E = \tilde{E} \sqrt{\frac{2}{t(1-t)}}$ .

Of course, the same relation holds for the  $b_i$ 's. For instance, a typical situation is to take  $\beta_1 = -\beta_p$  and have all the remaining target points in arbitrary position between  $\beta_1$  and  $\beta_p$ . This case will be discussed in Section 8.

The natural initial or rather “*final condition*” for the transition probability (1.3) is given by what happens when  $t_m \rightarrow 1$ , keeping  $t_1, \dots, t_{m-1}$ , away from 0 or 1; namely,

$$\lim_{t_m \rightarrow 1} \mathbb{P}_n^{(\beta)}(t, \tilde{\mathbb{E}}) = 0, \text{ when } \tilde{E}_m \not\supset \{\beta_1, \dots, \beta_p\}. \quad (1.6)$$

It is also known (see ([19])) that the probability above  $\mathbb{P}_n^{(\beta)}(t_1, \dots, t_m, \tilde{E}_1 \times \dots \times \tilde{E}_m) = \det(\mathbb{1} - \chi_{\tilde{E}_i^c}(x) H_{t_i t_j}^{(N)}(x, y) \chi_{\tilde{E}_j^c}(y))$  can be expressed as a matrix Fredholm determinant of a matrix kernel <sup>7</sup>

$$\begin{aligned} H_{t_k, t_\ell}^{(N)}(x, y; \beta_1, \dots, \beta_p) dy &= - \frac{dy}{2\pi^2 \sqrt{(1-t_k)(1-t_\ell)}} \int_{\mathcal{C}} dV \int_{\Gamma_L} dU \frac{e^{-\frac{t_k V^2}{1-t_k} + \frac{2xV}{1-t_k}}}{e^{-\frac{t_\ell U^2}{1-t_\ell} + \frac{2yU}{1-t_\ell}}} \\ &\quad \times \prod_{r=1}^p \left( \frac{U - \beta_r}{V - \beta_r} \right)^{n_r} \frac{1}{U - V} \\ &- \begin{cases} 0, & \text{for } t_k \geq t_\ell \\ \frac{dy}{\sqrt{\pi(t_\ell - t_k)}} e^{-\frac{(x-y)^2}{t_\ell - t_k}} e^{\frac{x^2}{1-t_k} - \frac{y^2}{1-t_\ell}}, & \text{for } t_k < t_\ell \end{cases} \end{aligned} \quad (1.7)$$

where  $\mathcal{C}$  is a closed contour enclosing all the points  $\beta_r$ , which is to the left of the line  $\Gamma_L := L + i\mathbb{R}$  by picking  $L$  large enough, guaranteeing  $\Re(U - V) > 0$ .

These non-intersecting Brownian motions  $x_1(t) < \dots < x_N(t)$  describe a diffusion process in a sector  $\{x_1 < x_2 < \dots < x_N\}$  of  $\mathbb{R}^N$  and thus satisfy a diffusion equation. When the number  $N$  of particles tends to  $\infty$ , the transition

<sup>7</sup> The Fredholm determinant of a matrix kernel  $\hat{H}_{t_i t_j}(x, y) := \chi_{E_i}(x) H_{t_i t_j}(x, y) \chi_{E_j}(y)$ :

$$\begin{aligned} &\det \left( I - z(\hat{H}_{t_i t_j})_{1 \leq i, j \leq m} \right) \\ &= 1 + \sum_{n=1}^{\infty} (-z)^n \sum_{\substack{0 \leq r_i \leq n \\ \sum_1^m r_i = n}} \int_{\mathcal{R}} \prod_1^{r_1} d\alpha_i^{(1)} \dots \prod_1^{r_m} d\alpha_i^{(m)} \det \left( \left( \hat{H}_{t_k t_\ell}(\alpha_i^{(k)}, \alpha_j^{(\ell)}) \right)_{\substack{1 \leq i \leq r_k \\ 1 \leq j \leq r_\ell}} \right)_{1 \leq k, \ell \leq m}, \end{aligned}$$

where the  $n$ -fold integral in each term above is taken over the range

$$\mathcal{R} = \left\{ \begin{array}{l} -\infty < \alpha_1^{(1)} \leq \dots \leq \alpha_{r_1}^{(1)} < \infty \\ \vdots \\ -\infty < \alpha_1^{(m)} \leq \dots \leq \alpha_{r_m}^{(m)} < \infty \end{array} \right\}.$$

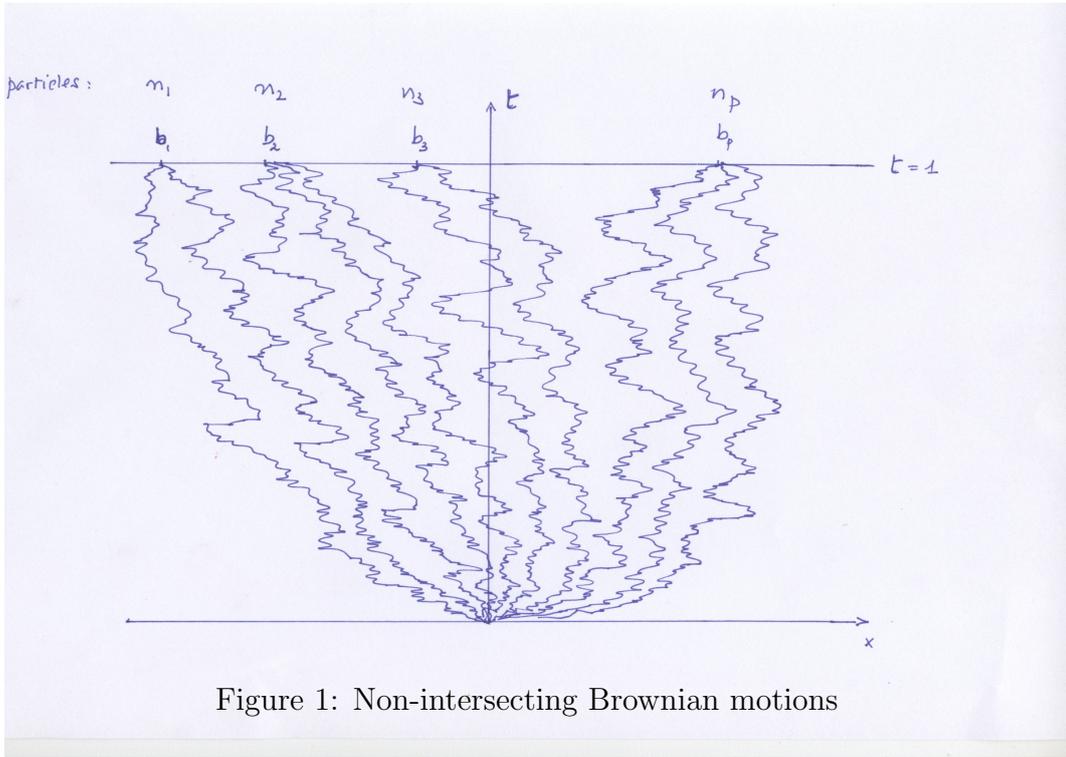


Figure 1: Non-intersecting Brownian motions

probability would have to satisfy an “*infinite-dimensional diffusion equation*”, which however would be very difficult to use. The main result of this paper is to show that this transition probability  $\mathbb{P}_n^A(c, \mathbb{E})$  satisfies a non-linear PDE in the boundary points of  $E_1, \dots, E_m$ , the target points  $b_1, \dots, b_p$ , and the couplings  $c_1, \dots, c_{m-1}$ . It is the determinant of a certain matrix of size  $p + 1$ ;  $p$  being the number of target points; so, when the number of particles tends to  $\infty$ , the form of this equation remains the same, which will be exploited in the limit discussed in Theorem 1.3. Moreover, this determinant misses to be a *Wronskian* by the last column only.

The PDE for the transition probability stems largely from integrable theory; this at least is our approach in the present paper. The integrable theory behind non-intersecting Brownian motions has been developed by us in [8]; the latter contains many different ingredients; among them, multi-component KP hierarchies [18, 6] and multiple-orthogonal polynomials [4, 9, 10]. It is – in our opinion – an *interesting open question* to understand the PDE from a *more probabilistic point of view* and to use more conventional probabilistic tools to derive them.

Throughout the paper, we shall use, without further warning, the following **notation**: (i) The inverse of the following Jacobi matrix will play an important



for the operator  $' := \partial_0$ ,

$$\det \begin{pmatrix} F_1 & F_2 & F_3 & \dots & F_p & G_0 \\ F_1' & F_2' & F_3' & \dots & F_p' & G_1 \\ F_1'' & F_2'' & F_3'' & \dots & F_p'' & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \dots & F_p^{(p)} & G_p \end{pmatrix} = 0, \quad (1.12)$$

where the  $F_\ell$  and  $G_\ell$  are given by

$$\begin{aligned} F_\ell &= -\partial_0 \partial_\ell \ln \mathbb{P}_n - n_\ell \mathcal{J}_{1m}, \\ G_{\ell+1} &:= \partial_0 G_\ell + \sum_{i=1}^p (\partial_0)^\ell F_i \left( \partial_0 \frac{H_i^{(1)}}{F_i} - \partial_i \frac{H_i^{(2)}}{F_i} \right), \quad G_0 := 0, \\ H_\ell^{(1)} &:= (\kappa_\ell (\delta_{1,m} - \varepsilon_m) \partial_0 + 2\mathcal{J}_{1m} \partial_b^{(\ell)}) \ln \mathbb{P}_n + C_\ell, \\ H_\ell^{(2)} &:= (\delta_{1,m} - \varepsilon_0 + 2\mathcal{J}_{1m} b_\ell \partial_0) \partial_\ell \ln \mathbb{P}_n, \end{aligned} \quad (1.13)$$

with

$$C_\ell := 2n_\ell \mathcal{J}_{1m} \left( \mathcal{J}_{mm} b_\ell - \sum_{i \neq \ell} \frac{n_i}{b_\ell - b_i} \right). \quad (1.14)$$

The final condition (1.6) translates into an “initial condition” near  $c_{m-1} \rightarrow 0$  and  $b_\ell \rightarrow \infty$ , upon using the fact that

$$c_{m-1} \simeq \sqrt{1 - t_m}, \quad c_{m-1} b_\ell \simeq O(1).$$

As a special case, we consider the *one-time probability*  $\mathbb{P}_n^{(\beta)}(t, \tilde{E})$  for  $0 < t_1 = t < 1$ . For this case, (1.3) becomes a one-matrix model with external potential  $\mathbb{P}_n := \mathbb{P}_n^A(E)$ , thus with no coupling. The expressions for (1.13) can be replaced by simpler expressions; note that the  $H_\ell^{(1)}$  in (1.15) below are not obtained from the  $H_\ell^{(1)}$ , as in (1.13), by setting  $m = 1$ ; in fact, a further simplification occurs in the equations; also the functions  $G_\ell$  are only specializations of the above  $G_\ell$  up to a sign  $-(-1)^\ell$  and  $'$  now denotes  $\partial_E$  instead of  $\partial_0 = -\partial_E$ . In this statement, we use the operator  $\partial_b^{(\ell)}$  as in (1.11), and we use the following simple operator, in accord with (1.9):

$$\varepsilon := \varepsilon_E - \varepsilon_b, \quad \text{with } \varepsilon_b = \sum_1^{p-1} b_i \frac{\partial}{\partial b_i}.$$

**Corollary 1.2** *When  $m = 1$  (the one-time case), then  $\ln \mathbb{P}_n = \ln \mathbb{P}_n^A(E)$  satisfies the same non-linear PDE (1.12), but with simpler expressions  $F_\ell$  and*

$H_\ell^{(1)}$  and with  $' = \partial_E$ ,

$$\begin{aligned}
F_\ell &:= \left( \partial_b^{(\ell)} + \kappa_\ell \partial_E \right) \partial_E \ln \mathbb{P}_n + n_\ell, \\
\bar{H}_\ell^{(1)} &:= \left( -\kappa_\ell \partial_E \varepsilon + (\kappa_\ell (\varepsilon - 1) + 2) (\partial_b^{(\ell)} + \kappa_\ell \partial_E) \right) \ln \mathbb{P}_n + \bar{C}_\ell, \\
H_\ell^{(2)} &:= (1 - \varepsilon + 2b_\ell \partial_E) \left( \partial_b^{(\ell)} + \kappa_\ell \partial_E \right) \ln \mathbb{P}_n, \\
G_{\ell+1} &:= \partial_E G_\ell + \sum_{i=1}^p (\partial_E)^\ell F_i \left( \partial_E \frac{\bar{H}_i^{(1)}}{F_i} - \partial_b^{(i)} \frac{H_i^{(2)}}{F_i} \right), \quad G_0 = 0, \\
\bar{C}_\ell &:= -2n_\ell \left( (1 - \kappa_\ell) b_\ell + \sum_{j \neq \ell} \frac{n_j}{b_\ell - b_j} \right).
\end{aligned} \tag{1.15}$$

In section 7, we shall work out two examples, immediate applications of the equations in Theorem 1.1 and Corollary 1.2. In the first example, we describe nonintersecting Brownian motions, leaving from 0 and forced back to 0. The second example deals with the situation of several target points with the extreme ones being symmetric with regard to the origin. That model will also be used later in Section 8.

**Pearcey process with inliers:** In section 8, we consider non-intersecting Brownian motions leaving from 0 and forced to  $p$  target points at time  $t = 1$ , with the only condition that the left-most and right-most target points are symmetric with respect to the origin, with  $p - 2$  intermediate target points thrown in totally arbitrarily; it is convenient to rename the target points  $\beta_1 < \dots < \beta_p$ , as follows:

$$\begin{array}{ccccccc}
\tilde{a} & < & -\tilde{c}_1 & < & \dots & < & -\tilde{c}_{p-2} & < & -\tilde{a} \\
n_+ & & n_1 & & \dots & & n_{p-2} & & n_-
\end{array} \tag{1.16}$$

with the corresponding number of particles forced to those points at time  $t = 1$ . The purpose of this section is to identify the critical process obtained by letting  $n := n_+ = n_- \rightarrow \infty$  and by rescaling  $\tilde{a}$  and the  $\tilde{c}_i$  accordingly, while keeping  $n_1, \dots, n_{p-2}$  fixed. We let  $\tilde{a}$  go to  $-\infty$  like  $-\sqrt{n}$  and  $-\tilde{a}$  to  $\infty$  like  $\sqrt{n}$ . The target points  $-\tilde{c}_1, \dots, -\tilde{c}_{p-2}$  of the inliers move to  $\infty$  as well, but at a much slower rate, namely like  $-u_\ell \left(\frac{n}{2}\right)^{1/4}$ . A new process will appear at the point of bifurcation, where the bulk of the particles forced to  $-\sqrt{n}$  depart from those going to  $\sqrt{n}$ , namely the *Pearcey process with inliers*, which generalizes the Pearcey process found by C. Tracy and H. Widom [19]. It describes the statistical fluctuations near that point of bifurcation; it will be sensitive to the presence of inliers and will be different in the absence of inliers (Pearcey process). We will compute the kernel governing the transition probabilities and also apply the formulae obtained in Corollary 1.2 to compute a PDE for the gap probability, which, to our surprise, appears to be an *exact*  $p \times p$  *Wronskian*. This is the content of Theorem 1.3.

**Theorem 1.3** Pick times  $\tau_1 < \dots < \tau_m$ , subsets  $E_j \subset \mathbb{R}$  for  $j = 1, \dots, m$  and parameters  $u_\ell$  for  $\ell = 1, \dots, p-2$ . Consider  $2n + \sum_{\ell=1}^{p-2} n_\ell$  non-intersecting Brownian motions, such that

- (i) all particles leave from 0 at time  $t = 0$ ,
- (ii)  $n = n_\pm$  particles are forced to  $\pm\sqrt{n}$  at time  $t = 1$ ,
- (iii)  $n_\ell$  paths are forced to points<sup>8</sup>  $-u_\ell \left(\frac{n}{2}\right)^{1/4}$  at time  $t = 1$  ( $1 \leq \ell \leq p$ ).

Then the following Brownian motion limit holds for the gap probability, about time  $t = 1/2$ , keeping  $n_\ell$  fixed,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \left( \bigcap_{j=1}^m \left\{ \text{all } x_i \left( \frac{1}{2} + \frac{\tau_j}{4\sqrt{2n}} \right) \in \frac{E_j^c}{4(n/2)^{1/4}} \right\} \right) \\ = \mathbb{P}^{\mathcal{P}}(u_1, \dots, u_{p-2}) \left( \bigcap_{j=1}^m \{ \mathcal{P}(\tau_j) \cap E_j = \emptyset \} \right) \\ = \det \left( \mathbb{1} - \left( \chi_{E_i} K_{\tau_i \tau_j}^{\mathcal{P}} \chi_{E_j} \right)_{1 \leq i, j \leq m} \right), \end{aligned} \quad (1.17)$$

where this probability is given by the Fredholm determinant of the Pearcey matrix kernel with inliers, which is a rational perturbation of the customary Pearcey kernel<sup>9</sup>, namely

$$\begin{aligned} K_{s,t}^{\mathcal{P}}(X, Y; u_1, \dots, u_{p-2}) \\ = -\frac{1}{4\pi^2} \int_{\mathbf{X}} dV \int_{-i\infty}^{i\infty} dU \frac{1}{U-V} \frac{e^{-\frac{U^4}{4} + \frac{iU^2}{2} - UY}}{e^{-\frac{V^4}{4} + \frac{sV^2}{2} - VX}} \prod_{\ell=1}^{p-2} \left( \frac{U + u_\ell}{V + u_\ell} \right)^{n_\ell} \\ - \begin{cases} 0 & \text{for } t - s \leq 0 \\ \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(X-Y)^2}{2(t-s)}} & \text{for } t - s > 0. \end{cases} \end{aligned} \quad (1.18)$$

The log of the gap probability ( $\mathbb{E} = E_1 \times \dots \times E_m$ )

$$\mathbb{Q}(\tau_1, \dots, \tau_m; u_1, \dots, u_{p-2}; \mathbb{E}) := \ln \mathbb{P}^{\mathcal{P}}(u_1, \dots, u_{p-2}) \left( \bigcap_{j=1}^m \{ \mathcal{P}(\tau_j) \cap E_j = \emptyset \} \right)$$

satisfies a partial differential equation, which is a  $p \times p$  Wronskian with respect to the operator  $\partial_{\mathbb{E}} = \sum_{i=1}^m \partial_{E_i}$ :

$$\mathcal{W}_p \left[ \partial_{\mathbb{E}}^2 \partial_\tau \mathbb{Q}, \partial_{\mathbb{E}}^2 \frac{\partial \mathbb{Q}}{\partial u_1}, \dots, \partial_{\mathbb{E}}^2 \frac{\partial \mathbb{Q}}{\partial u_{p-2}}, \mathbb{X} \right]_{\partial_{\mathbb{E}}} = 0, \quad (1.19)$$

<sup>8</sup>Note that those points belong to the interval  $[-\sqrt{n}, \sqrt{n}]$  for large enough  $n$ .

<sup>9</sup> $\mathbf{X}$  stands for the contour



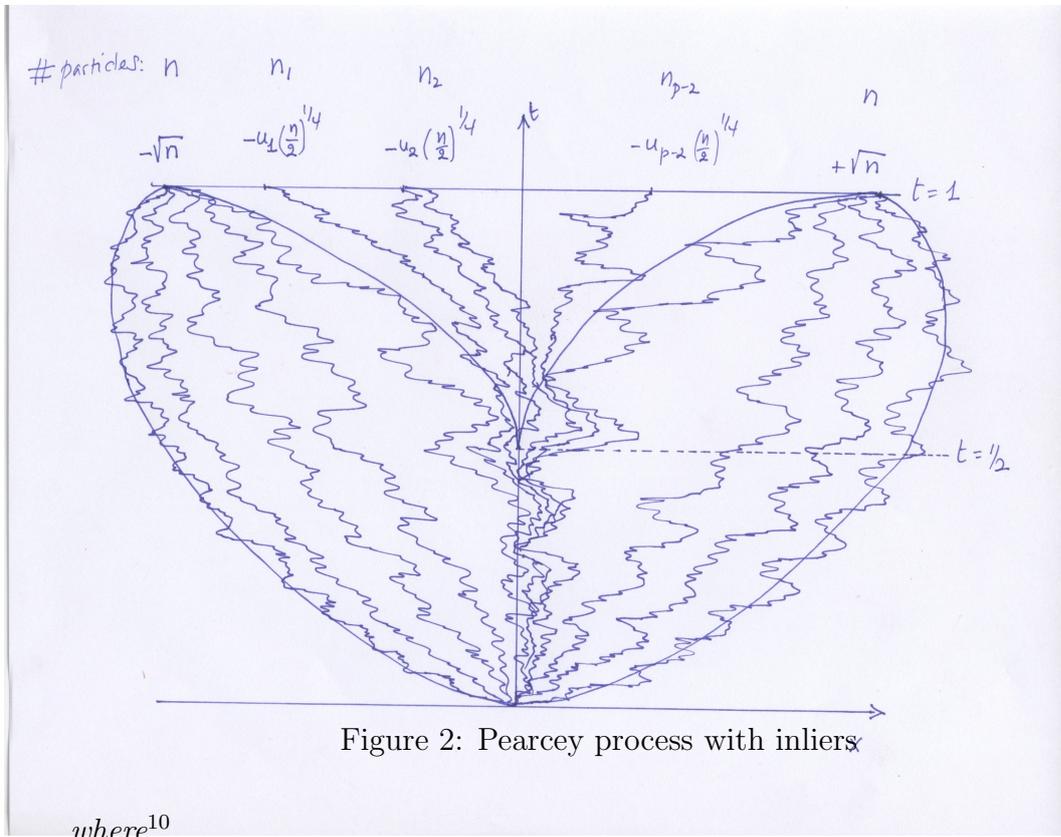


Figure 2: Pearcey process with inliers

where<sup>10</sup>

$$\mathbb{X} := (\varepsilon_{\mathbb{E}} - \varepsilon_u + 2\varepsilon_{\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 4\partial_u\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q} + 8\partial_{\tau}^3\mathbb{Q} - 4\tilde{\partial}_{\mathbb{E}}\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q} + 4\left\{\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q}, \partial_{\mathbb{E}}^2\mathbb{Q}\right\}_{\partial_{\mathbb{E}}}. \quad (1.20)$$

For one-time ( $m = 1$ ), the expression  $\mathbb{X}$  reads as follows:

$$\mathbb{X} := (\varepsilon_{\mathbb{E}} - \varepsilon_u - 2\tau\frac{\partial}{\partial\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 4\partial_u\partial_{\mathbb{E}}\frac{\partial\mathbb{Q}}{\partial\tau} + 8\frac{\partial^3\mathbb{Q}}{\partial\tau^3} + 4\left\{\partial_{\mathbb{E}}\frac{\partial\mathbb{Q}}{\partial\tau}, \partial_{\mathbb{E}}^2\mathbb{Q}\right\}_{\partial_{\mathbb{E}}}. \quad (1.21)$$

**Remark:** The term  $\varepsilon_u\partial_{\mathbb{E}}^2\mathbb{Q}$  could be omitted in the definition of  $\mathbb{X}$ , since it is a linear combination of  $(p-1)$  columns in the matrix (1.19) (from the second  $\times u_1$  to the  $(p-1)$ st column  $\times u_{p-2}$ ). We nevertheless keep this term in the expression, in view of Conjecture 1.5.

In the absence of inliers, one obtains, in particular, the PDE for the transition probability of the Pearcey process: it is a  $2 \times 2$  Wronskian with  $\mathbb{X}$  as in (1.20) and (1.21), but without the  $u$ -partials. In [3], it is shown that the transition probability of the Pearcey process satisfies the simpler equation  $\mathbb{X} = 0$ .

**Corollary 1.4** [3] *In the absence of inliers ( $p = 2$ ),*

$$\mathbb{Q}(\tau_1, \dots, \tau_m; \mathbb{E}) := \ln \mathbb{P}^{\mathcal{P}} \left( \bigcap_{j=1}^m \{\mathcal{P}(\tau_j) \cap E_j = \emptyset\} \right)$$

<sup>10</sup>Remember for  $u = (u_1, \dots, u_{p-2})$  and  $\tau = (\tau_1, \dots, \tau_m)$ , one has  $\partial_u = \sum_1^{p-2} \frac{\partial}{\partial u_i}$ ,  $\varepsilon_u = \sum_1^{p-2} u_i \frac{\partial}{\partial u_i}$ ,  $\partial_{\tau} = \sum_1^m \frac{\partial}{\partial \tau_i}$ . One also needs  $\varepsilon_{\mathbb{E}} := \sum_1^m \varepsilon_{E_i}$  and the mixed time-space derivative  $\tilde{\partial}_{\mathbb{E}} := \sum_1^m \tau_i \partial_{E_i}$ .

satisfies

$$(\varepsilon_{\mathbb{E}} + 2\varepsilon_{\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 8\partial_{\tau}^3\mathbb{Q} - 4\tilde{\partial}_{\mathbb{E}}\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q} + 4\{\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q}, \partial_{\mathbb{E}}^2\mathbb{Q}\}_{\partial_{\mathbb{E}}} = 0,$$

and for the one-time case ( $m = 1$ ),

$$(\varepsilon_{\mathbb{E}} - 2\tau\frac{\partial}{\partial\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 8\frac{\partial^3\mathbb{Q}}{\partial\tau^3} + 4\left\{\partial_{\mathbb{E}}\frac{\partial\mathbb{Q}}{\partial\tau}, \partial_{\mathbb{E}}^2\mathbb{Q}\right\}_{\partial_{\mathbb{E}}} = 0.$$

We now formulate a conjecture, stating that, even with inliers, the equation for the transition probability reads  $\mathbb{X} = 0$ , where  $\mathbb{X}$  is given by (1.20) and (1.21):

**Conjecture 1.5** *Even with inliers ( $p > 2$ ), we conjecture that the function*

$$\mathbb{Q}(\tau_1, \dots, \tau_m; u_1, \dots, u_{p-2}; \mathbb{E}) := \ln \mathbb{P}^{\mathcal{P}(u_1, \dots, u_{p-2})} \left( \bigcap_{j=1}^m \{\mathcal{P}(\tau_j) \cap E_j = \emptyset\} \right)$$

satisfies

$$\mathbb{X} = (\varepsilon_{\mathbb{E}} - \varepsilon_u + 2\varepsilon_{\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 4\partial_u\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q} + 8\partial_{\tau}^3\mathbb{Q} - 4\tilde{\partial}_{\mathbb{E}}\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q} + 4\{\partial_{\mathbb{E}}\partial_{\tau}\mathbb{Q}, \partial_{\mathbb{E}}^2\mathbb{Q}\}_{\partial_{\mathbb{E}}} = 0, \quad (1.22)$$

and for the one-time case ( $m = 1$ ),

$$\mathbb{X} = (\varepsilon_{\mathbb{E}} - \varepsilon_u - 2\tau\frac{\partial}{\partial\tau} - 2)\partial_{\mathbb{E}}^2\mathbb{Q} + 4\partial_u\partial_{\mathbb{E}}\frac{\partial\mathbb{Q}}{\partial\tau} + 8\frac{\partial^3\mathbb{Q}}{\partial\tau^3} + 4\left\{\partial_{\mathbb{E}}\frac{\partial\mathbb{Q}}{\partial\tau}, \partial_{\mathbb{E}}^2\mathbb{Q}\right\}_{\partial_{\mathbb{E}}} = 0. \quad (1.23)$$

The PDE's play a *prominent role* in obtaining certain approximations which would be very hard to obtain without that technology. An example will be given here, without proof, for the Pearcey process without inliers. At the point of bifurcation, mentioned above, there appears a cusp in the Pearcey scale  $\xi = \pm\frac{2}{27}(3\tau)^{3/2}$ , such that, roughly speaking, most Pearcey process paths stay completely to the left or to the right of this cusp. Upon comparing the Pearcey process with, say, the right branch of the cusp in the new (crude) space-scale  $(3\tau)^{1/6}$ , and letting two different times  $\tau_1$  and  $\tau_2$  tend to  $\infty$  in a very specific way, one is led to the so-called Airy process  $\mathcal{A}(t)$ . The exact approximation is given in the Theorem below taken from [1]:

**Theorem 1.6** *Let  $\tau_1, \tau_2 \rightarrow \infty$ , such that*

$$\frac{\tau_2 - \tau_1}{2(t_2 - t_1)} = (3\tau_1)^{1/3} + \frac{t_2 - t_1}{(3\tau_1)^{1/3}} + \frac{2t_1t_2}{3\tau_1} + O\left(\frac{1}{\tau_1^{5/3}}\right);$$

this specifies two new times  $t_1$  and  $t_2$ . The following approximation, far out along the cusp, of the Pearcey process by the Airy process holds:

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{i=1}^2 \left\{ \frac{\mathcal{P}(\tau_i) - \frac{2}{27}(3\tau_i)^{3/2}}{(3\tau_i)^{1/6}} \cap (-E_i) = \emptyset \right\} \right) \\ &= \mathbb{P} \left( \bigcap_{i=1}^2 \{ \mathcal{A}(t_i) \cap (-E_i) = \emptyset \} \right) \left( 1 + O\left(\frac{1}{\tau_1^{4/3}}\right) \right). \end{aligned}$$

*Remark:* The  $O(\tau_1^{-4/3})$ -approximation, obtained via the PDE is much better than any rough estimate one might predict. Also one expects that, in this precise limit, the Pearcey process with inliers tends to the Airy process with outliers; see [2].

## 2 Non-intersecting Brownian motions and a chain of Coupled Random Matrices

Setting

$$\tau_k := t_{k+1} - t_k \quad \text{and} \quad \frac{1}{\sigma_k} := \frac{1}{t_k - t_{k-1}} + \frac{1}{t_{k+1} - t_k}, \quad \text{for } 1 \leq k \leq m,$$

and taking in (1.1) the limit  $\gamma_i \rightarrow 0$ , for  $i = 1, \dots, N$ , leads to

$$\begin{aligned} & \mathbb{P} \left( \bigcap_{k=1}^m \left\{ \text{all } x_i(t_k) \in \tilde{E}_k \right\} \mid \begin{array}{l} x_j(0) = 0, \quad x_j(1) = \delta_j, \\ \text{for } j = 1, \dots, N \end{array} \right) \\ &= \frac{1}{Z'_n} \int_{\mathbb{E}^N} \Delta_N(u_1) \prod_{k=1}^m \left[ \det \left( e^{\frac{2u_{k;i}u_{k+1;j}}{\tau_k}} \right)_{1 \leq i, j \leq N} \prod_{1 \leq i \leq N} e^{-\frac{u_{k;i}^2}{\sigma_k}} du_{k;i} \right], \quad (2.1) \end{aligned}$$

where  $\Delta_N(u_1)$  stands for the Vandermonde determinant in the variables  $u_1 = (u_{1;1}, \dots, u_{1;N})$ . Notice that each of the sets of variables  $u_1, \dots, u_m$  appears in exactly two of the determinants in the above integrand and that the other factors are insensitive to a permutation, for fixed  $k$  with  $1 \leq k \leq m$ , of the variables  $u_k = u_{k;1}, \dots, u_{k;N}$ . Therefore, taking the limit  $u_{m+1;i} = \delta_i \rightarrow \beta_j$ , for  $i = 1, \dots, N$ , with  $n_\ell$  of the  $\delta_i$  going to  $\beta_\ell$ , namely  $u_{m+1;1}, \dots, u_{m+1;n_1} \rightarrow \beta_1$ , and so on, making  $m$  synchronized changes of variables, and using the

symmetry of the integration ranges vis-à-vis these variables  $u_{k;1}, \dots, u_{k;N}$ ,

$$\begin{aligned}
& \mathbb{P} \left( \bigcap_{k=1}^m \left\{ \text{all } x_i(t_k) \in \tilde{E}_k \right\} \left| \begin{array}{l} x_j(0) = 0, (j = 1, \dots, N), \\ x_1(1) = \dots = x_{n_1}(1) = \beta_1, \\ \vdots \\ x_{N-n_p+1}(1) = \dots = x_N(1) = \beta_p \end{array} \right. \right) \\
&= \frac{1}{Z_n''} \int_{\tilde{\mathbb{E}}^N} \Delta_N(u_1) \prod_{\ell=1}^p \left( \Delta_{n_\ell}(u_m^{(\ell)}) \prod_{i=1}^{n_\ell} e^{-\sum_{k=1}^m \frac{u_{k;i}^{(\ell)2}}{\sigma_k} + \sum_{k=1}^{m-1} \frac{2u_{k;i}^{(\ell)} u_{k+1;i}^{(\ell)}}{\tau_k} + \frac{2\beta_\ell u_{m;i}^{(\ell)}}{\tau_m}} \right) \prod_{\substack{1 \leq i \leq N \\ 1 \leq k \leq m}} du_{k;i}, \\
&= \frac{1}{Z_n'''} \int_{\mathbb{E}^N} \Delta_N(v_1) \prod_{\ell=1}^p \left( \Delta_{n_\ell}(v_m^{(\ell)}) \prod_{i=1}^{n_\ell} e^{-\sum_{k=1}^m \frac{1}{2} v_{k;i}^{(\ell)2} + \sum_{k=1}^{m-1} c_k v_{k;i}^{(\ell)} v_{k+1;i}^{(\ell)} + b_\ell v_{m;i}^{(\ell)}} \right) \prod_{\substack{1 \leq i \leq N \\ 1 \leq k \leq m}} dv_{k;i}, \\
&=: \frac{1}{Z_n'''} \int_{\mathbb{E}^N} I_n(v) \prod_{k=1}^m dv_k, \tag{2.2} \\
&= \frac{1}{\tilde{Z}_n} \int_{\text{spec}(M_k) \in E_k} e^{-\frac{1}{2} \text{tr} \left( \sum_{k=1}^m M_k^2 - 2 \sum_{k=1}^{m-1} c_k M_k M_{k+1} - 2A M_m \right)} \prod_{k=1}^m dM_k, \tag{2.3}
\end{aligned}$$

where the diagonal matrix  $A$ ,  $c_k$ ,  $\tilde{b}_\ell$  and  $\tilde{E}_k$  were defined in (1.4) or alternatively expressed below in terms of the  $\sigma_k$ 's and  $\tau_k$ 's. The last integration is taken over Hermitian matrices, with  $\text{spec}(M_k) \in E_k$ . Also the change of integration variables  $u_{k;i}^{(\ell)} \mapsto v_{k;i}^{(\ell)}$  above is given by

$$v_{k;i}^{(\ell)} = \sqrt{\frac{2}{\sigma_k}} u_{k;i}^{(\ell)}, \quad c_k = \frac{\sqrt{\sigma_k \sigma_{k+1}}}{\tau_k}, \quad b_\ell = \frac{\sqrt{2\sigma_m}}{\tau_m} \beta_\ell, \quad E_k = \sqrt{\frac{2}{\sigma_k}} \tilde{E}_k.$$

For  $k = 1, \dots, m$  and for  $\ell = 1, \dots, p$ , the vector  $u_k^{(\ell)} = (u_{k;1}^{(\ell)}, \dots, u_{k;n_\ell}^{(\ell)})$  is defined by

$$(u_{k;1}, \dots, u_{k;N}) = (u_{k;1}^{(1)}, \dots, u_{k;n_1}^{(1)}, u_{k;1}^{(2)}, \dots, u_{k;n_2}^{(2)}, \dots, u_{k;1}^{(p)}, \dots, u_{k;n_p}^{(p)}).$$

Concerning the Jacobi matrix (1.8), one needs the following formulas for derivatives of  $\mathcal{J}$ ; they can be shown by recurrence:

$$c_1 \frac{\partial}{\partial c_1} \mathcal{J}_{mm} = -2\mathcal{J}_{m1}^2, \quad c_{m-1} \frac{\partial}{\partial c_{m-1}} \mathcal{J}_{m1} = -\mathcal{J}_{m1}(2\mathcal{J}_{mm} + 1). \tag{2.4}$$

### 3 Integrable deformations

In this section, we introduce a time deformation  $\tilde{I}_n(v)$  of the integrand  $I_n(v)$ , introduced in (2.3). The deformation is chosen such that the resulting integral

is on the one hand a solution to the multi-component KP hierarchy (see [8] and Proposition 3.1 below) and satisfies on the other hand a set of Virasoro constraints. We will impose on the rescaled target points  $b_1, \dots, b_p$ , which we henceforth denote by  $b_1^{(1)}, \dots, b_1^{(p)}$ , a non-trivial linear constraint

$$\sum_{\ell=1}^p \kappa_\ell b_1^{(\ell)} = 0. \quad (3.1)$$

Without loss of generality, we may assume (upon reordering) that  $\kappa_p \neq 0$  and impose if  $\sum_1^p \kappa_\ell \neq 0$  that  $\sum_{\ell=1}^p \kappa_\ell = 1$ ; also define  $\kappa_0 := -1$ . Thus, the non-deformed integral which we will consider is

$$\int_{\mathbb{E}^N} I_n(v) \Big|_{\sum_1^p \kappa_\ell b_{1,2}^{(\ell)} = 0} \prod_{k=1}^m dv_k. \quad (3.2)$$

The integrand  $I_n(v)$  will be deformed by four sets of parameters: **(i)** A first set, denoted by  $b_2^{(1)}, \dots, b_2^{(p)}$ , deforms the parameters  $b_1^{(\ell)}$ . They are subjected to the same constraint (3.1) as the parameters  $b_1^{(\ell)}$ , namely<sup>11</sup>

$$\sum_{\ell=1}^p \kappa_\ell b_2^{(\ell)} = 0. \quad (3.3)$$

**(ii)** A second set of deformations consists of parameters corresponding to the KP time variables; they are denoted by  $s_r^{(0)}$  ( $r \in \mathbb{Z}_{>0}$ ) for the parameters going with the starting point 0 of the Brownian motion and  $s_r^{(\ell)}$  ( $1 \leq \ell \leq p$  and  $r \in \mathbb{Z}_{>0}$ ) for the parameters going with the  $\ell$ -th end point of the Brownian motion. **(iii)** There is furthermore a set of parameters  $\gamma_r^{(k)}$  ( $2 \leq k \leq m-1$  and  $r \in \mathbb{Z}_{>0}$ ) going with the intermediate times  $t_2, \dots, t_{m-1}$  and **(iv)** a set of parameters  $c_{r,q}^{(k)}$  ( $k = 1, \dots, m-1$  and<sup>12</sup>  $(r, q) > (1, 1)$ ), going with consecutive times  $t_k, t_{k+1}$ .

For  $n = (n_1, \dots, n_p)$  and  $\mathbb{E} = E_1 \times E_2 \times \dots \times E_m$ , where each  $E_k$  is the union of a finite number of intervals in  $\mathbb{R}$ , define

$$\tau_n(\mathbb{E}) := \int_{\mathbb{E}^N} \tilde{I}_n(v) \Big|_{\sum_1^p \kappa_\ell b_{1,2}^{(\ell)} = 0} \prod_{k=1}^m dv_k, \quad (3.4)$$

<sup>11</sup>The combination of the two constraints (3.1) and (3.3) will in the formulas below be denoted by  $\sum_1^p \kappa_\ell b_{1,2}^{(\ell)} = 0$ .

<sup>12</sup>The inequality  $(r, q) > (1, 1)$  means by definition that  $r \geq 1, q \geq 1$  and  $(r, q) \neq (1, 1)$ .

where

$$\tilde{I}_n(v) = I_n(v) \times \prod_{\ell=1}^p \prod_{i=1}^{n_\ell} e^{b_2^{(\ell)} v_{m;i}^{(\ell)2} + \sum_{r \geq 1} (s_r^{(0)} v_{1;i}^{(\ell)r} - s_r^{(\ell)} v_{m;i}^{(\ell)r}) + \sum_{k=1}^{m-1} \sum_{(r,q) > (1,1)} c_{rq}^{(k)} v_{k;i}^{(\ell)r} v_{k+1;i}^{(\ell)q} + \sum_{k=2}^{m-1} \sum_{r \geq 1} \gamma_r^{(k)} v_{k;i}^{(\ell)r}},$$

with

$$I_n(v) = \frac{\Delta_N(v_1)}{\prod_{\ell=1}^p n_\ell!} \prod_{\ell=1}^p \left( \Delta_{n_\ell}(v_m^{(\ell)}) \prod_{i=1}^{n_\ell} e^{\sum_{k=1}^{m-1} c_k v_{k;i}^{(\ell)} v_{k+1;i}^{(\ell)} - \frac{1}{2} \sum_{k=1}^m v_{k;i}^{(\ell)2} + b_1^{(\ell)} v_{m;i}^{(\ell)}} \right).$$

We denote by  $\mathcal{L}$  the locus corresponding to setting all deformation parameters equal to zero, so that  $\tilde{I}_n|_{\mathcal{L}} = I_n$ ,

$$\mathcal{L} = \left\{ \begin{array}{l} s_r^{(0)}, \dots, s_r^{(p)} = 0, \quad r \in \mathbb{Z}_{>0}, \\ b_2^{(1)}, \dots, b_2^{(p)} = 0, \\ \gamma_r^{(2)}, \dots, \gamma_r^{(m-1)} = 0, \quad r \in \mathbb{Z}_{>0}, \\ c_{rq}^{(1)}, \dots, c_{rq}^{(m-1)} = 0, \quad (r, q) > (1, 1) \end{array} \right\}. \quad (3.5)$$

We list a number of operator identities, valid when acting on  $\tau_n(\mathbb{E})$ ,

$$\frac{\partial}{\partial b_h^{(\ell)}} = -\frac{\partial}{\partial s_h^{(\ell)}} + \frac{\kappa_\ell}{\kappa_p} \frac{\partial}{\partial s_h^{(p)}}, \quad 1 \leq \ell \leq p-1, \quad h = 1, 2, \quad (3.6)$$

$$\sum_{\ell=1}^p b_j^{(\ell)} \frac{\partial}{\partial s_h^{(\ell)}} = -\sum_{\ell=1}^{p-1} b_j^{(\ell)} \frac{\partial}{\partial b_h^{(\ell)}}, \quad h, j \in \{1, 2\}, \quad (3.7)$$

$$\begin{aligned} \frac{\partial}{\partial s_h^{(\ell)}} &= -(1 - \delta_{\ell,p}) \frac{\partial}{\partial b_h^{(\ell)}} + \kappa_\ell \left( \sum_{i=1}^p \frac{\partial}{\partial s_h^{(i)}} + \sum_{i=1}^{p-1} \frac{\partial}{\partial b_h^{(i)}} \right) \\ &= \partial_{b_h}^{(\ell)} + \kappa_\ell \sum_{i=1}^p \frac{\partial}{\partial s_h^{(i)}}, \quad h = 1, 2, \quad 1 \leq \ell \leq p, \end{aligned} \quad (3.8)$$

where for  $h = 1, 2$  and  $1 \leq \ell \leq p$  we define

$$\partial_{b_h}^{(\ell)} := -(1 - \delta_{\ell,p}) \frac{\partial}{\partial b_h^{(\ell)}} + \kappa_\ell \sum_{i=1}^{p-1} \frac{\partial}{\partial b_h^{(i)}} = \sum_{i=1}^{p-1} (\kappa_\ell - \delta_{\ell,i}) \frac{\partial}{\partial b_h^{(i)}}, \quad (3.9)$$

implying

$$\sum_{\ell=1}^p \partial_{b_h}^{(\ell)} = 0. \quad (3.10)$$

Using  $\sum_{\ell=1}^p \kappa_\ell b_h^{(\ell)} = 0$ , one first establishes identity (3.6) and then (3.7), while the first equality in (3.8) is obtained by computing  $\sum_{i=1}^{p-1} \frac{\partial}{\partial b_h^{(i)}}$  from (3.6) and by using  $\sum_{\ell=1}^p \kappa_\ell = 1$  and the identity (3.6).

From section 7.3 in [8], it follows that  $\tau_n(\mathbb{E})$  can be written as

$$\tau_n(\mathbb{E}) = \det \begin{pmatrix} (\langle y^i \varphi_1(y) | x^j \psi(x) \rangle)_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq N}} \\ \vdots \\ (\langle y^i \varphi_p(y) | x^j \psi(x) \rangle)_{\substack{0 \leq i \leq n_p \\ 0 \leq j \leq N}} \end{pmatrix}, \quad (3.11)$$

where

$$\begin{aligned} \psi(x) &:= \exp \left( -\frac{1}{2}x^2 + \sum_{r \geq 1} s_r^{(0)} x^r \right), \\ \varphi_\ell(y) &:= \exp \left( -\frac{1}{2}y^2 + b_1^{(\ell)} y + b_2^{(\ell)} y^2 - \sum_{r \geq 1} s_r^{(\ell)} y^r \right), \end{aligned}$$

for  $\ell = 1, \dots, p$ , and where the inner product  $\langle \cdot | \cdot \rangle$  is defined by

$$\langle f(y) | g(x) \rangle := \iint_{E_1 \times E_m} f(y) g(x) \mu(x, y) dx dy,$$

with

$$\begin{aligned} \mu(x, y) &:= \int_{\prod_{k=2}^{m-1} E_k} \exp \sum_{k=2}^{m-1} \left( -\frac{1}{2} w_k^2 + \sum_{r \geq 1} \gamma_r^{(k)} w_k^r \right) \times \\ &\quad \exp \sum_{k=1}^{m-1} \left( c_k w_k w_{k+1} + \sum_{(r,q) > (1,1)} c_{rq}^{(k)} w_k^r w_{k+1}^q \right) \prod_{k=2}^{m-1} dw_k, \end{aligned}$$

$w_1 := x$  and  $w_m := y$ . For  $m = 2$  the latter formula for  $\mu$  should be interpreted as  $\mu(x, y) := 1$ , while  $\mu(x, y) := \delta(x - y)e^{x^2/2}$  (the delta distribution) in the case of  $m = 1$ .

The above representation (3.11) of  $\tau_n$  implies, in view of [8, Prop. 6.2], that  $\tau_n$  is a tau function of the  $p + 1$  component KP hierarchy, in particular we have the following Proposition.

**Proposition 3.1** *The function  $\tau_n = \tau_n(\mathbb{E})$ , as in (3.4), satisfies for  $1 \leq \ell \leq p$*

$$\frac{\partial}{\partial s_1^{(0)}} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} = \frac{\frac{\partial^2}{\partial s_2^{(0)} \partial s_1^{(\ell)}} \ln \tau_n}{\frac{\partial^2}{\partial s_1^{(0)} \partial s_1^{(\ell)}} \ln \tau_n}, \quad \frac{\partial}{\partial s_1^{(\ell)}} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} = -\frac{\frac{\partial^2}{\partial s_1^{(0)} \partial s_2^{(\ell)}} \ln \tau_n}{\frac{\partial^2}{\partial s_1^{(0)} \partial s_1^{(\ell)}} \ln \tau_n}, \quad (3.12)$$

where  $n \pm e_\ell = (n_1, \dots, n_p) \pm e_\ell := (n_1, \dots, n_{\ell-1}, n_\ell \pm 1, n_{\ell+1}, \dots, n_p)$ .

Both equations will play an important role in Section 6 below.

## 4 The Virasoro constraints

Remembering the definition (1.10) of the operators  $\partial_E$  and  $\varepsilon_E$  and the definition (3.9) of the operators  $\partial_{b_k}^{(\ell)}$ , define for  $\ell = 1, \dots, p$  the operators:

$$\mathcal{B}_1^{(0)} := \sum_{k=1}^m \mathcal{J}_{1k} \partial_{E_k} - 2\mathcal{J}_{1m} \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_1^{(i)}}, \quad (4.1)$$

$$\mathcal{B}_1^{(\ell)} := \partial_{b_1}^{(\ell)} - \kappa_\ell \left( \sum_{k=1}^m \mathcal{J}_{mk} \partial_{E_k} - 2\mathcal{J}_{mm} \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_1^{(i)}} \right), \quad (4.2)$$

$$\mathcal{B}_2^{(0)} := -\varepsilon_{E_1} + c_1 \frac{\partial}{\partial c_1} + \delta_{1,m} \left( \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} + 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_2^{(i)}} \right), \quad (4.3)$$

$$\mathcal{B}_2^{(\ell)} := \partial_{b_2}^{(\ell)} - \kappa_\ell \left( -\varepsilon_{E_m} + c_{m-1} \frac{\partial}{\partial c_{m-1}} + \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} + 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_2^{(i)}} \right) \quad (4.4)$$

We show in the following proposition, how the action of these operators on the tau function can be represented by time derivatives.

**Proposition 4.1** *The integral  $\tau_n(\mathbb{E})$ , as in (3.4), satisfies<sup>13</sup>, for  $\ell = 0, \dots, p$  and  $h = 1, 2$ ,*

$$\mathcal{B}_h^{(\ell)} \ln \tau_n = \left( \frac{\partial}{\partial s_h^{(\ell)}} + \kappa_\ell \Sigma_h^{(\ell)} \right) \ln \tau_n + \kappa_\ell T_h^{(\ell)}, \quad (4.5)$$

where

$$T_1^{(\alpha)} = \begin{cases} -\mathcal{J}_{11} N s_1^{(0)} - \mathcal{J}_{1m} \sum_{\ell=1}^p n_\ell (b_1^{(\ell)} - s_1^{(\ell)}) & \alpha = 0, \\ -\mathcal{J}_{1m} N s_1^{(0)} - \mathcal{J}_{mm} \sum_{\ell=1}^p n_\ell (b_1^{(\ell)} - s_1^{(\ell)}) & \alpha \neq 0, \end{cases} \quad (4.6)$$

$$T_2^{(\alpha)} = \begin{cases} \sum_{1 \leq i \leq j \leq p} n_i n_j & m = 1, \\ N(N+1)/2 & \alpha = 0 \text{ and } m > 1, \\ \sum_{\ell=1}^p n_\ell (n_\ell + 1)/2 & \alpha \neq 0 \text{ and } m > 1, \end{cases} \quad (4.7)$$

and each  $\Sigma_h^{(\alpha)}$  is a homogeneous first order differential operator in all deformation parameters, except for the deformation parameters  $b_2^{(\ell)}$ , so that  $\Sigma_k^{(\alpha)} \Big|_{\mathcal{L}} = 0$ , and moreover, for  $k = 1, 2$  and for  $\ell = 1, \dots, p$ ,

$$\left[ \frac{\partial}{\partial s_1^{(\ell)}}, \Sigma_h^{(0)} \right] = \delta_{h,2} \delta_{1,m} \frac{\partial}{\partial s_1^{(\ell)}}, \quad \left[ \frac{\partial}{\partial s_1^{(0)}}, \Sigma_2^{(\ell)} \right] = \delta_{1,m} \frac{\partial}{\partial s_1^{(0)}}. \quad (4.8)$$

<sup>13</sup>Recall that  $\kappa_0 = -1$ .

*Proof:* We give a detailed proof for the case of  $m = 2$  (see remark 4.2 for the case of  $m > 2$  and see remark 4.3 for the special case of  $m = 1$ ). Then  $c_{m-1} = c_1$ , which we simply write as  $c$ . Also,  $\mathcal{J}$  is the  $2 \times 2$  matrix

$$\mathcal{J} = \begin{pmatrix} -1 & c \\ c & -1 \end{pmatrix}^{-1} = \frac{-1}{1-c^2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}.$$

In this case, referring to (3.4), there are two sets of variables  $v_1$  and  $v_2$ , which we denote by  $x$  and  $y$ , there are no deformation parameters  $\gamma_r^{(k)}$  and there is a single set of deformation parameters  $c_{rq}^{(1)}$ , which we will denote by  $c_{rq}$ . For  $E_1, E_2 \subset \mathbb{R}$ , and taking into account the usual constraint  $\sum_{\ell=1}^p \kappa_\ell b_{1,2}^{(\ell)} = 0$ ,

$$\tau_n(E_1, E_2) := \frac{1}{\prod_{\ell=1}^p n_\ell!} \iint_{E_1^N \times E_2^N} \tilde{I}_n(x, y) dx dy, \quad (4.9)$$

where

$$\tilde{I}_n(x, y) = \Delta_N(x) \prod_{\ell=1}^p \left( \Delta_{n_\ell}(y) \prod_{i=1}^{n_\ell} e^{-\frac{1}{2}x_i^{(\ell)2} - \frac{1}{2}y_i^{(\ell)2} + cx_i^{(\ell)}y_i^{(\ell)} + b_1^{(\ell)}y_i^{(\ell)}} \times \right. \quad (4.10)$$

$$\left. e^{b_2^{(\ell)}y_i^{(\ell)2} + \sum_{r \geq 1} (s_r^{(0)}x_i^{(\ell)r - s_r^{(\ell)}y_i^{(\ell)r}) + \sum_{(r,q) > (1,1)} c_{rq}x_i^{(\ell)r}y_i^{(\ell)q}} \right).$$

We first compute the action of the operators  $\partial_{E_k}$  and  $\varepsilon_{E_k}$  on the tau function (4.9). We start with  $\partial_{E_2}$ . Using the fundamental theorem of calculus and

the fact that  $\sum_{i=1}^N \frac{\partial}{\partial y_i} \Delta_N(y) = 0$ , we compute from (4.10) that

$$\begin{aligned}
\partial_{E_2} \tau_n &= \iint_{E_1^N \times E_2^N} \sum_{i=1}^N \frac{\partial \tilde{I}_n}{\partial y_i}(x, y) dx dy \\
&= \iint_{E_1^N \times E_2^N} \sum_{\ell=1}^p \sum_{j=1}^{n_\ell} \left( -y_j^{(\ell)} + c x_j^{(\ell)} + b_1^{(\ell)} - \sum_{k \geq 1} k s_k^{(\ell)} (y_j^{(\ell)})^{k-1} + 2b_2^{(\ell)} y_j^{(\ell)} \right. \\
&\quad \left. + \sum_{(r,q) > (1,1)} q c_{rq} (x_j^{(\ell)})^r (y_j^{(\ell)})^{q-1} \right) \tilde{I}_n(x, y) dx dy \\
&= \iint_{E_1^N \times E_2^N} \left( \sum_{\ell=1}^p \frac{\partial}{\partial s_1^{(\ell)}} + c \frac{\partial}{\partial s_1^{(0)}} + \sum_{\ell=1}^p n_\ell (b_1^{(\ell)} - s_1^{(\ell)}) + \sum_{\ell=1}^p \sum_{k \geq 2} k s_k^{(\ell)} \frac{\partial}{\partial s_{k-1}^{(\ell)}} \right. \\
&\quad \left. - 2 \sum_{\ell=1}^p b_2^{(\ell)} \frac{\partial}{\partial s_1^{(\ell)}} + \sum_{r \geq 2} c_{r1} \frac{\partial}{\partial s_r^{(0)}} + \sum_{\substack{(r,q) > (1,1) \\ q \geq 2}} q c_{rq} \frac{\partial}{\partial c_{r,q-1}} \right) \tilde{I}_n(x, y) dx dy \\
&= \left( \sum_{\ell=1}^p \frac{\partial}{\partial s_1^{(\ell)}} + c \frac{\partial}{\partial s_1^{(0)}} + \sum_{\ell=1}^p n_\ell (b_1^{(\ell)} - s_1^{(\ell)}) + \sum_{\ell=1}^p \sum_{k \geq 2} k s_k^{(\ell)} \frac{\partial}{\partial s_{k-1}^{(\ell)}} + 2 \sum_{\ell=1}^{p-1} b_2^{(\ell)} \frac{\partial}{\partial b_1^{(\ell)}} \right. \\
&\quad \left. + \sum_{r \geq 2} c_{r1} \frac{\partial}{\partial s_r^{(0)}} + \sum_{\substack{(r,q) > (1,1) \\ q \geq 2}} q c_{rq} \frac{\partial}{\partial c_{r,q-1}} \right) \tau_n,
\end{aligned}$$

where we have used the identity (3.7), which follows from the constraint  $\sum_1^p \kappa_\ell b_{1,2}^{(\ell)} = 0$ , in the last step. The computation for  $\partial_{E_1}$  is similar, but simpler:

$$\begin{aligned}
\partial_{E_1} \tau_n &= \iint_{E_1^N \times E_2^N} \sum_{i=1}^N \frac{\partial \tilde{I}_n}{\partial x_i}(x, y) dx dy \\
&= \iint_{E_1^N \times E_2^N} \sum_{\ell=1}^p \sum_{j=1}^{n_\ell} \left( -x_j^{(\ell)} + c y_j^{(\ell)} + \sum_{k \geq 1} k s_k^{(0)} (x_j^{(\ell)})^{k-1} + \sum_{(r,q) > (1,1)} r c_{rq} (x_j^{(\ell)})^{r-1} (y_j^{(\ell)})^q \right) \tilde{I}_n(x, y) dx dy \\
&= \left( -\frac{\partial}{\partial s_1^{(0)}} - c \sum_{\ell=1}^p \frac{\partial}{\partial s_1^{(\ell)}} + N s_1^{(0)} + \sum_{k \geq 2} k s_k^{(0)} \frac{\partial}{\partial s_{k-1}^{(0)}} - \sum_{\ell=1}^p \sum_{q \geq 2} c_{1q} \frac{\partial}{\partial s_q^{(\ell)}} + \sum_{\substack{(r,q) > (1,1) \\ r \geq 2}} r c_{rq} \frac{\partial}{\partial c_{r-1,q}} \right) \tau_n.
\end{aligned}$$

For the computation of the action of  $\varepsilon_{E_1}$  and  $\varepsilon_{E_2}$  on the tau function, note

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} (x_i f) = N f + \sum_{i=1}^N x_i \frac{\partial f}{\partial x_i}, \quad \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} \Delta_N(x) = \frac{N(N-1)}{2} \Delta_N(x),$$

and so from (4.10), compute using (3.7) and the constraints  $\sum_1^p \kappa_\ell b_{1,2}^{(\ell)} = 0$ ,

$$\begin{aligned}
\varepsilon_{E_2} \tau_n &= \iint_{E_1^N \times E_2^N} \sum_{i=1}^N \frac{\partial}{\partial y_i} (y_i \tilde{I}_n(x, y)) dx dy \\
&= \iint_{E_1^N \times E_2^N} \left( N + \sum_{\ell=1}^p \frac{n_\ell(n_\ell - 1)}{2} + \sum_{\ell=1}^p \sum_{j=1}^{n_\ell} \left( -y_j^{(\ell)2} + c x_j^{(\ell)} y_j^{(\ell)} + b_1^{(\ell)} y_j^{(\ell)} - \sum_{k \geq 1} k s_k^{(\ell)} (y_j^{(\ell)})^k + 2b_2^{(\ell)} y_j^{(\ell)2} \right. \right. \\
&\quad \left. \left. + \sum_{(r,q) > (1,1)} q c_{rq} (x_j^{(\ell)})^r (y_j^{(\ell)})^q \right) \right) \tilde{I}_n(x, y) dx dy \\
&= \iint_{E_1^N \times E_2^N} \left( \sum_{\ell=1}^p \frac{n_\ell(n_\ell + 1)}{2} + \sum_{\ell=1}^p \frac{\partial}{\partial s_2^{(\ell)}} + c \frac{\partial}{\partial c} - \sum_{\ell=1}^p b_1^{(\ell)} \frac{\partial}{\partial s_1^{(\ell)}} + \sum_{\ell=1}^p \sum_{k \geq 1} k s_k^{(\ell)} \frac{\partial}{\partial s_k^{(\ell)}} \right. \\
&\quad \left. - 2 \sum_{\ell=1}^p b_2^{(\ell)} \frac{\partial}{\partial s_2^{(\ell)}} + \sum_{(r,q) > (1,1)} q c_{rq} \frac{\partial}{\partial c_{rq}} \right) \tilde{I}_n(x, y) dx dy
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&= \left( \sum_{\ell=1}^p \frac{n_\ell(n_\ell + 1)}{2} + \sum_{\ell=1}^p \frac{\partial}{\partial s_2^{(\ell)}} + c \frac{\partial}{\partial c} + \sum_{\ell=1}^{p-1} b_1^{(\ell)} \frac{\partial}{\partial b_1^{(\ell)}} + \sum_{\ell=1}^p \sum_{k \geq 1} k s_k^{(\ell)} \frac{\partial}{\partial s_k^{(\ell)}} \right. \\
&\quad \left. + 2 \sum_{\ell=1}^{p-1} b_2^{(\ell)} \frac{\partial}{\partial b_2^{(\ell)}} + \sum_{(r,q) > (1,1)} q c_{rq} \frac{\partial}{\partial c_{rq}} \right) \tau_n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varepsilon_{E_1} \tau_n &= \iint_{E_1^N \times E_2^N} \sum_{i=1}^N \frac{\partial}{\partial x_i} (x_i \tilde{I}_n(x, y)) dx dy \\
&= \left( \frac{N(N+1)}{2} - \frac{\partial}{\partial s_2^{(0)}} + c \frac{\partial}{\partial c} + \sum_{k \geq 1} k s_k^{(0)} \frac{\partial}{\partial s_k^{(0)}} + \sum_{(r,q) > (1,1)} r c_{rq} \frac{\partial}{\partial c_{rq}} \right) \tau_n.
\end{aligned}$$

In order to deduce (4.5) from these formulas it suffices, for  $h = 1$ , to substitute in the first line the definitions (4.1), (4.2) for  $\mathcal{B}_1^{(\ell)}$  and in the second line the

expressions for  $\partial_{E_1}\tau_n$  and  $\partial_{E_2}\tau_n$  in<sup>14</sup>

$$\begin{aligned}
\begin{pmatrix} \mathcal{B}_1^{(0)} \\ \mathcal{B}_1^{(\ell)} \end{pmatrix} \tau_n &= \left\{ - \begin{pmatrix} \kappa_0 & 0 \\ 0 & \kappa_\ell \end{pmatrix} \mathcal{J} \left( \partial_{E_2} - 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_1^{(i)}} \right) + \begin{pmatrix} 0 \\ \partial_{b_1}^{(\ell)} \end{pmatrix} \right\} \tau_n \\
&= \left\{ \begin{pmatrix} \frac{\partial}{\partial s_1^{(0)}} \\ \partial_{b_1}^{(\ell)} + \kappa_\ell \sum_{i=1}^p \frac{\partial}{\partial s_1^{(i)}} \end{pmatrix} - \begin{pmatrix} \kappa_0 & 0 \\ 0 & \kappa_\ell \end{pmatrix} \mathcal{J} \left( \sum_{i=1}^p n_i (b_1^{(i)} - s_1^{(i)}) \right) + \begin{pmatrix} \kappa_0 \Sigma_1^{(0)} \\ \kappa_\ell \Sigma_1^{(\ell)} \end{pmatrix} \right\} \tau_n \\
&= \left\{ \begin{pmatrix} \frac{\partial}{\partial s_1^{(0)}} \\ \frac{\partial}{\partial s_1^{(\ell)}} \end{pmatrix} + \begin{pmatrix} \kappa_0 (T_1^{(0)} + \Sigma_1^{(0)}) \\ \kappa_\ell (T_1^{(\ell)} + \Sigma_1^{(\ell)}) \end{pmatrix} \right\} \tau_n,
\end{aligned}$$

where we used (3.8) (for  $k = 1$ ) in the third line, and where we set

$$\begin{pmatrix} \Sigma_1^{(0)} \\ \Sigma_1^{(\ell)} \end{pmatrix} := -\mathcal{J} \left( \begin{array}{l} \sum_{k \geq 2} k s_k^{(0)} \frac{\partial}{\partial s_{k-1}^{(0)}} - \sum_{i=1}^p \sum_{q \geq 2} c_{1q} \frac{\partial}{\partial s_q^{(\ell)}} + \sum_{\substack{(r,q) > (1,1) \\ r \geq 2}} r c_{rq} \frac{\partial}{\partial c_{r-1,q}} \\ \sum_{\ell=1}^p \sum_{k \geq 2} k s_k^{(\ell)} \frac{\partial}{\partial s_{k-1}^{(\ell)}} + \sum_{r \geq 2} c_{r1} \frac{\partial}{\partial s_r^{(0)}} + \sum_{\substack{(r,q) > (1,1) \\ q \geq 2}} q c_{rq} \frac{\partial}{\partial c_{r,q-1}} \end{array} \right). \quad (4.12)$$

Thus we see that  $\Sigma_1^{(0)}$  and  $\Sigma_1^{(\ell)}$  are homogeneous first order differential operators in the deformation parameters, and that they are independent of  $s_1^{(1)}, \dots, s_1^{(p)}$ , and of  $b_2^{(1)}, \dots, b_2^{(p)}$ , leading to the stated properties of  $\Sigma_1^{(0)}$  and  $\Sigma_1^{(\ell)}$ . For  $k = 2$ , it suffices to substitute the found expressions for  $\varepsilon_{E_1}$  and  $\varepsilon_{E_2}$ , acting on  $\tau_n$ , in the definitions (4.3) and (4.4) of  $\mathcal{B}_2^{(0)}$  and  $\mathcal{B}_2^{(\ell)}$ , to wit:

$$\begin{aligned}
\mathcal{B}_2^{(0)} \tau_n &= \left( -\varepsilon_{E_1} + c \frac{\partial}{\partial c} \right) \tau_n \\
&= \left( \frac{\partial}{\partial s_2^{(0)}} - \frac{N(N+1)}{2} - \sum_{k \geq 1} k s_k^{(0)} \frac{\partial}{\partial s_k^{(0)}} - \sum_{(r,q) > (1,1)} r c_{rq} \frac{\partial}{\partial c_{rq}} \right) \tau_n, \\
&=: \left( \frac{\partial}{\partial s_2^{(0)}} + \kappa_0 \frac{N(N+1)}{2} + \kappa_0 \Sigma_2^{(0)} \right) \tau_n,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_2^{(\ell)} \tau_n &= \left( \partial_{b_2}^{(\ell)} + \kappa_\ell \left( \varepsilon_{E_2} - c \frac{\partial}{\partial c} - \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} - 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_2^{(i)}} \right) \right) \tau_n \\
&= \left( \partial_{b_2}^{(\ell)} + \kappa_\ell \sum_{i=1}^p \frac{\partial}{\partial s_2^{(i)}} + \right. \\
&\quad \left. \kappa_\ell \left( \sum_{i=1}^p \frac{n_i(n_i+1)}{2} + \sum_{i=1}^p \sum_{k \geq 1} k s_k^{(i)} \frac{\partial}{\partial s_k^{(i)}} + \sum_{(r,q) > (1,1)} q c_{rq} \frac{\partial}{\partial c_{rq}} \right) \right) \tau_n \\
&=: \left( \frac{\partial}{\partial s_2^{(\ell)}} + \kappa_\ell \sum_{i=1}^p \frac{n_i(n_i+1)}{2} + \kappa_\ell \Sigma_2^{(\ell)} \right) \tau_n,
\end{aligned} \quad (4.13)$$

<sup>14</sup>Recall that  $m = 2$  and that  $\kappa_0 = -1$ .

where<sup>15</sup>

$$\begin{aligned}\Sigma_2^{(0)} &= \sum_{k \geq 1} k s_k^{(0)} \frac{\partial}{\partial s_k^{(0)}} + \sum_{(r,q) > (1,1)} r c_{rq} \frac{\partial}{\partial c_{rq}}, \\ \Sigma_2^{(\ell)} &= \sum_{i=1}^p \sum_{k \geq 1} k s_k^{(i)} \frac{\partial}{\partial s_k^{(i)}} + \sum_{(r,q) > (1,1)} q c_{rq} \frac{\partial}{\partial c_{rq}}.\end{aligned}\quad (4.14)$$

■

**Remark 4.2** For  $m > 2$  the proof goes along the same line, but it has extra terms, coming from the deformation parameters  $\gamma_r^{(k)}$ . As it turns out,

$$\frac{\partial}{\partial \gamma_1^{(k)}} \tau_n = \sum_{i=1}^m \mathcal{J}_{ki} \left( \partial_{E_i} - \delta_{i,m} \left( 2 \sum_{\ell=1}^{p-1} b_2^{(\ell)} \frac{\partial}{\partial b_1^{(\ell)}} + \sum_{\ell=1}^p n_\ell b_1^{(\ell)} \right) \right) \tau_n + O(\mathcal{L}), \quad (4.15)$$

while  $\frac{\partial}{\partial s_1^{(i)}} \tau_n$  are as before,  $\text{mod } O(\mathcal{L})$ , so the  $\frac{\partial}{\partial \gamma_1^{(k)}} \tau_n$  are only needed to solve for  $\frac{\partial}{\partial s_1^{(i)}} \tau_n$  in terms of the  $(\partial_{E_i} - \delta_{i,m}(\star))\tau_n$ , but they do not enter into the actual solution of  $\frac{\partial}{\partial s_1^{(i)}} \tau_n \text{ mod } O(\mathcal{L})$ .

**Remark 4.3** For  $m = 1$  (one time) the proof of Proposition 4.1 is simpler, but a few adjustments are needed. Denoting the subset  $E_1 \subset \mathbb{R}$  by  $E$ , setting  $\kappa_0 := -1$  and  $\partial_{b_1}^{(0)} := \partial_{b_2}^{(0)} := 0$ , the operators  $\mathcal{B}_1^{(\ell)}$  and  $\mathcal{B}_2^{(\ell)}$  can for  $\ell = 0, \dots, p$ , be written as

$$\begin{aligned}\mathcal{B}_1^{(\ell)} &= \partial_{b_1}^{(\ell)} + \kappa_\ell \left( \partial_E - 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_1^{(i)}} \right), \\ \mathcal{B}_2^{(\ell)} &= \partial_{b_2}^{(\ell)} + \kappa_\ell \left( \varepsilon_E - \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} - 2 \sum_{i=1}^{p-1} b_2^{(i)} \frac{\partial}{\partial b_2^{(i)}} \right),\end{aligned}\quad (4.16)$$

while  $T_k := T_k^{(\alpha)}$  and  $\Sigma_k := \Sigma_k^{(\alpha)}$  are independent of  $\alpha$  and take the simple form

$$T_1 = N s_1^{(0)} + \sum_{\ell=1}^p n_\ell (b_1^{(\ell)} - s_1^{(\ell)}), \quad T_2 = \sum_{1 \leq i \leq j \leq p} n_i n_j, \quad (4.17)$$

$$\Sigma_1 = \sum_{\ell=0}^p \sum_{k \geq 2} k s_k^{(\ell)} \frac{\partial}{\partial s_{k-1}^{(\ell)}}, \quad \Sigma_2 = \sum_{\ell=0}^p \sum_{k \geq 1} k s_k^{(\ell)} \frac{\partial}{\partial s_k^{(\ell)}}. \quad (4.18)$$

---

<sup>15</sup>Notice that  $\Sigma_2^{(\ell)}$  is independent of  $\ell$  for  $1 \leq \ell \leq p$ .

## 5 Virasoro constraints, restricted to the locus $\mathcal{L}$

Restricting the operators  $\mathcal{B}_i$ ,  $T_i$  and  $\Sigma_i$  ((4.1) – (4.7)) to the locus  $\mathcal{L}$ , defined by setting all deformation parameters equal to zero (see (3.5)), yields new operators for  $\ell = 0, \dots, p$ ,

$$\begin{aligned}\hat{\mathcal{B}}_1^{(\ell)} &:= \partial_{b_1}^{(\ell)} - \kappa_\ell \sum_{i=1}^m \partial_{E_i} \times \begin{cases} \mathcal{J}_{1i} & \text{for } \ell = 0 \\ \mathcal{J}_{mi} & \text{for } 1 \leq \ell \leq p \end{cases} \\ \hat{\mathcal{B}}_2^{(0)} &:= -\varepsilon_{E_1} + c_1 \frac{\partial}{\partial c_1} + \delta_{1,m} \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} \\ \hat{\mathcal{B}}_2^{(\ell)} &:= \partial_{b_2}^{(\ell)} - \kappa_\ell \left( -\varepsilon_{E_m} + c_{m-1} \frac{\partial}{\partial c_{m-1}} + \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} \right), \text{ for } \ell \geq 1. \end{aligned} \quad (5.1)$$

while all  $\Sigma_k^{(\ell)}$ , defined in (4.12) and (4.14), restrict to zero,  $\hat{T}_2^{(\ell)} = T_2^{(\ell)}$  for  $0 \leq \ell \leq p$  and

$$\hat{T}_1^{(0)} = -\mathcal{J}_{1m} N(b_1), \quad \hat{T}_1^{(\ell)} = -\mathcal{J}_{mm} N(b_1), \text{ for } 1 \leq \ell \leq p, \quad (5.2)$$

where  $N(b_1) := \sum_{\ell=1}^p n_\ell b_1^{(\ell)}$ . It leads, on the locus  $\mathcal{L}$ , to the identities:

**Proposition 5.1** *For  $\ell = 0 \dots, p$  and  $h = 1, 2$ , the following formulas hold on the locus  $\mathcal{L}$ :*

$$\frac{\partial}{\partial s_h^{(\ell)}} \ln \tau_n = \hat{\mathcal{B}}_h^{(\ell)} \ln \tau_n - \kappa_\ell \hat{T}_h^{(\ell)}, \quad (5.3)$$

while for second derivatives and  $\ell = 1, \dots, p$ , also on the locus  $\mathcal{L}$ ,

$$\begin{aligned} \frac{\partial^2}{\partial s_1^{(0)} \partial s_1^{(\ell)}} \ln \tau_n &= \hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n + n_\ell \mathcal{J}_{1m} =: -F_\ell, \\ \frac{\partial^2}{\partial s_2^{(0)} \partial s_1^{(\ell)}} \ln \tau_n &= (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m}) \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n - 2\mathcal{J}_{1m}^2 \kappa_\ell N(b_1), \\ \frac{\partial^2}{\partial s_1^{(0)} \partial s_2^{(\ell)}} \ln \tau_n &= (\hat{\mathcal{B}}_2^{(\ell)} - \kappa_\ell \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} \ln \tau_n - 2\mathcal{J}_{1m} (\mathcal{J}_{mm} \kappa_\ell N(b_1) + \partial_{b_1}^{(\ell)} \ln \tau_n). \end{aligned} \quad (5.4)$$

*Proof:* The first set of identities (5.3) follows at once from restricting the identities (4.5) of Proposition 4.1 to the locus  $\mathcal{L}$  and using that  $\frac{\partial}{\partial s_k^{(\ell)}}$  and  $\mathcal{B}_k^{(\ell)}$  are first order differential operators. The identities (5.4) involving second derivatives are shown as follows. Concerning the first one, observe from Proposition

4.1 that

$$\begin{aligned}
\hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} &= \mathcal{B}_1^{(0)} \mathcal{B}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} = \mathcal{B}_1^{(0)} \left( \frac{\partial}{\partial s_1^{(\ell)}} + \kappa_\ell \Sigma_1^{(\ell)} \right) \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell \mathcal{B}_1^{(0)} T_1^{(\ell)} \Big|_{\mathcal{L}} \\
&= \left( \frac{\partial}{\partial s_1^{(\ell)}} + \kappa_\ell \Sigma_1^{(\ell)} \right) \mathcal{B}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} = \frac{\partial}{\partial s_1^{(\ell)}} \mathcal{B}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \frac{\partial}{\partial s_1^{(\ell)}} \left( \left( \frac{\partial}{\partial s_1^{(0)}} + \kappa_0 \Sigma_1^{(0)} \right) \ln \tau_n + \kappa_0 T_1^{(0)} \right) \Big|_{\mathcal{L}} = \frac{\partial^2}{\partial s_1^{(\ell)} \partial s_1^{(0)}} \ln \tau_n \Big|_{\mathcal{L}} - \mathcal{J}_{1m} n_\ell,
\end{aligned}$$

where we used in the last equality the relations  $\frac{\partial}{\partial s_1^{(\ell)}} T_1^{(0)} = \mathcal{J}_{1m} n_\ell$  (see (4.6)) and  $\left[ \frac{\partial}{\partial s_1^{(\ell)}}, \Sigma_1^{(0)} \right] = 0$ , (see (4.8)). This yields the first identity in (5.4). To prove the third one, we use that

$$\sum_{i=1}^{p-1} \partial_{b_2}^{(\ell)}(b_2^{(i)}) \frac{\partial}{\partial b_1^{(i)}} = \sum_{i=1}^{p-1} (\kappa_\ell - \delta_{\ell,i}) \frac{\partial}{\partial b_1^{(i)}} = \partial_{b_1}^{(\ell)},$$

as follows from (3.9), and

$$\begin{aligned}
\mathcal{B}_2^{(\ell)} T_1^{(0)} \Big|_{\mathcal{L}} &= \kappa_\ell c_{m-1} \frac{\partial \mathcal{J}_{1m}}{\partial c_{m-1}} N(b_1) + \kappa_\ell \mathcal{J}_{1m} \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} N(b_1) \\
&= -\kappa_\ell \mathcal{J}_{1m} (2\mathcal{J}_{mm} + 1) N(b_1) + \kappa_\ell \mathcal{J}_{1m} N(b_1) = -2\kappa_\ell \mathcal{J}_{1m} \mathcal{J}_{mm} N(b_1),
\end{aligned}$$

by using (2.4), when  $m > 1$ , and  $\mathcal{B}_2^{(\ell)} T_1^{(0)} \Big|_{\mathcal{L}} = -\kappa_\ell N(b_1)$ , by Remark 4.3 for  $m = 1$ , so that

$$\mathcal{B}_2^{(\ell)} T_1^{(0)} \Big|_{\mathcal{L}} = -\kappa_\ell N(b_1) (2\mathcal{J}_{1m} \mathcal{J}_{mm} - \delta_{1,m}),$$

for all  $m$ . Using these identities, (4.1), (4.4), Proposition 4.1, (4.8) and (5.3), compute

$$\begin{aligned}
& \hat{\mathcal{B}}_2^{(\ell)} \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \mathcal{B}_2^{(\ell)} \mathcal{B}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} + 2\mathcal{J}_{1m} \sum_{i=1}^{p-1} \partial_{b_2}^{(\ell)}(b_2^{(i)}) \frac{\partial}{\partial b_1^{(i)}} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \mathcal{B}_2^{(\ell)} \left( \frac{\partial}{\partial s_1^{(0)}} + \kappa_0 \Sigma_1^{(0)} \right) \ln \tau_n \Big|_{\mathcal{L}} + \kappa_0 \mathcal{B}_2^{(\ell)} T_1^{(0)} \Big|_{\mathcal{L}} + 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \left( \frac{\partial}{\partial s_1^{(0)}} + \kappa_0 \Sigma_1^{(0)} \right) \mathcal{B}_2^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m} \mathcal{J}_{mm} - \delta_{1,m}) + 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \frac{\partial}{\partial s_1^{(0)}} \left( \left( \frac{\partial}{\partial s_2^{(\ell)}} + \kappa_\ell \Sigma_2^{(\ell)} \right) \ln \tau_n + \kappa_\ell T_2^{(\ell)} \right) \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m} \mathcal{J}_{mm} - \delta_{1,m}) + 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \frac{\partial^2}{\partial s_1^{(0)} \partial s_2^{(\ell)}} \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell \left[ \frac{\partial}{\partial s_1^{(0)}}, \Sigma_2^{(\ell)} \right] \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m} \mathcal{J}_{mm} - \delta_{1,m}) + 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \frac{\partial^2}{\partial s_1^{(0)} \partial s_2^{(\ell)}} \ln \tau_n \Big|_{\mathcal{L}} + \delta_{1,m} \kappa_\ell \left( \frac{\partial}{\partial s_1^{(0)}} \ln \tau_n \Big|_{\mathcal{L}} - N(b_1) \right) + 2\mathcal{J}_{1m} (\kappa_\ell N(b_1) \mathcal{J}_{mm} + \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}}) \\
&= \frac{\partial^2}{\partial s_1^{(0)} \partial s_2^{(\ell)}} \ln \tau_n \Big|_{\mathcal{L}} + \delta_{1,m} \kappa_\ell \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} + 2\mathcal{J}_{1m} (\kappa_\ell N(b_1) \mathcal{J}_{mm} + \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}}),
\end{aligned}$$

which yields the third relation (5.4). Using  $\mathcal{B}_2^{(0)} T_1^{(\ell)} \Big|_{\mathcal{L}} = N(b_1) (2\mathcal{J}_{1m}^2 - \delta_{1,m})$ , which follows from (4.3), (4.6) and (2.4), the second identity in (5.4) is proven in a similar fashion, using (4.8) and (5.3), namely

$$\begin{aligned}
\hat{\mathcal{B}}_2^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} &= \mathcal{B}_2^{(0)} \mathcal{B}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} \\
&= \mathcal{B}_2^{(0)} \left( \frac{\partial}{\partial s_1^{(\ell)}} + \kappa_\ell \Sigma_1^{(\ell)} \right) \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell \mathcal{B}_2^{(0)} T_1^{(\ell)} \Big|_{\mathcal{L}} \\
&= \left( \frac{\partial}{\partial s_1^{(\ell)}} + \kappa_\ell \Sigma_1^{(\ell)} \right) \mathcal{B}_2^{(0)} \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m}^2 - \delta_{1,m}) \\
&= \frac{\partial}{\partial s_1^{(\ell)}} \left( \left( \frac{\partial}{\partial s_2^{(0)}} + \kappa_0 \Sigma_2^{(0)} \right) \ln \tau_n + \kappa_0 T_2^{(0)} \right) \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m}^2 - \delta_{1,m}) \\
&= \frac{\partial^2}{\partial s_1^{(\ell)} \partial s_2^{(0)}} \ln \tau_n \Big|_{\mathcal{L}} - \left[ \frac{\partial}{\partial s_1^{(\ell)}}, \Sigma_2^{(0)} \right] \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m}^2 - \delta_{1,m})
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial^2}{\partial s_1^{(\ell)} \partial s_2^{(0)}} \ln \tau_n \Big|_{\mathcal{L}} - \delta_{1,m} \frac{\partial}{\partial s_1^{(\ell)}} \ln \tau_n \Big|_{\mathcal{L}} + \kappa_\ell N(b_1) (2\mathcal{J}_{1m}^2 - \delta_{1,m}) \\
&= \frac{\partial^2}{\partial s_1^{(\ell)} \partial s_2^{(0)}} \ln \tau_n \Big|_{\mathcal{L}} + 2\kappa_\ell N(b_1) \mathcal{J}_{1m}^2 - \delta_{1,m} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}}.
\end{aligned}$$

■

## 6 A PDE for the transition probability

This section aims at proving Theorem 6.3, which leads at once to Theorem 1.1. In order to do so, we shall need two propositions:

**Proposition 6.1** *For  $1 \leq \ell \leq p$ , the function  $X_\ell := \partial_{b_2}^{(\ell)} \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}}$  satisfies the equation*

$$\{X_\ell, F_\ell\}_{\hat{\mathcal{B}}_1^{(0)}} = \{H_\ell^{(1)}, F_\ell\}_{\hat{\mathcal{B}}_1^{(0)}} - \{H_\ell^{(2)}, F_\ell\}_{\hat{\mathcal{B}}_1^{(\ell)}}, \quad (6.1)$$

where  $(n = (n_1, \dots, n_p))$

$$\begin{aligned}
F_\ell &= -\hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \mathbb{P}_n - n_\ell \mathcal{J}_{1m}, \\
H_\ell^{(1)} &:= (\kappa_\ell (\delta_{1,m} - \varepsilon_m) \hat{\mathcal{B}}_1^{(0)} + 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)}) \ln \mathbb{P}_n + C_\ell, \\
H_\ell^{(2)} &:= (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m} + 2\mathcal{J}_{1m} b_1^{(\ell)} \hat{\mathcal{B}}_1^{(0)}) \hat{\mathcal{B}}_1^{(\ell)} \ln \mathbb{P}_n, \\
\varepsilon_m &= \varepsilon_{E_m} - c_{m-1} \frac{\partial}{\partial c_{m-1}} - \sum_{\ell=1}^{p-1} b_1^{(\ell)} \frac{\partial}{\partial b_1^{(\ell)}}, \quad (6.2)
\end{aligned}$$

$$C_\ell := 2n_\ell \mathcal{J}_{1m} \left( \mathcal{J}_{mm} b_1^{(\ell)} - \sum_{i \neq \ell} \frac{n_i}{b_1^{(\ell)} - b_1^{(i)}} \right). \quad (6.3)$$

*Proof:* From (5.3) and (5.2), one finds, along  $\mathcal{L}$ , for  $\ell = 1, \dots, p$ ,

$$\frac{\partial}{\partial s_1^{(0)}} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} = \hat{\mathcal{B}}_1^{(0)} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} - 2\mathcal{J}_{1m} b_1^{(\ell)}, \quad (6.4)$$

$$\frac{\partial}{\partial s_1^{(\ell)}} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} = \hat{\mathcal{B}}_1^{(\ell)} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} + 2\kappa_\ell \mathcal{J}_{mm} b_1^{(\ell)}. \quad (6.5)$$

A direct substitution of these formulas, as well as the formulas (5.3) and (5.4), in (3.12), leads, along  $\mathcal{L}$ , for  $\ell = 1, \dots, p$ , to

$$\begin{aligned}
\hat{\mathcal{B}}_1^{(0)} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} - 2\mathcal{J}_{1m} b_1^{(\ell)} &= -\frac{1}{F_\ell} \left( (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m}) \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n - 2\mathcal{J}_{1m}^2 \kappa_\ell N(b_1) \right), \\
\hat{\mathcal{B}}_1^{(\ell)} \ln \frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}} + 2\kappa_\ell \mathcal{J}_{mm} b_1^{(\ell)} &= \frac{1}{F_\ell} \left( (\hat{\mathcal{B}}_2^{(\ell)} - \kappa_\ell \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \right. \\
&\quad \left. - 2\mathcal{J}_{1m} (\mathcal{J}_{mm} \kappa_\ell N(b_1) + \partial_{b_1}^{(\ell)} \ln \tau_n) \right),
\end{aligned}$$

where  $F_\ell := -\hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} - n_\ell \mathcal{J}_{1m}$  (see (5.4)). Eliminating from these equations the term which contains  $\frac{\tau_{n+e_\ell}}{\tau_{n-e_\ell}}$ , which can be done by applying  $\hat{\mathcal{B}}_1^{(\ell)}$  to the first equation and  $\hat{\mathcal{B}}_1^{(0)}$  to the second equation, and using that these operators commute, we get the single equation

$$\begin{aligned} & \hat{\mathcal{B}}_1^{(\ell)} \left( \frac{2\mathcal{J}_{1m} b_1^{(\ell)} F_\ell - (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m}) \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} + 2\mathcal{J}_{1m}^2 \kappa_\ell N(b_1)}{F_\ell} \right) \\ &= \hat{\mathcal{B}}_1^{(0)} \left( \frac{(\hat{\mathcal{B}}_2^{(\ell)} - \kappa_\ell \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}} - 2\mathcal{J}_{1m} (\mathcal{J}_{mm} \kappa_\ell N(b_1) + \partial_{b_1}^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}})}{F_\ell} \right). \end{aligned}$$

Using the fact that the derivative of a ratio amounts to a Wronskian, by clearing the denominator, and writing  $\hat{\mathcal{B}}_2^{(\ell)}$  as  $\hat{\mathcal{B}}_2^{(\ell)} = \partial_{b_2}^{(\ell)} + \kappa_\ell \varepsilon_m$  (see (6.2) and (5.1)) and using the formula for  $F_\ell$ , one can rewrite the latter equation as

$$\begin{aligned} & - \left\{ \partial_{b_2}^{(\ell)} \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}}, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(0)}} \tag{6.6} \\ &= \left\{ (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m} + 2\mathcal{J}_{1m} b_1^{(\ell)} \hat{\mathcal{B}}_1^{(0)}) \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n \Big|_{\mathcal{L}} + 2\mathcal{J}_{1m}^2 (n_\ell b_1^{(\ell)} - \kappa_\ell N(b_1)), F_\ell \right\}_{\hat{\mathcal{B}}_1^{(\ell)}} \\ &+ \left\{ (\kappa_\ell (\varepsilon_m - \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} - 2\mathcal{J}_{1m} \partial_{b_1}^{(\ell)}) \ln \tau_n \Big|_{\mathcal{L}} - 2\kappa_\ell \mathcal{J}_{1m} \mathcal{J}_{mm} N(b_1), F_\ell \right\}_{\hat{\mathcal{B}}_1^{(0)}}. \end{aligned}$$

Finally the integral  $\tau_n$  (as in (3.4)), but integrated over the full range  $\mathbb{R}$ , equals (see the Appendix)

$$\tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} = g_n(c) e^{-\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^p n_\ell b_1^{(\ell)2}} \prod_{1 \leq i < j \leq p} (b_1^{(j)} - b_1^{(i)})^{n_i n_j}, \tag{6.7}$$

with  $g_n(c)$  a function, depending on  $c_1, \dots, c_{m-1}$  and  $n$  only. Thus one has, restricted to  $\mathcal{L}$ ,

$$\begin{aligned} \ln \tau_n(\mathbb{E}) \Big|_{\mathcal{L}} &= \ln \mathbb{P}_n(\mathbb{E}) + \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}}, \tag{6.8} \\ \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} &= -\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^p n_\ell (b_1^{(\ell)})^2 + \sum_{1 \leq i < j \leq p} n_i n_j \ln(b_1^{(j)} - b_1^{(i)}) + \ln g(c). \end{aligned}$$

When (6.8) is substituted in (6.6), a few terms will appear where  $\ln \tau_n(\mathbb{R}^m)$  is acted upon by a differential operator. We derive the formulas which will be used. First, it is clear that  $\hat{\mathcal{B}}_1^{(0)} \tau_n(\mathbb{R}^m) = 0$ . Therefore, since  $[\hat{\mathcal{B}}_1^{(0)}, \hat{\mathcal{B}}_1^{(\ell)}] = 0$ ,

$$F_\ell = -\hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n(\mathbb{E}) \Big|_{\mathcal{L}} - n_\ell \mathcal{J}_{1m} = -\hat{\mathcal{B}}_1^{(0)} \hat{\mathcal{B}}_1^{(\ell)} \ln \mathbb{P}_n(\mathbb{E}) - n_\ell \mathcal{J}_{1m}. \tag{6.9}$$

Also, using  $\partial_{b_1^{(\ell)}} b_1^{(i)} = \kappa_\ell - \delta_{\ell,i}$ , valid for  $i = 1, \dots, p$ , one computes

$$\partial_{b_1^{(\ell)}} \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} = \mathcal{J}_{mm}(n_\ell b_1^{(\ell)} - \kappa_\ell N(b_1)) - n_\ell \sum_{i \neq \ell} \frac{n_i}{b_1^{(\ell)} - b_1^{(i)}}$$

and therefore, since  $\hat{\mathcal{B}}_1^{(0)} \ln \tau_n(\mathbb{R}^m) = 0$ , and by (5.1) and (2.4)

$$\begin{aligned} & (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m} + 2\mathcal{J}_{1m} b_1^{(\ell)} \hat{\mathcal{B}}_1^{(0)}) \hat{\mathcal{B}}_1^{(\ell)} \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} \\ &= \left( c_1 \frac{\partial}{\partial c_1} + \delta_{1,m} + \delta_{1,m} \sum_{i=1}^{p-1} b_1^{(i)} \frac{\partial}{\partial b_1^{(i)}} \right) \partial_{b_1^{(\ell)}} \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} \\ &= 2\mathcal{J}_{1m}^2 (\kappa_\ell N(b_1) - n_\ell b_1^{(\ell)}) \end{aligned}$$

and

$$\begin{aligned} & (\kappa_\ell (\varepsilon_m - \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} - 2\mathcal{J}_{1m} \partial_{b_1^{(\ell)}}) \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} = -2\mathcal{J}_{1m} \partial_{b_1^{(\ell)}} \ln \tau_n(\mathbb{R}^m) \Big|_{\mathcal{L}} \\ &= -2\mathcal{J}_{1m} \left( \mathcal{J}_{mm}(n_\ell b_1^{(\ell)} - \kappa_\ell N(b_1)) - n_\ell \sum_{i \neq \ell} \frac{n_i}{b_1^{(\ell)} - b_1^{(i)}} \right). \end{aligned}$$

Substituted in (6.6), yields the identity

$$\begin{aligned} & - \left\{ \partial_{b_2^{(\ell)}} \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}}, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(0)}} \\ &= \left\{ (\hat{\mathcal{B}}_2^{(0)} + \delta_{1,m} + 2\mathcal{J}_{1m} b_1^{(\ell)} \hat{\mathcal{B}}_1^{(0)}) \hat{\mathcal{B}}_1^{(\ell)} \ln \mathbb{P}_n, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(\ell)}} \\ &+ \left\{ (\kappa_\ell (\varepsilon_m - \delta_{1,m}) \hat{\mathcal{B}}_1^{(0)} - 2\mathcal{J}_{1m} \partial_{b_1^{(\ell)}}) \ln \mathbb{P}_n - C_\ell, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(0)}}. \end{aligned} \quad (6.10)$$

This ends the proof of Proposition 6.1.  $\blacksquare$

For  $\ell = 1, \dots, p$ , using the shorthand notation,

$$X_\ell = \partial_{b_2^{(\ell)}} \hat{\mathcal{B}}_1^{(0)} \ln \tau_n \Big|_{\mathcal{L}}, \quad H_\ell := \left\{ H_\ell^{(1)}, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(0)}} - \left\{ H_\ell^{(2)}, F_\ell \right\}_{\hat{\mathcal{B}}_1^{(\ell)}} \quad (6.11)$$

and  $' := \hat{\mathcal{B}}_1^{(0)}$ , the equations (6.1) become (taking into account  $\sum_{\ell=1}^p \partial_{b_2^{(\ell)}} = 0$ )

$$\{X_\ell, F_\ell\} = H_\ell, \quad 1 \leq \ell \leq p, \quad \text{with} \quad \sum_{\ell=1}^p X_\ell = 0.$$

**Proposition 6.2** *Given for  $\ell = 1, \dots, p$  functions  $H_\ell$  and  $F_\ell$ , such that the Wronskian of the derivatives  $F'_1, \dots, F'_p$  is non-zero, the system of ODE's*

$$\{X_\ell, F_\ell\} = H_\ell, \quad 1 \leq \ell \leq p$$

subjected to the condition  $\sum_{\ell=1}^p X_\ell = 0$ , has a unique solution  $(X_1, \dots, X_p)$ , where  $X_\ell$  is given by

$$X_\ell = \frac{F_\ell}{D} \det \begin{pmatrix} F'_1 & F'_2 & F'_3 & \cdots & \overset{\ell}{\downarrow} -G_1 & \cdots & F'_p \\ F''_1 & F''_2 & F''_3 & \cdots & -G_2 & \cdots & F''_p \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \cdots & -G_p & \cdots & F_p^{(p)} \end{pmatrix}. \quad (6.12)$$

In this formula,  $D$  is the Wronskian of the functions  $F'_1, \dots, F'_p$ ,

$$D := \det \begin{pmatrix} F'_1 & F'_2 & F'_3 & \cdots & F'_p \\ F''_1 & F''_2 & F''_3 & \cdots & F''_p \\ \vdots & \vdots & \vdots & & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \cdots & F_p^{(p)} \end{pmatrix} \neq 0$$

and the  $G_i$ 's are defined inductively as

$$G_{i+1} = G'_i + \sum_{\ell=1}^p \frac{H_\ell F_\ell^{(i)}}{F_\ell^2}, \quad G_0 = 0, \quad G_1 = \sum_1^p \frac{H_\ell}{F_\ell}. \quad (6.13)$$

Moreover

$$\det \begin{pmatrix} F_1 & F_2 & F_3 & \cdots & F_p & G_0 \\ F'_1 & F'_2 & F'_3 & \cdots & F'_p & G_1 \\ F''_1 & F''_2 & F''_3 & \cdots & F''_p & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \cdots & F_p^{(p)} & G_p \end{pmatrix} = 0. \quad (6.14)$$

*Proof:* If  $X_\ell$  is a solution of the equation  $\{X_\ell, F_\ell\} = H_\ell$ , subjected to the condition  $\sum_{\ell=1}^p X_\ell = 0$ , then its derivatives are given by

$$X_\ell^{(i)} = G_{\ell,i} + X_\ell \frac{F_\ell^{(i)}}{F_\ell}, \quad (6.15)$$

where for a fixed  $\ell$ , the  $G_{\ell,i}$  are defined inductively as

$$G_{\ell,0} := 0, \quad G_{\ell,1} := \frac{H_\ell}{F_\ell}, \quad \dots, \quad G_{\ell,i+1} := G'_{\ell,i} + \frac{H_\ell F_\ell^{(i)}}{F_\ell^2}.$$

Indeed, starting with (6.15) and using  $X'_\ell = \frac{1}{F_\ell}(H_\ell + X_\ell F'_\ell)$ , one computes inductively

$$\begin{aligned} X_\ell^{(i+1)} &= G'_{\ell,i} + X'_\ell \frac{F_\ell^{(i)}}{F_\ell} + X_\ell \frac{F_\ell^{(i+1)}}{F_\ell} - X_\ell \frac{F'_\ell F_\ell^{(i)}}{F_\ell^2} \\ &= \left( G'_{\ell,i} + \frac{H_\ell F_\ell^{(i)}}{F_\ell^2} \right) + X_\ell \frac{F_\ell^{(i+1)}}{F_\ell} = G_{\ell,i+1} + X_\ell \frac{F_\ell^{(i+1)}}{F_\ell}, \end{aligned}$$

establishing (6.15). Summing up (6.15) for  $\ell$  from 1 to  $p$ , one finds

$$0 = G_i + \sum_{\ell=1}^p X_\ell \frac{F_\ell^{(i)}}{F_\ell}, \quad \text{where } G_i := \sum_{\ell=1}^p G_{\ell,i}.$$

Then solving this linear system for the  $X_\ell$ 's, one finds the ratio (6.12) above. Then using that solution and expressing  $\sum_{\ell=1}^p X_\ell = 0$  establishes (6.14) and thus the proof of Proposition 6.2.  $\blacksquare$

This enables us to make the following statement, remembering the operators  $\hat{\mathcal{B}}_1^{(\ell)}$ , with  $' = \hat{\mathcal{B}}_1^{(0)} = \sum_{i=1}^m \mathcal{J}_{1i} \partial_{E_i}$ , and  $\partial_{b_1}^{(\ell)}$  with  $\sum_{\ell=1}^p \partial_{b_1}^{(\ell)} = 0$ .

**Theorem 6.3** *The probability  $\mathbb{P}_n = \mathbb{P}_n^A(c, \mathbb{E})$  as in (1.3), with the linear constraint  $\sum_{\ell=1}^p \kappa_\ell b_1^{(\ell)} = 0$ , with  $\sum_{\ell=1}^p \kappa_\ell = 1$ , satisfies a non-linear PDE in the boundary points of the subsets  $E_1, \dots, E_m$  and in the target points  $b_1^{(1)}, \dots, b_1^{(p)}$ , given by the determinant of a  $(p+1) \times (p+1)$  matrix*

$$\det \begin{pmatrix} F_1 & F_2 & F_3 & \dots & F_p & G_0 \\ F_1' & F_2' & F_3' & \dots & F_p' & G_1 \\ F_1'' & F_2'' & F_3'' & \dots & F_p'' & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \dots & F_p^{(p)} & G_p \end{pmatrix} = 0, \quad (6.16)$$

where the  $F_\ell$ ,  $H_\ell^{(i)}$  and  $C_\ell$  are given by in Proposition 6.1 and the  $G_\ell$  inductively by

$$G_{\ell+1} := G_\ell' + \sum_{i=1}^p F_i^{(\ell)} \left( \hat{\mathcal{B}}_1^{(0)} \frac{H_i^{(1)}}{F_i} - \hat{\mathcal{B}}_1^{(i)} \frac{H_i^{(2)}}{F_i} \right), \quad G_0 := 0.$$

*Proof of Theorem 1.1:* It follows immediately from Theorem 6.3 by noticing that in the notation of (1.11), the  $\hat{\mathcal{B}}_i^{(\ell)}$  are expressed as

$$\hat{\mathcal{B}}_1^{(\ell)} = \partial_\ell, \quad \hat{\mathcal{B}}_2^{(0)} = -\varepsilon_0.$$

*Proof of Corollary 1.2:* The simplification comes from the fact that for one-time (i.e.,  $m = 1$ ) the operators  $\partial_0$  and  $\partial_\ell$  differ by very little, namely:

$$\partial_0 = -\partial_E, \quad \partial_\ell = \partial_b^{(\ell)} + \kappa_\ell \partial_E, \quad \varepsilon = \varepsilon_0 = \varepsilon_m.$$

This means that the expression in brackets in the definition of  $G_{i+1}$  in (1.13) can be re-expressed as follows,

$$\partial_0 \frac{H_\ell^{(1)}}{F_\ell} - \partial_\ell \frac{H_\ell^{(2)}}{F_\ell} = \partial_E \frac{-H_\ell^{(1)} - \kappa_\ell H_\ell^{(2)} + 2\kappa_\ell b_\ell F_\ell}{F_\ell} - \partial_b^{(\ell)} \frac{H_\ell^{(2)}}{F_\ell} = \partial_E \frac{\bar{H}_\ell^{(1)}}{F_\ell} - \partial_b^{(\ell)} \frac{H_\ell^{(2)}}{F_\ell},$$

upon setting  $\bar{H}_\ell^{(1)} := -H_\ell^{(1)} - \kappa_\ell H_\ell^{(2)} + 2\kappa_\ell b_\ell F_\ell$ , which one checks<sup>16</sup> to be the expression  $\bar{H}_\ell^{(1)}$  announced in (1.15) and one repeats the proof of Proposition 6.2 with  $' = \partial_E$  (instead of  $' = \partial_0 = -\partial_E$ ) and  $X_\ell \mapsto -X_\ell$ , ending the proof of Corollary 1.2.  $\blacksquare$

## 7 Examples

### 7.1 One target point at the origin

In this case,  $m = 2$ ,  $p = 1$  and the diagonal matrix  $A = 0$ . The matrix  $\mathcal{J}$  reads

$$\mathcal{J} = \frac{1}{1-c^2} \begin{pmatrix} -1 & -c \\ -c & -1 \end{pmatrix}$$

and one checks

$$\begin{aligned} \kappa_0 = -1, \quad b_1 = 0, \quad \kappa_1 = 1, \quad \partial_b^{(0)} = \partial_b^{(1)} = \varepsilon_b = 0, \quad \varepsilon_0 = \varepsilon_{E_1} - c \frac{\partial}{\partial c}, \quad \varepsilon_2 = \varepsilon_{E_2} - c \frac{\partial}{\partial c}, \\ \partial_0 = -\frac{1}{1-c^2} (\partial_{E_1} + c\partial_{E_2}), \quad \partial_1 = \frac{1}{1-c^2} (c\partial_{E_1} + \partial_{E_2}), \quad C_\ell = 0 \end{aligned} \quad (7.1)$$

So, for  $\ell = 1$ , one has

$$\begin{aligned} F_1 &= -\partial_0 \partial_1 \log \mathbb{P}_n + \frac{nc}{1-c^2} = \frac{1}{(1-c^2)^2} (\partial_{E_1} + c\partial_{E_2}) (c\partial_{E_1} + \partial_{E_2}) \log \mathbb{P}_n + \frac{nc}{1-c^2} \\ H_1^{(1)} &= -\varepsilon_2 \partial_0 \log \mathbb{P}_n = (\varepsilon_{E_2} - c \frac{\partial}{\partial c}) \frac{1}{1-c^2} (\partial_{E_1} + c\partial_{E_2}) \log \mathbb{P}_n \\ H_1^{(2)} &= -\varepsilon_0 \partial_1 \log \mathbb{P}_n = -(\varepsilon_{E_1} - c \frac{\partial}{\partial c}) \frac{1}{1-c^2} (c\partial_{E_1} + \partial_{E_2}) \log \mathbb{P}_n \end{aligned} \quad (7.2)$$

and thus

$$G_0 = 0, \quad G_1 = F_1 \left( \partial_0 \frac{H_1^{(1)}}{F_1} - \partial_1 \frac{H_1^{(2)}}{F_1} \right) = \frac{1}{F_1} \left( \left\{ H_1^{(1)}, F_1 \right\}_{\partial_0} - \left\{ H_1^{(2)}, F_1 \right\}_{\partial_1} \right)$$

leading to the PDE, with  $\partial_0$  and  $\partial_1$  as in (7.1) and  $H_1^{(j)}$  and  $F_i$  as in (7.2): (see [5] and [6])

$$\det \begin{pmatrix} F_1 & G_0 \\ \partial_0 F_1 & G_1 \end{pmatrix} = \left\{ H_1^{(1)}, F_1 \right\}_{\partial_0} - \left\{ H_1^{(2)}, F_1 \right\}_{\partial_1} = 0$$

<sup>16</sup>Upon using the commutation relation  $[\varepsilon_E, \partial_E] = -\partial_E$  and  $\varepsilon = \varepsilon_0 = \varepsilon_m$ .

## 7.2 Target points with some symmetry

Consider non-intersecting Brownian motions leaving from 0 and forced to  $p$  target points at time  $t = 1$ , with the only condition that the left-most and right-most target points are symmetric with respect to the origin, with  $p - 2$  intermediate target points thrown in totally arbitrarily; this example will be used in section 8. It is convenient to rename the target points  $\beta_1 < \dots < \beta_p$ , as follows:

$$\begin{array}{ccccccc} \tilde{a} & < & -\tilde{c}_1 & < & \dots & < & -\tilde{c}_{p-2} & < & -\tilde{a} \\ & & n_+ & & n_1 & & \dots & & n_{p-2} & & n_- \end{array} \quad (7.3)$$

with the corresponding number of particles forced to those points at time  $t = 1$ . Using the change of variables (1.4) from  $\beta_i$ 's to

$$b = (b_1, \dots, b_p) = (a, -c_1, -c_2, \dots, -c_{p-2}, -a), \quad (7.4)$$

one is led to the diagonal matrix of the form<sup>17</sup>:

$$A := \text{diag}(\overbrace{a, \dots, a}^{n_+}, \overbrace{-c_1, \dots, -c_1}^{n_1}, \dots, \overbrace{-c_{p-2}, \dots, -c_{p-2}}^{n_{p-2}}, \overbrace{-a, \dots, -a}^{n_-}) \quad (7.5)$$

with the obvious constraint  $\sum_1^p \kappa_i b_i = \frac{1}{2}a + \frac{1}{2}(-a) = 0$ , as in (1.5), and thus

$$\kappa_1 = \kappa_p = \frac{1}{2} \quad \text{and} \quad \kappa_i = 0 \quad \text{for} \quad 2 \leq i \leq p-1.$$

Moreover, setting  $c = (c_1, \dots, c_{p-2})$ , formulae (1.11) become

$$\partial_b^{(1)} = \frac{1}{2} \left( -\frac{\partial}{\partial a} - \partial_c \right), \quad \partial_b^{(p)} = \frac{1}{2} \left( \frac{\partial}{\partial a} - \partial_c \right), \quad \partial_b^{(\ell)} = \frac{\partial}{\partial c_{\ell-1}}, \quad 2 \leq \ell \leq p-1, \quad (7.6)$$

and  $\varepsilon = \varepsilon_E - a \frac{\partial}{\partial a} - \varepsilon_c$ ; also set  $' = \partial_E$ . Besides the renaming  $n_1 = n_+$ ,  $n_p = n_-$  and  $n_k \mapsto n_{k-1}$  for  $2 \leq k \leq p-1$ , already mentioned, one also has, referring to formulae (1.15), the following renaming:

$$\begin{aligned} F_1 &\mapsto F_+, & F_p &\mapsto F_-, & F_k &\mapsto F_{k-1}, & \text{for } 2 \leq k \leq p-1, \\ \bar{H}_1^{(1)} &\mapsto H_+^{(1)}, & \bar{H}_p^{(1)} &\mapsto H_-^{(1)}, & H_1^{(2)} &\mapsto H_+^{(2)}, & H_p^{(2)} &\mapsto H_-^{(2)}, \\ \bar{H}_\ell^{(1)} &\mapsto H_{\ell-1}^{(1)}, & H_\ell^{(2)} &\mapsto H_{\ell-1}^{(2)}, & \text{for } 2 \leq \ell \leq p-1. \end{aligned}$$

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<sup>17</sup>Note the  $c_i$  have nothing to do with the couplings  $c_i$  appearing in (1.3).

Then, one checks from Corollary 1.2, formulae (1.15), that<sup>18</sup> for  $1 \leq \ell \leq p-2$  and for  $\mathbb{P} := \mathbb{P}_n^A(E)$ , with  $\varepsilon = \varepsilon_E - a \frac{\partial}{\partial a} - \varepsilon_c$  (as in (1.3) for  $m = 1$ )

$$\begin{aligned} F_{\pm} &= \frac{1}{2}(\mp \frac{\partial}{\partial a} - \partial_c + \partial_E) \partial_E \ln \mathbb{P} + n_{\pm}, & F_{\ell} &= \frac{\partial}{\partial c_{\ell}} \partial_E \ln \mathbb{P} + n_{\ell}, \\ H_{\pm}^{(1)} &= \frac{1}{4} \left( -2\partial_E \varepsilon + (\varepsilon + 3) \left( \mp \frac{\partial}{\partial a} - \partial_c + \partial_E \right) \right) \ln \mathbb{P} + C_{\pm}, & H_{\ell}^{(1)} &= 2 \frac{\partial}{\partial c_{\ell}} \ln \mathbb{P} + C_{\ell}, \\ H_{\pm}^{(2)} &= \frac{1}{2}(1 - \varepsilon \pm 2a\partial_E) \left( \mp \frac{\partial}{\partial a} - \partial_c + \partial_E \right) \ln \mathbb{P}, & H_{\ell}^{(2)} &= (1 - \varepsilon - 2c_{\ell}\partial_E) \frac{\partial}{\partial c_{\ell}} \ln \mathbb{P}. \end{aligned} \quad (7.7)$$

In accordance with formulae (6.11), adapted to the case  $m = 1$ , one defines for later use:

$$\begin{aligned} H_{\pm} &:= \{H_{\pm}^{(1)}, F_{\pm}\}_{\partial_E} - \{H_{\pm}^{(2)}, F_{\pm}\} \frac{1}{2}(\mp \frac{\partial}{\partial a} - \partial_c) \\ H_{\ell} &:= \{H_{\ell}^{(1)}, F_{\ell}\}_{\partial_E} - \{H_{\ell}^{(2)}, F_{\ell}\} \frac{\partial}{\partial c_{\ell}} \end{aligned} \quad (7.8)$$

and one checks that, with this notation (7.8) and upon decoding formula (1.15) for the  $G_k$ 's,

$$G_{k+1} = \partial_E G_k + \frac{H_+ F_+^{(k)}}{F_+^2} + \frac{H_- F_-^{(k)}}{F_-^2} + \sum_{\ell=1}^{p-2} \frac{H_{\ell} F_{\ell}^{(k)}}{F_{\ell}^2},$$

where  $F^{(k)}$  is a shorthand for  $(\partial_E)^k F$ . With these expressions in mind,  $\mathbb{P} := \mathbb{P}_n^A(E)$  satisfies the (near-Wronskian) PDE (1.12), i.e.,

$$\det \begin{pmatrix} F_+ & F_- & F_1 & \dots & F_{p-2} & G_0 \\ F'_+ & F'_- & F'_1 & \dots & F'_{p-2} & G_1 \\ F''_+ & F''_- & F''_1 & \dots & F''_{p-2} & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_+^{(p)} & F_-^{(p)} & F_1^{(p)} & \dots & F_{p-2}^{(p)} & G_p \end{pmatrix} = 0, \quad (7.9)$$

**Special case:** For Brownian motions forced to  $a$  and  $-a$ , without the intermediate points, the formula (7.9) turns into the following determinant, with  $F_{\pm}$  and  $H_{\pm}^{(i)}$  as in (7.7), but with all  $c$ -partials removed:

$$\begin{aligned} &F_+ F_- \det \begin{pmatrix} F_+ & F_- & G_0 \\ F'_+ & F'_- & G_1 \\ F''_+ & F''_- & G_2 \end{pmatrix} \\ &= F_+ F_- \det \begin{pmatrix} F_+ & F_- & 0 \\ F'_+ & F'_- & \frac{H_+}{F_+} + \frac{H_-}{F_-} \\ F''_+ & F''_- & \frac{H_+'}{F_+} + \frac{H_-'}{F_-} \end{pmatrix} \\ &= (H_+ F_- + H_- F_+) \{F_+, F_-\}' - (H_+' F_- + H_-' F_+) \{F_+, F_-\} = 0. \end{aligned}$$

<sup>18</sup>In the formulae below (7.7), the constants  $C_{\pm}$  and  $C_{\ell}$  have the value:

$$C_{\pm} = -n_{\pm} \left( \pm a \pm \frac{n_{\mp}}{a} + 2 \sum_{r=1}^{p-2} \frac{n_r}{\pm a + c_r} \right) \quad \text{and} \quad C_{\ell} = 2n_{\ell} \left( c_{\ell} + \frac{n_+}{c_{\ell} + a} + \frac{n_-}{c_{\ell} - a} + \sum_{\substack{r=1 \\ r \neq \ell}}^{p-2} \frac{n_r}{c_{\ell} - c_r} \right)$$

## 8 Pearcey process with inliers

In this section, we consider non-intersecting Brownian motions leaving from 0 and forced to  $p$  target points at time  $t = 1$ , with the only condition that the left-most and right-most target points are symmetric with respect to the origin, with  $p - 2$  intermediate target points thrown in totally arbitrarily, exactly as in section 7.1. The purpose of this section is to identify the critical process obtained by letting  $n := n_+ = n_- \rightarrow \infty$  and by rescaling  $\tilde{a}$  and the  $\tilde{c}_i$  accordingly, while keeping  $n_1, \dots, n_{p-2}$  fixed. This is the content of Theorem 1.3.

*Proof of Theorem 1.3:* The proof consists of letting  $n = n_+ = n_- \rightarrow \infty$  in the kernel (1.7) and in the PDE (1.12). In the proof, which requires several steps, we shall restrict ourselves to  $m = 1$  (one-time), except for *Step 2*, which deals with the kernel.

*Step 1: The PDE.* The probability  $\mathbb{P} := \mathbb{P}_n^A(E)$  satisfies the (near-Wronskian) PDE (7.9); see section 7.2.

*Step 2: The scaling limit of the Brownian kernel.* Non-intersecting Brownian motions leaving from 0, such that  $n_r$  particles are forced to  $\beta_r$  at time  $t = 1$ , are given by the kernel (1.7), which is, in this instance, conveniently rewritten as

$$\begin{aligned}
 & H_{t_k, t_\ell}^{(n)}(x, y; \tilde{a}, -\tilde{c}_1, \dots, -\tilde{c}_{p-2}, -\tilde{a}) dy \\
 &= -\frac{dy}{2\pi^2 \sqrt{(1-t_k)(1-t_\ell)}} \int_C dV \int_{\Gamma_L} dU \frac{1}{U-V} \\
 &\quad \times \frac{e^{-\frac{t_k V^2}{1-t_k} + \frac{2xV}{1-t_k} - n_+ \ln(V-\tilde{a}) - n_- \ln(V+\tilde{a})}}{e^{-\frac{t_\ell U^2}{1-t_\ell} + \frac{2yU}{1-t_\ell} - n_+ \ln(U-\tilde{a}) - n_- \ln(U+\tilde{a})}} \prod_{r=1}^{p-2} \left( \frac{U + \tilde{c}_r}{V + \tilde{c}_r} \right)^{n_r} \\
 &\quad - \begin{cases} 0, & \text{for } t_k \geq t_\ell, \\ \frac{dy}{\sqrt{\pi(t_\ell - t_k)}} e^{-\frac{(x-y)^2}{t_\ell - t_k} - \frac{x^2}{1-t_k} - \frac{y^2}{1-t_\ell}}, & \text{for } t_k < t_\ell. \end{cases} \quad (8.1)
 \end{aligned}$$

One then uses the same steepest descent method as for the case without inliers; the so-called steepest descent  $F$ -function is the one (depending on  $U$  or  $V$ ) appearing in the exponential, with three consecutive derivatives being  $= 0$  at the origin; the change of integration variables  $U = U'(n/2)^{1/4}$  and  $V = V'(n/2)^{1/4}$  then leads, in the limit for  $n = n_+ = n_- \rightarrow \infty$  about the saddle point, to the kernel (1.18) (see for instance [19] and in the asymmetric

case [3]). So, the limit is

$$\lim_{n \rightarrow \infty} H_{t_i, t_j}^{(n)}(\tilde{x}, \tilde{y}; \tilde{a}, \tilde{c}_1, \dots, \tilde{c}_{p-2}, -\tilde{a}) d\tilde{y} \left| \begin{array}{l} t_k = \frac{1}{2} + \frac{\tau_k}{4\sqrt{2n}} \\ \tilde{x} = \frac{X}{4(n/2)^{1/4}} \\ \tilde{y} = \frac{Y}{4(n/2)^{1/4}} \\ \tilde{a} = \sqrt{n} \\ \tilde{c}_\ell = u_\ell \left(\frac{n}{2}\right)^{1/4} \end{array} \right. = K_{\tau_i, \tau_j}^{\mathcal{P}}(X, Y; u_1, \dots, u_{p-2}) dY, \quad (8.2)$$

where  $K_{\tau_i, \tau_j}^{\mathcal{P}}(X, Y; u_1, \dots, u_{p-2})$  is the Pearcey kernel with inliers (1.18).

*Step 3: The scaling limit of the PDE.* As mentioned, for the proof we limit ourselves to the one-time case, i.e.,  $m = 1$ . We now proceed in two steps:

**(i)** The change of variables (1.4) (especially footnote 6) from the non-intersecting Brownian motion probability to the matrix model (1.3); this change of variables appears in the first column of the table (8.3) below. In other terms, it is the time-dependent change from the variables  $(\tilde{x}, \tilde{a}, \tilde{c})$  to the variables  $(x, a, c)$ , yielding in particular the diagonal matrix  $A$  as in (7.5).

**(ii)** Subsequently apply the scaling given by (8.2) with  $z := n^{-1/4}$  and a very small renaming  $s := \tau/\sqrt{8}$ ,  $v_j := 2^{1/4}u_j$ ,  $\xi := X/2^{5/4}$  for computational convenience. This appears in the second column of table (8.3) below.

$$\begin{array}{l} x \\ a \\ c_\ell \end{array} = \begin{array}{l} \tilde{x} \sqrt{\frac{2}{t(1-t)}} \\ \tilde{a} \sqrt{\frac{2t}{1-t}} \\ \tilde{c}_\ell \sqrt{\frac{2t}{1-t}} \end{array} \left| \begin{array}{l} t = \frac{1}{2} + \frac{\tau}{4\sqrt{2n}} = \frac{1}{2} \left(1 + \left(\frac{\tau}{\sqrt{8}}\right)z^2\right) =: \frac{1}{2}(1 + s_k z^2) \\ \tilde{x} = \frac{X}{4(n/2)^{1/4}} = \left(\frac{X}{2^{5/4}}\right) \frac{z}{\sqrt{2}} =: \frac{\xi z}{\sqrt{2}} \\ \tilde{a} = \sqrt{n} = \frac{1}{z^2} \\ \tilde{c}_\ell = u_\ell \left(\frac{n}{2}\right)^{1/4} = \frac{(u_\ell 2^{1/4})}{\sqrt{2}z} =: \frac{v_\ell}{\sqrt{2}z} \end{array} \right. \quad (8.3)$$

Concatenating these two scalings leads to the following; in the string of equalities below, the change corresponding to **(i)** is indicated by  $*$ , whereas the second change **(ii)** is indicated by  $**$ :

$$P(E, s, v) := \ln \mathbb{P}_n^{(\tilde{a}, -\tilde{c}_2, \dots, -\tilde{c}_{p-1}, -\tilde{a})}(\text{all } x_i(t) \in \tilde{E}) \left| \begin{array}{l} t = \frac{1}{2}(1 + s z^2) \\ \tilde{a} = 1/z^2 \\ \tilde{c}_i = v_i/(\sqrt{2}z) \\ \tilde{E} = E z/\sqrt{2} \end{array} \right.$$

$$\begin{aligned}
&\stackrel{*}{=} \ln \mathbb{P}_n^A \left( \begin{array}{c} \text{entries of diagonal matrix } A \\ \tilde{E} \sqrt{\frac{2}{t(1-t)}; \quad \sqrt{\frac{2t}{1-t}}(\tilde{a}, \tilde{c}, -\tilde{a})} \end{array} \right) \Big| \begin{array}{l} t = \frac{1}{2}(1 + sz^2) \\ \tilde{a} = 1/z^2 \\ \tilde{c}_i = v_i/(\sqrt{2}z) \\ \tilde{E} = Ez/\sqrt{2} \end{array} \\
&\stackrel{**}{=} \ln \mathbb{P}_n^A \left( \begin{array}{c} \text{entries of diagonal matrix } A \\ \frac{2zE}{\sqrt{1-s^2z^4}}; \quad \sqrt{\frac{1+sz^2}{1-sz^2}} \left( \frac{\sqrt{2}}{z^2}, \frac{v_i}{z}, -\frac{\sqrt{2}}{z^2} \right) \end{array} \right) \\
&=: \ln \mathbb{P}_n^A(E'; \underbrace{a, c, -a}_{\substack{\text{entries of diagonal} \\ \text{matrix } A}}) =: Q(E'; a, c). \tag{8.4}
\end{aligned}$$

Note that in the rest of this section,  $E$  and  $E'$  refer to complement of compact intervals; i.e., we shall be dealing with gap probabilities. The identity (8.4) suggests the  $z$ -dependent map:

$$T_z^{-1} : (E, s, v_j) \mapsto (E', a, c_j), \quad 1 \leq j \leq p-2,$$

given by

$$E' = \frac{2zE}{\sqrt{1-s^2z^4}}, \quad a = \frac{\sqrt{2}}{z^2} \sqrt{\frac{1+sz^2}{1-sz^2}}, \quad c_j = \frac{v_j}{z} \sqrt{\frac{1+sz^2}{1-sz^2}}, \tag{8.5}$$

with inverse map

$$T_z : (E', a, c) \mapsto (E, s, v_j), \quad 1 \leq j \leq p-2,$$

given by

$$E = \frac{\sqrt{2}azE'}{a^2z^4 + 2}, \quad s = \frac{a^2z^4 - 2}{z^2(a^2z^4 + 2)}, \quad v_j = \frac{\sqrt{2}c_j}{az}. \tag{8.6}$$

Then summarizing the above, one has

$$Q(E'; a, c) := \log \mathbb{P}_n^A(E'; a, c, -a) = \log \mathbb{P}_n^A(T_z^{-1}(E; s, v_j)) =: P(E; s, v),$$

and thus

$$Q(E', a, c) = P \left( \frac{\sqrt{2}azE'}{a^2z^4 + 2}, \frac{a^2z^4 - 2}{z^2(a^2z^4 + 2)}, \frac{\sqrt{2}c_j}{az} \right)$$

satisfies the PDE (1.12) in the variables  $E', a, c$ , in terms of the operators specified in (7.6), with  $F_{\pm}$ ,  $F_{\ell}$ ,  $H_{\pm}^{(i)}$ ,  $H_{\ell}^{(i)}$  given by (7.7). In order to express

the PDE in terms of the function  $P(E; s, v)$ , one must express all partials of  $Q(E'; a, c)$  in terms of partials of  $P(E; s, v)$  in  $E, s, v$ ; e.g.,

$$\partial_{E'} Q(E'; a, c) \Big|_{T_z} = \frac{\sqrt{2}azE'}{a^2z^4 + 2} \partial_E P \Big|_{T_z} = \frac{\sqrt{1-s^2z^4}}{2z} \partial_E P(E; s, v)$$

and thus the operators  $\partial_{E'}$  and  $\partial_E$ , as acting on  $Q$  and  $P$  respectively, and similarly for the others, are related by the following; we also indicate what the relationship becomes for  $z \rightarrow 0$ :<sup>19</sup>

$$\begin{aligned} \partial_{E'} \Big|_{T_z} &= \frac{\sqrt{1-s^2z^4}}{2z} \partial_E = \left( \frac{1}{2z} - \frac{1}{4}s^2z^3 - \frac{1}{16}s^4z^7 + O(z^9) \right) \partial_E \\ \varepsilon_{E'} \Big|_{T_z} &= \varepsilon_E \\ \partial_{c_i} \Big|_{T_z} &= z \sqrt{\frac{1-sz^2}{1+sz^2}} \partial_{v_i} = \left( z - sz^3 + \frac{1}{2}s^2z^5 + O(z^7) \right) \partial_{v_i} \\ \sqrt{2} \frac{\partial}{\partial a} \Big|_{T_z} &= (1-sz^2)^2 \sqrt{\frac{1+sz^2}{1-sz^2}} \frac{\partial}{\partial s} - z^2 \sqrt{\frac{1-sz^2}{1+sz^2}} (\varepsilon_v + sz^2 \varepsilon_E) \\ &= \frac{\partial}{\partial s} - z^2 (\varepsilon_v + s \frac{\partial}{\partial s}) - sz^4 \left( \frac{1}{2}s \frac{\partial}{\partial s} + \varepsilon_E - \varepsilon_v \right) + O(z^6). \end{aligned} \quad (8.7)$$

For notational simplicity, derivatives will often be abbreviated in the obvious way:

$$(\partial_{E'})^j F_i \mapsto F_i^{(j)}, \quad (\partial_E)^j P \mapsto P^{(j)}, \quad \frac{\partial}{\partial s} P \mapsto \dot{P}, \dots, \quad (8.8)$$

while keeping in mind from (8.7) that  $\partial_{E'}$ , acting on functions of  $(E', a, c)$ , as  $F_{\pm}$ ,  $H_{\ell}$  and  $G_{\ell}$ , translates, *to leading order*, into  $\partial_E/(2z)$  acting on functions of  $(E, s, v)$ ; also notice the *big gaps* in the first few terms of the series for  $\partial_{E'}$ . In view of the PDE (1.12), one needs the series expansion in  $z$  of the  $F$ 's, the  $H$ 's and the  $G$ 's and their derivatives  $\partial_{E'}$ . This is the content of:

**Lemma 8.1** *Introducing the expression  $\mathbb{Y}$ , with  $\varepsilon = \varepsilon_E - \varepsilon_v$ , and  $v = (v_1, \dots, v_{p-2})$ ,*

$$\frac{1}{2} \mathbb{Y} := 4(\varepsilon - 2s \frac{\partial}{\partial s} - 2) \partial_E^2 P + 16 \partial_v \partial_E \dot{P} + 8 \ddot{P} + \left\{ \partial_E \dot{P}, \partial_E^2 P \right\}_{\partial_E}, \quad (8.9)$$

*one checks, (remember from (7.8) the definition of  $H_{\pm}$  and  $H_{\ell}$ )*

$$\partial_{E'}^i F_{\pm} = \left( \frac{\partial_E}{2z} \right)^i \left( \frac{1}{z^4} + \frac{1}{8z^2} \partial_E^2 P \mp \frac{1}{4\sqrt{2}z} \partial_E \dot{P} \right) + O(z^{-i})$$

<sup>19</sup>Since  $\partial_c = \sum_1^{p-2} \partial_{c_i}$  and  $\partial_v = \sum_1^{p-2} \partial_{v_i}$ , the third relation is valid for  $\partial_c$  and  $\partial_v$  as well.

$$\begin{aligned}
\partial_{E'}^i F_\ell &= \left(\frac{\partial_E}{2z}\right)^i \left(\frac{1}{2}\partial_{v_\ell}\partial_E P + n_\ell - \frac{sz^2}{2}\partial_{v_\ell}\partial_E P\right) + O(z^{3-i}) \\
\frac{H_+}{F_+} + \frac{H_-}{F_-} + \sum_{\ell=1}^{p-2} \frac{H_\ell}{F_\ell} &\stackrel{*}{=} \frac{1}{64z^2} \left(\mathbb{Y} - 3(\partial_E^2 P)(\partial_E^2 \dot{P})\right) + O(1) \\
\frac{H_+\partial_{E'}^i F_+}{F_+^2} + \frac{H_-\partial_{E'}^i F_-}{F_-^2} + \sum_{\ell=1}^{p-2} \frac{H_\ell\partial_{E'}^i F_\ell}{F_\ell^2} &\stackrel{*}{=} \frac{3}{32z} (\partial_E^3 P) \left(\frac{\partial_E}{2z}\right)^{1+i} \dot{P} + O(z^{-i-1})
\end{aligned} \tag{8.10}$$

and also, for  $k = 0, 1, \dots$ , one has

$$G_{k+1} + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(k+1)}P'' = \frac{\mathbb{Y}^{(k)}}{16(2z)^{k+2}} + O(z^{-k-1}). \tag{8.11}$$

*Proof:* The formulae (8.10) are straightforward computations; one of them involves the expression  $\mathbb{Y}$  introduced in (8.9). The big gaps in the series (8.7) of  $\partial_{E'}$  is responsible for the mere action of  $(\partial_E/2z)^i$ , in computing higher derivatives. Moreover, in the third formula, one notices that the sums  $\sum_{\ell=1}^{p-2} H_\ell/F_\ell$  on the left hand side of  $\stackrel{*}{=}$  actually do not play any role in the leading terms, because  $H_\ell$  and  $F_\ell$  both are  $O(1)$ . Formula (8.11) is shown by induction; namely for  $k = 0$ , one checks, using the formulae (8.10),

$$\begin{aligned}
G_1 + \frac{3\sqrt{2}}{16}(F_- - F_+)P'' &= \sum_{\ell=1}^p \frac{H_\ell}{F_\ell} + \frac{3\sqrt{2}}{16}(F_- - F_+)P'' \\
&= \frac{1}{64z^2}(\mathbb{Y} - 3P''\dot{P}) + \frac{3\sqrt{2}}{16z^2} \left(\frac{1}{4\sqrt{2}}\dot{P}''P''\right) + O(1) = \frac{\mathbb{Y}}{64z^2} + O(1).
\end{aligned}$$

Assume inductively

$$G_i + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(i)}P'' = \frac{\mathbb{Y}^{(i-1)}}{16(2z)^{i+1}} + O(z^{-i}) \text{ for } 1 \leq i \leq k, \tag{8.12}$$

and prove it for  $i = k + 1$ . Then, using the general definition (1.15) of  $G_{k+1}$  in terms of  $G_k$ , formula (8.12), the derivatives  $\partial_{E'}$  of  $F_\pm$  as in (8.10) and the

last formula of (8.10), one checks

$$\begin{aligned}
& G_{k+1} + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(k+1)}P'' \\
&= \partial_{E'}G_k + \frac{H_+F_+^{(k)}}{F_+^2} + \frac{H_-F_-^{(k)}}{F_-^2} + \sum_{\ell=1}^{p-2} \frac{H_\ell F_\ell^{(k)}}{F_\ell^2} + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(k+1)}P'' \\
&= \frac{\partial_E}{2z} \left( \frac{\mathbb{Y}^{(k-1)}}{16(2z)^{k+1}} - \frac{3\sqrt{2}}{16}(F_- - F_+)^{(k)}P'' + O(z^{-k}) \right) \\
&\quad + \frac{H_+F_+^{(k)}}{F_+^2} + \frac{H_-F_-^{(k)}}{F_-^2} + \sum_{\ell=1}^{p-2} \frac{H_\ell F_\ell^{(k)}}{F_\ell^2} + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(k+1)}P'' \\
&= \frac{\mathbb{Y}^{(k)}}{16(2z)^{k+2}} + O(z^{-k-1}),
\end{aligned}$$

establishing Lemma 8.1. ■

By Corollary 1.2,  $Q(E'; a, c) = \ln \mathbb{P}_n^A(E'; a, c)$  satisfies the PDE (7.9), which induces a PDE for  $P(E; s, v) = \ln \mathbb{P}_n^A(T_z^{-1}(E; s, v_j))$ , remembering (8.4) and (8.5). As pointed out, the PDE for  $Q(E'; a, c)$  misses to be a Wronskian by the last column. It is appropriate to do some column operations; e.g., subtracting the first from the second and then adding the second, multiplied with  $P''$ , to the last one; also it is convenient to multiply the columns with 2's and  $\sqrt{2}$ 's. This gives us the determinant below, which vanishes according to Corollary 1.2. The second equality  $\stackrel{*}{=}$  uses in a straightforward way the series expansion of Lemma 8.1 above,

$$0 = \det \begin{pmatrix} 2F_+ & \sqrt{2}(F_- - F_+) & 2F_1 & \dots & 2F_{p-2} & G_0 + \frac{3\sqrt{2}}{16}(F_- - F_+)P'' \\ 2F'_+ & \sqrt{2}(F_- - F_+)' & 2F'_1 & \dots & 2F'_{p-2} & G_1 + \frac{3\sqrt{2}}{16}(F_- - F_+)'P'' \\ 2F''_+ & \sqrt{2}(F_- - F_+)'' & 2F''_1 & \dots & 2F''_{p-2} & G_2 + \frac{3\sqrt{2}}{16}(F_- - F_+)''P'' \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2F_+^{(p)} & \sqrt{2}(F_- - F_+)^{(p)} & 2F_1^{(p)} & \dots & 2F_{p-2}^{(p)} & G_p + \frac{3\sqrt{2}}{16}(F_- - F_+)^{(p)}P'' \end{pmatrix},$$

$$\stackrel{*}{=} \det \begin{pmatrix} \frac{2}{z^4} + \frac{P''}{(2z)^2} + O\left(\frac{1}{z}\right) & \frac{\dot{P}'}{(2z)} + O(z) & \frac{\partial P'}{\partial v_1} + 2n_1 + O(z^2) & \dots \\ \frac{P''}{(2z)^3} + O\left(\frac{1}{z^2}\right) & \frac{\dot{P}''}{(2z)^2} + O(1) & \frac{1}{(2z)} \frac{\partial P''}{\partial v_1} + O(z) & \dots \\ \frac{P^{iv}}{(2z)^4} + O\left(\frac{1}{z^3}\right) & \frac{\dot{P}'''}{(2z)^3} + O\left(\frac{1}{z}\right) & \frac{1}{(2z)^2} \frac{\partial P'''}{\partial v_1} + O(1) & \dots \\ \vdots & \vdots & \vdots & \dots \\ \frac{P^{(p+2)}}{(2z)^{p+2}} + O\left(\frac{1}{z^{p+1}}\right) & \frac{\dot{P}^{(p+1)}}{(2z)^{p+1}} + O\left(\frac{1}{z^{p-1}}\right) & \frac{1}{(2z)^p} \frac{\partial P^{(p+1)}}{\partial v_1} + O\left(\frac{1}{z^{p-2}}\right) & \dots \end{pmatrix}$$

$$\left. \begin{array}{l} \dots \quad \frac{\partial P'}{\partial v_{p-2}} + 2n_{p-2} + O(z^2) \quad \frac{3}{16(2z)} \dot{P}' P'' + O(z) \\ \dots \quad \frac{1}{(2z) \partial v_{p-2}} \frac{\partial P''}{\partial v_{p-2}} + O(z) \quad \frac{\mathbb{Y}}{16(2z)^2} + O(1) \\ \dots \quad \frac{1}{(2z)^2 \partial v_{p-2}} \frac{\partial P'''}{\partial v_{p-2}} + O(1) \quad \frac{\mathbb{Y}'}{16(2z)^3} + O\left(\frac{1}{z}\right) \\ \dots \quad \vdots \quad \vdots \\ \dots \quad \frac{1}{(2z)^p} \frac{\partial P^{(p+1)}}{\partial v_{p-2}} + O\left(\frac{1}{z^{p-2}}\right) \quad \frac{\mathbb{Y}^{(p-1)}}{16(2z)^{p+1}} + O\left(\frac{1}{z^{p-1}}\right) \end{array} \right)$$

$$\stackrel{**}{=} \frac{C}{z^{6+p(p+1)/2}} \mathcal{W}_p \left( \partial_E^2 \frac{\partial}{\partial s} P, \frac{\partial}{\partial v_1} \partial_E^2 P, \dots, \frac{\partial}{\partial v_{p-2}} \partial_E^2 P, \mathbb{Y} \right) + O\left(\frac{1}{z^{4+p(p+1)/2}}\right).$$

The last equality  $\stackrel{**}{=}$  stems from the fact that the matrix consists of columns with increasing powers in  $1/z$ , except for the element  $(1, 1)$ , whose leading term is  $2/z^4$ . Therefore the leading contribution of the determinant of the matrix will be given by

$$\frac{2}{z^4} \times \text{the determinant of the } (1, 1)\text{-minor,}$$

which indeed leads to equality  $\stackrel{**}{=}$ . Also the term  $-8s \partial_E^2 \dot{P}_n$  could be removed by adding  $8s \times$  (the first column); but we prefer not to do this, in view of the conjecture 1.5. Taking the limit, when  $n \rightarrow \infty$ , leads to the PDE for  $P = \lim \log \mathbb{P}_n$ . In the end, one must undo the slight renaming (8.3) of the variables  $s = \tau/\sqrt{8}$ ,  $v_j = 2^{1/4} u_j$ ,  $x_i = \xi_i/2^{5/4}$  and go back to the  $(\tau, u_j, \xi)$ -variables, yielding  $\frac{1}{2} \mathbb{Y} = 8^{3/2} \mathbb{X}$ , with  $\mathbb{X}$  as defined in (1.20). This yields PDE (1.19), which ends the proof of Theorem 1.3.  $\blacksquare$

*Very sketchy Proof of Corollary 1.4:* A detailed proof appears in Adler-Orantin-van Moerbeke [3]. In the absence of inliers ( $p = 2$ ), the Wronskian (1.19) is the determinant of a  $2 \times 2$  matrix:

$$0 = \mathcal{W}_2 \left[ \partial_E^2 \partial_\tau \ln \mathbb{P}^P, \mathbb{X} \right]_{\partial_E} = \left\{ \partial_E^2 \partial_\tau \ln \mathbb{P}^P, \mathbb{X} \right\}_{\partial_E}. \quad (8.13)$$

Performing the same scaling limit on an *asymmetric* situation, with  $2nq$  particles forced to  $-\sqrt{n}$  and  $2n(1-q)$  particles forced to  $\sqrt{n}$  for  $0 < q < 1$ , with  $q \neq 1/2$ , leads to a PDE for the leading term having the form

$$\left\{ \partial_E^3 \ln \mathbb{P}^P, \mathbb{X} \right\}_{\partial_E} = 0. \quad (8.14)$$

Thus  $\ln \mathbb{P}^P$  satisfies two different PDE's, (8.13) and (8.14), given by two Wronskians of  $\mathbb{X}$  with  $\partial_E^2 \partial_\tau \ln \mathbb{P}^P$  and  $\partial_E^3 \ln \mathbb{P}^P$ . Then a functional-theoretical argument explained in [3] implies  $\mathbb{X} = 0$ .  $\blacksquare$

For inliers, we further conjecture -in analogy with the result in Corollary 1.4- the validity of equations (1.22) and (1.23), as stated in Conjecture 1.5.

## 9 Appendix: evaluation of the integral over the full range

In this section we prove formula (6.7), i.e., we show that

$$\begin{aligned} \int_{\mathbb{R}^{mN}} \Delta_N(v_1) \prod_{\ell=1}^p \left( \frac{\Delta_{n_\ell}(v_m^{(\ell)})}{n_\ell!} \prod_{i=1}^{n_\ell} e^{-\frac{1}{2} \sum_{k=1}^m v_{k;i}^{(\ell)2} + \sum_{k=1}^{m-1} c_k v_{k;i}^{(\ell)} v_{k+1;i}^{(\ell)} + b_1^{(\ell)} v_{m;i}^{(\ell)}} \right) \prod_{k=1}^m dv_k \\ = g_n(c) e^{-\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^p n_\ell b_1^{(\ell)2}} \prod_{1 \leq i < j \leq p} (b_1^{(j)} - b_1^{(i)})^{n_i n_j}, \end{aligned}$$

with  $g_n(c)$  a function, depending on  $c_1, \dots, c_{m-1}$  and  $n$  only, computed below. In view of the representation of the above integral as the determinant of a moment matrix, as in (3.11), it suffices to prove that

$$\det \begin{pmatrix} M^{(1)} \\ \vdots \\ M^{(p)} \end{pmatrix} = g_n(c) e^{-\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^p n_\ell b_1^{(\ell)2}} \prod_{1 \leq i < j \leq p} (b_1^{(j)} - b_1^{(i)})^{n_i n_j}, \quad (9.1)$$

where, for  $\ell = 1, \dots, p$ , the  $n_\ell \times N$  matrix  $M^{(\ell)}$  is defined by

$$M^{(\ell)} := \left( \int_{\mathbb{R}^m} w_1^j w_m^i e^{-\frac{1}{2} \sum_{k=1}^m w_k^2 + \sum_{k=1}^{m-1} c_k w_k w_{k+1} + b_1^{(\ell)} w_m} \prod_{k=1}^m dw_k \right)_{\substack{0 \leq i \leq n_\ell \\ 0 \leq j \leq N}}.$$

Introducing for  $a, b \in \mathbb{R}$  the zero moment<sup>20</sup>

$$\begin{aligned} m(a, b) &:= \int_{\mathbb{R}^m} e^{-\frac{1}{2} (\sum_{k=1}^m w_k^2 - 2 \sum_{k=1}^{m-1} c_k w_k w_{k+1}) + a w_1 + b w_m} \prod_{k=1}^m dw_k \\ &= (2\pi)^{m/2} \sqrt{-\det \mathcal{J}} e^{-\frac{1}{2} (\mathcal{J}_{11} a^2 + 2 \mathcal{J}_{1m} a b + \mathcal{J}_{mm} b^2)}, \end{aligned} \quad (9.2)$$

we can express all the entries of  $M^{(\ell)}$  as

$$M_{ij}^{(\ell)} = \frac{\partial^j}{\partial a^j} \frac{\partial^i}{\partial b^i} m(0, b_1^{(\ell)}). \quad (9.3)$$

Let us first prove (9.1) in the case in which all  $n_\ell$  are equal to 1 (so that  $p = N$ ). Then, it follows from (9.2) and (9.3) that, for  $\ell = 1, \dots, p$ , the vector  $M^{(\ell)}$  is, modulo a constant which depends on  $c_1, \dots, c_{m-1}$  and  $N$ , but not on  $\ell$ , of the form

$$M^{(\ell)} \sim e^{-\frac{\mathcal{J}_{mm}}{2} b_1^{(\ell)2}} \left( 1, \alpha_1(b_1^{(\ell)}), \dots, \alpha_{N-1}(b_1^{(\ell)}) \right),$$

<sup>20</sup>Using

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2} \langle Qw, w \rangle + \langle \ell, w \rangle} dw_1 \dots dw_m = \frac{(2\pi)^{m/2}}{\sqrt{\det Q}} e^{\frac{1}{2} \langle Q^{-1} \ell, \ell \rangle},$$

for  $Q := -\mathcal{J}^{-1}$  and  $\ell := (a, 0, \dots, 0, b)$ .

where  $\alpha_j(b_1^{(\ell)})$  is a polynomial in  $b_1^{(\ell)}$  of degree  $j$ , with leading term  $(-\mathcal{J}_{1m}b_1^{(\ell)})^j$ , and whose coefficients are independent of  $\ell$ , but depend on  $c$ . It follows that, if all  $n_\ell$  are equal to 1, then<sup>21</sup>

$$\det \begin{pmatrix} M^{(1)} \\ \vdots \\ M^{(p)} \end{pmatrix} = g_N(c) e^{-\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^p b_1^{(\ell)2}} \prod_{1 \leq i < j \leq p} (b_1^{(j)} - b_1^{(i)}), \quad (9.4)$$

proving (9.1) in that case. Let us show how the other extreme case, where there is only one  $n_\ell$  (so that  $p = 1$  and  $n_1 = N$ ), is derived from it. Let  $f : \mathbb{R} \rightarrow \mathbb{R}^N$  be a smooth function and let  $\beta \in \mathbb{R}$ . Then

$$\det \begin{pmatrix} f(\beta) \\ f'(\beta) \\ \vdots \\ f^{(N-1)}(\beta) \end{pmatrix} = \lim_{\beta_1, \dots, \beta_N \rightarrow \beta} \frac{\prod_{k=1}^{N-1} k!}{\prod_{1 \leq i < j \leq N} (\beta_j - \beta_i)} \det \begin{pmatrix} f(\beta_1) \\ f(\beta_2) \\ \vdots \\ f(\beta_N) \end{pmatrix}, \quad (9.5)$$

as follows by writing each  $f(\beta_k)$  as a Taylor series around  $f(\beta)$ . Applied to

$$f(\beta) := e^{-\frac{\mathcal{J}_{mm}}{2} \beta^2} (1, \alpha_1(\beta), \dots, \alpha_{N-1}(\beta)),$$

and  $\beta_1, \dots, \beta_N = b_1^{(\ell)}, \dots, b_N^{(\ell)}$  we conclude using (9.3) and (9.4) that, when  $p = 1$ , then

$$\det(M^{(1)}) = \lim_{\beta_1, \dots, \beta_N \rightarrow b_1^{(1)}} \prod_{k=1}^{N-1} k! g_N(c) e^{-\frac{\mathcal{J}_{mm}}{2} \sum_{\ell=1}^N \beta_\ell^2} = g_N(c) e^{-\frac{\mathcal{J}_{mm}}{2} N b_1^{(1)2}},$$

proving (9.1) in this case. The proof of formula (9.1) in the intermediate case, when there are several  $n_\ell$ , which are not equal to 1, follows in a similar way from (9.4), taking the limit  $\beta_i \rightarrow b_1^{(j)}$ , for  $i = 1, \dots, N$ , with  $n_\ell$  of the  $\beta_i$  going to  $b_1^{(\ell)}$ , namely  $\beta_1, \dots, \beta_{n_1} \rightarrow b_1^{(1)}$ , and  $\beta_{n_1+1}, \dots, \beta_{n_1+n_2} \rightarrow b_1^{(2)}$ , and so on, where now one divides by a product of  $p$  Vandermonde determinants, each going with a collapsing group.

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<sup>21</sup>Note  $g_N(c) = (-\mathcal{J}_{1m})^{N(N-1)/2} (-\det \mathcal{J})^{N/2} (2\pi)^{Nm/2}$ , as easily follows from the argument, and finally in the full case  $g'_N(c) = \prod_{\ell=1}^p \prod_{k=1}^{n_\ell-1} k! g_N(c)$ .

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