POISSON COHOMOLOGY OF THE AFFINE PLANE

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ABSTRACT. We compute the Poisson cohomology of homogeneous Poisson structures on the plane. The singular locus Γ of such a Poisson structure consists of a family of lines passing through O and we show how the dimensions of the first and second cohomology groups are related to the weight of O as a singular point of Γ.

CONTENTS

1. INTRODUCTION

Poisson structures appear naturally in the study of rigidity/deformations of associative commutative algebras, in Lie theory and in classical mechanics. Poisson cohomology in turn appears when one considers rigidity/deformations of Poisson algebras, it generalizes Lie algebra cohomology and the basic concepts of Hamiltonian mechanics are conveniently expressed in terms of Poisson cohomology.

In order to justify the latter three claims, let $(\mathbf{A}, \{\cdot\,,\cdot\})$ be a Poisson algebra over a field $\mathbb{F}$ of characteristic 0 and let us introduce for $k > 0$ the vector space $\Lambda^k(\mathbf{A})$ of antisymmetric k-derivations: a $Q \in \Lambda^k(\mathbf{A})$ is a multilinear antisymmetric map from $\mathbf{A}^k$ to $\mathbf{A}$ such that for any $a_1, \dots, a_{k-1}$ the map $a \mapsto Q(a, a_1, \dots, a_{k-1})$ is a derivation. We set $\Lambda^0(\mathbf{A}) = \mathbf{A}$. These spaces are the elements of a complex whose coboundary operator $\delta : \Lambda^k(\mathbf{A}) \to \Lambda^{k+1}(\mathbf{A})$ is defined for $Q \in \Lambda^k(\mathbf{A})$ by

$$(\delta Q)(q_0, q_1, \dots, q_k) = \sum_{i=0}^{k} (-1)^i \{q_i\,, Q(q_0, \dots, \hat{q}_i, \dots, q_k)\} + \\ + \sum_{0 \le i < j}^{k} (-1)^{i+j} Q(\{q_i\,, q_j\}, q_0, \dots, \hat{q}_i, \dots, \hat{q}_j, \dots, q_k),$$

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where $q_0,\dots,q_k$ are arbitrary elements of $\mathbf{A}$. In terms of the Schouten bracket $[\cdot\,,\cdot]_S$ we have that $\delta Q=[\{\cdot\,,\cdot\},Q]_S$, yielding $\delta^2=0$, an immediate consequence of the graded Jacobi identity for $[\cdot\,,\cdot]_S$. Recall that this bracket is the natural bracket on the graded Lie algebra of derivations of the exterior algebra of $\mathbf{A}$. The k-th cohomology group of this complex is called the k-th *Poisson cohomology group* of $(\mathbf{A},\{\cdot\,,\cdot\})$ and is denoted by $H^k(\mathbf{A},\{\cdot\,,\cdot\})$.

(1) The relevance of $H^2(\mathbf{A},\{\cdot\,,\cdot\})$ and $H^3(\mathbf{A},\{\cdot\,,\cdot\})$ for the deformation theory of Poisson algebras comes from the following. Suppose that $\{\cdot\,,\cdot\}_\star=\sum_{i=0}^n\{\cdot\,,\cdot\}_ih^i$ is a n-th order deformation, i.e. $(\mathbf{A}[[h]]/(h^{n+1}),\{\cdot\,,\cdot\}_\star)$ is a Poisson algebra (over $\mathbb{C}[[h]]$), with $\{\cdot\,,\cdot\}_0=\{\cdot\,,\cdot\}$ (on $\mathbf{A}$). Then $\{\cdot\,,\cdot\}_\star$ can be extended to an $(n+1)$-th order deformation if and only if the three-cocycle

$$C_{n+1}=\sum_{\substack{i+j=n+1\\ i,j>0}}[\{\cdot\,,\cdot\}_i,\{\cdot\,,\cdot\}_j]_S$$

is a coboundary. The extension of the k-th order deformation is then given by $\pi_\star+\{\cdot\,,\cdot\}_{k+1}h^{k+1}$, where $\delta\{\cdot\,,\cdot\}_{k+1}=C_{k+1}$. Moreover, any two such extensions differ by a 2-cocycle and this cocycle is a coboundary if and only if the two extensions define equivalent $(n+1)$-th order deformations.

(2) Suppose that $\mathfrak{g}$ is a (finite-dimensional) Lie algebra. Sym $\mathfrak{g}$ becomes a Poisson algebra, simply by defining $\{x,y\}=[x,y]$ for any $x,y\in\mathfrak{g}$, and extending $\{\cdot\,,\cdot\}$ to a biderivation. Let us denote by $\mathrm{Cas}(\{\cdot\,,\cdot\})$ the algebra of Casimirs of $\{\cdot\,,\cdot\}$, which is the central part of the enveloping algebra and consists of the symmetric invariants of $\mathfrak{g}$. If $\mathfrak{g}$ is reductive then Poisson cohomology $H^\star(\mathrm{Sym}\,\mathfrak{g},\{\cdot\,,\cdot\})$ is related to Lie algebra cohomology $H^\star(\mathfrak{g})$ by

$$H^k(\mathrm{Sym}\,\mathfrak{g},\{\cdot\,,\cdot\})=H^k(\mathfrak{g})\otimes_{\mathbb{F}}\mathrm{Cas}(\{\cdot\,,\cdot\}),$$

where k is any non-negative integer.

(3) The phase space of a classical mechanical system comes always equipped with a Poisson structure (which is not necessarily symplectic). The algebra of Casimirs of $(\mathbf{A},\{\cdot\,,\cdot\})$ is precisely $H^0(\mathbf{A},\{\cdot\,,\cdot\})$ and corresponds to the Hamiltonians with trivial (zero) dynamics. The 1-coboundaries are the Hamiltonian derivations, i.e., the Hamiltonian vector fields in the smooth case. Since the coboundary of a vector field X is the Lie derivative of $\{\cdot\,,\cdot\}$ with respect to X the 1-cocycles are the symmetries of the Poisson structure. Furthermore, a 2-cocycle which defines a Poisson structure is compatible with $\{\cdot\,,\cdot\}$, leading to a multi-Hamiltonian structure, and a 2-coboundary is the Lie derivative of $\{\cdot\,,\cdot\}$ with respect to some vector field.

We also wish to point out that the Schouten bracket, which defines a graded Lie algebra structure on the space of antisymmetric derivations, induces a graded Lie algebra structure $[\cdot\,,\cdot]_S$ in Poisson cohomology,

$$[\cdot\,,\cdot]_S:H^k(\mathbf{A},\{\cdot\,,\cdot\})\times H^l(\mathbf{A},\{\cdot\,,\cdot\})\to H^{k+l-1}(\mathbf{A},\{\cdot\,,\cdot\}).$$

For $k=l=1$ this bracket is precisely the commutator of derivations. There exists moreover another algebra structure on $H^\star(\mathbf{A},\{\cdot\,,\cdot\})$: exterior product defines a commutative graded algebra structure on the space of cochains $\Lambda^\star\mathbf{A}$ and induces a cup-product in Poisson cohomology,

$$\wedge:H^k(\mathbf{A},\{\cdot\,,\cdot\})\times H^l(\mathbf{A},\{\cdot\,,\cdot\})\to H^{k+l}(\mathbf{A},\{\cdot\,,\cdot\}).$$

These two different graded products define on $H^\star(\mathbf{A},\{\cdot\,,\cdot\})$ a Gerstenhaber algebra structure, induced from the one on $\Lambda^\star\mathbf{A}$; the latter algebra structure can, in the case when $\mathbf{A}$ is the algebra of smooth functions on a differentiable manifold, be identified with the Schouten algebra of antisymmetric contravariant tensors on a manifold.

As was noticed by many authors, the computation of the Poisson cohomology of a given space is very difficult. Exceptions are the Poisson cohomology of a symplectic manifold, which is precisely its de Rham cohomology, and the Poisson cohomology of a linear Poisson structure, as discussed in (2) above. Indeed, already the calculation in the case of the Poisson structure on $\mathbb{R}^2$, defined by

$$\{x,y\} = x^2 + y^2$$

has been the subject of several papers! The purpose of the present paper is to compute the Poisson cohomology for all Poisson structures on $\mathbb{F}^2$ which are homogeneous, in the sense that they are given by $\{x,y\}^\varphi = \varphi(x,y)$, where φ is a homogeneous polynomial of degree $n \in \mathbb{N}$. Notice that the singular locus $\Gamma_\varphi \subset \bar{\mathbb{F}}^2$ of $\{\cdot\,,\cdot\}^\varphi$ consists of $m \leq n$ distinct lines through the origin, a singular curve (if $m \geq 2$). It is easy to show (Lemma 2.1) that $\dim H^2(\mathbf{A},\varphi)$ is infinite-dimensional when $m \neq n$ and that $\dim H^2(\mathbf{A},\varphi) > 0$ when $m \geq 2$. A more precise statement, obtained in Proposition 2.3, states that if $m = n$ then $\dim H^2(\mathbf{A},\varphi) = n(n-1)$, a number which is precisely twice the number of singularities of Γ_φ; indeed, the origin is the only singular point, but it has weight $\begin{pmatrix} n \\ 2 \end{pmatrix}$. Similarly we show that under the same assumption $\dim H^1(\mathbf{A},\varphi) = n$. In the more general case where φ admits a complete factorization into factors of degree 1, which means that Γ_φ consists of arbitrary lines in the plane (assumed non-parallel) we show that $2\begin{pmatrix} n \\ 2 \end{pmatrix}$ is an upper bound for the dimension of $H^2(\mathbf{A},\varphi)$. We conjecture that the inequality, given by this bound, is actually an equality. Notice that when the n lines of Γ_φ are in general position then $\begin{pmatrix} n \\ 2 \end{pmatrix}$ is precisely the number of singular points of Γ_φ.

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2. Poisson cohomology of $\mathbb{F}^2$

In this section we will study the Poisson cohomology of $(\mathbf{A},\cdot,\{\cdot\,,\cdot\})$, in the case of $\mathbf{A} = \mathbb{F}[x,y]$, the algebra of regular functions on $\mathbb{F}^2$, where $\mathbb{F}$ is a field of characteristic 0, and $\{\cdot\,,\cdot\}$ is a *homogeneous* Poisson structure on $\mathbf{A}$, as will be defined below. Notice that if φ is any polynomial in two variables then there is a unique antisymmetric biderivation $\{\cdot\,,\cdot\}^\varphi$ of $\mathbb{F}[x,y]$ such that $\{x\,,y\}^\varphi = \varphi(x,y)$ and this biderivation automatically satisfies the Jacobi identity because $\Lambda^3(\mathbf{A}) = 0$, hence $\{\cdot\,,\cdot\}^\varphi$ is a Poisson bracket on $\mathbf{A}$. Conversely every Poisson bracket on $\mathbf{A}$ is obtained in this way for a unique φ, so that the vector space of Poisson brackets on $\mathbf{A}$ is isomorphic to $\mathbf{A}$ (as well as to $\Lambda^2(\mathbf{A})$). In the sequel we freely use the identification $\{\cdot\,,\cdot\}^\varphi \leftrightarrow \varphi$. In particular we write $H^k(\mathbf{A},\varphi)$ for $H^k(\mathbf{A},\{\cdot\,,\cdot\}^\varphi)$, we denote the coboundary operator corresponding to $\{\cdot\,,\cdot\}^\varphi$ by δ_φ and we say that $\{\cdot\,,\cdot\}^\varphi$ is a *homogeneous* Poisson structure (of degree n) when φ is a homogeneous polynomial (of degree n). We have that $H^i(\mathbf{A},\varphi) = 0$ for $i \geq 3$, because $\Lambda^i(\mathbf{A}) = 0$ for $i \geq 3$. Also $H^0(\mathbf{A},0) = \mathbf{A}$ and $H^0(\mathbf{A},\varphi) = \mathbb{F}$ if $\varphi \neq 0$. Thus we are left with the computation of $H^1(\mathbf{A},\varphi)$ and $H^2(\mathbf{A},\varphi)$. As we will see the properties of $B^2(\mathbf{A},\varphi)$ are reflected in the properties of the plane algebraic curve Γ_φ defined by

$$\Gamma_\varphi = \{(x,y) \in \bar{\mathbb{F}}^2 \mid \varphi(x,y) = 0\}, \tag{1}$$

where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. Notice that the points on this curve are those points on the plane $\bar{\mathbb{F}}^2$ where the rank of the Poisson structure vanishes; it is the *singular locus* of $\{\cdot\,,\cdot\}^\varphi$. We stress the fact that although we compute the cohomology of $\mathbf{A}$ and not of the Poisson algebra $\bar{\mathbf{A}} = \mathbf{A} \times \bar{\mathbb{F}}$, it is $\bar{\mathbb{F}}$ and not $\mathbb{F}$ which is relevant in the computation.

2.1. **The second Poisson cohomology space.** Under the above identification of antisymmetric biderivations and polynomials the vector space of 2-cocycles is just $\mathbf{A} = \mathbb{F}[x,y]$. On the other hand an antisymmetric biderivation $\{\cdot\,,\cdot\}^\psi$ is a 2-coboundary if and only if there exists a derivation X of $\mathbf{A}$ such that $\delta_\varphi X = \{\cdot\,,\cdot\}^\psi$. In terms of polynomials, ψ is a 2-coboundary if and only if there exist $f,g \in \mathbf{A}$ such that $\psi = \Phi(f,g)$, where $\Phi : \mathbf{A}\times\mathbf{A} \to \mathbf{A}$ is the linear map defined by

$$\Phi(f,g) = f\frac{\partial\varphi}{\partial x} + g\frac{\partial\varphi}{\partial y} - \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\varphi. \tag{2}$$

In order to determine $H^2(\mathbf{A},\varphi)$ it is thus sufficient to explicitly describe the vector space of polynomials given by (2). We denote this space by $B^2(\mathbf{A},\varphi)$ and we let $\mathcal{I}(\varphi)$ denote the ideal (of $(\mathbf{A},\cdot)$) generated by φ, $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$. According to (2) we have that $B^2(\mathbf{A},\varphi) \subset \mathcal{I}(\varphi)$, but in general $B^2(\mathbf{A},\varphi)$ is not an ideal of $\mathbf{A}$ and hence it is strictly contained in $\mathcal{I}(\varphi)$. Moreover, since Φ is linear, its image $B^2(\mathbf{A},\varphi)$ is generated by the images $\Phi(m,0)$ and $\Phi(0,m)$, where m runs over the set of all (monic) monomials.

Lemma 2.1.

(1) *If* Γ_φ *is singular then* $\dim H^2(\mathbf{A},\varphi) > 0$.
(2) *If* Γ_φ *is non-reduced then* $H^2(\mathbf{A},\varphi)$ *is infinite-dimensional.*

Proof. If Γ_φ is singular then all elements of $\mathcal{I}(\varphi)$ have a common zero in $\bar{\mathbb{F}}$, hence $\mathcal{I}(\varphi)$ does not contain the constants, $\mathcal{I}(\varphi) \neq \mathbf{A}$. A fortiori $B^2(\mathbf{A},\varphi) \neq \mathbf{A}$ so that $\dim H^2(\mathbf{A},\varphi) > 0$. Similarly, if Γ_φ is not reduced, i.e., if φ contains a factor of multiplicity at least two (in $\bar{\mathbb{F}}[x,y]$), then all elements of $\mathcal{I}(\varphi)$ have a common factor in $\bar{\mathbb{F}}[x]$, hence $\mathcal{I}(\varphi)$ is of infinite codimension in $\mathbf{A}$, a fortiori $\dim H^2(\mathbf{A},\varphi) = \infty$. □

We next consider the case in which $\{\cdot\,,\cdot\}^\varphi$ is a homogeneous Poisson structure on $\mathbf{A}$ of degree n, i.e., φ is a homogeneous polynomial of degree n. Then the curve Γ_φ, which is defined by (1), is a singular curve consisting of n lines that pass through the origin. Notice that Γ_φ is reduced if and only if these n lines are distinct. The homogeneous case becomes feasible thanks to the fact that $\Phi(f,g)$ is homogeneous when f and g are homogeneous of the same degree; more precisely, in this case $\deg\Phi(f,g) = \deg f + n - 1$ and the subspace of $B^2(\mathbf{A},\varphi)$ which consists of homogeneous elements of degree i is generated by the images $\Phi(m,0)$ and $\Phi(0,m)$, where m runs over the set of all (monic) monomials of degree $i+1-n$. Let us denote for $i \in \mathbb{N}$ the linear map

$$\Phi_i : \mathbf{A}_i \times \mathbf{A}_i \to \mathbf{A}_{i+n-1}, \tag{3}$$

which is the restriction of Φ to $\mathbf{A}_i \times \mathbf{A}_i$, with $\mathbf{A}_i$ the subspace of $\mathbf{A}$ of homogeneous polynomials of total degree i. Notice that $\mathbf{A}_i$ has dimension $i+1$ so that Φ_i is a map between equi-dimensional spaces precisely when $i = n-2$. In terms of the maps Φ_i the dimension of $H^2(\mathbf{A},\{\cdot\,,\cdot\})$ is given by

$$\begin{aligned}\dim H^2(\mathbf{A},\{\cdot\,,\cdot\}) &= \sum_{i=0}^{n-2}\dim \mathbf{A}_i + \sum_{i\in\mathbb{N}}(\dim \mathbf{A}_{i+n-1} - \operatorname{rk}\Phi_i)\\ &= \frac{n(n-1)}{2} + \sum_{i\in\mathbb{N}}(i+n-\operatorname{rk}\Phi_i)\end{aligned}$$

In order to compute the rank of the maps Φ_i we will use the following lemma. The lemma can easily be generalized to the case of a pair of polynomials but we will not need it in that degree of generality. For $i \in \mathbb{N}$ we denote by $\mathbb{F}_i[x]$ the vector space of polynomials of degree at most i. For a linear map $f : V \to W$ between finite-dimensional vector spaces we say that the rank of f is maximal when $\operatorname{rk} f = \min\{\dim V, \dim W\}$. Moreover we define $\operatorname{cork} f = \min\{\dim V, \dim W\} - \operatorname{rk} f$.

Lemma 2.2. *Let $\psi \in \mathbb{F}[x]$ be a polynomial of degree n and for any $i \in \mathbb{N}$ let Ψ_i be the linear map $\Psi_i : \mathbb{F}_i[x] \times \mathbb{F}_{i+1}[x] \to \mathbb{F}_{n+i}[x]$ defined by $\Psi_i(f,g) = f\psi + g\psi'$, where ψ' denotes the derivative of ψ. The rank of Ψ_i is given by*

$$\operatorname{rk}\Psi_i = \begin{cases} 2i+3 & i \le m-2\\ i+m+1 & i \ge m-2\end{cases}$$

where m is the number of distinct *roots of ψ in $\bar{\mathbb{F}}$. In particular, if ψ is square-free then Ψ_i has maximal rank for all $i \in \mathbb{N}$.*

Proof. We have that $\operatorname{rk}\Psi_i = 2i+3-\dim\operatorname{Ker}\Psi_i$. Therefore it suffices to show that the dimension of $\operatorname{Ker}\Psi_i$ is given by $\max\{i+2-m, 0\}$. Let us denote by r the greatest common divisor (in $\mathbb{F}[x]$) of ψ and ψ'. Since the degree of r is $n-m$ the degree of the polynomial ψ/r (resp. ψ'/r) is m (resp. $m-1$). Since ψ/r and ψ'/r are coprime any pair (U,V) such that $U\psi + V\psi' = 0$ is of the form $(F\psi'/r, -F\psi/r)$. It follows that

$$\operatorname{Ker}\Psi_i = \left\{\left(\frac{F\psi'}{r}, -\frac{F\psi}{r}\right) \mid \deg F \le i+1-m\right\}.$$

The above claim about the dimension of $\operatorname{Ker}\Psi_i$ follows. □

We can now give a complete description of $H^2(\mathbf{A},\varphi)$ in case φ is a homogeneous polynomial.

Proposition 2.3. *Suppose that φ is homogeneous of degree $n \ge 1$ and that φ has m distinct factors in $\bar{\mathbb{F}}[x]$.*

(1) *The rank of Φ_i is given by*

$$\operatorname{cork}\Phi_i = \begin{cases} \max\{i-m+2, 0\} & 0 \le i \le n-2\\ n-1 & i = n-1\\ n-m & i \ge n\end{cases} \tag{4}$$

(2) *If Γ_φ is reduced then the rank of Φ_i is maximal for all $i \ne n-1$, and* $\operatorname{rk}\Phi_{n-1} = n$*;*

(3) *If Γ_φ is reduced then* $\dim H^2(\mathbf{A},\varphi) = n(n-1)$.

Proof. Suppose that we have established the first claim. Then 2. follows from the fact that $m = n$ if Γ_φ is reduced. Also $\operatorname{rk}\Phi_i = \min\{2i+2, i+n\}$ for $i \neq n-1$, hence

$$\dim H^2(\mathbf{A}, \{\cdot\,,\cdot\}) = \frac{n(n-1)}{2} + \sum_{0\le i\le n-2}(n-i-2) + n - 1 = n(n-1).$$

In order to show 1, we write $\varphi = \sum_{i=0}^n \sigma_i x^{n-i}y^i$. By a linear change on x, y we may assume that $\sigma_0 = 1$. We will show that the maps Ψ_i and Φ_i are intimately related. The matrix of Ψ_i is given, in terms of natural bases, by

$$\tag{5} \begin{pmatrix} 1 & & & & n & & & \\ \sigma_1 & 1 & & & (n-1)\sigma_1 & n & & \\ \vdots & \sigma_1 & \ddots & & \vdots & (n-1)\sigma_1 & \ddots & \\ \sigma_n & \vdots & \ddots & 1 & \sigma_{n-1} & \vdots & \ddots & n \\ & \sigma_n & & \sigma_1 & & \sigma_{n-1} & & (n-1)\sigma_1 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & \sigma_n & & & & \sigma_{n-1} \end{pmatrix}$$

with $i+1$ columns in the leftmost block and $i+2$ columns in the rightmost block. Now subtract n times the first column from the $(i+2)$-th column and remove the first row and the first column from the resulting matrix. The resulting $(n+i)\times(2i+2)$ matrix is given by

$$\tag{6} M_i = \begin{pmatrix} 1 & & & \sigma_1 & n & & \\ \sigma_1 & \ddots & & 2\sigma_2 & (n-1)\sigma_1 & \ddots & \\ \vdots & \ddots & 1 & \vdots & \vdots & \ddots & n \\ \sigma_n & & \sigma_1 & n\sigma_n & \sigma_{n-1} & & (n-1)\sigma_1 \\ & \ddots & \vdots & & & \ddots & \vdots \\ & & \sigma_n & & & & \sigma_{n-1} \end{pmatrix}$$

and the rank of M_i is one less than the rank of Ψ_i,

$$\operatorname{rk} M_i = \begin{cases} 2i+2 & i \le m-2 \\ i+m & i \ge m-2 \end{cases}$$

We will show that when $i \neq n-1$ then M_i is the matrix of $\Phi_i : \mathbf{A}_i \times \mathbf{A}_i \to \mathbf{A}_{i+n-1}$ with respect to appropriate bases for $\mathbf{A}_i \times \mathbf{A}_i$ and for $\mathbf{A}_{i+n-1}$. One easily computes that

$$\Phi_i(x^{i-j+1}y^{j-1}, x^{i-j}y^j) = (n-i-1)x^{i-j}y^{j-1}\varphi(x,y),$$
$$\Phi_i((n-j)x^{i-j+1}y^{j-1}, (i+1-j)x^{i-j}y^j) = (n-i-1)x^{i-j+1}y^{j-1}\frac{\partial\varphi}{\partial x}(x,y),$$

for $j = 1, \dots, i$, and $\Psi(y^i, 0) = y^i\frac{\partial\varphi}{\partial x}$, $\Psi(0, x^i) = x^i\frac{\partial\varphi}{\partial y}$. This produces (up to a non-zero factor) precisely all columns of M_i when $i \neq n-1$, yielding Formula (4) for $i \neq n-1$. For $i = n-1$ the preceeding computation shows that $n-1$ of the columns of the matrix of Φ_{n-1} are dependent. The other $n+1$ columns lead to the following $(2n-1)\times(n+1)$ matrix

$$
\begin{pmatrix}
1 & & & & & \sigma_1 \\
0 & 2 & & & & 2\sigma_2 \\
-\sigma_2 & \sigma_1 & 3 & & & 3\sigma_3 \\
\vdots & \vdots & 2\sigma_1 & \ddots & & \vdots \\
(1-n)\sigma_n & (3-n)\sigma_{n-1} & \vdots & \ddots & n & n\sigma_n \\
0 & (2-n)\sigma_n & (4-n)\sigma_{n-1} & & (n-1)\sigma_1 & 0 \\
\vdots & \ddots & & \ddots & \vdots & \vdots \\
0 & & 0 & & \sigma_{n-1} & 0
\end{pmatrix} \tag{7}
$$

It is easy to see that the last column is a linear combination of the n other columns, which are linearly independent. It follows that $\operatorname{rk}\Phi_{n-1}=n$ which establishes formula (4) for all $i\in\mathbb{N}$. □

Notice that the number $n(n-1)$ that appears here is precisely twice the number of singularities (with multiplicities) of Γ_φ. We conjecture that the number of singularities Γ_φ is in general a lower bound for the dimension of $H^2(\mathbf{A},\{\cdot\,,\cdot\}^\varphi)$.

2.2. **The first Poisson cohomology space.** We proceed to compute the dimension of the first Poisson cohomology space for the case in which φ is homogeneous of degree $n\geq 1$. We have in this case a bijective correspondence between $\Lambda^1(\mathbf{A})$ and $\mathbf{A}\times\mathbf{A}$, given by $\Lambda^1(\mathbf{A})\ni X\mapsto (X(x),X(y))\in\mathbf{A}\times\mathbf{A}$. Since $H^1(\mathbf{A},\varphi)$ is the space of Poisson vector fields modulo the space of Hamiltonian vector fields we have, using this correspondence,

$$H^1(\mathbf{A},\varphi)=\frac{\left\{(f,g)\in\mathbf{A}\times\mathbf{A}\mid f\frac{\partial\varphi}{\partial x}+g\frac{\partial\varphi}{\partial y}-\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)\varphi=0\right\}}{\left\{\left(\varphi\frac{\partial f}{\partial y},-\varphi\frac{\partial f}{\partial x}\right)\mid f\in\mathbf{A}\right\}}.$$

It follows that

$$\dim H^1(\mathbf{A},\varphi)=\sum_{i=0}^{\infty}(\dim\operatorname{Ker}\Phi_i-\dim\Im\chi_{i+1-n}),$$

where $\chi_i:\mathbf{A}_i\to\mathbf{A}_{i+n-1}\times\mathbf{A}_{i+n-1}$ is defined, for $i\geq 0$, by $\chi_i(f)=\left(\varphi\frac{\partial f}{\partial y},-\varphi\frac{\partial f}{\partial x}\right)$, and $\chi_i=0$ for $i<0$. We obviously have that

$$\begin{aligned}\dim\Im\chi_{i+1-n}&=\dim\mathbf{A}_{i+1-n}-\dim\operatorname{Ker}\chi_{i+1-n}\\&=i-n+2-\delta_{i,n-1}.\end{aligned}$$

On the other hand, since $\operatorname{cork}\Phi_i=\min\{2i+2,i+n\}-\operatorname{rk}\Phi_i$ we find that

$$\dim\operatorname{Ker}\Phi_i=\operatorname{cork}\Phi_i-\min\{0,n-i-2\},$$

and we find a formula for $\dim\operatorname{Ker}\Phi_i$ by using the formula (4) for $\operatorname{cork}\Phi_i$, giving

$$\dim\operatorname{Ker}\Phi_i=\begin{cases}0 & 0\leq i\leq m-2\\ i+2-m & m-1\leq i\leq n-2\\ n & i=n-1\\ i+2-m & i\geq n.\end{cases}$$

By a direct substitution we find that $\dim H^1(\mathbf{A},\varphi)$ is infinite-dimensional when $m\neq n$, i.e., when Γ_φ is non-reduced, that $\dim H^1(\mathbf{A},\varphi)=0$ when φ is constant, and that $\dim H^1(\mathbf{A},\varphi)=n$ when $m=n\geq 1$. Notice that this number equals the number of irreducible components of the curve Γ_φ and that the modular vector field

$$\left(-\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial x}\right)$$

defines a non-trivial cohomology class at level $i = n-1$, corresponding to the special term $\delta_{i,n-1}$ which appears in the computation. We conjecture that the number of irreducible components of the curve Γ_φ is in general a lower bound for the dimension of the first Poisson cohomology space.

3. Finite dimensionality of the second Poisson cohomology space

It follows from Section 2 that $H^2(\mathbf{A},\varphi)$ is finite-dimensional when φ is a homogeneous polynomial which is square-free. In the present section we will generalize this result to a general class of polynomials. It will follow in particular that $H^2(\mathbf{A},\varphi)$ is finite-dimensional when φ is a generic polynomial of degree n.

For $i\in\mathbb{N}$ let us denote by $\mathbf{A}_{\leq i}$ the subspace of $\mathbf{A}$ consisting of all polynomials of total degree at most i. We also introduce for $i\geq 0$ the vector space $\mathcal{A}_i = \mathbf{A}_{\leq i}/\mathbf{A}_{\leq i-1}$. We have a natural isomorphism $\mathcal{A}_i\cong\mathbf{A}_i$, in particular $\dim\mathcal{A}_i = i+1$. We denote for a polynomial $f\in\mathbf{A}$ of total degree i its projection on $\mathcal{A}_i$ as well as the corresponding element of $\mathbf{A}_i$ by $\hat f$. Let φ be a polynomial of total degree n and let $\Phi:\mathbf{A}\times\mathbf{A}\to\mathbf{A}$ denote the linear map given by (2). Φ induces for any $i\in\mathbb{N}$ a linear map

$$\begin{array}{rcl}\hat\Phi_i:\mathcal{A}_i\times\mathcal{A}_i & \to & \mathcal{A}_{i+n-1}\\ (\hat f,\hat g) & \mapsto & \widehat{\Phi_i(f,g)}.\end{array}\tag{8}$$

Under the above isomorphism $\mathcal{A}_i\cong\mathbf{A}_i$ the map Ψ is precisely the linear map $\mathbf{A}_i\times\mathbf{A}_i\to\mathbf{A}_{n+i-1}$ associated to the leading term $\hat\varphi\in\mathbf{A}_n$, so that

$$\operatorname{cork}\hat\Phi_i = \begin{cases}\max\{i-m+2,0\} & 0\leq i\leq n-2\\ n-1 & i=n-1\\ n-m & i\geq n,\end{cases}\tag{9}$$

where m denotes the number of different roots of the polynomial $\hat\varphi$. Notice that this number is the number of points at infinity of Γ_φ and that $m<n$ if and only if Γ_φ has a multiple point at infinity if and only if $\hat\varphi$ contains a square factor. We have that

$$\dim H^2(\mathbf{A},\varphi)\leq\sum_{0\leq i\leq n-2}\dim\mathcal{A}_i+\sum_{i\in\mathbb{N}}(\dim\mathcal{A}_{i+n-1}-\operatorname{rk}\hat\Phi_i),$$

so that $\dim H^2(\mathbf{A},\varphi)\leq\dim H^2(\mathbf{A},\hat\varphi) = 2\begin{pmatrix}n\\2\end{pmatrix}$ when $\hat\varphi$ is square-free, an inequality which is by the preceeding section an equality for all homogeneous polynomials φ that are square-free. We see in particular that when φ is a generic polynomial of degree n then $\dim H^2(\mathbf{A},\varphi)\leq 2\begin{pmatrix}n\\2\end{pmatrix}$. Examples of this are given by polynomials φ that factorize completely into terms of degree at most one, so that Γ_φ consists of n lines in the plane: if the linear parts of each factor are all different (so that no two lines of Γ_φ are parallel) then the above inequality holds and it admits an interpretation in terms of the number of intersection points of these lines, as in the case of a homogeneous polynomial φ. We conjecture that in this case the inequality is actually an equality.

4. An example

The non-homogeneous case not being tractable in full generality at this point we treat a simple example, which shows that even when Γ_φ is as simple as a circle (no singularities, genus zero) the dimension of $\dim H^2(\mathbf{A},\varphi)$ needs not be zero. The techniques that we use may be useful to study more general examples. We take $\varphi = x^2+y^2-1$, and write $\varphi=\varphi_1+\varphi_2$, where $\varphi_1 = x^2+y^2$ and $\varphi_2=-1$. The corresponding coboundary operators then satisfy $\delta=\delta_1+\delta_2$. Since $\delta^2=\delta_1^2=\delta_2^2=0$ we have that $\delta_1\delta_2=-\delta_2\delta_1$, so that δ_2 induces a coboundary operator $\hat\delta_2$ on $H^\star(\mathbf{A},\varphi_1)$, making the latter into a complex. Explicitly, if we denote the cohomology class of an element in $H^\star(\mathbf{A},\varphi_1)$ by square brackets, then $H^0(\mathbf{A},\varphi_1)$ is generated by [1], $H^1(\mathbf{A},\varphi_1)$ is generated by $\{[(x,y)],[(y,-x)]\}$ and $H^2(\mathbf{A},\varphi_1)$ by $\{[1],[x^2+y^2]\}$. By direct computation we have $\hat\delta_2[(y,-x)]=0$ and $\hat\delta_2[(x,y)]=[1]$. It follows that 1 defines a trivial class in $H^2(\mathbf{A},\varphi)$ and that the image of $\hat\delta_2$ is generated by [1]. It is easy to see that x^2+y^2 defines a non-trivial cohomology class in $H^2(\mathbf{A},\varphi)$, so that $H^2(\mathbf{A},\varphi)$ is one-dimensional. Indeed, if we suppose that there exists a pair $(f,g)\in\mathbf{A}\times\mathbf{A}$ such that $\delta(f,g)=x^2+y^2$ then

$$\hat\delta_2[(f,g)]=[\delta_2(f,g)]=[\delta(f,g)]=[x^2+y^2],$$

a contradiction because the image of $\hat\delta_2$ is generated by [1].

5. Final remarks

(1) There is also Poisson *homology*, which is in a sense dual to the cohomology that we considered. The complex $H_\star(\mathbf{A},\{\cdot\,,\cdot\})$ is defined by using the $\mathbf{A}$-modules $\Omega^p_{\mathbf{A}}$ of differential p-forms on $\mathbf{A}$ with the differential defined as the Lie derivative with respect to the Poisson bracket. In our polynomial setting $\delta:\Omega^p_{\mathbf{A}}\to\Omega^{p-1}_{\mathbf{A}}$ is given by the formula

$$\delta(f_0df_1\wedge\ldots\wedge df_p)=\sum_{i=1}^p(-1)^{i+1}\{f_0,f_i\}df_1\wedge\cdots\wedge\widehat{df_i}\wedge\cdots\wedge df_p$$
$$+\sum_{1\le i<j}^p(-1)^{i+j}f_0d\{f_i,f_j\}\wedge df_1\wedge\cdots\wedge\widehat{df_i}\wedge\cdots\wedge\widehat{df_j}\wedge\cdots\wedge df_p,$$

for any polynomials $f_0,f_1,\ldots,f_p$ (cfr. [1], [8]). If one translates this complex in a contravariant setting, using the volume form (if it exists) then one obtains a differential $\tilde\delta:\Lambda^p(\mathbf{A})\to\Lambda^{p+1}(\mathbf{A})$, which reads

$$\tilde\delta Q=\delta Q+Q\wedge\mathrm{Div}(\Lambda),$$

where $\mathrm{Div}(\Lambda)$ denotes the modular vector field. In the case where the class $[\mathrm{Div}(\Lambda)]$ is cohomologically trivial, Poisson cohomology and Poisson homology are canonically dual to each other. In our case this class never vanishes (cfr. Section 2) and it is not hard to compute the homology: one has that $H_0(\mathbf{A},\{\cdot\,,\cdot\})=\mathbb{F}$ and that $H_1(\mathbf{A},\{\cdot\,,\cdot\})$ and $H_2(\mathbf{A},\{\cdot\,,\cdot\})$ are canonically isomorphic to $\mathbb{F}[x,y]/(\varphi)$, the ring of regular functions on Γ_φ. In general there exists a kind of duality theorem between Poisson homology and cohomology with non-trivial coefficients (see [8]).

(2) Poisson structures and their cohomology classes are a particular case of the theory of Lie algebroids, initiated by J. Pradines in the differential geometric setting. Maybe the techniques we use here can be extended to the simplest case of Lie algebroids.

(3) We already mentioned that the Poisson cohomology we consider here can be used to study deformation theory of Poisson algebras, more precisely to study deformations where the Poisson bracket is deformed without changing the associative structure. It is however also possible to deform both structures (Lie and associative) preserving their compatibility; the corresponding cohomology has been studied by Flato, Gerstenhaber and Voronov (see [3]). Analogous cohomologies for Poisson algebras have been settled in the general framework of the theory of operads.

(4) Ph. Monnier undertook in his thesis [5] the computation of Poisson cohomology in cases analogous to ours, but at the level of jets, i.e., he is computing the *local* Poisson cohomology (in a differential geometric setting). His approach is based on differentiable singularity theory and the theory of normal forms. The quadratic case was previously worked out by N. Nakanishi [6].

(5) We conclude this article with some indications about its relations with deformation quantization. A fundamental result by M. Kontsevich (see [4]) establishes a quasi-isomorphism between moduli spaces of Poisson tensors and of associative multiplications on functions, for any infinitesimal. One deduces from this quasi-isomorphism its infinitesimal (linearized) part to obtain a multiplicative isomorphism between Poisson cohomology for a given Poisson tensor and Hochschild cohomology of the deformed associative algebra ($\star$-product) canonically associated to it. See the article of Voronov [7] for a deduction of this isomorphism from the formality theorem. So our results provide information about the Hochschild cohomology for $\star$-products on the plane; recall that Hochschild cohomology for $\star$-products is a natural non-commutative analog of the De Rham complex.

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