The moduli problem for integrable systems: the example of a geodesic flow on SO(4)

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Abstract. We introduce the moduli problem for (algebraic completely) integrable systems. This problem consists in constructing a moduli space of affine algebraic varieties and explicitly describing a map which associates to a generic affine variety, which appears as a level set of the first integrals of the system (or, equivalently, a generic affine variety which is preserved by the flows of the integrable vector fields), a point in this moduli space. As an illustration, we work out the example of an integrable geodesic flow on $\mathbf{SO}(4)$. In this case, the generic invariant variety is an affine part of the Jacobian of a Riemann surface of genus two. Our construction relies heavily on the fact that these affine parts have the additional property of being $4:1$ unramified covers of Abelian surfaces of type $(1,4)$.

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1. Introduction

An important property which distinguishes integrable systems from generic dynamical systems is that the flow of an integrable vector field, starting from an arbitrary point, is not dense in the corresponding energy level but is constrained to a subvariety of half the dimension (or less) of phase space. These subvarieties are the level sets of the Poisson commuting functions that make up the integrable system and the above property follows from the fact that $X_G F = \{F, G\} = 0$, where F and G are any two Poisson commuting functions (such as the Hamiltonian and any of its first integrals).

In a real setting the relevance of these invariant manifolds for understanding and describing the mechanics of the integrable system can be seen from the Arnold-Liouville theorem and from the existence of action-angle variables: if such an invariant manifold is compact then it is diffeomorphic to a real torus and the flow is a linear flow on this torus; moreover, such a linearizing diffeomorphism can be constructed semi-locally (on a neighborhood of a compact invariant manifold) onto a product of a torus with a linear space, the latter being equipped with the standard symplectic or Poisson structure. In particular, it entails quasi-periodicity of the solutions, hence of the motion of the mechanical system. See [4] or [5].

In a complex analytic or a complex algebraic setting the structure of the invariant manifolds is more complicated. First, while two real tori are diffeomorphic if and only if they have the same dimension, the conditions for two complex tori (quotients of $\mathbf{C}^n$ by a lattice) to be biholomorphic (or, in an algebraic setting, isomorphic) are much more involved, as becomes already apparent in the one-dimensional case (Riemann surfaces of genus 1; elliptic curves). Second, since complex tori are compact they cannot live in an affine variety and the complex invariant manifolds are at best affine parts of complex algebraic tori and this fact puts an extra condition on the possibility of two invariant manifolds being biholomorphic or isomorphic: such an isomorphism must preserve the divisor at infinity, i.e., the divisor to be glued to the affine variety in order to complete it into a compact complex torus. Both aspects have their relevance for mechanics. First, while the solutions corresponding to non-isomorphic complex tori can in both cases be written in terms of theta functions, the characteristics of these quasi-periodic functions will be different; and second, even when the tori are isomorphic, if their affine parts are not isomorphic then the behaviour of the system for finite (complex) time will be different because in an a.c.i. system on an affine variety every (complex) integral curve hits the divisor at infinity after a finite time (see [2] or [3]).

These considerations lead us to what we call the *moduli problem* for integrable systems. In order to give a precise definition we will restrict ourselves to algebraic completely integrable systems (a.c.i. systems) on an affine variety; the definition can easily be adapted to other situations when needed. We recall from [14] that the generic invariant manifold of an a.c.i. system is an affine part of an Abelian variety (a complex algebraic torus) and that a large amount of explicit information about these tori (such as equations for an affine part, embeddings in projective space and equations for the divisor at infinity) can be obtained by studying the Laurent solutions to the differential equations, which describe the vector field defined by the Hamiltonian. The divisor which is needed to complete the affine part into an Abelian variety induces a polarization on the corresponding torus; in all known cases the polarization type, which is a discrete invariant of the Abelian variety, is the same for all these invariant manifolds. We are thus led to consider, on one side, the family of affine parts of Abelian varieties that appear as invariant manifolds in the a.c.i. system, on the other side, the moduli space[1] of Abelian varieties with a prescribed

[1] In order to have a good moduli space (e.g. one that admits an algebraic structure),

divisor (hence polarization type) and, finally, the map between these two spaces, which sends an invariant manifold to its isomorphism class. The moduli problem for an a.c.i. system consists in explicitly constructing the moduli space and the canonical map.

To illustrate our point, we will treat a non-trivial example, namely an integrable system that appears in Adler and van Moerbeke's classification of integrable geodesic flows on $\mathbf{SO}(4)$ as the case of metric II (see [1]). It has the following geometric description (for further details, see Section 2). Phase space is $\mathbf{C}^6$ and there are four independent quadratic polynomial functions $H_1, \ldots, H_4$ which Poisson commute; in fact we will exhibit a tri-Hamiltonian structure for this integrable system. The affine surfaces that appear as the fibers of the map

$$\mu : \mathbf{C}^6 \to \mathbf{C}^4 : z = (z_1, \ldots, z_6) \mapsto (H_1(z), H_2(z), H_3(z), H_4(z))$$

are invariant for the flow of the two commuting vector fields X_1 and X_2 and for generic $h \in \mathbf{C}^4$ the invariant surface $\mu^{-1}(h)$ is isomorphic to an affine part of the Jacobian of a (compact) Riemann surface $\bar{\Gamma}_h$. We will construct such an isomorphism; in classical terminology we separate the variables of the integrable system (leading to explicit solutions; as far as we could check a separation of variables for this integrable system was not known). From this isomorphism we can read off, in terms of the Weierstrass points on $\bar{\Gamma}_h$, the relative position of the four curves which are missing in the affine part $\mathcal{A}_h$. On the one hand we can deduce from it that the quotient $\mathcal{A}_h/\mathfrak{T}_h$ is an affine part of an Abelian surface of type $(1,4)$. On the other hand it will allow us to set up a basic correspondence between the affine varieties $\mathcal{A}_h$ (modulo isomorphism) and the space of Riemann surfaces of genus two, equipped with a decomposition of their Weierstrass points (modulo isomorphism). As the latter space is in turn isomorphic to a moduli space of polarized Abelian surfaces of type $(1,4)$ (see [13]) this reduces the moduli problem to a question of determining the moduli (i.e., the moduli space and the corresponding map) for the underlying Abelian surfaces of type $(1,4)$. This reduction will be done in Section 3.

Abelian surfaces of type $(1,4)$ admit a holomorphic map to projective space $\mathbf{P}^3$ and the image is (in general) an octic surface with four singular points of order four, as was shown in [6]. In order to find an equation of this octic we first compute the Laurent solutions to the integrable vector field X_1 and deduce from it four independent sections, invariant for the group action, of the line bundle $[\mathcal{D}_h]$, which corresponds to the divisor at infinity. When the four singular points of order four are taken as base points in $\mathbf{P}^3$, the coefficients of the equation for the octic surface are closely related to moduli for Abelian surfaces of type $(1,4)$. Thus we need to reduce the equation of the octic that we have found to its more symmetric form. This computation will be feasible thanks to the richness of the underlying geometry.

When the octic is reduced to its symmetric form its coefficients are expressed in terms of the coordinates of the Weierstrass points of $\bar{\Gamma}_h$ rather than h, an explicit dependence which disappears in the final step of the computation. Indeed, it remains then to pass from the coefficients of this octic to the moduli space of Abelian surfaces of type $(1,4)$. Incidently this moduli space was constructed by the second author when studying the Garnier system, another a.c.i. system, whose invariant manifolds are Abelian surfaces of type $(1,4)$ (see [13]). The moduli space is described there as a Zariski open subset of a cone in weighted projective space and we can use the explicitly given map from the space of parameters of the octic to this cone to complete our example: we end up with five explicit polynomials in the values $h = (h_1, h_2, h_3, h_4)$ of the constants of motion which take the same value (as an element of weighted projective space) on two sets of constants

in general one also has to throw out a few bad elements.

of motion if and only if the corresponding level sets are isomorphic affine varieties. This final step will be done in Section 4.

It would be interesting to study the moduli problem for other integrable systems, such as the Toda lattices and the classical integrable tops.

2. An integrable geodesic flow on $\mathbf{SO}(4)$

It was shown by Adler and van Moerbeke (see [1, Theorem 4]) that there exist three classes of left-invariant metrics on $\mathbf{SO}(4)$ for which the geodesic flow reduces to an algebraic completely integrable system (a.c.i. system) on its Lie algebra $\mathfrak{so}(4)$. In the sequel, we will consider the second case, known as the case of metric II. In suitable coordinates, the first vector field X_1 of this a.c.i. system is given by the differential equations

$$\begin{aligned} \dot{z_1} &= 2z_5z_6, & \dot{z_2} &= 2z_3z_4, & \dot{z_3} &= z_5(z_1+z_4), \\ \dot{z_4} &= 2z_2z_3, & \dot{z_5} &= z_3(z_1+z_4), & \dot{z_6} &= 2z_1z_5. \end{aligned} \tag{1}$$

The second vector field X_2, commuting with X_1, is given by the differential equations

$$\begin{aligned} \dot{z_1} &= z_2z_6, & \dot{z_2} &= z_4(2z_3-z_6), & \dot{z_3} &= z_4z_5, \\ \dot{z_4} &= z_2(2z_3-z_6), & \dot{z_5} &= z_3z_4, & \dot{z_6} &= z_1z_2; \end{aligned} \tag{2}$$

the vector fields X_1 and X_2 admit four independent quadratic invariants, given by the following expressions:

$$\begin{aligned} H_1 &= z_3^2 - z_5^2, \\ H_2 &= z_1^2 - z_6^2, \\ H_3 &= z_2^2 - z_4^2, \\ H_4 &= (z_1+z_4)^2 + 4(z_3^2 - z_2z_5 - z_3z_6). \end{aligned} \tag{3}$$

It is easy to verify that there exist precisely three linearly independent linear Poisson structures on $\mathbf{C}^6$ with respect to which X_1 and X_2 are Hamiltonian; moreover, these Poisson structures are compatible, implying that the integrable system admits a tri-Hamiltonian structure. Explicitly, for any $(\alpha,\beta,\gamma)\in\mathbf{C}^3$, the matrix

$$\begin{pmatrix} 0 & \alpha z_6 & -\beta z_5 & 0 & -\beta z_3 - 2\gamma z_6 & \beta(z_2 - 2z_5) \\ -\alpha z_6 & 0 & 2\gamma z_4 & \alpha(z_6 - 2z_3) & 0 & -\alpha z_1 - \beta z_4 \\ \beta z_5 & -2\gamma z_4 & 0 & -\alpha z_5 - 2\gamma z_2 & -\gamma(z_1+z_4) & 0 \\ 0 & \alpha(2z_3 - z_6) & \alpha z_5 + 2\gamma z_2 & 0 & \alpha z_3 & -\beta z_2 \\ \beta z_3 + 2\gamma z_6 & 0 & \gamma(z_1+z_4) & -\alpha z_3 & 0 & 2\gamma z_1 \\ \beta(2z_5 - z_2) & \alpha z_1 + \beta z_4 & 0 & \beta z_2 & -2\gamma z_1 & 0 \end{pmatrix}$$

is the Poisson matrix of a Poisson structure $P_{\alpha\beta\gamma}$ on $\mathbf{C}^6$. If $(\alpha,\beta,\gamma)\neq(0,0,0)$ then $P_{\alpha\beta\gamma}$ generates the Hamiltonian vector fields X_1 and X_2 as described in the following table; generators for the algebra of Casimirs of these structures $P_{\alpha\beta\gamma}$ also follow from the table.

	H_1	H_2	H_3	H_4
P_{100}	0	0	$2X_2$	$-2X_1$
P_{010}	0	$2(X_1 - X_2)$	0	$2X_1$
P_{001}	$2X_1$	0	0	$8X_2$

Table 1

It was shown by Adler and van Moerbeke in [2, Section 4] that, for any $h = (h_1, h_2, h_3, h_4)$ which belongs to some[2] Zariski open subset $\mathcal{H}$ of $\mathbf{C}^4$, the affine surface

$$\mathcal{A}_h = \{z \in \mathbf{C}^6 \mid H_i(z) = h_i,\ i = 1, \dots, 4\}$$

is isomorphic to an affine part of the Jacobian of a compact Riemann surface $\bar{\Gamma}_h$ of genus two (which depends on $h \in \mathcal{H}$), $\mathcal{A}_h \cong \operatorname{Jac}(\bar{\Gamma}_h) \setminus \mathcal{D}_h$ and that the vector fields X_1 and X_2 are linear[3] when restricted to these surfaces $\mathcal{A}_h$, thereby proving that the above system is algebraic completely integrable (see [2, Section 4]). The affine part $\mathcal{A}_h$, the divisor $\mathcal{D}_h$ and the Riemann surface $\bar{\Gamma}_h$ can be described as follows. First notice that the group $\mathfrak{T}$ of involutions, generated by

$$\begin{aligned} \sigma_1(z_1, \dots, z_6) &= (-z_1, -z_2, z_3, -z_4, -z_5, z_6), \\ \sigma_2(z_1, \dots, z_6) &= (-z_1, z_2, -z_3, -z_4, z_5, -z_6), \end{aligned} \tag{4}$$

commutes with the vector fields X_1 and X_2 and leaves the affine surfaces $\mathcal{A}_h$ invariant; in fact they generate, for any $h \in \mathcal{H}$, a group $\mathfrak{T}_h$ of translations over half periods in the tori $\operatorname{Jac}(\bar{\Gamma}_h)$. As a consequence, the divisors $\mathcal{D}_h$ are also stable under these translations. For a more precise description of the divisors $\mathcal{D}_h$ one applies Painlevé analysis to the vector field X_1 (or any combination of X_1 and X_2). To do this one searches Laurent solutions to the differential equations (1), depending on five free parameters (principal balances). There are precisely four such families, labeled by $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, and they are explicitly given as follows ($a, b, \dots, e$ are the free parameters).

$$\begin{aligned} z_1 &= \frac{(a-1)\epsilon_1}{t}\left(1 - bt + (b^2 - d - e)t^2 + O(t^3)\right), \\ z_2 &= \frac{\epsilon_1\epsilon_2}{t}\left(a - abt + ((a-1)(ae - c - ab^2) + a^2 d)t^2 + O(t^3)\right), \\ z_3 &= \frac{\epsilon_2}{2t}\left(1 + bt - ((a-1)e + ad - c - ab^2)t^2 + O(t^3)\right), \\ z_4 &= \frac{\epsilon_1}{t}\left(-a + abt + ct^2 + O(t^3)\right), \\ z_5 &= \frac{\epsilon_1\epsilon_2}{2t}\left(1 + bt + dt^2 + O(t^3)\right), \\ z_6 &= \frac{(a-1)\epsilon_2}{t}\left(-1 + bt - et^2 + O(t^3)\right). \end{aligned} \tag{5}$$

When any of these families of Laurent solutions is substituted in the equations $H_i(z) = h_i$, $i = 1, \dots, 4$, the resulting expressions are independent of t. This leads to four algebraic equations in the five free parameters, giving explicit equations for an affine part Γ_h of $\bar{\Gamma}_h$. Each of these equations is easily rewritten as

$$y^2 = x(1-x)\left[4x^3 h_1 - (4h_1 + h_4)x^2 + (h_4 - h_3 - h_2)x + h_3\right]. \tag{6}$$

In what follows, we will refer to the curve in $\mathbf{C}^2$, given by (6), as the curve Γ_h. In order to recover the Riemann surface $\bar{\Gamma}_h$ from it one has to adjoin one point which we denote by ∞_h. Since there are four families of Laurent solutions (5), the divisor $\mathcal{D}_h$ consists of four copies $\bar{\Gamma}_h(\epsilon_1, \epsilon_2)$, $\epsilon_1^2 = \epsilon_2^2 = 1$, of the curve $\bar{\Gamma}_h$, i.e.,

$$\mathcal{D}_h = \bar{\Gamma}_h(1,1) + \bar{\Gamma}_h(1,-1) + \bar{\Gamma}_h(-1,1) + \bar{\Gamma}_h(-1,-1).$$

[2] Explicit equations for $\mathcal{H}$ will be given in the next section.

[3] Recall e.g. from [7, Chapter 2.7] or [8, Chapter 11] that the Jacobian of an algebraic curve is a complex torus.

The Laurent solutions can also be used to compute an explicit embedding of the tori $\mathrm{Jac}(\bar{\Gamma}_h)$ in $\mathbf{P}^{15}$: the sections of the line bundle on $\mathrm{Jac}(\bar{\Gamma}_h)$, defined by $\mathcal{D}_h$, correspond to the meromorphic functions on $\mathrm{Jac}(\bar{\Gamma}_h)$ with a simple pole (at worst) at the divisor $\mathcal{D}_h$ and, in turn, these are found by constructing those polynomials on $\mathbf{C}^6$ which have a simple pole in t (at worst) when any of the four families of Laurent solutions are substituted in them (see [14, Chapter V]). Apart from the constant function $z_0 = 1$ and the functions z_i, $i = 1, \dots, 6$, one easily finds the following independent functions with this property:

$$
\begin{aligned}
z_7 &= z_5(2z_3 - z_6) - z_2 z_3, \\
z_8 &= z_1(2z_3 - z_6) - z_4 z_6, \\
z_9 &= z_4(2z_5 - z_2) - z_1 z_2, \\
z_{10} &= (2z_5 - z_2)^2 - z_6^2, \\
z_{11} &= (2z_3 - z_6)^2 - z_2^2,
\end{aligned}
\qquad
\begin{aligned}
z_{12} &= z_1 z_2 z_3 - z_4 z_5 z_6, \\
z_{13} &= z_2 z_3 z_6 - z_1 z_4 z_5, \\
z_{14} &= z_2 z_5 z_6 - z_1 z_3 z_4, \\
z_{15} &= z_1 z_2 z_5 - z_3 z_4 z_6.
\end{aligned}
\tag{7}
$$

The embedding of $\mathrm{Jac}(\bar{\Gamma}_h)$ in $\mathbf{P}^{15}$ is given on the affine part $\mathcal{A}_h$ by the map

$$\phi : \mathcal{A}_h \to \mathbf{P}^{15} : P = (z_1, \dots, z_6) \mapsto (1 : z_1(P) : \dots : z_{15}(P)).$$

These sections will be used later to construct two maps which are similar to ϕ and which map two different quotients of $\mathcal{A}_h$ birationally into $\mathbf{P}^3$.

3. Linearizing variables

In this section we show that from the point of view of moduli, the family of affine surfaces $\mathcal{A}_h$, $h \in \mathcal{H}$, can be replaced by a family of polarized Abelian surfaces of type $(1, 4)$. In order to do this we will first construct an explicit map from the affine surface $\mathcal{A}_h$ ($h \in \mathcal{H}$) to an affine part of $\mathrm{Jac}(\bar{\Gamma}_h)$. We do this by following an algorithm, outlined in [12, Section 3], which leads to linearizing variables for any two-dimensional a.c.i. system. We note that although the characterization of the affine surfaces $\mathcal{A}_h$ as affine parts of hyperelliptic Jacobians was already given by Adler and van Moerbeke in [2, Section 4], neither an explicit map nor linearizing variables follow from their results.

We define $\mathcal{H}$ to be the set of those $h = (h_1, h_2, h_3, h_4) \in \mathbf{C}^4$ for which the curve (6) is a non-singular curve of genus two, i.e., that its right hand side is of degree 5 and has no multiple roots; notice that this entails in particular that $h_1 h_2 h_3 \neq 0$ for all $h \in \mathcal{H}$. It will follow from our construction that, for every $h \in \mathcal{H}$, $\mathcal{A}_h$ is indeed an affine part of the Jacobian, thereby justifying the notation $\mathcal{H}$. In order to apply the procedure described in [12, Section 3], we fix an arbitrary element $h \in \mathcal{H}$ and we choose one component, say $C = \bar{\Gamma}_h(1, -1)$, of the divisor $\mathcal{D}_h$ on $\mathrm{Jac}(\bar{\Gamma}_h)$. The meromorphic functions on $\mathrm{Jac}(\bar{\Gamma}_h)$ which have at worst a double pole along the divisor C can be obtained by constructing those polynomials on $\mathbf{C}^6$ which have at worst a double pole in t when the Laurent solutions (5) corresponding to $\epsilon_1 = 1$, $\epsilon_2 = -1$ are substituted into them (and no poles when the other solutions are substituted). It is easily computed that the space of such polynomials is spanned by

$$\theta_0 = 1, \quad \theta_1 = (z_2 + z_4)(z_3 + z_5), \quad \theta_2 = (z_3 + z_5)(z_1 + z_6), \quad \theta_3 = (z_1 + z_6)(z_2 + z_4), \tag{8}$$

where we think of these polynomials as being restricted to $\mathcal{A}_h$. The mapping ϕ, given on $\mathrm{Jac}(\bar{\Gamma}_h) \setminus C$ by

$$\phi : \mathrm{Jac}(\bar{\Gamma}_h) \setminus C \to \mathbf{P}^3 : P = (z_1, z_2, \dots, z_6) \mapsto (\theta_0(P) : \theta_1(P) : \theta_2(P) : \theta_3(P))$$

maps the surface $\mathrm{Jac}(\bar{\Gamma}_h)$ to its Kummer surface, which is a singular quartic in $\mathbf{P}^3$. An equation for this quartic surface can be computed by eliminating the variables $z_1, \dots, z_6$

from the equations (3) and (8): solving the equations (8) and the first three equations in (3) for the variables $z_1, z_2, \ldots, z_6$ and substituting these values in the remaining equation, the equation for the Kummer surface of $\mathrm{Jac}(\bar{\Gamma}_h)$ can be written in the form

$$\theta_3^2((\theta_1+\theta_2-2h_1)^2+8h_1\theta_1)+f_3(\theta_1,\theta_2)\theta_3+f_4(\theta_1,\theta_2)=0, \tag{9}$$

where f_3 (respectively f_4) is a polynomial of degree three (respectively four) in θ_1 and θ_2.

It follows from (9) (see [12, Theorem 9]) that a system of linearizing variables (x_1, x_2) is given by the equations

$$-2h_1(x_1+x_2)=\theta_1+\theta_2-2h_1, \quad -2h_1x_1x_2=\theta_1. \tag{10}$$

This is checked in the present case as follows. First make use of (8), to rewrite the equations (10) as

$$(z_3+z_5)(z_2+z_4)=-2h_1x_1x_2, \quad (z_3+z_5)(z_1+z_6)=2h_1(x_1-1)(x_2-1). \tag{11}$$

Since $h \in \mathcal{H}$ the variables x_1 and x_2 are both different from 1 and from 0 so that below we can divide by x_i and by $x_i - 1$ as necessary. Deriving the equations (11) with respect to the vector field X_1 given by (1) we find that

$$\begin{aligned} &\dot{x}_1x_1^{-1}+\dot{x}_2x_2^{-1}=z_1+z_4+2z_3, \\ &\dot{x}_1(x_1-1)^{-1}+\dot{x}_2(x_2-1)^{-1}=z_1+z_4+2z_5. \end{aligned} \tag{12}$$

Then we can solve the first three equations of (3), together with (11) and the difference of the two equations in (12) for $z_1, \ldots, z_6$. Substituting these values in the second equation of (12) we find that

$$\left(\frac{\dot{x}_1}{x_1(x_1-1)}\right)^2-\left(\frac{\dot{x}_2}{x_2(x_2-1)}\right)^2=\frac{1}{x_1-x_2}\left[4h_1+\frac{h_2}{(x_1-1)(x_2-1)}+\frac{h_3}{x_1x_2}\right]. \tag{13}$$

Notice that this equation is linear in $\dot{x}_1^2$ and $\dot{x}_2^2$. Finally we substitute the values for $z_1, \ldots, z_6$ in the fourth equation of (3) to find another equation which is linear in $\dot{x}_1^2$ and $\dot{x}_2^2$, leading to

$$\dot{x}_i^2=\frac{f(x_i)}{(x_1-x_2)^2}, \qquad i=1,2,$$

where

$$f(x)=x(1-x)[4h_1x^3-(4h_1+h_4)x^2+(h_4-h_2-h_3)x+h_3].$$

(We note that the curve $y^2 = f(x)$ is precisely the curve Γ_h given by (6).) It follows that, in terms of the coordinates x_1, x_2 given by (10), the differential equations (1) reduce to the Jacobi form

$$\frac{\dot{x}_1}{\sqrt{f(x_1)}}+\frac{\dot{x}_2}{\sqrt{f(x_2)}}=0, \qquad \frac{x_1\dot{x}_1}{\sqrt{f(x_1)}}+\frac{x_2\dot{x}_2}{\sqrt{f(x_2)}}=1, \tag{14}$$

so that x_1 and x_2 are indeed linearizing variables.

The construction of these linearizing variables leads to an explicit map into the Jacobian $\mathrm{Jac}(\bar{\Gamma}_h)$ as follows. Recalling that we denote by ∞_h the point which is added to Γ_h in order to complete it into a compact Riemann surface, the map $P \mapsto [P + \infty_h]$ defines an embedding of $\bar{\Gamma}_h$ into its Jacobian; we have defined here $\mathrm{Jac}(\bar{\Gamma}_h)$ as the space of divisors of degree two on $\bar{\Gamma}_h$ modulo linear equivalence. We denote the image of this map by Θ_h and call it the theta divisor. It follows from Mumford's description of hyperelliptic Jacobians (see [9, Section 3.1]) that the affine surface $\mathrm{Jac}(\bar\Gamma_h)\setminus\Theta_h$ is isomorphic to the space of pairs of polynomials $(u(x),v(x))$ such that $u(x)$ is monic of degree two, $v(x)$ is of degree less than two and $f(x)-v^2(x)$ is divisible by $u(x)$. Let us describe the map from $\mathcal{A}_h$ into $\mathrm{Jac}(\bar\Gamma_h)$ in terms of these polynomials. We define the polynomial $u(x)$ by demanding that its roots are x_1 and x_2, i.e.,

$$u(x)=x^2+\left(\frac{z_1+z_2+z_4+z_6}{2(z_3-z_5)}-1\right)x-\frac{z_2+z_4}{2(z_3-z_5)}. \tag{15}$$

The polynomial $v(x)$ is defined as the derivative of $u(x)$ in the direction of X_1 and can be most easily described by the following formulas:

$$v(0)=u(0)(z_1+z_4+2z_3),\quad v(1)=u(1)(z_1+z_4+2z_5). \tag{16}$$

It is easy to check that $f(x)-v^2(x)$ is divisible by $u(x)$ so that the above formulas indeed define a point of $\mathrm{Jac}(\bar\Gamma_h)\setminus\Theta_h$. Since $h\in\mathcal{H}$, $h_1\neq 0$ and hence $z_3-z_5\neq 0$, showing that the above map is regular; moreover it is birational because (16) gives

$$z_3-z_5=\frac{1}{2}\left(\frac{v(0)}{u(0)}-\frac{v(1)}{u(1)}\right), \tag{17}$$

while, using (15), z_2+z_4 and z_1+z_6 can be rewritten as follows:

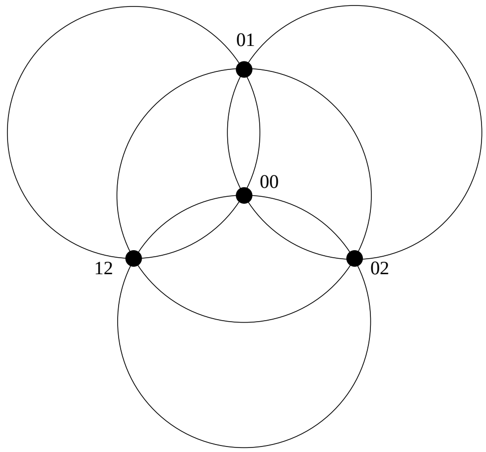
$$\begin{aligned} z_2+z_4&=\left(\frac{v(1)}{u(1)}-\frac{v(0)}{u(0)}\right)u(0),\\ z_1+z_6&=\left(\frac{v(0)}{u(0)}-\frac{v(1)}{u(1)}\right)u(1). \end{aligned} \tag{18}$$

Using the invariants H_1,H_2 and H_3 one easily finds formulas for z_3+z_5, z_2-z_4 and z_1-z_6 showing that the map is birational. On the one hand this proves that when $h\in\mathcal{H}$, i.e., when Γ_h is a non-singular curve of genus two, then $\mathcal{A}_h$ is isomorphic to an affine part of $\mathrm{Jac}(\bar\Gamma_h)$. On the other hand it leads to explicit solutions for (1) with respect to initial conditions which correspond to a point $h\in\mathcal{H}$, in terms of theta functions, in view of the following formulas

$$u(0)=c_0\left(\frac{\theta[\delta_0](At+B)}{\theta[\delta](At+B)}\right)^2\qquad u(1)=c_1\left(\frac{\theta[\delta_1](At+B)}{\theta[\delta](At+B)}\right)^2;$$

$v(0)$ and $v(1)$ are the derivatives of $u(0)$ and $u(1)$ with respect to t (see [9, page 3.81]). The constants c_0 and c_1 can be written explicitly in terms of the coordinates of the Weierstrass points and theta constants (see [9, page 3.113]) and the rational vectors δ,δ_0 and δ_1 are half-characteristics; the values of A and B depend in a transcendental way on h and on the initial conditions.

We see that the inverse map, given by (17) and (18), is holomorphic away from the divisors $u(0)=0$, $u(1)=0$ and $u(1)v(0)-u(0)v(1)=0$. When $u(0)=0$ then 0 is one of the roots of u so that the corresponding divisors are of the form W_0+P, where W_0 stands for the Weierstrass point over 0, $x(W_0)=0$ and $P\in\bar\Gamma_h$. Similarly, $u(1)=0$ corresponds to the divisors W_1+P, where W_1 stands for the Weierstrass point over 1. In order to avoid a rather involved explicit computation for the third divisor we appeal to the fact that the divisor at infinity $\mathcal{D}_h$ is invariant for the group $\mathfrak{T}_h$. Knowing that $\mathcal{D}_h$ consists of the theta divisor (consisting of divisors ∞_h+P) besides the two divisors that we have just determined we can identify the elements of $\mathfrak{T}_h$ as translations over $[W_1-W_0]$, $[\infty_h-W_1]$ and $[\mathcal{W}_0-\infty_h]$. Thus, the divisor $u(1)v(0)-u(0)v(1)=0$ corresponds to the effective divisors in $[W_0 + W_1 + P - \infty_h]$. It is now easy to see that the four points $2\infty_h, \infty_h + W_0, \infty_h + W_1$ and $W_0 + W_1$ (which constitute a single $\mathfrak{T}_h$ orbit) each belong to exactly three of the four curves and that these four curves have no other intersection points. Thus, as a byproduct, we have recovered[4] the following intersection pattern of the components of the divisor $\mathcal{D}_h$.



We will now use the above results to study the moduli space $\mathcal{M}$ defined by

$$\mathcal{M} = \{\mathcal{A}_h \mid h \in \mathcal{H}\}/\text{isomorphism},$$

where isomorphism means isomorphism of affine algebraic surfaces. We will relate this moduli space to a moduli space $\mathcal{M}_{(1,4)}$ of Abelian surfaces of type $(1,4)$ which is defined as follows. If $\mathcal{T}$ is an Abelian surface and $\mathcal{L}$ is a line bundle which induces a polarization $\omega = c_1(\mathcal{L})$ of type $(1,4)$ on $\mathcal{T}$ then the induced map $\phi_{\mathcal{L}} : \mathcal{T} \to \mathbf{P}^3$ is birational onto an octic surface (the generic case), or it is a double cover of a quartic surface. We define

$$\mathcal{M}_{(1,4)} = \{(\mathcal{T}, \mathcal{L}) \mid \phi_{\mathcal{L}} : \mathcal{T} \to \mathbf{P}^3 \text{ is birational onto an octic}\}/\text{isomorphism},$$

in which an isomorphism $(\mathcal{T}, \mathcal{L}) \cong (\mathcal{T}', \mathcal{L}')$ is a biholomorphic map $\Psi : \mathcal{T} \to \mathcal{T}'$ which preserves the polarization, $\Psi^*(c_1(\mathcal{L}')) = c_1(\mathcal{L})$. This moduli space was explicitly described in [13] as a Zariski open subset of a cone in weighted projective space $\mathbf{P}^{(1,2,2,3,4)}$ (see Section 4 below). In the following two propositions we show how $\mathcal{M}$ and $\mathcal{M}_{(1,4)}$ are related.

Proposition 1. *For any $h \in \mathcal{H}$ the quotient $\mathcal{A}_h/\mathfrak{T}_h$ is an affine part of an Abelian surface $\mathcal{T}_h$. The line bundle $\mathcal{L}_h = [\mathcal{D}_h/\mathfrak{T}_h]$ induces a polarization of type $(1,4)$ on $\mathcal{T}_h$ and the induced map $\phi_{\mathcal{L}_h} : \mathcal{T}_h \to \mathbf{P}^3$ is birational onto an octic surface.*

Proof. It is a general fact that the quotient of $\mathrm{Jac}(\bar{\Gamma}_h)$ by a group of half periods is an Abelian surface. For a group of half periods of order four it was shown in [13, Section 5] that the quotient is an Abelian surface of type $(1,4)$ if and only if the group of half periods is of the form $\{0, [W_2 - W_1], [W_1 - W_0], [W_0 - W_2]\}$, where W_0, W_1 and W_2 are Weierstrass points on the underlying curve; we have shown above that $\mathfrak{T}_h$ is indeed of this form. The divisor $\mathcal{D}_h$ descends to the irreducible divisor $\mathcal{D}_h/\mathfrak{T}_h$ which has a triple point which corresponds to the singular points of $\mathcal{D}_h$. Since $\mathcal{D}_h$ induces a polarization of type $(4,4)$ on $\mathrm{Jac}(\bar{\Gamma}_h)$, $\mathcal{D}_h/\mathfrak{T}_h$ induces a polarization of type $(1,4)$ on $\mathcal{T}_h$. In order to see

[4] This intersection pattern was first determined in [2, Figure 4.3] by using the Laurent solutions to the vector field X_1.

that the induced map $\phi_{\mathcal{L}_h}$ is birational onto its image one considers $\mathcal{T}_h/K_h$ where K_h is the group of two-torsion elements inside the kernel of the natural isogeny from $\mathcal{T}_h$ to its dual Abelian surface $\hat{\mathcal{T}}_h$. Since $\mathcal{T}_h = \mathrm{Jac}(\bar{\Gamma}_h)/\mathfrak{T}_h$ the map $\mathrm{Jac}(\bar{\Gamma}_h) \to \mathcal{T}_h/K_h$ is an isogeny whose kernel consists of the sixteen half periods of $\mathrm{Jac}(\bar{\Gamma}_h)$. This means that this isogeny is multiplication by 2 in $\mathrm{Jac}(\bar{\Gamma}_h)$ and hence that $\mathcal{T}_h/K_h$ is a Jacobi surface. This implies that the map $\phi_{\mathcal{L}_h} : \mathcal{T}_h \to \mathbf{P}^3$ is birational onto its image (see [6, Section 4]).

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Proposition 2. *The above correspondence between affine surfaces $\mathcal{A}_h$ and Abelian surfaces $\mathcal{T}$ induces a bijection $\chi : \mathcal{M} \to \mathcal{M}_{(1,4)}$.*

Proof. For $h \in \mathcal{H}$ we know that $\mathfrak{T}_h$ is a group of four translations of $\mathrm{Jac}(\bar{\Gamma}_h)$ over half periods leaving $\mathcal{D}_h$ invariant. Since the group of translations over half periods acts transitively on the set of theta curves (translates of the theta divisor over half periods) this property characterizes $\mathfrak{T}_h$. It follows that isomorphic surfaces $\mathcal{A}_h$ and $\mathcal{A}_k$ lead to isomorphic quotients $\mathcal{A}_h/\mathfrak{T}_h$ and $\mathcal{A}_k/\mathfrak{T}_k$ and hence to isomorphic polarized Abelian surfaces $(\mathcal{T}_h, \mathcal{L}_h)$ and $(T_k, \mathcal{L}_k)$. This shows that the given correspondence between affine surfaces $\mathcal{A}_h$ and Abelian surfaces $\mathcal{T}$ induces a map $\chi : \mathcal{M} \to \mathcal{M}_{(1,4)}$.

Starting from any polarized Abelian surface $(\mathcal{T}, \phi_{\mathcal{L}})$ of type $(1,4)$ for which the induced map is birational there exists a Riemann surface $\bar{\Gamma}$ and a partition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 = \{W_0, W_1, W_2\} \cup \{W_3, W_4, W_5\}$ of its Weierstrass points such that $\mathcal{T} = \mathrm{Jac}(\bar{\Gamma})/\mathfrak{T}$, where $\mathfrak{T}$ is the group of translations, given by $\mathfrak{T} = \{0, [W_0 - W_1], [W_1 - W_2], [W_2 - W_0]\}$. Moreover the triple $(\bar{\Gamma}, \mathcal{W}_1, \mathcal{W}_2)$ is uniquely determined up to isomorphism (see [13, Theorem 4]). Let us pick one particular triple $(\bar{\Gamma}, \mathcal{W}_1, \mathcal{W}_2)$ and let us choose coordinates for $\mathbf{P}^1$ such that the image of $\mathcal{W}_1$ under the natural double cover $\bar{\Gamma} \to \mathbf{P}^1$ is given by $0, 1$ and ∞ (in some order). Then we find an equation of the form

$$y^2 = x(1-x)(Ax^3 + Bx^2 + Cx + D)$$

in which the right hand side has no double roots. Obviously then we can find at least one $h \in \mathcal{H}$ such that this above curve corresponds to the curve Γ_h, given by (6). By construction (the isomorphism class of) the affine surface $\mathcal{A}_h$ is contained in the fiber $\chi^{-1}(\mathcal{T}, \mathcal{L})$, showing the surjectivity of χ. Finally, a triple $(\bar{\Gamma}', \mathcal{W}_1', \mathcal{W}_2')$ which is isomorphic to $(\bar{\Gamma}, \mathcal{W}_1, \mathcal{W}_2)$ leads to an isomorphic surface $\mathcal{A}_k$ because $\mathcal{A}_h$ is intrinsically described in terms of the triple $(\bar{\Gamma}, \mathcal{W}_1, \mathcal{W}_2)$ as being the affine part of the Jacobian of $\bar{\Gamma}$, obtained by removing the translates of the theta divisor, corresponding to the half periods $\{0, [W_0 - W_1], [W_1 - W_2], [W_2 - W_0]\}$, where $\mathcal{W}_1 = \{W_0, W_1, W_2\}$.

∎

4. The map to moduli space

It follows from Section 2 that for any $h \in \mathcal{H}$ the line bundle $\mathcal{L}_h$ which corresponds to $\mathcal{D}_h/\mathfrak{T}_h$ defines a birational map $\phi_{\mathcal{L}_h}$ from $\mathcal{T}_h$ to an octic surface in $\mathbf{P}^3$. We will compute an equation of this octic because the coefficients of this equation, which depend on h, will allow us to solve the moduli problem. Since $\mathcal{T}_h = \mathrm{Jac}(\bar{\Gamma}_h)/\mathfrak{T}_h$ the vector space of functions which provide this map consists of the $\mathfrak{T}_h$-invariant functions on $\mathrm{Jac}(\bar{\Gamma}_h)$ with a simple pole along $\mathcal{D}_h$ (at worst), i.e., the $\mathfrak{T}$-invariant functions in the span of $\{z_0, \ldots, z_{15}\}$. Using (4) and (7) one finds the following four independent invariant functions:

$$\begin{aligned}
\theta_0 &= z_0 = 1, \\
\theta_1 &= z_{10} = (z_2 - 2z_5)^2 - z_6^2, \\
\theta_2 &= z_{11} = (2z_3 - z_6)^2 - z_2^2, \\
\theta_3 &= z_{12} = z_1 z_2 z_3 - z_4 z_5 z_6.
\end{aligned} \tag{19}$$

In order to compute an equation for the octic it suffices — in principle — to eliminate the variables $z_1, \ldots, z_6$ from the equations (3) and (19). In practice, doing the calculation in a straightforward way leads to disastrous results, even when using a computer algebra package such as Maple. Therefore we will describe in some detail how this computation can be done. As a first step we notice that the octic which we want to compute is isomorphic to the variety defined by the following equations:

$$\begin{aligned}
h_1 &= X_3 - X_5, & 0 &= X_1X_4 - Z_1^2,\\
h_2 &= X_1 - X_6, & 0 &= X_2X_5 - Z_2^2,\\
h_3 &= X_2 - X_4, & 0 &= X_3X_6 - Z_3^2,\\
h_4 &= X_1 + X_4 + 2Z_1 + 4X_3 - 4Z_2 - 4Z_3, &&\\
\theta_1 &= 4X_5 - 4Z_2 + X_2 - X_6, &&\\
\theta_2 &= 4X_3 - 4Z_3 + X_6 - X_2, &&\\
\theta_3^2 &= X_1X_2X_3 + X_4X_5X_6 - 2Z_1Z_2Z_3. &&
\end{aligned} \tag{20}$$

To see this, we consider a regular map φ from the variety given by (3) and (19) to the variety given by (20). The map φ is given by $X_i = z_i^2$ and $Z_j = z_j z_{j+3}$, where $i = 1, \ldots, 6$ and $j = 1, 2, 3$. On the one hand φ is constant on the orbits of $\mathfrak{T}$ because all X_i and Z_j are $\mathfrak{T}$-invariant; on the other hand it is easy to check that every fiber of φ contains precisely four points, hence the degree of φ is four. This shows that (20) represents the image of $\mathcal{A}_h/\mathfrak{T}_h$ in projective space, obtained by using the sections of the line bundle associated to $\mathcal{D}_h/\mathfrak{T}_h$.

Six of the equations in (20) are linear and we can use these equations to eliminate X_2, X_3, X_5, X_6, Z_2 and Z_3 from the four non-linear equations. Apart from $X_1X_4 = Z_1^2$, this leaves us with the following three equations (we have used $X_1X_4 = Z_1^2$ to simplify them)

$$\begin{aligned}
&2\left[h_3X_1^2 - h_2X_4^2 - (h_2 - h_3 - \theta_1 - \theta_2)Z_1^2\right] - 2(4h_1 + h_2 - h_3 - h_4 + \theta_1)X_4Z_1\\
&\qquad - 2(h_2 - h_3 - h_4 + \theta_2)X_1Z_1 + 2h_3(4h_1 - h_4 + \theta_1 + \theta_2)X_1 + 2h_2(h_4 - \theta_1 - \theta_2)X_4\\
&\qquad - (h_2 + h_3 - h_4 + \theta_2)(4h_1 - h_2 - h_3 - h_4 + \theta_1)Z_1 - 8\theta_3^2 = 0,\\
&4h_3X_1 - 4(h_2 - \theta_1)(h_3 + X_4) + 4X_1X_4 - (h_2 - h_3 - h_4 + \theta_2 + 2Z_1)^2 = 0,\\
&4(h_3 + \theta_2)(X_1 - h_2) - 4X_4h_2 + 4X_1X_4 - (4h_1 + h_2 - h_3 - h_4 + \theta_1 + 2Z_1)^2 = 0.
\end{aligned} \tag{21}$$

The first trick that we use to make the rest of the computation feasible stems from the following observation. If we multiply the second equation by X_1 and the third equation by X_4 to remove from the first equation in (21) those terms which contain X_1^2 and X_4^2, then the resulting equation is a linear equation in X_1, X_4 and Z_1 (the relation $X_1X_4 = Z_1^2$ is again used to simplify this expression) so that (21) is equivalent to a linear system of equations in X_1, X_4 and Z_1, which is solved at once. An equation for the octic is then given by substituting the expressions for X_1, X_4 and Z_1 in the only remaining equation $X_1X_4 = Z_1^2$.

The resulting equation is monstrous (it has 2441 terms), in contrast with the following equation for the octic, corresponding to an Abelian surface of type $(1,4)$ proposed in [6, Section 2][5]:

$$\begin{aligned}
&\mu^2 y_0^2y_1^2y_2^2y_3^2 + \mu_1^2(y_0^4y_1^4 + y_2^4y_3^4) + \mu_2^2(y_1^4y_3^4 + y_0^4y_2^4) + \mu_3^2(y_0^4y_3^4 + y_1^4y_2^4)+\\
&- 2\mu_1\mu_2(y_0^2y_1^2 + y_2^2y_3^2)(y_0^2y_2^2 - y_3^2y_1^2) - 2\mu_2\mu_3(y_0^2y_2^2 + y_3^2y_1^2)(y_0^2y_3^2 - y_1^2y_2^2)\\
&- 2\mu_3\mu_1(y_0^2y_3^2 + y_1^2y_2^2)(y_0^2y_1^2 - y_2^2y_3^2) = 0.
\end{aligned} \tag{22}$$

[5] We have rescaled some of the coordinates by roots of -1 so as to obtain a more symmetric equation.

The difference between these two equations lies of course in the choice of coordinates. In order to compute the coordinate transformation which reduces our equation to the symmetric form (22) we use the following geometric fact. Since the octic that we obtained has the form $A\theta_3^4+B\theta_3^2+C^2=0$ the octic has a singular point of order four at $(0:0:0:1)$ and such a singular point necessarily comes from four of the sixteen half periods on $\mathrm{Jac}(\bar{\Gamma}_h)$. Clearly, (22) also has a singular point of order four at $(0:0:0:1)$. On the other hand, we remark that the tangent cone to (22) at $(0:0:0:1)$, is the union of four hyperplanes because the zero locus of the coefficient of y_3^4 in (22) has the form

$$(Y_0+Y_1+Y_2)(Y_0-Y_1+Y_2)(Y_0+Y_1-Y_2)(Y_0-Y_1-Y_2)$$

where $Y_0=\sqrt{\mu_3}y_0$, $Y_1=\sqrt{\mu_2}y_1$ and $Y_2=i\sqrt{\mu_1}y_2$ (the particular choices made for each square root are irrelevant). The coefficient A of θ_3^4 in our equation for the octic must also factor in four linear factors, but these are harder to determine because this can only be done by passing to an extension field of the field $\mathbf{C}[h_1,h_2,h_3,h_4]$. However, if one uses the sections of a symmetric line bundle to map a Jacobian in projective space, then symmetric equations for the image are usually obtained by explicitly introducing the Weierstrass points on the curve, rather than working with the coefficients of a polynomial that defines the underlying curve (see [11, Section 6]). In view of the equation (6) for Γ_h we are therefore led to defining[6]

$$4x^3h_1-(4h_1+h_4)x^2+(h_4-h_3-h_2)x+h_3=\lambda(x-\lambda_1)(x-\lambda_2)(x-\lambda_3).$$

Indeed, in terms of the λ_i one finds the following factorization for A,

$$A=\theta_0\prod_{i=1}^{3}[\lambda\lambda_i(\lambda_i-1)\theta_0-\lambda_i\theta_1-(\lambda_i-1)\theta_2].$$

In order to find the required coordinate transformation, we can now use the following ansatz:

$$\begin{aligned}
Y_0+Y_1+Y_2&=\theta_0,\\
-Y_0-Y_1+Y_2&=\kappa_1(\lambda\lambda_1(\lambda_1-1)\theta_0-\lambda_1\theta_1-(\lambda_1-1)\theta_2),\\
-Y_0+Y_1-Y_2&=\kappa_2(\lambda\lambda_2(\lambda_2-1)\theta_0-\lambda_2\theta_1-(\lambda_2-1)\theta_2),\\
Y_0-Y_1-Y_2&=\kappa_3(\lambda\lambda_3(\lambda_3-1)\theta_0-\lambda_3\theta_1-(\lambda_3-1)\theta_2).
\end{aligned}\tag{23}$$

The coefficients κ_i are uniquely determined by the compatibility equations, which stem from the vanishing of the sum of the left hand sides of these four equations. If we denote $\Lambda(x)=\lambda(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ then the solution to the compatibility equations is given by $\kappa_i=-1/\Lambda'(\lambda_i)$, $(i=1,\ldots,3)$. Substituting these values for κ_i in (23) we can rewrite our equation for the octic in terms of the coordinates $Y_0,\ldots,Y_3$. Putting $Y_i=\rho_iy_i$ we can determine the ρ_i such that we obtain precisely (22). It gives the following values for μ,μ_1,μ_2,μ_3:

$$\begin{aligned}
\mu_i^2&=\lambda_i(1-\lambda_i)(\lambda_{i+1}-\lambda_{i+2})^3,\\
\mu^2&=12(\sigma_2^2-\sigma_1^2\sigma_3)+2(\sigma_2-\sigma_1)(\sigma_1\sigma_2+9\sigma_3),
\end{aligned}\tag{24}$$

where σ_i is the i-th symmetric function of $\lambda_1,\lambda_2,\lambda_3$ and $\lambda_4=\lambda_1$, $\lambda_5=\lambda_2$. This determines the parameters μ_i explicitly in terms of the Weierstrass points of the curve $\bar{\Gamma}_h$. The sign of the parameters μ and μ_i is not important. Indeed, the coefficients (μ,μ_1,μ_2,μ_3) are only intermediate moduli for Abelian surfaces of type $(1,4)$, the moduli themselves being

[6] The final result will be symmetric in $\lambda_1,\lambda_2,\lambda_3$, hence does not depend on the order of these parameters.

given by the following expressions which realize the moduli space as the cone $\mathcal{C} : f_4^2 = f_1(4f_2^3 - 27f_3^2)$ in weighted projective space $\mathbf{P}^{(1,2,2,3,4)}$ (see [13, Theorem 3]):

$$\begin{aligned}
f_0 &= \mu^2, \\
f_1 &= (\mu_1^2 + \mu_2^2 + \mu_3^2)^2, \\
f_2 &= \mu_1^4 + \mu_2^4 + \mu_3^4 - \mu_1^2\mu_2^2 - \mu_2^2\mu_3^2 - \mu_3^2\mu_1^2, \\
f_3 &= (\mu_2^2 - \mu_1^2)(\mu_3^2 - \mu_2^2)(\mu_1^2 - \mu_3^2), \\
f_4 &= (\mu_1^2 + \mu_2^2 + \mu_3^2)(\mu_1^2 + \mu_2^2 - 2\mu_3^2)(\mu_2^2 + \mu_3^2 - 2\mu_1^2)(\mu_3^2 + \mu_1^2 - 2\mu_2^2).
\end{aligned}$$

The standard action of the symmetric group S_3 on $\mathbf{C}[\lambda_1, \lambda_2, \lambda_3]$ induces on $\mathbf{C}[\mu_1^2, \mu_2^2, \mu_3^2]$ an action which is determined by $(1,2) \cdot (\mu_1^2, \mu_2^2, \mu_3^2) = (-\mu_2^2, -\mu_1^2, -\mu_3^2)$ and $(1,2,3) \cdot (\mu_1^2, \mu_2^2, \mu_3^2) = (\mu_2^2, \mu_3^2, \mu_1^2)$. Therefore, every symmetric function in $\mathbf{C}[\mu_1^2, \mu_2^2, \mu_3^2]$ is either invariant or anti-invariant with respect to this induced action and it follows that the above polynomials $f_0, \ldots, f_4$ are symmetric in $\lambda_1, \lambda_2, \lambda_3$. They are easily expressed in terms of $h = (h_1, h_2, h_3, h_4)$; the resulting map $\mathcal{M} \to \mathcal{C} \subset \mathbf{P}^{(1,2,2,3,4)}$ solves the moduli problem, posed in the introduction.

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