COMMUTING MATRIX DIFFERENTIAL OPERATORS AND LOOP ALGEBRAS

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Abstract. We consider for fixed positive integers p and q which are coprime the space of all pairs (P,Q) of commuting matrix differential operators (of a fixed size n), where P is monic of order p and Q is normalized of order q. We use the vector valued Sato Grassmannian to construct a natural bijection to an affine subspace of the loop algebra $\mathfrak{gl}(nq)((\lambda^{-1}))$. In the scalar case ($n=1$) the KP flows on the Grassmannian, which are known to trace out Jacobians, lead to commuting flows on this affine space. These flows are Hamiltonian with respect to a family of Poisson structures which are obtained from a family of Lie brackets on the loop algebra.

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1. Introduction

The key fact in the description of the algebraic solutions to the Kadomtsev-Petviashvili equation (KP equation) is that a pair of commuting differential operators (in one variable) defines an algebraic curve with a privileged smooth point on it, a local coordinate at that point, and a rank 1 torsion free sheaf with trivial cohomology. This fact was discovered by Krichever (see [3] and see [6] for generalizations) and was exploited by him to provide a large class of solutions to the KP equation, written explicitly in terms of theta functions. The main properties of pairs of commuting differential operators were already known to Burchnall and Chaundy (see [1]) and to Schur (see [11]), but not in the context of integrable systems. When Sato showed that the KP equation(s) linearizes on an infinite Grassmannian (the Sato Grassmannian), the picture became even more geometric, for example the commutativity of two operators became a stabilization condition of an infinite-dimensional plane (a point in the Sato Grassmannian) and the theta function was shown to be a special case of the tau function, which is a section of the dual of the determinant bundle on the Grassmannian (see [9]).

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The above description of the algebraic solutions can be called local in the sense that one always deals with one particular orbit in the Grassmannian, with one particular solution to the KP equation(s) and with one particular Jacobian (i.e., one particular curve). We will show in this paper that when one studies algebraic solutions in families, then one finds that (for fixed orders of the differential operators) these solutions fit perfectly together in one affine space, i.e., there is a natural bijection to an affine space of matrices, and all ingredients of the KP equation can be transferred to this affine space, making the study of the algebraic solutions the study of a finite-dimensional integrable system. Among other things this gives explicit equations for the Abelian varieties that are traced out by the integrable flows: they are the fibers of the moment map, which maps in the case at hand a matrix to its characteristic polynomial. For applications of this, see [12] and [7].

In this paper we will describe this natural bijection in the general case of matrix differential operators. We fix p and q which are coprime and we take any n. Let Q be a normalized matrix differential operator of size n and of order q, i.e.,

$$Q = I_n\partial^q + Q_{q-2}\partial^{q-2} + \cdots + Q_1\partial + Q_0$$

where I_n is the $n \times n$ identity matrix and Q_i, $(i = 0, \dots, n-2)$ is a square matrix of size n whose entries are formal power series in one variable (x). We can find a monic zeroth order matrix pseudo-differential operator T such that

$$Q = T^{-1}\partial^q T$$

which leads to a point $W_T = T \cdot H_n$ in the Sato Grassmannian. A quick description of W_T is this: for every constant coefficient matrix differential operator S of size n, write the product QS in left normal form and then evaluate it at zero; the resulting linear space of Laurent series in ∂^{-1} is W_T. Suppose now that we have another matrix differential operator P of size n, which is monic and of order p. If P commutes with Q then $\tilde{P} = TPT^{-1}$ is a constant coefficient matrix differential operator and it stabilizes W_T, i.e., $\tilde{P}W_T \subset W_T$. In fact, W_T is also stabilized by ∂^q and these two conditions characterize points in the Sato Grassmanian that come from pairs of commuting matrix differential operators with prescribed orders (Theorem 3.1). This first step does not require that p and q are coprime and the arguments that we use are matrix generalizations of standard arguments that can e.g., be found in [5]. One should note however that Schur's lemma, which says that the commutant of a monic differential operator is always commutative, is not valid for matrix differential operators: indeed, constant coefficient matrix differential operators need not commute!

The second step of the construction is more delicate. Since $\tilde{P}$ is an endomorphism of W_T we can describe this endomorphism with an infinite periodic matrix (upon taking a periodic basis) and this matrix becomes a matrix of finite size nq, but it depends now on a formal variable $\lambda = \partial^q$. Due to the freedom in the choice of basis we have that the matrix is only well-defined up to conjugation by a block lower triangular matrix (the blocks have size n); we denote the group of those matrices by $N^-_{n,q}$. In λ these matrices have a degree which is represented by the following block matrix:

$$
\begin{pmatrix}
\leq [p/q] & & & & [p/q] & \leq [p/q]-1 \\
 & \ddots & & & \ddots & \\
 & & & & & [p/q] \\
 & & & \leq [p/q] & & \\
{[p/q]+1} & & & & & \\
 & \ddots & & & \ddots & \\
\leq [p/q]+1 & & [p/q]+1 & & & \leq [p/q]
\end{pmatrix}. \tag{1}
$$

In this matrix every integer stands for the degree of the corresponding block matrix; $[p/q]$ means that the corresponding block, which is a polynomial in λ whose coefficients are $n \times n$ matrices, is monic (has leading coefficient I_n) and has degree $[p/q]$. Similarly $\leq [p/q]$ means that the corresponding block is a polynomial of degree $[p/q]$ at most. If we denote this space of matrices by $\mathcal{M}_n^{p,q}$ then we have associated to the commuting pair (P, Q) an element of $\mathcal{M}_n^{p,q}/N_{n,q}^-$. Moreover, we will show (Proposition 3.3) that this quotient space is in a natural way an affine subspace $\tilde{\mathcal{M}}_n^{p,q}$ of $\mathcal{M}_n^{p,q}$. This is the first time that we use the fact that p and q are coprime, the result being false when p and q have a common divisor.

In the scalar case ($n = 1$), the space $\tilde{\mathcal{M}}_n^{p,q}$ already appears in [7] as a multi-Hamiltonian manifold, obtained by a reduction on the loop algebra of $\mathfrak{gl}_q$. It will be shown that the integrable systems described in [7] are precisely the KP flows, transferred to $\tilde{\mathcal{M}}_n^{p,q}$.

The second time where the fact that p and q are coprime comes in is when we prove that every element of $\mathcal{M}_n^{p,q}$ (or $\tilde{\mathcal{M}}_n^{p,q}$) comes from a commuting pair (P, Q) of matrix differential operators (Proposition 4.2). Our proof uses only linear algebra and is elementary once one realizes that the proper gradation to be used to solve the linear equations recursively is the one coming from the Kac-Moody Lie algebra of (block) $\mathfrak{gl}(q)$, as is apparent from (1). The fact that p and q should be coprime comes from the fact that at each step we need to solve q equations, subject to one consistency equation, with each equation containing (essentially) the difference $x_i - x_{i+p}$ of the unknowns x_i (the indices i are taken here mod q).

In the scalar case ($n = 1$) $\tilde{\mathcal{M}}_n^{p,q}$ is essentially the space of Burchnall-Chaundy matrices (of fixed size), see [1] and [8]. The Burchnall-Chaundy construction obscures however the Hamiltonian structures, which are non-linear and are most naturally obtained by a reduction: in the Burchnall-Chaundy construction the Hamiltonian structure gets lost because the reduction gets bypassed by picking particular matrices representing the pair of commuting differential operators. Still in the scalar case a different construction was given in [10], when studying solutions to the string equation $[P, Q] = 1$. The first step of our construction is a matrix generalization of this construction.

The structure of this paper is as follows. We state some preliminaries about matrix differential operators and the vector valued Sato Grassmannian in Section 2, referring to [2] and [4] for proofs. We show in Section 3 how to associate to a pair of commuting operators a point in the Sato Grassmannian with a large stabilizer and we show how on the one hand the pair (P, Q) is reconstructed from such a point and how on the other hand to such a point one can associate a unique element of an affine subspace of the loop algebra of $\mathfrak{gl}(nq)$. We show in Section 4 that every element in this affine subspace comes from a unique pair of commuting matrix differential operators, yielding the bijectivity of our correspondence. We also give a generalization of our construction to equations, other than the commutativity equation, such as the matrix string equation ([P,Q]=1) and the equations $[P, Q] = f(Q)$ where f is any polynomial. In the final section we give explicit formulas for the vector fields on $\tilde{\mathcal{M}}_n^{p,q}$ that correspond to the KP vector fields.

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2. Matrix differential operators

We start by reviewing the basic definitions and properties of (formal) matrix differential operators. Throughout this paper a field $\mathbb{F}$ of characteristic zero will be fixed. For any $\mathbb{F}$-algebra $\mathcal{A}$ we denote by $\mathbf{Mat}_n(\mathcal{A})$ the $\mathbb{F}$-algebra of $(n\times n)$-matrices with coefficients in $\mathcal{A}$ and we denote its Lie algebra by $\mathfrak{gl}_n(\mathcal{A})$. We write $\mathbf{Mat}_n$ for $\mathbf{Mat}_n(\mathbb{F})$ and $\mathfrak{gl}_n$ for $\mathfrak{gl}(n,\mathbb{F})$. We denote by $\mathbb{F}[[x]]$ the ring of formal power series in x with coefficients in $\mathbb{F}$ and we let $\mathcal{D}$ be the non-commutative algebra $\mathbb{F}[[x]][\partial]$ of differential operators, the multiplication being given by juxtaposition and applying the commutation rule $[\partial, a(x)] = a'(x)$; here $a(x)$ is any formal power series, $a(x) \in \mathbb{F}[[x]]$, and $a'(x)$ its (formal) derivative. $\mathcal{D}$ has as a distinguished maximal commutative subalgebra the algebra $\mathcal{D}^c = \mathbb{F}[\partial]$ of constant coefficient differential operators. We will denote the algebra $\mathbf{Mat}_n(\mathcal{D})$ by $\mathcal{D}_n$ and we call the elements of $\mathbf{Mat}_n(\mathcal{D})$ *matrix differential operators of size n* or *$(n\times n)$–differential operators*. A significant difference between $\mathcal{D}_n$ and $\mathcal{D}$ is that the subalgebra $\mathcal{D}_n^c = \mathfrak{gl}(n, \mathcal{D}^c)$ is non-commutative (for $n > 1$). It is often convenient to see elements of $\mathcal{D}_n$ as differential operators, whose coefficients lie in $\mathbf{Mat}_n(\mathbb{F}[[x]])$, e.g., writing $Q \in \mathcal{D}_n$ as

$$Q = \sum_{i=0}^{q} a_i(x)\partial^i, \qquad a_i(x) \in \mathbf{Mat}_n(\mathbb{F}[[x]])$$

we call q the *order* of Q if $a_q \neq 0$, we say that Q is *monic* when $a_q = I_n$ (the identity matrix of size n) and we call Q *normalized* when Q is monic and $a_{q-1} = 0$. Also we will identify $(n\times n)$-differential operators whose coefficients are scalar matrices (multiples of I_n) with elements of $\mathcal{D}$ (in particular we identify $\mathcal{D}_1$ with $\mathcal{D}$) and we call them *scalar differential operators.*

$\mathcal{D}$ and $\mathcal{D}_n$ embed in the larger algebras

$$\Psi = \mathbb{F}[[x]]((\partial^{-1})) \qquad \text{and} \qquad \Psi_n = \mathbf{Mat}_n(\mathbb{F}[[x]])((\partial^{-1}))$$

of (matrix) *pseudo-differential operators,* the multiplication being formally derived from the commutation rule $[\partial, a(x)] = a'(x)$, i.e.,

$$\partial^{-1}a(x) = \sum_{i=0}^{\infty}(-1)^i a^{(i)}(x)\partial^{-i-1};$$

the notion of order and the properties of being scalar, monic or normalized are defined in the same way as in the case of (matrix) differential operators. The subalgebras of Ψ and Ψ_n consisting of constant coefficient (matrix) pseudo-differential operators are respectively denoted by Ψ^c and Ψ_n^c. We have that

$$\Psi^c = \mathbb{F}((\partial^{-1})) \qquad \text{and} \qquad \Psi_n^c = \mathbf{Mat}_n(\mathbb{F}((\partial^{-1}))). \tag{2}$$

The crucial properties of matrix differential operators, which are well-known in the scalar case ($n = 1$), are listed in the following proposition (see [2]).

Proposition 2.1.

1. *Every monic $(n \times n)$-pseudo-differential operator Q of order q has a unique inverse Q^{-1} in Ψ_n. In particular the monic $(n \times n)$-pseudo-differential operators of order zero form a group, called the* Volterra group *and denoted by* Volt_n. *Its subgroup of constant coefficient operators is denoted by* Volt_n^c. *We will write* Volt *for* Volt_1.
2. *Every normalized $(n \times n)$–differential operator Q of order $q > 0$ has a unique monic q-th root $Q^{1/q}$ in Ψ. This root $Q^{1/q}$ is normalized and has order 1.*
3. *Every normalized $(n \times n)$-differential operator Q of order $q > 0$ is conjugated to ∂^q,*
$$Q = T^{-1}\partial^q T$$
where $T \in \mathrm{Volt}_n$ is unique up to left multiplication by an element of Volt_n^c.

Finally we recall the definition of the vector valued Sato Grassmannian Gr_n. The space $\Psi^{\oplus n}$ is a left Ψ_n-module in the obvious way; we will refer to this action also as *multiplication of matrix differential operators.* Let us denote by δ Dirac's delta function thought of as a zeroth order (1×1) or scalar differential operator. It has the fundamental property that for any $Q \in \Psi$ there exists a unique $Q^c \in \Psi^c$ such that $Q\delta = Q^c\delta$. The left coset
$$V_n := \mathbb{F}((\partial^{-1}))^{\oplus n}\delta = (\Psi^c)^{\oplus n}\delta$$
is a left Ψ_n-module in a natural way: for $P \in \Psi_n$ and $Q \in \mathbb{F}((\partial^{-1}))^{\oplus n} \subset \Psi^{\oplus n}$ we define $P \cdot (Q\delta) = (PQ)\delta$ where PQ is the above multiplication of matrix differential operators. A distinguished subspace of V_n is defined by
$$H_n = \mathbb{F}[\partial]^{\oplus n}\delta = (\mathcal{D}^c)^{\oplus n}\delta.$$
The multiplication allows us to associate to each element in Ψ_n a subspace of $\mathbb{F}((\partial^{-1}))^{\oplus n}\delta$ in a natural way, namely given $Q \in \Psi_n$ define $W_Q \subset \mathbb{F}((\partial^{-1}))^{\oplus n}\delta$ by
$$W_Q = Q \cdot H_n.$$
We call the set of all W_T which correspond to elements T in the Volterra group the *vector valued Sato Grassmannian*,
$$\mathrm{Gr}_n := \{W_T \mid T \in \mathrm{Volt}_n\}.$$
If $T \in \mathrm{Volt}_n$ the linear space W_T has a basis, similar to the standard basis of H_n, namely it has a basis whose elements have the form
$$T \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \partial^i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \delta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \partial^i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \delta + \mathcal{O}(\partial^{i-1}), \qquad (i \geq 0).$$
It follows that the map $\mathrm{Volt}_n \to \mathrm{Gr}_n$ given by $P \mapsto W_P$ is injective hence bijective. The following proposition gives a useful characterization of matrix differential operators in terms of this map (see [4]).

Proposition 2.2. *Let $Q \in \Psi_n$. Then $Q \in \mathcal{D}_n$ if and only if $W_Q \subset H_n$.*

Since no confusion is possible we remove δ from the elements of W_Q, i.e., we identify V_n with $\mathbb{F}((\partial^{-1}))^{\oplus n}$.

3. The big matrix associated to two commuting operators

In this section we assume that P and Q are monic $(n \times n)$-matrix differential operators, which commute, $[P, Q] = 0$. We will assume that one of them, say Q, is normalized. Finally we assume that the orders p and q of P and Q are positive and coprime. In this section we show how to associate a unique element $M \in \mathbf{Mat}_{nq}(\mathbb{F}[\lambda])$ to (P, Q); it will be shown in the next section how to reconstruct (P, Q) from M.

In a first step we associate to the pair (P, Q) a pair $(\tilde{P}, W)$ of elements $\tilde{P} \in \Psi_n^c$ and $W \in \mathrm{Gr}_n$ (this pair is unique up to an equivalence specified below). Since Q is normalized there exists by Proposition 2.1 an element $T \in \mathrm{Volt}_n$ such that $Q = T^{-1}\partial^q T$. Choosing such an element T we define $W = W_T = T \cdot H_n \in \mathrm{Gr}_n$. If we let

$$\tilde{P} = TPT^{-1}$$

then $\tilde{P} \in \Psi_n$ is monic of order p and $[\tilde{P}, \partial^q] = 0$. We claim that the latter implies that $\tilde{P} \in \Psi_n^c$. To show this it is sufficient to show that if $a(x) \in \mathbf{Mat}_n(\mathbb{F}[[x]])$ is such that $[\partial^q, a(x)] = 0$ for some $q > 0$ then $a(x)$ is constant. Since

$$0 = [\partial^q, a(x)] = \partial^q a(x) - a(x)\partial^q = qa'(x)\partial^{q-1} + \mathcal{O}(\partial^{q-2}),$$

we find indeed that $a'(x) = 0$.

The pair $(\tilde{P}, W)$ clearly depends on the choice of T: if T is replaced by $T^c T$ where $T^c \in \mathrm{Volt}_n^c$ then $(\tilde{P}, W)$ is replaced by

$$T^c \cdot (\tilde{P}, W) = (T^c \tilde{P}(T^c)^{-1}, T^c \cdot W).$$

The above equation defines an action of Volt_n^c on $\Psi_n^c \times \mathrm{Gr}_n$; we say that two elements in $\Psi_n^c \times \mathrm{Gr}_n$ are *equivalent* when they correspond under this action. Note that (only) in the scalar case ($n = 1$) the fact that $\tilde{P} \in \Psi^c$ implies that the action on $\tilde{P}$ is trivial, i.e., $\tilde{P}$ is independent of the choice of T. We now give a characterizing property of the pair $(\tilde{P}, W)$ and we use it to show that the pair (P, Q) can be reconstructed from it.

Proposition 3.1. *The element $W \in \mathrm{Gr}_n$ is stable under the action of ∂^q and $\tilde{P}$,*

$$\partial^q \cdot W \subset W, \tag{3}$$

$$\tilde{P} \cdot W \subset W. \tag{4}$$

Conversely, suppose that $(\tilde{P}, W) \in \Psi_n^c \times \mathrm{Gr}_n$ satisfies (3) and (4), $\tilde{P}$ being monic. The pair $(\tilde{P}, W)$ is associated to a unique pair (P, Q) of commuting matrix differential operators, such that P is monic and Q is normalized.

Proof. The verification of (3) and (4) is easy, for example

$$\partial^q \cdot W = T \cdot (Q \cdot H_n) \subset T \cdot H_n = W,$$

in which the inclusion $Q \cdot H_n \subset H_n$ holds because $Q \in \mathcal{D}_n$. Let us show now how to reconstruct (P, Q) from $(\tilde{P}, W)$. Since $W \in \mathrm{Gr}_n$ it is of the form $W = T \cdot H_n$

for a unique $T \in \mathrm{Volt}_n$. If we define

$$Q = T^{-1}\partial^q T,$$
$$P = T^{-1}\tilde{P}T,$$

then Q is normalized and has order q while P is monic; also $[P,Q]=0$. The crucial property is that (3) and (4) imply that P and Q are matrix differential operators. We have that $W_Q \subset H_n$ and $W_P \subset H_n$ because e.g. for P one computes

$$W_P = P \cdot H_n = T^{-1} \cdot (\tilde{P} \cdot W) \subset T^{-1} \cdot W = H_n.$$

Using this, the fact that P and Q are matrix differential operators follows from Lemma 2.2. Clearly, if we replace $(\tilde{P}, W)$ by an equivalent pair then the same pair (P,Q) is obtained. □

Notice that in the above proposition the orders of P and Q need not be coprime.

The next step is to associate to the pair $(\tilde{P}, W)$ (up to equivalence) a matrix $M(\lambda) \in \mathbf{Mat}_{nq}(\mathbb{F}[\lambda])$ from which $(\tilde{P}, W)$ can be reconstructed. Since $\tilde{P} \in \Psi_n^c$ and $\tilde{P} \cdot W \subset W$ it follows that $\tilde{P}$ is an endomorphism of W, hence can be represented by a semi-infinite matrix (with entries in $\mathbb{F}$) by choosing any basis for W. This matrix becomes a periodic matrix (with period nq) when a periodic basis is chosen, i.e., a basis such that for any basis element $b(\partial) \in W$ the element $\partial^q b(\partial)$ is also a basis element. Note that the existence of such a basis follows from the inclusion $\partial^q W \subset W$. Periodic matrices can be rewritten as square matrices at the price of allowing entries which are polynomials in $\lambda = \partial^q$. In the present case it amounts to defining a matrix $M = (m_{ij}) \in \mathbf{Mat}_{nq}(\mathbb{F}[\lambda])$ by

$$\tilde{P} \cdot e_i = \sum_{j=0}^{nq} m_{ij} e_j, \qquad (1 \le i \le nq), \tag{5}$$

where $e_1, \ldots, e_{nq}$ consists of the first nq vectors of any periodic basis for W. We will always choose these nq vectors of the form

$$f_v^u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \star \\ \vdots \\ \star \end{pmatrix} \partial^q + \mathcal{O}(\partial^{u-1}), \qquad (1 \le v \le n,\ 0 \le u \le q-1), \tag{6}$$

where the leading term 1 appears in the v-th slot. We introduce, for arbitrary $u \ge 0$ the vectors $f_v^{u+q} = \partial^q f_v^u$ and we order the vectors f_v^u as follows: $f_1^0 < f_2^0 < \cdots < f_n^0 < f_1^1, \cdots$. With this ordering it is easy to describe the degree (in $\lambda = \partial^q$) of all entries of M. Indeed,

$$\begin{aligned} \tilde{P} \cdot f_v^u &= f_v^{p+u} + \text{ lower order in } \partial \\ &= \lambda^{[\frac{p+u}{q}]} f_v^{(p+u) \bmod q} + \text{ lower order in } \partial. \end{aligned}$$

It follows that if we write the matrix M in block-form

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1q} \\ \vdots & \ddots & \vdots \\ M_{q1} & \cdots & M_{qq} \end{pmatrix}, \tag{7}$$

where $M_{\alpha\beta}$ is an element of $\mathbf{Mat}_n(\mathbb{F}[\lambda])$ then its elements have the following degree constraints:

$$\begin{pmatrix} \leq [p/q] & & & [p/q] & & \leq [p/q]-1 \\ & \ddots & & & \ddots & \\ & & \leq [p/q] & & & [p/q] \\ [p/q]+1 & & & \leq [p/q] & & \\ & \ddots & & & \ddots & \\ \leq [p/q]+1 & & [p/q]+1 & & & \leq [p/q] \end{pmatrix}. \tag{8}$$

We mean by this that

$$\deg_\lambda(M_{\alpha\beta}) \leq \left[\frac{p+\alpha-\beta}{q}\right], \tag{9}$$

and that when q divides $p+\alpha-\beta$ then $M_{\alpha\beta}$ (which is a constant coefficient $(n\times n)$–differential operator) is monic of degree $(p+\alpha-\beta)/q$. We denote the affine space of matrices of the form (8) by $\mathcal{M}_n^{p,q}$. We introduce the group $N_{n,q}^- \subset GL(nq)$ of block lower triangular matrices, i.e., matrices of the form

$$\begin{pmatrix} I_n & 0 & \cdots & 0 \\ \star & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & I_n \end{pmatrix},$$

where each entry above represents a square matrix of size n. $N_{n,q}^-$ acts on $\mathcal{M}_n^{p,q}$ by conjugation.

Proposition 3.2. *The above procedure associates to the pair (P,Q) a well-defined element of $\mathcal{M}_n^{p,q}/N_{n,q}^-$.*

Proof. Two bases of the form (6) are related by an element of $N_{n,q}^-$ hence their matrices are related by conjugation by an element of $N_{n,q}^-$. When $(\tilde{P},W)$ is replaced by any other representative $T^c\cdot(\tilde{P},W)$ then precisely the same matrix M is obtained when using the basis for $T^c\cdot W$ which is obtained by multiplying all elements of the basis for W by T^c. Notice that such a basis is always of the form (6). $\square$

As it turns out the quotient space $\mathcal{M}_n^{p,q}/N_{n,q}^-$ is in a natural way isomorphic to an affine subspace of $\mathcal{M}_n^{p,q}$. This subspace, which will be denoted by $\tilde{\mathcal{M}}_n^{p,q}$, consists of all square matrices of size nq of the form $\sum_{i=0}^{[p/q]+1} M_i\lambda^i$ for which $M_{[p/q]+1}$ and $M_{[p/q]}$ have the special form

$$M_{[p/q]+1} = \begin{pmatrix} 0 & 0 \\ I_{q-d}\otimes I_n & 0 \end{pmatrix} \qquad M_{[p/q]} = \begin{pmatrix} 0 & I_d\otimes I_n \\ \star & \star \end{pmatrix},$$

where d is defined by $d = p \mod q$, $0 \leq d < q$, and the stars denote arbitrary matrices of the appropriate size. The dimension of $\tilde{\mathcal{M}}_n^{p,q}$ is $n^2q(p+q-2d)$.

Proposition 3.3. *The quotient space $\mathcal{M}_n^{p,q}/N_{n,q}^-$ is isomorphic to the affine space $\tilde{\mathcal{M}}_n^{p,q}$.*

Proof. We show that every element in $\mathcal{M}_n^{p,q}$ is $N_{n,q}^-$ conjugate to a unique element of $\tilde{\mathcal{M}}_n^{p,q}$. The proof then follows from the fact that the corresponding map $\mathcal{M}_n^{p,q} \to \tilde{\mathcal{M}}_n^{p,q}$ is regular. We introduce a gradation $\bigoplus_{\gamma=1-q}^{q-1} \mathfrak{g}_\gamma$ on $\mathbf{Mat}_{nq}$ as follows. For $1 \le \alpha, \beta \le q$ let $e_{\alpha\beta}$ denote the $(q \times q)$-matrix with a 1 at position (α, β) and zeros elsewhere and define $E_{\alpha\beta} = e_{\alpha\beta} \otimes I_n$. Then $\mathfrak{g}_\gamma$ is spanned by those block matrices $e_{\alpha\beta} \otimes \mathbf{Mat}_n$ for which $\beta - \alpha = \gamma$. When $Z \in \mathfrak{g}_\gamma$ we also write $\deg Z = \gamma$. The projection $\mathfrak{g} \to \mathfrak{g}_\gamma$ will be denoted by Π_γ. Let $M \in \mathcal{M}_n^{p,q}$, $M = \sum_{i=0}^{[p/q]+1} M_i \lambda^i$. We first show that $M_{[p/q]+1}$ is conjugate to

$$S = \sum_{\alpha-\beta=d} E_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ I_{q-d} \otimes I_n & 0 \end{pmatrix}$$

Notice that $M_{[p/q]+1} \in S + \bigoplus_{\gamma<-d} \mathfrak{g}_\gamma$ and that $\deg S = -d$. Take any element $\xi \in \mathfrak{g}_{-1}$ and let $g = \exp \xi \in N_{n,q}^-$. Then

$$\mathrm{Ad}_g\, M_{[p/q]+1} = \exp \mathrm{ad}_\xi\, M_{[p/q]+1} = M_{[p/q]+1} + [\xi, M_{[p/q]+1}] + \dots,$$

which after projection on $\mathfrak{g}_{-d-1}$ becomes

$$\Pi_{-d-1}\left(\mathrm{Ad}_g\, M_{[p/q]+1}\right) = \Pi_{-d-1}\left(M_{[p/q]+1}\right) + [\xi, S].$$

Now $\mathrm{ad}_S : \mathfrak{g}_{-1} \to \mathfrak{g}_{-d-1}$ is surjective because

$$\left[S, \sum_{\alpha=1}^{q-1} a_\alpha \otimes e_{\alpha+1,\alpha}\right] = \sum_{\alpha=1}^{q-d-1} (a_\alpha - a_{\alpha+d}) \otimes e_{\alpha+d+1,\alpha},$$

for any elements $a_1, \dots, a_{q-1} \in \mathbf{Mat}_n$. It follows that we can pick g (i.e. ξ) such that $\Pi_{-d-1}\left(\mathrm{Ad}_g\, M_{[p/q]+1}\right) = 0$. Repeating this procedure with $\xi \in \mathfrak{g}_{-\gamma}$ where $\gamma = 2, 3, \dots$ and using the surjectivity of $\mathrm{ad}_S : \mathfrak{g}_{-\gamma} \to \mathfrak{g}_{-d-\gamma}$ we find that $M_{[p/q]+1}$ is conjugate to S. For the rest of the proof we may assume that $M_{[p/q]+1} = S$.

Let us denote the Lie algebra of $N_{n,q}^-$ by $\mathfrak{n}_{n,q}^-$ and let $\mathfrak{g}_S$ denote the isotropy algebra of S,

$$\mathfrak{g}_S = \{X \in \mathfrak{n}_{n,q}^- \mid [X, S] = 0\}, \tag{10}$$

which is the Lie algebra of $G_S = \exp \mathfrak{g}_S$, the stabilizer of S. Notice that $\mathfrak{g}_S$ is given explicitly as the algebra of block strictly lower triangular matrices $X_{\alpha\beta}$ for which $X_{\alpha+d,\beta+d} = X_{\alpha\beta}$ for all $1 \le \beta < \alpha \le q$. We need to show that $M_{[p/q]}$ is G_S conjugate to a unique element of the form $\begin{pmatrix} 0 & I_d \otimes I_n \\ \star & \star \end{pmatrix}$. Notice that $M_{[p/q]} \in R + \bigoplus_{\gamma<q-d} \mathfrak{g}_\gamma$ where

$$R = \sum_{\beta-\alpha=q-d} E_{\alpha\beta} = \begin{pmatrix} 0 & I_d \otimes I_n \\ 0 & 0 \end{pmatrix},$$

and that $\deg R = q - d$. Fix any $\delta \in \{1, \dots, q-1\}$ and consider the space of matrices of the form

$$\left(\begin{array}{cccccccccc} \star & \dots & \star & \star_1 & 0 & \dots & 0 & I_n & & 0 \\ \vdots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ \star & \dots & \star & \dots & \star & \star_s & 0 & \dots & 0 & I_n \\ \hline \star & & & \dots & & & \dots & & & \star \\ \vdots & & & & & & & & & \vdots \\ \star & & & \dots & & & \dots & & & \star \end{array}\right) \tag{11}$$

where the diagonal with the stars $\star_1, \dots, \star_s$ and the diagonal with the I_n are precisely a (block) distance δ apart. The number s is given in terms of δ by $s = \min\{d, q-\delta\}$. It is easy to see that the adjoint action of $G_\delta = \exp(\mathfrak{g}_{-\delta} \cap \mathfrak{g}_S)$ on $\mathbf{Mat}_{nq}$ leaves the space (11) invariant. Also, at level $q-d-\delta$ (the diagonal where the $\star_i$ in (11) are) the adjoint action induces for any element $g = \exp \xi \in G_\delta$ the affine map (translation) $\mathfrak{g}_{q-d-\delta} \to \mathfrak{g}_{q-d-\delta} : Z \mapsto Z + [R, \xi]$. In turn this map induces on $\mathbf{Mat}_n^{\oplus s}$ the translation by $\Pi[R, \xi]$ where Π is the linear map $\mathbf{Mat}_{nq} \to \mathbf{Mat}_n^{\oplus s}$ which maps the matrix (11) to $(\star_1, \dots, \star_s)$. We denote this map by χ_ξ, thus $\chi_\xi(z) = z + \Pi[R, \xi]$ for $z \in \mathbf{Mat}_n^{\oplus s}$. We wish to show that there exists for any $z \in \mathbf{Mat}_n^{\oplus s}$ a unique $\xi \in \mathfrak{g}_{-\delta} \cap \mathfrak{g}_S$ such that $\chi_\xi(z) = 0$. Equivalently, that for any $z \in \mathbf{Mat}_n^{\oplus s}$ the affine map $\chi_z : \mathfrak{g}_{-\delta} \cap \mathfrak{g}_S \to \mathbf{Mat}_n^{\oplus s} : \xi \mapsto \chi_\xi(z)$ is a bijection. Taking the corresponding linear map this means that we need to show that

$$\chi : \mathfrak{g}_{-\delta} \cap \mathfrak{g}_S \to \mathbf{Mat}_n^{\oplus s} : \xi \mapsto \Pi[R, \xi]$$

is an isomorphism. Since both spaces have dimension $sn^2 = \min\{d, q-\delta\}n^2$ it suffices to show that χ is injective. Let $\xi = \sum_{\gamma=1}^{s} e_{\gamma+\delta,\gamma} \otimes \xi_\gamma$ be in the kernel of χ. If $\delta \geq q-d$ then $s = q - \delta > d - \delta$ and

$$\chi(\xi) = (0, 0, \dots, 0, \xi_1, \dots, \xi_{d-\delta}) - (\xi_1, \dots, \xi_s)$$

so that χ is injective. When $\delta < q - d$ the proof is more delicate and depends on the fact that p and q are relatively prime (indeed if p and q have a common divisor then χ is not injective for some values of δ). Then

$$\chi(\xi) = (0, \dots, 0, \xi_1, \xi_2, \dots, \xi_{d-\delta}) - (\xi_{q-d-\delta+1}, \dots, \xi_{q-\delta}), \tag{12}$$

where ξ_1 appears at position $\delta+1$, the length of these vectors being $s = d$. If ξ is in the kernel of χ then $\xi_{q-d-\delta+1} = \dots = \xi_{q-d} = 0$ and $\xi_k = \xi_{q-d+k}$ for $k = 1, \dots, d-\delta$. But remember that $\xi_k = \xi_{k+d}$ because $\xi \in \mathfrak{g}_S$. This means that the indices of ξ may be thought of as lying in $\mathbb{Z}_d$. Now the fact that p and q are relatively prime implies that d and q are coprime, hence also $q-d$ and q. The fact that $\xi_k = \xi_{q-d+k}$ for $k = 1, \dots, d-\delta$ then implies that all ξ_γ are equal, hence they are all equal to 0. Thus χ is injective in all cases and we can make all $\star_j$ in (11) equal to zero by using a unique element of G_i; doing this consecutively for $\delta = 1, 2, \dots, q-1$ leads to the desired result. $\square$

We say that $M \in \tilde{\mathcal{M}}_n^{p,q}$ is the *big matrix* of (P, Q) or of $(\tilde{P}, W)$.

4. The inverse construction

We will now show that every element $M \in \tilde{\mathcal{M}}_n^{p,q}$ (with p and q coprime) is the big matrix of a pair (P, Q) of commuting matrix differential operators such that P is monic of degree p and Q is normalized of degree q. By Proposition 3.1 it suffices to show that M is the big matrix of a pair $(\tilde{P}, W) \in \Psi_n^c \times \mathrm{Gr}_n$ satisfying $\partial^q \cdot W \subset W$, $\tilde{P} \cdot W \subset W$ and $\tilde{P}$ monic. Equivalently we need to show that there exist a monic element $\tilde{P} \in \Psi_n^c$ of degree p and vectors f_v^u of the form (6) such that

$$\tilde{P} \cdot f_v^u = \sum_{t=1}^{n} \sum_{s=0}^{q-1} m_{nu+v,ns+t} f_t^s. \tag{13}$$

In order to do this we expand $\tilde{P}$, f_v^u and M in terms of ∂,

$$\begin{aligned}
&\tilde{P} = p_0\partial^p + p_1\partial^{p-1} + p_2\partial^{p-2} + \cdots, && p_i \in \mathbf{Mat}_n,\ p_0 = I_n,\\
&f_v^u = g_{v,0}^u\partial^u + g_{v,1}^u\partial^{u-1} + \cdots, && g_{v,i}^u \in \mathbb{F}^n,\ (g_{1,0}^u, \ldots, g_{n,0}^u) = I_n\\
&M_{\alpha\beta} = \textstyle\sum_r M_{\alpha\beta}^{(r)}\partial^{p+\alpha-\beta-r}, &&\\
&M^{(r)} = \left(M_{\alpha\beta}^{(r)}\right)_{1\le\alpha,\beta\le q}. &&
\end{aligned}$$

In order to expand the individual entries $m_{ij}\,(1 \le i, j \le nq)$ of M in terms of ∂ it is essential to use the above expansion of the blocks $M_{\alpha\beta}\,(1 \le \alpha, \beta \le q)$ of M as follows. If m_{ij} is located in the block $M_{\alpha\beta}$ (i.e., $\alpha = \left[\frac{i-1}{n}\right] + 1$) and $\beta = \left[\frac{j-1}{n}\right] + 1$ then

$$m_{ij} = \sum_r m_{ij}^{(r)}\partial^{p+\alpha-\beta-r}.$$

Notice that the matrices $M^{(r)}$ admit the following alternative definition: $M^{(r)} = \left(m_{ij}^{(r)}\right)_{1\le i,j\le nq}$.

Lemma 4.1.

1. $M^{(r)} = 0$ *for* $r < 0$*;*
2. $M_{\alpha\beta}^{(0)} = \begin{cases} I_n & \text{if } p+\alpha-\beta = 0 \mod q,\\ 0 & \text{if } p+\alpha-\beta \ne 0 \mod q;\end{cases}$
3. $M^{(0)} = \sigma \otimes I_n$ *where* $\sigma \in \mathbf{Mat}_q$ *is the matrix of a cyclic permutation of order* q*.*

Proof. If $r < 0$ then

$$p + \alpha - \beta - r > q\left[\frac{p+\alpha-\beta}{q}\right] \ge \mathrm{ord}(M_{\alpha\beta})$$

and so $M_{\alpha\beta}^{(r)}$, which is the coefficient of $\partial^{p+\alpha-\beta-r}$, is zero. The same inequality holds when q does not divide $p+\alpha-\beta$ so that again $M_{\alpha\beta}^{(0)} = 0$. If q divides $p+\alpha-\beta$ then $M_{\alpha\beta}$ is monic of order $p+\alpha-\beta$ hence $M_{\alpha\beta}^{(0)} = I_n$. It implies that $M^{(0)}$ can be written as $\sigma \otimes I_n$ for some permutation matrix. The fact that σ corresponds to a cyclic permutation of order q follows from the fact that p and q are coprime; indeed, this permutation corresponds to the translation over p in $\mathbb{Z}_q$. $\square$

If we plug (4) into (13) then we find that for any $\gamma \geq 0$

$$\sum_{r=0}^{\gamma} p_r g^u_{v,\gamma-r} = \sum_{r=0}^{\gamma}\sum_{t=1}^{n}\sum_{s=0}^{q-1} m^{(r)}_{nu+v,ns+t} g^s_{t,\gamma-r}. \tag{14}$$

In order to simplify the notation we introduce for $1 \leq u \leq n$ and $w \in \mathbb{N}$ the column vector (of size n^2)

$$h^u_w = (g^u_{1,w}, \dots, g^u_{n,w})^T.$$

The subscript w in h^u_w, which ranges over $\mathbb{N}$ is not to be confused with the subscript v in f^u_v which ranges over $\{0, \dots, q-1\}$. Notice that

$$h^0_0 = h^1_0 = \cdots = h^{q-1}_0 = ((1,0,\dots,0),(0,1,\dots,0),\dots,(0,0,\dots,1))^T \tag{15}$$

in view of the restriction on $g^u_{v,0}$ given in (4). In terms of the vectors h^u_w the equation (14) becomes

(16)

$$\sum_{r=0}^{\gamma} \begin{pmatrix} p_r \otimes I_n & 0 & \dots & 0 \\ 0 & p_r \otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_r \otimes I_n \end{pmatrix} \cdot \begin{pmatrix} h^0_{\gamma-r} \\ h^1_{\gamma-r} \\ \vdots \\ h^{q-1}_{\gamma-r} \end{pmatrix} = \sum_{r=0}^{\gamma} M^{(r)} \begin{pmatrix} h^0_{\gamma-r} \\ h^1_{\gamma-r} \\ \vdots \\ h^{q-1}_{\gamma-r} \end{pmatrix}.$$

We show that the above equation can be solved recursively for p_i and h^u_w. Since we have picked $p_0 = I_n$ and since all h^u_0 are equal, as given by (15), the equation (16) is satisfied identically for $\gamma = 0$. Let us assume that we have constructed p_r and h^u_r for all $r < \gamma$ and $u \in \{0, \dots, q-1\}$. Then equation (16) can be written as

$$\begin{aligned} &\begin{pmatrix} p_\gamma \otimes I_n & 0 & \dots & 0 \\ 0 & p_\gamma \otimes I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_\gamma \otimes I_n \end{pmatrix} \cdot \begin{pmatrix} h^0_0 \\ h^1_0 \\ \vdots \\ h^{q-1}_0 \end{pmatrix} \\ &\qquad\qquad = (M^{(0)} - I_q \otimes I_n) \begin{pmatrix} h^0_\gamma \\ h^1_\gamma \\ \vdots \\ h^{q-1}_\gamma \end{pmatrix} + (\text{known stuff}). \end{aligned}$$

If we sum up the q rows of this block matrix equation then we find the following matrix equation,

$$q(p_\gamma \otimes I_n) h^0_0 = \text{known stuff}; \tag{17}$$

we used (15) and we used the fact that $M^{(0)}$ is a permutation matrix of the form $\sigma \otimes I_n$, where $\sigma \in \mathbf{Mat}_q$. Since h^0_0 has the special form (15), equation (17) defines p_γ. We take $h^0_\gamma = h^0_0$ and use (16) to solve for $h^1_\gamma, \dots, h^{q-1}_\gamma$. To see how this is done, rewrite (16) as follows,

$$(\sigma - I_q) \otimes I_n \begin{pmatrix} h^0_\gamma \\ h^1_\gamma \\ \vdots \\ h^{q-1}_\gamma \end{pmatrix} = (\text{known stuff}).$$

Now σ is a permutation matrix which corresponds to a permutation of the set $\{0,1,\dots,q-1\}$; this permutation will also be denoted by σ. Since σ is a cyclic permutation of order q we can solve for the h^i_γ in the following order: solve first for $i=\sigma(0)$, then for $i=\sigma^2(0)$ and so on. Notice that the last equation is precisely the equation (17) showing that the solution exists and is unique once the vector h^0_γ has been chosen. Clearly the freedom in choice for h^0_γ corresponds to the left action of Volt^c_n. Notice that it is only in the very last step that we used that p and q are coprime.

Summarizing we have shown the following proposition.

Proposition 4.2. *Let $M \in \tilde{\mathcal{M}}^{p,q}_n$, where p and q are coprime. Then there exists a pair $(\tilde{P},W) \in \Psi^c_n \times \mathrm{Gr}_n$ such that $\tilde{P}\cdot W \subset W$ and $\partial^q \cdot W \subset W$ and such that M is the big matrix of $\tilde{P}$. The pair $(\tilde{P},W)$ is unique up to the left action of Volt^c_n. The correspondence which associates to the pair (P,Q) its big matrix is a bijection between $\tilde{\mathcal{M}}^{p,q}_n$ and the space of all pairs (P,Q) of commuting $(n\times n)$–differential operators with P monic of degree p and Q normalized of degree q.*

Amplification 4.3. Variants to the equation $[P,Q]=0$ have been considered in the literature. For example $[P,Q]=1$ is known as the string equation. Also the more general equation $[P,Q]=f(Q)$, with f being any polynomial, is of interest because of its close relation to the double bracket equation, $[[P,Q],Q]=0$. Our constructions generalize to this more general case as follows. Let f be given, $f(t)=\sum_{i=0}^N s_i t^i$, $(s_i \in \mathbb{F})$ and look for all pairs (P,Q) of matrix differential operators, with P monic, Q normalized and $[P,Q]=f(Q)$. Writing $Q=T^{-1}\partial^q T$ and defining $\tilde{P}=TPT^{-1}$ we find that

$$[\tilde{P},\partial^q]=\sum_{i=0}^N s_i\partial^{iq},$$

whose general solution is given by

$$\tilde{P}=\sum_{i\le p} c_i\partial^i-\sum_{i=0}^N \frac{s_i x}{q}\partial^{q(i-1)+1},$$

so we see that the solution space is non-empty if and only if $p>q(N-1)+1$ (because $\tilde{P}$ has to be monic). If this is the case then $\tilde{P}$ stabilizes the point $W=W_T=T\cdot H_n$ in Gr_n and its matrix, with respect to any periodic basis f^j_i of the type considered before, is precisely of the form (9). For the inverse construction, note that the non-constant terms in $\tilde{P}$ only play a role in the known part of the equations. It follows that for given p and q the spaces obtained for f of low enough degree are all the same, while for f of higher degree, are empty.

5. The KP vector fields

In this section we will realize the KP vector fields, which are a natural collection of commuting vector fields on the Sato Grassmannian $\mathrm{Gr}=\mathrm{Gr}_1$, as a collection of commuting vector fields on the affine space $\tilde{M}^{p,q}=\tilde{M}^{p,q}_1$. We restrict ourselves here to the scalar case because our computation does not lead to explicit Lax equations in the matrix case.

In order to write down the KP vector fields on the Grassmannian, let us first show that the tangent space at a point W of the *finite-dimensional* Grassmannian

$G = G(k,n)$ of k planes in $\mathbb{F}^n$ is naturally given by $\mathrm{Hom}(W, \mathbb{F}^n/W)$. To see this, we consider G as the homogeneous space $GL_n/\operatorname{Stab}(W)$, where $GL_n = GL(n, \mathbb{F})$ and $\operatorname{Stab}(W) \subset GL_n$ is the stabilizer of W, with Lie algebra

$$\mathfrak{stab}(W) = \{\phi \in \mathfrak{gl}_n \mid \phi(W) \subset W\}.$$

Then $T_W G = T_W(GL_n/\operatorname{Stab}(W)) = \mathfrak{gl}_n/\mathfrak{stab}(W) = \mathrm{Hom}(W, \mathbb{F}^n/W)$; for the last equality one associates to a representative ϕ of an element of $\mathfrak{gl}_n/\mathfrak{stab}(W)$ the composite map

$$W \hookrightarrow \mathbb{F}^n \xrightarrow{\phi} \mathbb{F}^n \to V/W.$$

In the case of the Sato Grassmannian (which is infinite-dimensional) we *define* the tangent space at a point $W \in \mathrm{Gr}$ to be given by $\mathrm{Hom}(W, \Psi^c/W)$, where we consider W in the last equation as a subspace of Ψ^c. In this language the i-th KP vector field is given by $V_i : W \to \Psi^c/W : w \mapsto \partial^i w \mod W$.

We first transfer these vector fields to the space of pairs (P, Q) of commuting scalar differential operators with P monic of degree p and Q normalized of degree q. To do this we use the bijection $\mathrm{Volt} = \mathrm{Volt}_1 \to \mathrm{Gr} : T \mapsto W_T$, which gives the following vector fields on Volt

$$\frac{dT}{dt_i} = T(T^{-1}\partial^i T)_-,$$

(see [5]). If we have a constant coefficient pseudo-differential operator $\tilde{U}$ and we define $U = U(t) = T^{-1}\tilde{U}T$ then

$$\frac{dU}{dt_i} = [T^{-1}\tilde{U}T, (T^{-1}\partial^i T)_-] = [U, Q_-^{i/q}] = [Q_+^{i/q}, U].$$

Applying this for $\tilde{U}$ given by ∂^q and by $\tilde{P}$ we find the following Lax representation for the KP vector fields on the above space of commuting operators (P, Q),

$$\frac{dQ}{dt_i} = [Q_+^{i/q}, Q], \qquad \frac{dP}{dt_i} = [Q_+^{i/q}, P].$$

We proceed to write these vector fields down on $\mathcal{M}^{p,q}$. More precisely, we will write down the vector fields that correspond to the constant coefficient scalar differential operators $\left[\tilde{P}^j/\lambda^k\right]_+$; the i-th KP vector field is then a linear combination of these vector fields. We fix j, k and denote the derivative in the direction of the vector field corresponding to $\left[\tilde{P}^j/\lambda^k\right]_+$ by a dot. We choose a periodic basis E for W and denote the column vector containing its first q elements by $\vec{e}$. By the above interpretation of the tangent space at W to the Grassmannian, we can write

$$\dot{\vec{e}} = \left[\tilde{P}^j/\lambda^k\right]_+ \vec{e} - A(\partial^q)\vec{e}, \tag{18}$$

where A is the polynomial matrix (in $\lambda = \partial^q$) such that the order of the i-th component of the right hand side of (18) is smaller than i. Also equation (5), which is the defining equation of the matrix $M \in \mathcal{M}^{p,q}$ with respect to the periodic basis E, can be rewritten as

$$\tilde{P}\vec{e} = M\vec{e}. \tag{19}$$

If we differentiate (19) then we find $\tilde{P}\dot{\vec{e}} = \dot{M}\vec{e} + M\dot{\vec{e}}$, which is easily rewritten as

$$\dot{M}\vec{e} = (MA - \tilde{P}A)\vec{e} = [M, A]\vec{e},$$

(one uses that elements of $\mathcal{D}^c$ commute among themselves and with matrices which are independent of x, such as M and A). From the last equality we can conclude that $\dot{M} = [M, A]$, because $\dot{M}$, M and A are polynomials in $\lambda = \partial^q$ (rather than in ∂). We claim that A can be taken as $(M^j/\lambda^k)_+$. To prove this we must show that the i-th component of

$$\left(\tilde{P}^j/\lambda^k\right)_+ \vec{e} - \left(M^j/\lambda^k\right)_+ \vec{e} \tag{20}$$

has order smaller than i (for $i = 1, \dots, q$). Since $\tilde{P}$ commutes with M we have that

$$\left(\tilde{P}^j/\lambda^k\right) \vec{e} = \left(M^j/\lambda^k\right) \vec{e} \tag{21}$$

and it suffices to show that the order of the i-th component of

$$\left(\tilde{P}^j/\lambda^k\right)_- \vec{e} - \left(M^j/\lambda^k\right)_- \vec{e} \tag{22}$$

is smaller than i. Since the i-th component of $\vec{e}$ has order i this is clearly the case for the first term; in the second term every component has negative order because M depends on $\lambda = \partial^q$ only. Thus we have shown that the KP vector fields lead to the following vector fields on $\mathcal{M}^{p,q}$:

$$\frac{dM}{dt_{ij}} = [M, [M^i/\lambda^j]_+].$$

Given $M \in \mathcal{M}^{p,q}$ we have shown in Section 3 that there exists a unique $g \in N^-_{1,q}$ such that $Y = gMg^{-1} \in \tilde{M}^{p,q}$. Differentiating $Y = gMg^{-1}$ we find for any i, j the following vector fields on $\tilde{M}^{p,q}$:

$$\frac{dY}{dt_{ij}} = [Y, [Y^i/\lambda^j]_+ - \dot{g}g^{-1}].$$

It was shown in [7] that each of these vectors fields is multi-Hamiltonian and that they are related by a symmetry vector field. It would be interesting to figure out whether the generalizations to arbitrary simple Lie algebras, which were also described in [7], can be obtained in an analogous way from a generalization of the KP hierarchy.

References

1. J.L. Burchnall and T.W. Chaundy, *Commutative ordinary differential operators*, Proc. Roy. Soc. London Ser. A **118** (1928), 557–583.
2. M. Kimura, *Matrix differential operators and Prym varieties*, preprint (1997).
3. I.M. Krichever, *Algebro-geometric construction of the Zaharov-Shabat equations and their periodic solutions*, Doklady Akad. Nauk SSSR, 1976.
4. Y. Li and M. Mulase, *Prym varieties and integrable systems*, Communications in Analysis and Geometry, **5 (2)**, (1997), 279–332.
5. M. Mulase, *Algebraic theory of the KP equations*, Perspectives in Mathematical Physics, ed. R. Penner and S.T. Yau 1994, pp. 151–217.
6. D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations*, Proceedings of the international symposium on algebraic geometry, Kyoto 1977, Kinokuniya Publ. (1978), pp. 115–153.
7. M. Pedroni and P. Vanhaecke, *A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure*, Regular and Chaotic dynamics **3** (1987), 132–160.
8. E. Previato, *Generalized Weierstrass $\wp$-functions and KP flows in affine space*, Comment. Math. Helvetici **62** (1987), 292–310.

9. A. Pressley and G.B. Segal, *Loop groups*, Oxford, 1986.
10. A. Schwarz, *On solutions to the string equation*, Modern Physics Letters A **6 (29)** (1991), 2731–2725.
11. I. Schur, *Über vertauschbare lineare Differentialausdrücke*, Sitzungsber. der Berliner Math. Gesel. **4** (1905), 2–8.
12. P. Vanhaecke, *Integrable systems in the realm of algebraic geometry*, Lecture Notes in Mathematics, vol. 1638, Springer Verlag, 1996.
13. P. van Moerbeke, *Integrable foundations of string theory*, Lectures on Integrable systems (Sophia-Antipolis, 1991) pp. 163–267, Word Sci. Publishing, NJ, 1994.

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