INTEGRABLE SYSTEMS AND PROJECTIVE IMAGES OF KUMMER SURFACES

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Dedicated to our teachers Mark Adler and Pierre van Moerbeke

ABSTRACT. The (-1)-involution on the Jacobian $\mathcal{J}_\Gamma$ of an arbitrary Riemann surface Γ of genus two leads to a singular surface, the Kummer surface $\mathcal{K}_\Gamma$ of $\mathcal{J}_\Gamma$, which, after desingularization, defines a K-3 surface $\tilde{\mathcal{K}}_\Gamma$. We consider ample line bundles on $\tilde{\mathcal{K}}_\Gamma$ coming from the even or odd sections of $[n\Theta]$ with prescribed vanishing at the 2-division points of $\mathcal{J}_\Gamma$ (Θ is the theta divisor of $\mathcal{J}_\Gamma$). We use an integrable system to show that in the cases we study the linear system is base-point-free, to determine which curves are contracted to singular points and to compute an explicit equation for the surface in projective space. Our explicit methods apply to the Kummer surface of any Abelian surface, given as the fiber of the moment map of an algebraic completely integrable system.

CONTENTS

1. INTRODUCTION

When studying quartic surfaces in three-space with sixteen nodes, Kummer discovered a very beautiful geometry, relating such a surface on the one hand to the Jacobian of a hyperelliptic curve (of genus two) and on the other hand to the singular surface of a quadratic complex (for a modern account of this, see [10, Chapter 6]). These singular surfaces, which form a three-dimensional family, are called (singular) Kummer surfaces. They reappeared recently in the compactification of the

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moduli space of stable rank two bundles (of fixed determinant) on a Riemann surface (see [20]) and as the singular locus of a natural Poisson structure on a moduli space of flat $SU(2)$ connections on a Riemann surface (see [13]).

The easiest way to obtain *abstractly* the Kummer surface $\mathcal{K}_\Gamma$ which is associated to a compact Riemann surface Γ of genus two is as the singular quotient $\mathcal{J}_\Gamma/(-1)$ of the Jacobian $\mathcal{J}_\Gamma$ of Γ by the (-1)-involution $x \mapsto -x$ (recall that $\mathcal{J}_\Gamma$ is a two-dimensional complex torus). As such the Kummer surface has an obvious generalization to other Jacobians (i.e., to Riemann surfaces of higher genus) and to other complex algebraic tori (Abelian varieties) (see [16, Section 4.8]). To obtain the Kummer surface *concretely*, i.e., as an algebraic surface in projective space, one considers the image of the regular map

$$\phi_{[2\Theta]} : \mathcal{J}_\Gamma \to \mathbb{P}H^0(\mathcal{J}_\Gamma, [2\Theta])^*$$

yielding a quartic surface in $\mathbb{P}^3$; the divisor Θ which appears in this map is the divisor of Riemann's theta function, and the $2 : 1$ map $\phi_{[2\Theta]}$ assigns to any point $P \in \mathcal{J}_\Gamma$ the hyperplane of sections of the line bundle $[2\Theta]$ that vanish at P. For higher dimensional Jacobians the image of $\phi_{[2\Theta]}$ also provides a projective image of its Kummer varieties (see [16, Section 4.8]), but for other Abelian varieties, even for Abelian surfaces, the situation is more complicated (see [6]). Getting *explicit* equations for Kummer surfaces is still a different matter and relies in all situations that have been considered on arguments that depend heavily on the specific geometry of the Kummer surface at hand (for higher dimensional Kummer varieties no such equations are known at present). One classical computation of the equation of the Kummer surface $\mathcal{K}_\Gamma$ as a surface in $\mathbb{P}^3$ for example relies on the symmetries of the level two Heisenberg group (a central extension of the group of half periods (2-division points) of $\mathcal{J}_\Gamma$) (see [12, Chapter 8]); it is not clear how to adapt this approach to other Kummer surfaces. The other classical computation relies on the above mentioned fact that the minimal resolution of $\mathcal{K}_\Gamma$ is the singular surface of the quadratic complex (see [14, Sect. 82]) and is thus even more dependent on the specifics of the geometric situation.

The purpose of this paper is to show how equations for projective images of Kummer surfaces can be obtained in a systematic way. Although our techniques are valid for other Abelian varieties, we will restrict ourselves here to Kummer surfaces of two-dimensional Jacobians, but we will consider besides the classical Kummer surface in $\mathbb{P}^3$ also other, less singular, projective models in $\mathbb{P}^3$, $\mathbb{P}^4$, and $\mathbb{P}^5$. *Abstractly,* these Kummer surfaces are obtained by desingularizing $\mathcal{K}_\Gamma$ at some but not all of its singular points: note that on any Abelian surface the (-1)-involution has sixteen fixed points, hence the quotient $\mathcal{K}_\Gamma$ has sixteen singular points. The desingularization of $\mathcal{K}_\Gamma$ is a K-3 surface which is denoted by $\tilde{\mathcal{K}}_\Gamma$, and the partial desingularizations are called intermediate Kummer surfaces. *Concretely,* as algebraic surfaces in projective space, the K-3 surface and the intermediate Kummer surfaces are obtained by constructing line bundles on the (abstract) K-3 surface $\tilde{\mathcal{K}}_\Gamma$. We construct such line bundles as follows. Let

$$p : \tilde{\mathcal{J}}_\Gamma \to \mathcal{J}_\Gamma$$

be the blow-up of $\mathcal{J}_\Gamma$ at its sixteen half periods. The (-1)-involution on $\mathcal{J}_\Gamma$ induces an involution on $\tilde{\mathcal{J}}_\Gamma$ which leads to a non-singular quotient $\pi : \tilde{\mathcal{J}}_\Gamma \to \tilde{\mathcal{K}}_\Gamma$. We pick a symmetric line bundle $\mathcal{L}$ on $\mathcal{J}_\Gamma$ and denote the line bundle $p^*\mathcal{L}$ on $\tilde{\mathcal{J}}_\Gamma$ by $\tilde{\mathcal{L}}$. For any $\nu = (\nu_i)_{i=1,\ldots,16}$ we consider the space $|\tilde{\mathcal{L}}|_\nu^+$ (resp. $|\tilde{\mathcal{L}}|_\nu^-$) of even (resp. odd) sections

of $\tilde{\mathcal{L}}$ which vanish at least ν_i times at the exceptional divisor E_i which lies over the half period e_i. These linear systems descend to complete linear systems $|\mathcal{M}_\nu^+|$ (resp. $|\mathcal{M}_\nu^-|$) on $\tilde{\mathcal{K}}_\Gamma$. Using standard algebraic geometric arguments we will determine the dimension of such linear systems (Proposition 3.2), i.e., the dimension of the target space of the map

$$\phi_{\mathcal{M}_\nu^\pm} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}H^0(\tilde{\mathcal{K}}_\Gamma, \mathcal{M}_\nu^\pm).$$

The main focus of the paper is then on studying the map $\phi_{\mathcal{M}_\nu^\pm}$ and on obtaining *explicit* equations for the image of this map. We do this by using an algebraic completely integrable system (a.c.i. system) whose fibers of the (complex) moment map are affine parts of genus two Jacobians. Our methods do not depend on the particular a.c.i. system that we use and can hence be used to compute explicit equations for other Kummer varieties, as long as the corresponding Abelian varieties appear as the fiber of the moment map of some a.c.i. system. Let us explain shortly the role of this deus ex machina (for more information, see [3] or [28]). It was observed by Kowalewski that an a.c.i. system on an n-dimensional space M must have one or several families of Laurent solutions depending on $n - 1$ free parameters. A careful analysis shows that each such family $\mathcal{F}_i$ corresponds to an irreducible component $\mathcal{D}_i$ of the divisor $\mathcal{D}$ to be adjoined to a generic fiber $\mu^{-1}(c)$ of the moment map

$$\mu : M \to \operatorname{Spec} A$$

(A is the algebra of first integrals of the a.c.i. system; D depends on c) in order to complete it into an Abelian variety. Moreover, for any function f on M the restriction $f_{|\mu^{-1}(c)}$ of f to this fiber has a pole along $\mathcal{D}_i$ which equals the pole of the Laurent series of f, as computed from the family $\mathcal{F}_i$. Since (the first few terms of) the Laurent solutions of an a.c.i. system can be effectively computed, we have an effective way to compute a basis for the meromorphic functions having prescribed poles at a given divisor and hence an effective way to compute explicitly the sections of any of the line bundles $\mathcal{L} = [\sum n_i \mathcal{D}_i]$. Since the (-1)-involution reverses the signs of all the integrable vector fields of the a.c.i. system the splitting in even and odd sections can also be determined explicitly. Finally, having these sections at hand one expresses easily the condition that a section has a prescribed vanishing at some of the half periods. Summarizing, starting from an a.c.i. system which has a given Jacobian $\mathcal{J}_\Gamma$ (or, more generally, an Abelian variety) as one of its fibers, we can find an explicit basis for $H^0(\tilde{\mathcal{K}}_\Gamma, \mathcal{M}_\nu^\pm)$ and hence also explicit formulas for the (non-linear) relations which hold between those sections, i.e., for the equations that define the projective image of $\tilde{\mathcal{K}}_\Gamma$.

The integrable system comes in handy for many other things. We use it for example to determine the base locus of the linear system under consideration: in the cases of interest to us, this base locus will be shown to be empty, showing that our maps $\phi_{\mathcal{M}_\nu^\pm}$ are regular maps. Moreover we can use it to determine which divisors are contracted: in our case the only possible contractions will be divisors on $\tilde{\mathcal{K}}_\Gamma$ which correspond to translates of the theta divisor (theta curves) or to the exceptional divisors E_i. Our arguments have the advantage that they consist of an algebraic computation only, in contrast with the more geometric arguments, which are specific to the particular class of Abelian surfaces and to the linear system under consideration.

Finally, using the explicit sections we can compute the coordinates of the singular points of the image, which allows us to rewrite the equation(s) of the embedded intermediate Kummer surface in a very symmetric form. In those cases in which no vanishing at the half periods is prescribed we will also provide an equation whose coefficients are explicitly expressed in terms of the coefficients of the curve, defining the Riemann surface Γ; from the point of view of number theory such equations are more useful than equations that depend on the coordinates of the Weierstrass points of the surface. As far as we know such equations for Kummer surfaces do not appear in the classical or modern literature. When rewritten in a more symmetric form, depending on the coordinates of the Weierstrass points, we recover in some cases known equations and otherwise new equations for projective images of $\tilde{\mathcal{K}}_\Gamma$. In the following table we summarize some geometric information about the projective images that we consider.

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Table 1

$\mathcal{L}$	parity	ν	$\mathbb{P}^N$	sing. points	eq. 1	eq. 2
$[2\Theta]$	even	0	$\mathbb{P}^3$	16+0=16	(22)	(25)
$[3\Theta]$	even	0	$\mathbb{P}^3$	6+0=6	(27)	(29)
$[3\Theta]$	odd	0	$\mathbb{P}^4$	10+1=11	(30)	(31)
$[3\Theta]$	odd	(2,0, ... ,0)	$\mathbb{P}^3$	9+1=10	—	(34)
$[4\Theta]$	odd	0	$\mathbb{P}^5$	0+0=0	(35)	(36)
$[4\Theta]$	odd	(1, ... ,1,3)	$\mathbb{P}^3$	0+6=6	(27)	(29)
$[4\Theta]$	even	(2,2,2,0, ... ,0)	$\mathbb{P}^5$	12+0=12	—	(37)

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The meaning of the first three columns is clear. In the fourth column, $\mathbb{P}^N = \mathbb{P}H^0(\tilde{\mathcal{K}}_\Gamma, \mathcal{M}_\nu^\pm)^*$. The first number appearing in the sum in column five is the number of exceptional curves that get contracted while the second number is the number of theta curves that get contracted. The sum in column five is the total number of irreducible divisors that get contracted. In the last two columns we give a reference to the equations for the image of the Kummer surface in $\mathbb{P}^N$, the first equation being the one that does not involve the coordinates of the Weierstrass points explicitly, while the second equation is more symmetric but does depend on the coordinates of the Weierstrass points. Equations (25) and (36) appear already in [14] but all other equations are new. Using a related integrable system the second author has, in collaboration with José Bertin, obtained equations for a one-dimensional family of generalized Kummer surfaces in $\mathbb{P}^4$ (see [9]).

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2. ABELIAN AND K-3 SURFACES

In this section we consider some basic facts about complex Abelian surfaces and K-3 surfaces. These surfaces are nonsingular and their canonical bundles are trivial.

For any surface X we will write $\mathcal{O}_X$ for its structure sheaf and K_X for its canonical divisor. When X is non-singular then the line bundle $\mathcal{L}$ (invertible sheaf) which corresponds to a divisor D will be denoted by $[D]$ and the dimension of the i-th cohomology group $H^i(X, \mathcal{L})$ is written as $h^i(\mathcal{L})$ or $h^i(D)$. When D is effective we denote its complete linear system $\mathbb{P}H^0(X, [D])$ by $|D|$; for $\mathcal{L} = [D]$ we also write $|\mathcal{L}|$ for $|D|$. An effective reduced divisor on X will be called a *curve* on X. Linear equivalence of divisors is denoted by $\sim$.

For an Abelian or K-3 surface X the birational invariants are summarized in the following table.

Table 2

invariant	notation	definition	K-3	Abelian
irregularity	$q(X)$	$h^1(\mathcal{O}_X)$	0	2
arith. genus	$p_a(X)$	$\chi(\mathcal{O}_X) - 1$	1	-1
geom. genus	$p_g(X)$	$h^2(\mathcal{O}_X)$	1	1

We will use line bundles on Abelian and K-3 surfaces to construct images of Kummer surfaces and K-3 surfaces in projective space. Recall that to a line bundle $\mathcal{L} = [D]$ there is associated a holomorphic map

$$\phi_{\mathcal{L}} : X \setminus B(\mathcal{L}) \to \mathbb{P}H^0(X, \mathcal{L})^*$$

which assigns to any point P (which is not in the base locus $B(\mathcal{L})$ of $\mathcal{L}$) the space of sections of $\mathcal{L}$ that vanish at P. We call $\mathcal{L}$ (and D) *very ample* when ϕ is an embedding and $B(\mathcal{L}) = \emptyset$. If some positive power of $\mathcal{L}$ (multiple of D) provides an embedding then we call $\mathcal{L}$ (or D) *ample*. Explicitly, if $(s_0, \ldots, s_N)$ denotes a basis of $H^0(X, \mathcal{L})$ then $\phi_{\mathcal{L}}$ is given for $P \in X \setminus B(\mathcal{L})$ by

$$\phi_{\mathcal{L}}(P) = (s_0(P) : s_1(P) : \cdots : s_N(P)).$$

Let us assume that $B(\mathcal{L}) = \emptyset$ and that the image of $\phi_{\mathcal{L}}$ is a surface. Then, by Bertini's first theorem (see [26, p. 21]), the general member of $|\mathcal{L}|$ is irreducible and smooth. If $\phi_{\mathcal{L}}$ contracts a curve C ($\phi_{\mathcal{L}}(C)$ is a point p), then $\mathcal{L} \cdot C = 0$ since we can choose a curve $D' \in |\mathcal{L}| = |D|$ such that $\phi_{\mathcal{L}}(D')$ avoids the point p. By Bertini's second theorem ([26, p. 24]) such a curve is smooth and it is clear that D' does not intersect C. However, if C is not contracted then $D \cdot C$ is the degree of $\phi_{\mathcal{L}}(C)$ in $\mathbb{P}H^0(X, \mathcal{L})^*$, multiplied by the degree of $\phi_{\mathcal{L}}$.

The adjunction formula for nonsingular curves on a surface implies that the (virtual) genus of a curve C on an Abelian or K-3 surface is given by $g(C) = C^2/2 + 1$. On the other hand, the Riemann-Roch formula

$$\chi(D) = \frac{1}{2}D \cdot (D - K_X) + 1 + p_a(X)$$

simplifies for a curve C on an Abelian or K-3 surface to

$$h^0(C) = \frac{1}{2}C^2 + 1 + p_a(X) + h^1(C)$$

because $K_X = 0$ and the Euler characteristic of $[C]$ is given by $\chi(C) = h^0(C) - h^1(C) + h^0(K_X - C) = h^0(C) - h^1(C)$. In classical terminology $h^1(C)$ is called

the *superabundance* of C and is computed by using a theorem by Kodaira (see [15, Theorems 2.2 and 2.3]).

Theorem 2.1. *Let m be the number of connected components of a curve C on a surface X. Then $h^1(K+C) = m - 1 + k$, where the integer k denotes the dimension of the kernel of the homomorphism*

$$H^1(X, \mathcal{O}_X) \to H^1(C, \mathcal{O}_C).$$

In the case in which X is K-3 we have from Table 2 that $q(X) = h^1(\mathcal{O}_X) = 0$ so that $k = 0$ and $h^1(C) = m - 1$, leading to the final formula

$$(1) \qquad h^0(C) = \frac{1}{2}C^2 + m + 1 = g(C) + 1.$$

In this case conditions for an ample line bundle to lead to a birational map were given by Saint-Donat (see [25, Theorem 5.2]).

Theorem 2.2. *Let $\mathcal{L}$ be a line bundle on a K-3 surface X such that $\mathcal{L}^2 \geq 4$. If the linear system $|\mathcal{L}| = \mathbb{P}H^0(X, \mathcal{L})$ has no fixed components then $\mathcal{L} = [C]$ for an irreducible curve C of genus $g(C) = \mathcal{L}^2/2 + 1$ and the map*

$$\phi_{\mathcal{L}} : X \to \mathbb{P}H^0(X, \mathcal{L})^* = \mathbb{P}^{g(C)},$$

is regular. Moreover, ϕ is birational unless X contains an irreducible curve C' such that $g(C') = 1$ and $C' \cdot C = 2$ or such that $g(C') = 2$ and $C \sim 2C'$.

The following result, which is also due to Saint-Donat (see [25, Theorems 6.1 and 7.2]), gives some information about the equations which define the image.

Theorem 2.3. *Let $\mathcal{L} = [C]$ be a line bundle on a K-3 surface which satisfies the conditions of Theorem 2.2, excluding the exceptional cases, i.e., ϕ is birational. Then the natural map*

$$\psi : S^*H^0(X, \mathcal{L}) \longrightarrow \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$$

is surjective. If $\mathcal{L}^2 = 4$ then the kernel of ψ is generated by an element of degree four while if $\mathcal{L}^2 = 6$ it is generated by a pair of elements of degrees two and three. If $\mathcal{L}^2 \geq 8$ then the kernel of ψ is generated by its elements of degree two unless X contains an irreducible curve C' such that $g(C') = 1$ and $C' \cdot C = 3$ or X contains a pair of irreducible curves C', C'' such that $g(C') = 2$, $g(C'') = 0$, $C' \cdot C'' = 1$ and $C \sim 2C' + C''$.

3. PROJECTIVE IMAGES OF KUMMER SURFACES

A natural class of K-3 surfaces appears as follows. Let A be an Abelian surface. The (-1)-involution on A (reflection with respect to the origin), which will be denoted by $(-1)_A$, leads to a singular quotient $\mathcal{K}_A = A/(-1)_A$ which is called the *(singular) Kummer surface* of A. It has sixteen singular points which correspond to the half periods $e_1, \ldots, e_{16}$ of A. The desingularization of $\mathcal{K}_A$ can be described as follows. Let $p : \tilde{A} \to A$ be the blow-up of A at all its half periods and denote the corresponding exceptional divisors by E_i. $(-1)_A$ extends to an involution $(-1)_{\tilde{A}}$ on $\tilde{A}$ and the quotient $\tilde{\mathcal{K}}_A = \tilde{A}/(-1)_{\tilde{A}}$ is a K-3 surface (see [8, Proposition VIII.11]).

$\tilde{\mathcal{K}}_A$ is the desingularisation (minimal resolution) of $\mathcal{K}_A$ and we have the following commutative diagram.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\ p\ } & A \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \tilde{\mathcal{K}}_A & \longrightarrow & \mathcal{K}_A \end{array}$$

Associated to A there are also several *intermediate Kummer surfaces* which are desingularizations of $\mathcal{K}_A$ at some but not all singular points.

We will be interested in projective embeddings of smooth, singular and intermediate Kummer surfaces. Therefore we need to know how to construct ample line bundles on $\tilde{\mathcal{K}}_A$. Let $\mathcal{L}$ be a *symmetric* line bundle on A, $(-1)_A^*\mathcal{L} \cong \mathcal{L}$. Then $(-1)_A$ lifts uniquely to an involution $(-1)_{\mathcal{L}}$ on the total space of $\mathcal{L}$ which is $\mathbb{C}$-linear on the fibers of $\mathcal{L}$ and which is identity on the fiber over the origin of A (see [16, Lemma 4.6.3]). Since the involution which $(-1)_{\mathcal{L}}$ induces on the fiber over each half period is linear it is either identity or multiplication by -1. If it is identity the corresponding half period is called *even*, otherwise it is called *odd*; in particular the origin is always an even half period. The induced involution $s \to (-1)_{\mathcal{L}}s(-1)_A$ on $H^0(A, \mathcal{L})$ leads to a splitting of $H^0(A, \mathcal{L})$ into $(+1)$ and (-1) spaces, whose elements are called *even sections* and *odd sections*,

$$H^0(A, \mathcal{L}) = H^0(A, \mathcal{L})^+ \oplus H^0(A, \mathcal{L})^-.$$

Everything can be pulled back using p: we have a line bundle $\tilde{\mathcal{L}} = p^*\mathcal{L}$ on $\tilde{A}$ with an induced involution $(-1)_{\tilde{\mathcal{L}}}$ and an induced splitting of $H^0(\tilde{A}, \tilde{\mathcal{L}})$; clearly p^* realizes isomorphisms between the even resp. odd sections of $\tilde{\mathcal{L}}$ and those of $\mathcal{L}$. Most importantly, these even and odd sections of $\tilde{\mathcal{L}}$ correspond to the sections of two line bundles on $\tilde{\mathcal{K}}_A$: the rank two sheaf $\pi_*\tilde{\mathcal{L}}$ splits under the action $s \to (-1)_{\tilde{\mathcal{L}}}s(-1)_{\tilde{A}}$ into $(+1)$ and (-1) spaces

$$\pi_*\tilde{\mathcal{L}} = \mathcal{M}^+ \oplus \mathcal{M}^-,$$

and there are isomorphisms [6, Proposition 1.1]

$$H^0(\tilde{A}, \tilde{\mathcal{L}})^\pm \cong H^0(\tilde{\mathcal{K}}_A, \mathcal{M}^\pm).$$

So, we can realize odd (even) sections of $\mathcal{L}$ on the Abelian variety A as sections of $\mathcal{M}^-$ $(\mathcal{M}^+)$ on the smooth Kummer surface $\tilde{\mathcal{K}}_A$. The above construction can be generalized by defining for any vector $\nu = (\nu_1, \dots, \nu_{16}) \in \mathbb{N}^{16}$ the line bundle $\tilde{\mathcal{L}}_\nu$ by

$$(2) \qquad\qquad \tilde{\mathcal{L}}_\nu = p^*\mathcal{L} \otimes \left[\sum (-\nu_i)E_i \right].$$

We think of sections of $\tilde{\mathcal{L}}_\nu$ as sections of $\mathcal{L}$ with prescribed vanishing at the half periods e_i. The involution $(-1)_{\tilde{\mathcal{L}}_\nu}$ on the total space of $\tilde{\mathcal{L}}_\nu$ is defined as the tensor of $(-1)_{\tilde{\mathcal{L}}}$ with the identity on each $[E_i]$. Thus, $\pi_*\tilde{\mathcal{L}}_\nu$ splits under the action $\varphi(s) = (-1)_{\tilde{\mathcal{L}}_\nu}s(-1)_{\tilde{A}}$ into $(+1)$ and (-1) line bundles on $\tilde{\mathcal{K}}_A$, which we denote by $\mathcal{M}_\nu^+$ and $\mathcal{M}_\nu^-$.

When working out concrete examples it is useful to know in advance the dimension of $H^0(\tilde{\mathcal{K}}_A, \mathcal{M}_\nu^\pm)$, to know whether the map to projective space, given by the sections, is birational and whether some divisors (exceptional or not) are contracted

by this map. Since the symmetric line bundles $\mathcal{L}$ which we will consider come from explicitly given divisors, we will state the result in the language of divisors. A divisor (or curve) D on A is called *symmetric* if $(-1)_A^* D = D$. The line bundle of a symmetric divisor is symmetric and the even and odd sections of a symmetric line bundle are symmetric divisors ([16, Lemma 4.7.1]); therefore, working with symmetric divisors is just as general as working with symmetric line bundles. We will call a symmetric divisor even or odd according to whether it is defined by an even or odd section. It is easy to see that an even (resp. odd) divisor D has even (resp. odd) multiplicity precisely at the even half periods (in particular at the origin). We denote the multiplicity of D at the half period e_i by $\mu_i(D)$. Let us fix a symmetric curve D on A for which $[D] = \mathcal{L}$. By passing to a linearly equivalent divisor (if necessary) we may assume that D is such that the chosen numbers ν_i satisfy $\nu_i \leq \mu_i(D)$ for any i, because if a divisor with the required vanishing at the half periods does not exist, then $\tilde{\mathcal{L}}_\nu$ has no sections and is not of interest for our purposes. As we will show in the next proposition the divisor $p^* D - \sum_{i=1}^{16} \nu_i E_i$ is symmetric if and only if the multiplicities ν_i are either all even or all odd; we will say in these cases that ν is even or odd. If we denote the proper transform of D by $\hat{D}$ then $p^* D = \hat{D} + \sum \mu_i(D) E_i$ so that

$$\tilde{\mathcal{L}}_\nu = \left[\hat{D} + \sum (\mu_i(D) - \nu_i) E_i \right].$$

Let[1]

$$(3) \qquad\qquad C = \frac{1}{2} \pi_* \hat{D} + \frac{1}{2} \sum_{i=1}^{16} (\mu_i(D) - \rho_i) B_i,$$

where $B_i = \pi_* E_i$ and ρ_i is defined by $\rho_i = \nu_i$ if $\mu_i(D) - \nu_i$ is even, and $\rho_i = \nu_i + 1$ if $\mu_i(D) - \nu_i$ is odd ($i = 1, \ldots, 16$). The curves B_i are called (-2)-curves because

$$B_i^2 = \frac{1}{2} (\pi^* B_i)^2 = 2 E_i^2 = -2.$$

Proposition 3.1. *Suppose that $D \in |\mathcal{L}|$ be symmetric, let $\nu \in \mathbb{N}^{16}$ be even or odd and let the curve C on $\tilde{\mathcal{K}}_A$ be defined by (3). Then $[C] = \mathcal{M}_\nu^+$ if D and ν have the same parity, otherwise $[C] = \mathcal{M}_\nu^-$.*

Proof. Let s resp. s^* be sections that vanish at $D' = \hat{D} + \sum_{i=1}^{16} (\mu_i(D) - \nu_i) E_i$ resp. at $p^* D = \hat{D} + \sum_{i=1}^{16} \mu_i(D) E_i$. We choose a local defining equation x_i for E_i. Using the fact that $x_i(-1)_{\tilde{A}} = -x_i$ and that locally the two sections are related by $s = s^*/x_i^{\nu_i}$ we find

$$\varphi(s) = (-1)_{\tilde{\mathcal{L}}_\nu} s(-1)_{\tilde{A}} = (-1)_{\tilde{\mathcal{L}}_\nu} s^*(-1)_{\tilde{A}} / (-1)^{\nu_i} x_i^{\nu_i} = \pm s / (-1)^{\nu_i},$$

where $+/-$ corresponds to s^* even/odd. This implies that s is symmetric if and only if ν is even or odd, and it shows how the parity of s is related to the parity of s^* (i.e., of D) and of ν. We want to see how s descends to the Kummer surface $\tilde{\mathcal{K}}_A$. Let us assume that s is an even section, $s \in H^0(\tilde{A}, \tilde{\mathcal{L}}_\nu)^+$, so that $1/s$ locally generates the $\mathcal{O}_{\tilde{\mathcal{K}}_A}$-module $\mathcal{M}_\nu^+$; a proof for an odd section goes along similar lines.

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[1] Notice that since $\hat{D}$ is symmetric each irreducible component of its direct image $\pi_* \hat{D}$ appears an even number of times.

The canonical map $\mathcal{O}_{\bar{\mathcal{K}}_A} \to \pi_* \mathcal{O}_{\tilde{A}}$ leads, by taking a direct limit over neighbourhoods of $q \in \tilde{A}$, to a map

$$\pi_q : \mathcal{O}_{\bar{\mathcal{K}}_A, \pi(q)} \to \pi_* \mathcal{O}_{\tilde{A}, q} \to \mathcal{O}_{\tilde{A}, q}.$$

and induces an isomorphism $\mathcal{O}_{\bar{\mathcal{K}}_A} \cong (\pi_* \mathcal{O}_{\tilde{A}})^G$, where G is the group generated by $(-1)^*_{\tilde{A}}$ ([19, page 66]). Away from the ramification locus of π, which consists of the set of exceptional curves E_i, the map π_q is an isomorphism. Therefore, if $q \in \tilde{A}$ does not belong to an exceptional curve then π_q sends the local equation of C to the equation $s = 0$ of D'. It remains to be shown that this is also true when q belongs to an exceptional curve. Let q be a point in E_i; we may assume that q does not belong to any other E_j. If we denote a local defining equation for E_i by x_i, as before, and we consider a coordinate system (x_i, t), then $(u = x_i^2, t)$ is a coordinate system around $\pi(q)$ and the map π_q is the immersion $\mathbb{C}[[u, t]] = \mathbb{C}[[x^2, t]] \to \mathbb{C}[[x, t]]$. An equation for D' about the point q is given by $s = f(x, t) = x^{m_i} g(x, t)$, where g is a local equation for the proper transform $\hat{D}$ and $m_i = \mu_i(D) - \nu_i$. The local section g is even ([6, Proposition 1.2]), so that $g(x, t) = \tilde{g}(x^2, t)$, and

$$\varphi(s) = \varphi(x^{m_i} \tilde{g}) = (-1)^{m_i} x^{m_i} \tilde{g} = (-1)^{m_i} s.$$

Since s is an even section the above equation shows that e_i is an even half period precisely if m_i is even. Now, $1/f$ is a generator of the $\mathcal{O}_{\tilde{A}, q}$-module $\tilde{\mathcal{L}}_q$ and the linear map ϕ splits the rank two $\mathcal{O}_{\bar{\mathcal{K}}_A}$-module $\tilde{\mathcal{L}}_q$ into (± 1) spaces $\mathcal{M}^{\pm}_{\pi(q)}$. Then, for this generator

$$\varphi(1/f) = (-1)^{m_i}/f, \quad \varphi(x/f) = -(-1)^{m_i} x/f.$$

It follows that if e_i is an even half period then $1/f$ is a generator of $\mathcal{M}^+_\nu$ around q; otherwise $\mathcal{M}^+_\nu$ is generated by x_i/f around q. In the first case an equation for a divisor on $\tilde{\mathcal{K}}_\Gamma$ corresponding to the line bundle $\mathcal{M}^+_\nu$ is given by $f(x_i, t) = x_i^{2k} \tilde{g}(x_i^2, t) = u^k \tilde{g}(u, t) = 0$. In the second case such an equation is given by $f(x_i, t)/x_i = x_i^{2k+1} \tilde{g}(x_i^2, t)/x_i = u^k \tilde{g}(u, t) = 0$. In both cases this gives a local equation (around $q \in E_i$) for the divisor C, given by (3). $\qquad \square$

In the following proposition we use Kodaira's Theorem to compute $h^0(C)$. We also compute the intersection of C with other curves (in particular the (-2)-curves) because this allows us to determine which curves are contracted by the map $\phi : \tilde{\mathcal{K}}_A \to \mathbb{P}H^0(\tilde{\mathcal{K}}_A, [C])$ and to compute the degree of the image curve.

Proposition 3.2. *Let D a symmetric curve on an Abelian surface A which induces a polarization of type (δ_1, δ_2). Suppose that $\nu \in \mathbb{N}^{16}$ is symmetric and satisfies $\nu_i \leq \mu_i(D)$ for $i = 1, \ldots, 16$. Let C be the curve on $\tilde{\mathcal{K}}_A$ defined by (3) and assume that $|C|$ has no fixed components. Then*

$$(4) \qquad\qquad C^2 = \delta_1 \delta_2 - \frac{1}{2} \sum_{i=1}^{16} \rho_i^2,$$

$$(5) \qquad\qquad h^0(C) = \frac{\delta_1 \delta_2}{2} - \frac{1}{4} \sum_{i=1}^{16} \rho_i^2 + m + 1,$$

where $\rho_i = \nu_i$ if $\mu_i(D) - \nu_i$ is even and $\rho_i = \nu_i + 1$ otherwise; the integer m is the number of connected components of C. If C' is any curve in $\tilde{\mathcal{K}}_A$ which does not

contain any of the curves B_i as one of its irreducible components, then

$$(6) \qquad C \cdot C' = \frac{D \cdot D'}{2} - \frac{1}{2} \sum_{i=1}^{16} \rho_i \mu_i(D'),$$

where D' is the symmetric divisor on A such that $\pi^* C' = p^* D' - \sum_{i=1}^{16} \mu_i(D') E_i$.
Also $C \cdot B_i = \rho_i$ for any i.

Proof. We know from Formula (1) that

$$(7) \qquad h^0(C) = \frac{C^2}{2} + m + 1,$$

where m is the number of connected components of C. Since π is of degree two we get from (3) that

$$2C^2 = \left(\hat{D} + \sum_{i=1}^{16} (\mu_i(D) - \rho_i) E_i \right)^2 = \left(p^* D - \sum_{i=1}^{16} \rho_i E_i \right)^2.$$

Using the fact that $(p^* D)^2 = D^2 = 2\delta_1 \delta_2$ we find the announced formula (4). Combined with (7) this gives the right number for $h^0(C)$. The verification of (6) is similar:

$$\begin{aligned}
C \cdot C' &= \frac{1}{2} \left(\hat{D} + \sum_{i=1}^{16} (\mu_i(D) - \rho_i) E_i \right) \cdot \left(p^* D' - \sum_{i=1}^{16} \mu_i(D') E_i \right) \\
&= \frac{1}{2} \left(p^* D - \sum_{i=1}^{16} \rho_i E_i \right) \cdot \left(p^* D' - \sum_{i=1}^{16} \mu_i(D') E_i \right) \\
&= \frac{D \cdot D'}{2} - \frac{1}{2} \sum_{i=1}^{16} \rho_i \mu_i(D').
\end{aligned}$$

Finally,

$$C \cdot B_i = \frac{1}{2} \left(\hat{D} + \sum_{i=1}^{16} (\mu_i(D) - \rho_i) E_i \right) \cdot 2 E_i = \rho_i.$$

$\square$

Our formula for $h^0(C)$ generalizes the formula given in [6, Theorem 3.1]. In the latter formula all ν_i are zero which implies $m = 1$ because at any half period which belongs to two irreducible components of D we have $\mu_i(D) \geq 2$. If D is even (resp. odd) then $\rho_i = 1$ for the odd (resp. even) half periods and $\rho_i = 0$ for the even (resp. odd) half periods. Thus our formula specializes to Bauer's formula,

$$(8) \qquad h^0(C) = \frac{\delta_1 \delta_2}{2} - \frac{n}{4} + 2$$

where n is the number of even half periods if D is odd and n is the number of odd half periods if D is even.

4. The Mumford system

In this section we introduce an integrable system and we use it to compute explicit bases for the sections of different natural line bundles on the Jacobian as well as parametrizations of the divisors that are cut out by these sections. In the next section we will use these sections to compute several projective images of its Kummer surface.

Consider a hyperelliptic curve of genus two, given by the equation

$$(9) \qquad \mu^2 = f(\lambda) \quad \text{where} \quad f(\lambda) = \prod_{i=1}^{5} (\lambda - \lambda_i) = \sum_{i=0}^{5} \sigma_i \lambda^{5-i}$$

and assume that it is smooth, i.e., all λ_i are different. This curve can be completed into a non-singular complete curve (compact Riemann surface) Γ by adding a single point which we will denote by ∞. The map $\Gamma \to \mathbb{P}$ which is given on the affine part $\Gamma \setminus \{\infty\}$ by $(\lambda, \mu) \mapsto \lambda$ expresses Γ as a two-sheeted cover of $\mathbb{P}$. It has six ramification points ω_i $(i = 0, \ldots, 5)$ which are called Weierstrass points. They are the fixed points of the hyperelliptic involution $\imath$ which is given on $\Gamma \setminus \{\infty\}$ by $(\lambda, \mu) \mapsto (\lambda, -\mu)$. At ∞ the Riemann surface is described in terms of a uniformizing parameter t by

$$(10) \qquad \lambda(t) = t^{-2}, \qquad \mu(t) = t^{-5} \left(1 + \frac{\sigma_1}{2} t^2 + \frac{4\sigma_2 - \sigma_1^2}{8} t^4 + O(t^6) \right),$$

showing that ∞ is one of the Weierstrass points; we will always label these points such that $\infty = \omega_0$ and such that $\lambda(\omega_i) = \lambda_i$ for $1 \le i \le 5$. At ω_i the curve is parametrized by

$$(11) \qquad \lambda(t) = \lambda_i + t^2, \qquad \mu(t) = \sqrt{\prod_{j \neq i} (\lambda_i - \lambda_j)} \left(t + O(t^3) \right).$$

(the particular choice of square root is irrelevant because we can replace t by $-t$). We denote the Jacobian of Γ (its group of divisors of degree zero modulo linear equivalence; equivalently its group of line bundels of degree zero) by $\mathcal{J}_\Gamma$ and we denote the element of $\mathcal{J}_\Gamma$ that corresponds to a divisor D of degree zero on Γ by $[D]$. It is a fundamental fact that $\mathcal{J}_\Gamma$ is an Abelian surface and that the map $P \mapsto [P - \infty]$ is an embedding of the curve in its Jacobian. We denote the image of this map by Θ and call it the theta divisor; Θ is indeed a divisor and $\Theta^2 = 2$.

The hyperelliptic involution $\imath$ on Γ extends linearly to an involution on the group of divisors on Γ which in turn descends to the (-1)-involution on $\mathcal{J}_\Gamma$. It follows that the sixteen half periods on $\mathcal{J}_\Gamma$ are given by $e_{ij} = [\omega_i - \omega_j]$ and their group structure is governed by the formulas

$$e_{ij} + e_{jk} + e_{ki} = 0, \qquad \text{for any } i, j, k,$$
$$e_{ij} + e_{kl} + e_{mn} = 0, \qquad \text{for } i, j, k, l, m, n \text{ all different,}$$

(for the proof of the second formula, use the meromorphic function $(\lambda - \lambda_i)(\lambda - \lambda_k)(\lambda - \lambda_m)/\mu$ to realize the linear equivalence $\omega_i + \omega_k + \omega_m \sim \omega_j + \omega_l + \omega_n$). We also introduce the sixteen translates $\Theta_{ij} = \Theta + e_{ij}$ of the theta divisor which we will call *theta curves*. The theta curves Θ_{ij} are symmetric, the odd ones are the six curves Θ_{0i} which pass through the origin and the remaining ones are even.

To every point of $\mathcal{J}_\Gamma$ we can uniquely associate a matrix of polynomials (in λ)

$$(12) \qquad \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix} = \begin{pmatrix} v_1\lambda + v_2 & \lambda^2 + u_1\lambda + u_2 \\ \lambda^3 + w_0\lambda^2 + w_1\lambda + w_2 & -v_1\lambda - v_2 \end{pmatrix}$$

whose characteristic polynomial equals $\mu^2 - f(\lambda)$ as follows (see [18]). Every element of $\mathcal{J}_\Gamma$ is of the form $[P + Q - 2\infty]$ for some $P, Q \in \Gamma$ and the unorderd pair (P,Q) is unique if and only if $P \neq \imath(Q)$. In this case, if both P and Q are different form ∞ we take the entries of the matrix (12) to be given by (note that $w(\lambda)$ is indeed a polynomial because $v(\lambda(P)) = \mu(P)$ and $v(\lambda(Q)) = \mu(Q)$)

$$u(\lambda) = (\lambda - \lambda(P))(\lambda - \lambda(Q)),$$
$$(13) \qquad v(\lambda) = \frac{\mu(P) - \mu(Q)}{\lambda(P) - \lambda(Q)}\lambda + \frac{\lambda(P)\mu(Q) - \lambda(Q)\mu(P)}{\lambda(P) - \lambda(Q)},$$
$$w(\lambda) = \frac{f(\lambda) - v^2(\lambda)}{u(\lambda)}.$$

For example, for the ten half periods $e_{ij} = [\omega_i - \omega_j]$ ($1 \leq i < j \leq 5$), we get

$$(14) \qquad \begin{pmatrix} 0 & (\lambda - \lambda_i)(\lambda - \lambda_j) \\ \prod_{k\neq i,j}(\lambda - \lambda_k) & 0 \end{pmatrix}.$$

The above formula for $v(\lambda)$ is to be interpreted in the right way when $P = Q$: taking the limit $Q \to P$ in the above formula for $v(\lambda)$ we find the following formula for $v(\lambda)$ when $P = Q$,

$$(15) \qquad v(\lambda) = \frac{f'(\lambda(P))(\lambda - \lambda(P)) + 2f(\lambda(P))}{2\mu(P)}.$$

Note that the denominator does not vanish because $P \neq \imath P$, i.e., P is not a Weierstrass point. Still assuming that $P \neq \imath(Q)$, if $Q = \infty$ then the matrix is given by

$$(16) \qquad \begin{pmatrix} \mu(P) & \lambda - \lambda(P) \\ \dfrac{\prod_{i=1}^5(\lambda - \lambda_i) - \prod_{i=1}^5(\lambda(P) - \lambda_i)}{\lambda - \lambda(P)} & -\mu(P) \end{pmatrix}.$$

For example, for the five half periods $e_{i0} = [\omega_i - \infty]$, $i = 1, \ldots, 5$ we have

$$\begin{pmatrix} 0 & \lambda - \lambda_i \\ \prod_{j\neq i}(\lambda - \lambda_j) & 0 \end{pmatrix}.$$

The divisors $P + \imath(P) - 2\infty$ form a linear system that corresponds to the origin of the Jacobian; its matrix is given by

$$(17) \qquad \begin{pmatrix} 0 & 1 \\ \prod_{i=1}^5(\lambda - \lambda_i) & 0 \end{pmatrix}.$$

For future use we will now compute the set of matrices which correspond to the divisors Θ_{ij}; more precisely we will give a parametrization of all of the divisor minus one point. In order to make our formulas more compact we introduce the following

expressions in the λ_i which generalize the elementary symmetric polynomials σ_i (introduced in (9)),

$$\bar{\sigma}_{k,i_1\ldots i_n} = \sigma_{k|\lambda_{i_1}=\cdots=\lambda_{i_n}=0}, \qquad (1 \le n+k \le 5),$$

$$\sigma_{k,i_1\ldots i_n} = \bar{\sigma}_{k,j_1\ldots j_{5-n}}, \qquad (\{j_1,\ldots,j_{5-n}\} = \{1,\ldots,5\} \setminus \{i_1,\ldots,i_n\}).$$

For example $\sigma_{1,12} = -\lambda_1 - \lambda_2$ and $\bar{\sigma}_{3,12} = -\lambda_3\lambda_4\lambda_5$.

Clearly, a parametrization for the theta divisor $\Theta = \Theta_{00}$ is given by all matrices (16) where P runs over Γ. For the other divisors Θ_{0i} we get

$$(18) \qquad \begin{pmatrix} \mu(P)\dfrac{\lambda - \lambda_i}{\lambda(P) - \lambda_i} & (\lambda - \lambda(P))(\lambda - \lambda_i) \\[2ex] \star_i & -\mu(P)\dfrac{\lambda - \lambda_i}{\lambda(P) - \lambda_i} \end{pmatrix},$$

where $\star_i$ is found by expressing that the characteristic polynomial of the matrix is equal to $\mu^2 - f(\lambda)$,

$$\star_i = \lambda^3 + \lambda^2(\bar{\sigma}_{1,i} + \lambda(P)) + \lambda(\bar{\sigma}_{2,i} + \lambda(P)\bar{\sigma}_{1,i} + \lambda(P)^2)$$

$$- \frac{1}{\lambda(P) - \lambda_i} \left[\bar{\sigma}_{4,i} + \lambda_i\bar{\sigma}_{3,i} + \lambda_i\lambda(P)\bar{\sigma}_{2,i} + \lambda_i\lambda(P)^2\bar{\sigma}_{1,i} + \lambda_i\lambda(P)^3\right].$$

The formulas for computing the other Θ_{ij} (with $0 < i < j \le 5$) require some more work. The points on Θ_{ij} are of the form $[P + \omega_i + \omega_j - 3\infty]$, which we first need to rewrite in the standard form $[Q + R - 2\infty]$ (Q and R will depend on P). Consider for fixed P the following meromorphic function on Γ,

$$\varphi_P(\lambda,\mu) = \frac{\mu + \mu(P)\dfrac{(\lambda - \lambda_i)(\lambda - \lambda_j)}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)}}{(\lambda - \lambda(P))(\lambda - \lambda_i)(\lambda - \lambda_j)}.$$

It realises the linear equivalence $P + \omega_i + \omega_j \sim Q + R + \infty$, the points Q and R being given as the non-trivial zeros of the numerator. To find these zeros, multiply this numerator by

$$\mu - \mu(P)\frac{(\lambda - \lambda_i)(\lambda - \lambda_j)}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)}$$

to find the following equation in λ whose solutions are $\lambda(Q)$ and $\lambda(R)$,

$$\prod_{k=1}^{5}(\lambda - \lambda_k)(\lambda(P) - \lambda_i)^2(\lambda(P) - \lambda_j)^2 = \prod_{k=1}^{5}(\lambda(P) - \lambda_k)(\lambda - \lambda_i)^2(\lambda - \lambda_j)^2.$$

Note that we are not required to solve this for $\lambda(Q)$ and $\lambda(R)$ individually: we can solve it linearly for $\lambda(Q) + \lambda(R)$ and $\lambda(Q)\lambda(R)$ and this is enough to determine the polynomial $u(\lambda)$ which is associated to an arbitrary point on Θ_{ij}, in fact these are precisely the coefficients of $u(\lambda)$ since $u(\lambda) = (\lambda - \lambda(Q))(\lambda - \lambda(R))$. Solving linearly we get

$$u_1 = \frac{\lambda^2(P)\sigma_{1,ij} + \lambda(P)(\sigma_2 - 2\bar{\sigma}_{2,ij}) + \sigma_{2,ij}\bar{\sigma}_{1,ij} - \bar{\sigma}_{3,ij}}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)},$$

$$(19) \qquad u_2 = \frac{\lambda^2(P)\sigma_{2,ij} + \lambda(P)(\sigma_{2,ij}\bar{\sigma}_{1,ij} - \bar{\sigma}_{3,ij}) + \sigma_{2,ij}\bar{\sigma}_{2,ij} - \sigma_{1,ij}\bar{\sigma}_{3,ij}}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)}.$$

In order to find the polynomial $v(\lambda)$ which is associated to an arbitrary point on Θ_{ij} we use the vanishing of the numerator of φ_P to find

$$\frac{\mu(Q) - \mu(R)}{\lambda(Q) - \lambda(R)} = -\mu(P)\frac{\lambda(Q) + \lambda(R) - \lambda_i - \lambda_j}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)},$$

$$\frac{\lambda(Q)\mu(R) - \lambda(R)\mu(Q)}{\lambda(Q) - \lambda(R)} = \mu(P)\frac{\lambda(Q)\lambda(R) - \lambda_i\lambda_j}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)}.$$

The right hand side only involves $\lambda(P) + \lambda(Q)$ and $\lambda(P)\lambda(Q)$ hence it suffices to plug in the expressions (19) for these to find the polynomial $v(\lambda)$ associated to $[P + \omega_i + \omega_j - 3\infty]$,

$$v_1 = \mu(P)\frac{u_1 + \lambda_i + \lambda_j}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)},$$

(20)
$$v_2 = \mu(P)\frac{u_2 - \lambda_i\lambda_j}{(\lambda(P) - \lambda_i)(\lambda(P) - \lambda_j)}.$$

The corresponding polynomial $w(\lambda)$ is found from $w(\lambda) = (f(\lambda) - v^2(\lambda))/u(\lambda)$.

The above formulas for the divisors give a parametrization but do not describe the sections which cut them out. Nor do we have, at this point, a way to compute a basis for the odd or even sections of $[n\Theta]$ which lead to projective images of the Kummer surface. To get these we consider the (two-dimensional) Mumford system (see [18]), which consists of a pair of commuting vector fields on the seven dimensional affine space M of matrices (12). Coordinates on M are given by $u_1, u_2, v_1, v_2, w_0, w_1$ and w_2.

Let H denote the map

$$H : M \to \mathbb{C}[\lambda, \mu] : A(\lambda) \mapsto |A(\lambda) - \mu I|,$$

which associates to such a matrix $A(\lambda)$ its characteristic polynomial. Then the fiber of H over a polynomial $\mu^2 - f(\lambda)$ (f monic of degree five and square-free) is isomorphic to the affine variety $\mathcal{J}_\Gamma \setminus \Theta$ where Γ is the curve defined by $\mu^2 = f(\lambda)$; explicitly the isomorphism is given by (13). Equations for this affine variety thus follow from the equations of the fiber,

$$u_1 + w_0 = \sigma_1,$$

$$u_2 + u_1 w_0 + w_1 = \sigma_2,$$

(21)
$$u_2 w_0 + u_1 w_1 + w_2 + v_1^2 = \sigma_3,$$

$$u_2 w_1 + u_1 w_2 + 2v_1 v_2 = \sigma_4,$$

$$u_2 w_2 + v_2^2 = \sigma_5,$$

where we denoted the coefficients of the curve $\mu^2 = f(\lambda)$ by σ_i, as in (9). Two independent commuting vector fields on M are given by

$$\dot{u}_1 = v_1, \qquad\qquad u_1' = v_2,$$
$$\dot{u}_2 = v_2, \qquad\qquad u_2' = u_1 v_2 - u_2 v_1,$$
$$\dot{v}_1 = -\tfrac{1}{2}(w_1 + u_1^2 - u_1 w_0 - u_2), \qquad v_1' = -\tfrac{1}{2}(w_2 + u_1 u_2 - u_2 w_0),$$
$$\dot{v}_2 = -\tfrac{1}{2}(w_2 + u_1 u_2 - u_2 w_0), \qquad v_2' = -\tfrac{1}{2}(u_1 w_2 + u_2^2 - u_2 w_1),$$
$$\dot{w}_0 = -v_1, \qquad\qquad w_0' = -v_2,$$
$$\dot{w}_1 = u_1 v_1 - v_1 w_0 - v_2, \qquad w_1' = u_2 v_1 - v_2 w_0,$$
$$\dot{w}_2 = u_1 v_2 - v_2 w_0, \qquad\qquad w_2' = u_2 v_2 + v_1 w_2 - v_2 w_1.$$

Mumford shows that these vector fields restrict to linear vector fields on the Jacobians which appear as fibers of the map H (it is easy to check that these vector fields are indeed tangent to the fibers of H). Fixing the section which cuts out $n\Theta$, the sections of $[n\Theta]$ can be described by the meromorphic functions with a pole of order at most n at infinity, i.e., at Θ. To find these meromorphic functions one looks for Laurent solutions to the differential equations which describe one of the linear vector fields (see [28, Chapter 5.3]), more precisely one looks for all families of Laurent solutions of the maximal dimension (i.e., containing the maximal number of free parameters). In the case at hand we pick the first vector field (the Laurent solutions for this vector fields are easier to find because that vector field is weight homogeneous, see [28, loc. cit.]) We find that there is only one such family of Laurent solutions and that its dimension is six. We display here precisely as many terms of the Laurent solutions as we need for our computations below; moreover we only display them for u_1 and u_2 because the Laurent solutions for the other affine variables follow at once from them by using the differential equations (e.g., $v_1 = \dot{u}_1$, etc.).

$$u_1 = -\frac{4}{t^2} + a + 2ct^2 + 40dt^3 + et^4 + 3d(a + 2b)t^5 + ft^6 + \cdots,$$
$$u_2 = \frac{4b}{t^2} - b(a + b) - 240dt - 2bct^2 + 8d(3a + b)t^3 + (18f + 6c^2 - be)t^4 + \cdots.$$

A basis for the functions with a pole of order two at most at Θ is given by

$$z_0 = 1, \ z_1 = u_1, \ z_2 = u_2, \ z_3 = u_1 u_2 - w_2.$$

To see that the restriction of z_3 to $\mathcal{J}_\Gamma$ is linearly independent of the other functions it suffices to compute the leading term of z_3, which is given by $4b(3a + 2b)/t^2$. The corresponding sections embed the singular Kummer surface into $\mathbb{P}^3$ (see the next section). A basis for the functions with a triple pole along Θ is given by adding the following five functions

$$z_4 = v_1, \ z_5 = v_2, \ z_6 = u_1 v_2 - u_2 v_1, \ z_7 = v_2 w_1 - v_1 w_2 + u_1(u_1 v_2 - u_2 v_1),$$
$$z_8 = u_1 w_2 + u_2 w_1 + 2u_1 u_2 w_0.$$

Their leading terms are given by

$$(z_4, z_5, z_6, z_7, z_8) = \frac{8}{t^3}\left(1, -b, b^2, b^2(3a + 2b), 1440d\right),$$

showing their independence. These nine functions allow to embed the Jacobian into projective space $\mathbb{P}^8$. Finally, to get a basis for the space of functions with a pole of

order at most four along Θ, one also adds the following functions:

$$z_9 = u_1^2, \ z_{10} = u_1 u_2, \ z_{11} = u_2^2, \ z_{12} = v_1 w_2 + u_1 w_0 v_2,$$

$$z_{13} = u_1 u_2 v_2 - v_2 w_2 - u_2^2 v_1 - u_2 v_2 w_0, \ z_{14} = u_2(u_1 u_2 - w_2),$$

$$z_{15} = (u_1 u_2 - w_2)^2.$$

Their leading terms are given by

$$(z_9, z_{10}, \dots, z_{15}) = \frac{16}{t^4} \left(1, -b, b^2, -720d, -720bd, b^2(3a+2b), b^2(3a+2b)^2 \right),$$

hence these sections are also independent (on the Jacobian of any smooth curve Γ). Since the hyperelliptic involution on Γ is given by $(\lambda, \mu) \mapsto (\lambda, -\mu)$ the (-1)-involution on $\mathcal{J}_\Gamma$ is given by

$$(u_1, u_2, v_1, v_2, w_0, w_1, w_2) \mapsto (u_1, u_2, -v_1, -v_2, w_0, w_1, w_2),$$

and we easily see that each of the functions z_i, $i = 0, \dots, 15$ is either even or odd with respect to this involution. In order to consider such a function z_i as a section of $[n\Theta]$ we need to multiply z_i by the section that cuts out $n\Theta$, which is even for n even and which is odd for n odd. Therefore we find the following table for the functions z_i which represent the even and odd sections of $[2\Theta]$, $[3\Theta]$ and $[4\Theta]$. An explicit basis for the even and odd sections for $[n\Theta]$ with $n \geq 5$ are obtained in a completely analogous way but will not be used here.

Table 3

line bundle	even sections	odd sections
$[2\Theta]$	$1, z_1, z_2, z_3$	no
$[3\Theta]$	z_4, z_5, z_6, z_7	$1, z_1, z_2, z_3, z_8$
$[4\Theta]$	$1, z_1, z_2, z_3, z_8, z_9, z_{10}, z_{11}, z_{14}, z_{15}$	$z_4, z_5, z_6, z_7, z_{12}, z_{13}$

5. Kummer surfaces of Jacobians

In this section we will use the results of the previous section to compute different projective images of the Kummer surface $\mathcal{K}_\Gamma$ of $\mathcal{J}_\Gamma$. The linear systems which we will use consist of the even or odd sections of $[n\Theta]$ (with $n = 2, 3, 4$) with prescribed vanishing at the half periods. Recall from Section 3 that we denote the line bundle $p^* \mathcal{L} \otimes [\sum(-\nu_i) E_i]$ on $\tilde{\mathcal{J}}_\Gamma$ by $\tilde{\mathcal{L}}_\nu$ and that we denote the line bundles on $\tilde{\mathcal{K}}_\Gamma$ which correspond to the even and odd sections of $\tilde{\mathcal{L}}_\nu$ by $\mathcal{M}_\nu^\pm$. In order to compute these induced linear systems on $\tilde{\mathcal{K}}_\Gamma$ we will use divisors in $|n\Theta|$ which consist entirely of translates Θ_{ij} of Θ. We will call such divisors *totally reducible* divisors. These divisors have the convenient property of having large multiplicity at several half periods and it is for these divisors easy to figure out its multiplicity at any half period. The following lemma will tell us which divisors in $|n\Theta|$ are totally reducible.

Lemma 5.1. *The divisor* $\Theta_{i_1 j_1} + \cdots + \Theta_{i_n j_n}$ *is in* $|n\Theta|$ *if and only if* $e_{i_1 j_1} + \cdots + e_{i_n j_n} = 0$.

Proof. The proof of the only if part follows easily from [16, Lemma 4.1.5]. The if part then follows from the fact that two different translates of Θ are never be linearly equivalent. $\square$

We will in every case considered below show that the linear systems $\mathcal{M}_\nu^\pm$ have no base points, so that the corresponding map $\phi_{\mathcal{M}_\nu^\pm}$ is regular, we will compute an equation of its image and we will determine which curves are contracted (leading to a singular point of the image). We will denote the image of the (-2)-curve B_{ij} by $\mathcal{E}_{ij}$ and the image of $\pi_*(\hat{\Theta}_{ij})$ by $\mathcal{T}_{ij}$. These images can be points, straight lines or curves of higher degree. The incidence relations between the thirty-two objects $\mathcal{E}_{ij}$ and $\mathcal{T}_{ij}$ will follow easily from the incidence relations on $\tilde{\mathcal{J}}_\Gamma$ (see [12, Chapter 1]) which were classically represented in the following compact form.

Table 4

00	01	12	02
34	25	05	15
35	24	04	14
45	23	03	13

The way in which the incidence is encoded in this table is this: the divisors E_{ij} are pairwise disjoint as well as the divisors $\hat{\Theta}_{kl}$. Every divisor E_{ij} meets precisely six divisors $\hat{\Theta}_{kl}$ and vice versa. E_{ij} and $\hat{\Theta}_{kl}$ will meet precisely when the indices ij and kl appear in Table 4 either in the same row or in the same column (but not both!).

5.1. **The linear system $|2\Theta|$.** The first case is that of $D = 2\Theta$, $\mathcal{L} = [2\Theta]$. Some of the results in this paragraph are classical but the proofs that we give provide the reader with a good illustration of our approach, which also applies to the more complex situations studied in the subsequent paragraphs.

The divisor D has multiplicity two at the six half-periods e_{00}, e_{01}, ..., e_{05} and has multiplicity zero at the other half periods, in particular D is even and all half periods are even. We picture D as follows.
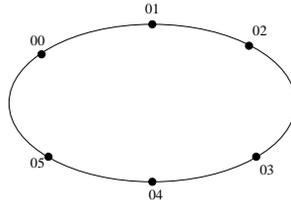
By (8) every section of $[2\Theta]$ is even, in agreement with Table 3, leading to a line bundle $\mathcal{M}^+$ on $\tilde{\mathcal{K}}_\Gamma$. If s denotes the section of $[2\Theta]$ that cuts out 2Θ then Table 3 tells us that $\theta_0 = s$, $\theta_1 = su_1$, $\theta_2 = su_2$ and $\theta_3 = s(u_1u_2 - w_2)$ span the space of sections of $[2\Theta]$.

Proposition 5.2. *The linear system $|2\Theta|$ is base-point-free hence leads to a regular map $\phi_{\mathcal{M}^+} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^3$. The image of $\phi_{\mathcal{M}^+}$ is a quartic surface whose equation is*

given, in terms of the coordinates θ_i, $i = 0, \ldots, 3$, by

$$0 = (4\sigma_3\sigma_5 - \sigma_4^2)\theta_0^4 + 2[-2\sigma_2\sigma_5\theta_1 + (\sigma_2\sigma_4 - 2\sigma_1\sigma_5)\theta_2 + 2\sigma_5\theta_3]\theta_0^3$$

(22)
$$+ [4\sigma_1\sigma_5\theta_1^2 - (2\sigma_4 + \sigma_2^2)\theta_2^2 + (4\sigma_5 - 2\sigma_1\sigma_4)\theta_1\theta_2 - 2\sigma_4\theta_1\theta_3 + 4\sigma_3\theta_2\theta_3]\theta_0^2$$

$$+ 2[-2\sigma_5\theta_1^3 + \sigma_2\theta_2^3 + 2\sigma_4\theta_1^2\theta_2 + (\sigma_1\sigma_2 - 2\sigma_3)\theta_1\theta_2^2 + 2\theta_2\theta_3^2$$

$$- \sigma_2\theta_1\theta_2\theta_3 - 2\sigma_1\theta_2^2\theta_3]\theta_0 - (\theta_2^2 - \sigma_1\theta_1\theta_2 + \theta_1\theta_3)^2.$$

The map $\phi_{\mathcal{M}^+}$ contracts the sixteen (-2)-curves B_{ij} and maps the sixteen theta curves to sixteen conics, leading to the classical 16_6-configuration on the Kummer surface $\mathcal{K}_\Gamma$. No other irreducible divisor is contracted by $\phi_{\mathcal{M}^+}$.

Proof. Let us show that there are no points on $\mathcal{J}_\Gamma$ where all sections of $[2\Theta]$ vanish. First, if such a point X exists then $s(X) = 0$ hence $X \in \Theta$. We know that the points on the theta divisor are of the form $[P - \infty]$ where $P \in \Gamma$. Let us suppose first that $P \neq \infty$ and consider the curve $X(t) = P + Q(t) - 2\infty$ where $Q(0) = \infty$ and $Q(t) = (\lambda(t), \mu(t))$ is given by (10) for t small and different from zero. The polynomials $u(\lambda)$, $v(\lambda)$ and $w(\lambda)$ which correspond to $X(t)$ are (for $t \neq 0$) computed from (13). The image of $X = X(0)$ in projective space is then given by

$$\lim_{t \to 0} (1 : u_1(t) : u_2(t) : u_1(t)u_2(t) - w_2(t)) = (0 : -1 : \lambda(P) : \lambda(P)(\sigma_1 + \lambda(P))),$$

in particular not all sections vanish at X. If X is the origin then we consider a curve $X(t) = [P(t) + Q(t) - 2\infty]$ where $P(t)$ and $Q(t)$ are given as $Q(t)$ above and we find in a similar way that the origin gets mapped to $(0 : 0 : 0 : 1)$. This shows that $|2\Theta|$ is base-point-free, hence $|\mathcal{M}^+|$ is also base-point-free.

We now indicate how the equation (22) was found. Since $\theta_0 = 0$ cannot be a component of the image it suffices to find a relation between the functions $z_0, \ldots, z_3$ (see Table 3). This is easily done from the equations (21) which define the affine part of the Jacobian: use the first two equations of (21) to eliminate w_0 and w_1 linearly from the other equations and eliminate v_1 and v_2 from these by expressing that the obvious identity $(v_1 v_2)^2 = v_1^2 v_2^2$ holds. The resulting equation for between u_1, u_2 and w_2 is rewritten at once in terms of $z_0, \ldots, z_3$. If we let $\theta_i = s z_i$, $i = 0, \ldots, 3$ then we find (22). In order to conclude from this computation that the image is always (i.e., for all values of the parameters σ_i which define a smooth curve) a quartic surface we only need to show that the quartic polynomial in equation (22) is not a complete square, because the image is certainly irreducible and has degree four. Let us suppose that the right hand side Q of (22) is a complete square, $Q = P^2$. Since the coefficient of θ_1^4 in Q vanishes there is no term θ_1^2 in P and hence no term $\theta_0\theta_1^3$ in Q, i.e., $\sigma_5 = 0$. But then also the coefficient of $\theta_0^2\theta_1^2$ in Q vanishes, hence the coefficient of $\theta_0\theta_1$ in P vanishes. This implies in turn that the coefficient $2\sigma_4$ of $\theta_0\theta_1^2\theta_2$ in Q vanishes. The two conditions $\sigma_4 = \sigma_5 = 0$ are however impossible when Γ is smooth.

Since $\rho_{ij} = 0$ for $0 \leq i, j \leq 5$ we have from Proposition 3.2 that $C \cdot B_{ij} = 0$ for any i, j, i.e., all (-2)-curves B_{ij} are contracted, so that every $\mathcal{E}_{ij}$ is a point. On the other hand, if we denote by C_{ij} the projection of the proper transform of any of the theta curves Θ_{ij} then $C \cdot C_{ij} = \Theta \cdot \Theta_{ij} = 2$ so the sixteen theta curves map to sixteen conics $\mathcal{T}_{ij}$ and we get Kummer's 16_6 configuration of lines and conics on $\mathcal{K}_\Gamma \subset \mathbb{P}^3$. Explicit coordinates of the points $\mathcal{E}_{ij}$ and the conics $\mathcal{T}_{ij}$ will be computed below.

Finally, we use the explicit sections to show that no other irreducible divisor in $\mathcal{J}_\Gamma$ is contracted by $\phi_{[2\Theta]}$. Since such a divisor lies in the affine part $\mathcal{J}_\Gamma \setminus \Theta$ we can write it as $[P(t) + Q(t) - 2\infty]$ where $P(t) = (\lambda_1(t), \mu_1(t))$ and $Q(t) = (\lambda_2(t), \mu_2(t))$. If we assume that this curve is contracted by ϕ then $u_1' = u_2' = (u_1 u_2 - w_2)' = 0$ where the prime denotes derivative with respect to t. Then

$$\lambda_1'(t) + \lambda_2'(t) = 0,$$
$$\lambda_2(t)\lambda_1'(t) + \lambda_1(t)\lambda_2'(t) = 0,$$

so that $\lambda_1'(t) = \lambda_2'(t) = 0$ or $\lambda_1(t) = \lambda_2(t)$. The first case does not correspond to a divisor. In the second case we have that $\mu_1(t) = \mu_2(t)$ because the pair of points (P, Q) which corresponds to any point of $\mathcal{J}_\Gamma$, different from the origin, has the property that $P \neq \imath Q$; from the explicit equations for ϕ it follows that ϕ does not map such a curve to a single point. $\qquad\square$

It should be remarked that the coefficients of the quartic (22) are expressed in terms of the coefficients σ_i of the equation $\mu^2 = f(\lambda)$ for Γ and not in terms of the roots λ_i of $f(\lambda)$. As far as we know such an equation does not appear in the classical literature.

In computing an equation for the quartic surface we could have used another basis for the sections of $[2\Theta]$; note that each such choice corresponds to the choice of a basis for $\mathbb{P}^3$. We will find a more symmetric equation by using the singular points $\mathcal{E}_{ij}$, which are the images of the sixteen (-2)-curves B_{ij}. For $0 < i < j \leq 5$ we find from (14) that the polynomials which correspond to e_{ij} are given by

$$u(\lambda) = \lambda^2 + u_1\lambda + u_2 = (\lambda - \lambda_i)(\lambda - \lambda_j),$$
$$v(\lambda) = 0,$$
$$w(\lambda) = \lambda^3 + w_0\lambda^2 + w_1\lambda + w_2 = \prod_{k \neq i,j}(\lambda - \lambda_k)$$

so that for $0 < i < j \leq 5$ the image $\mathcal{E}_{ij}$ of B_{ij} is given by the point

$$\mathcal{E}_{ij} = (1 : \sigma_{1,ij} : \sigma_{2,ij} : \sigma_{1,ij}\sigma_{2,ij} - \bar{\sigma}_{3,ij}).$$

The coordinates of the other six points $\mathcal{E}_{0i}$, $(0 \leq i \leq 5)$ are found as follows. The sixteen translations over half periods descend to sixteen automorphisms of the Kummer surface and of its image. Any such automorphism is induced by an automorphism of the ambient space $\mathbb{P}^3$. With the ten half periods at hand we can compute the matrices of these automorphisms: in order to compute the matrix τ_{0k} which goes with translation over e_{0k}, it suffices to express the fact that the translation interchanges the following three pairs of points: $\mathcal{E}_{ij} \leftrightarrow \mathcal{E}_{mn}$, $\mathcal{E}_{im} \leftrightarrow \mathcal{E}_{jn}$ and $\mathcal{E}_{in} \leftrightarrow \mathcal{E}_{jm}$ (here $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$). It leads to the following formula for τ_{0k}

$$\tau_{0k} = \begin{pmatrix} \lambda_k^2 & \lambda_k & 1 & 0 \\ \lambda_k\bar{\sigma}_{2,k} + \lambda_k^2\bar{\sigma}_{1,k} & -\lambda_k^2 & \sigma_1 & -1 \\ \bar{\sigma}_{4,k} + \lambda_k\bar{\sigma}_{3,k} & 0 & -\lambda_k\bar{\sigma}_{1,k} & \lambda_k \\ \sigma_1(\bar{\sigma}_{4,k} + \lambda_k\bar{\sigma}_{3,k}) & -\bar{\sigma}_{4,k} - \lambda_k\bar{\sigma}_{3,k} & 2\lambda_k^3 + (\sigma_2 - \sigma_1^2)\lambda_k & \lambda_k\bar{\sigma}_{1,k} \end{pmatrix}.$$

The matrices for the other translations τ_{kl} are found from $\tau_{kl} = \tau_{0k}\tau_{0l}$. From $\tau_{0k}(\mathcal{E}_{ik}) = \mathcal{E}_{0i}$ we find that $\mathcal{E}_{0i} = (0 : 1 : -\lambda_i : -\lambda_i\bar{\sigma}_{1,i})$ from which we also get

that the origin in $\mathcal{J}_\Gamma$ is mapped to $\mathcal{E}_{00} = (0 : 0 : 0 : 1)$. This provides us with the explicit coordinates of all singular points. Explicit equations for the hyperplanes which cut out the conics $\mathcal{T}_{0i}$ and $\mathcal{T}_{ij}$ are found from the explicit parametrization of these curves: using (18) we find at once that the section

$$(23) \qquad f_i = \lambda_i^2 \theta_0 + \lambda_i \theta_1 + \theta_2$$

vanishes once (hence twice) on Θ_{0i} giving the following equation for the conic $\mathcal{T}_{0k}$ (as sitting in the hyperplane $f_i = 0$).

$$(\theta_1 + \lambda_k \theta_0)(\theta_3 + \lambda_k \bar{\sigma}_{1,k} \theta_1 - \lambda_k (\lambda_k^2 - \lambda_k \bar{\sigma}_{1,k} + \bar{\sigma}_{2,k}) \theta_0)$$
$$+ \theta_0 (\lambda_k \theta_3 + (\bar{\sigma}_{4,k} + \lambda_k \bar{\sigma}_{3,k}) \theta_0) = 0.$$

Using (19) and (20) we find that

$$(24) \qquad f_{ij} = (\sigma_{2,ij} \bar{\sigma}_{1,ij} + \bar{\sigma}_{3,ij}) \theta_0 - \sigma_{2,ij} \theta_1 - \bar{\sigma}_{1,ij} \theta_2 + \theta_3$$

vanishes twice on Θ_{ij} giving the following equations for the quadrics $\mathcal{T}_{ij}$, $(0 < i, j \leq 5)$ (as sitting in the hyperplane $f_{ij} = 0$).

$$(\sigma_{2,ij} \theta_1 - \sigma_{1,ij} \theta_2)(\theta_1 - \bar{\sigma}_{1,ij} \theta_0) + (\theta_2 - \sigma_{2,ij} \theta_0)(\theta_2 - \bar{\sigma}_{2,ij} \theta_0)$$
$$+ \bar{\sigma}_{3,ij} \theta_0 (\theta_1 - \sigma_{1,ij} \theta_0) = 0.$$

A natural way to pick coordinates which make the equation of the quartic more symmetric is it take them such that four of the translations τ_{ij} correspond to interchanging the base points of $\mathbb{P}^3$ in pairs. Clearly these four translations must form a subgroup of the group of all translations over half periods. These subgroups come in two types: either one picks as generators two half periods on a single theta curve or one picks two generators on two distinct theta curves. If four half periods are linked by a subgroup of the first type they are classically said to form a *Rosenhain tetrad;* clearly there are eighty such tetrads. Otherwise they are said to form a *Göpel* tetrad; there are sixty such tetrads. There is a significant difference between these two types: if the vertices of a Rosenhain tetrad are taken as base points then each of the four coordinate planes contains one of the conics $\mathcal{T}_{ij}$, which is not true in the case of a Göpel tetrad. Indeed, since each coordinate plane of a Rosenhain tetrad contains three points of one of the conics $\mathcal{T}_{ij}$ it must contain the whole conic.

For example, the images of the half periods e_{00}, e_{0i}, e_{0j} and e_{ij} form a Rosenhain tetrad. If we take these as base points for $\mathbb{P}^3$ and we call t_0, t_1, t_2, t_3 the new coordinates and we write $\lambda_{ij} = \lambda_i - \lambda_j$ then

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sigma_{1,ij} & 1 & 1 & 0 \\ \sigma_{2,ij} & -\lambda_j & -\lambda_i & 0 \\ \sigma_{1,ij}\sigma_{2,ij} - \bar{\sigma}_{3,ij} & -\lambda_j \bar{\sigma}_{1,j} & -\lambda_i \bar{\sigma}_{1,i} & 1 \end{pmatrix} \begin{pmatrix} a_0 t_0 \\ a_1 t_1 \\ a_2 t_2 \\ a_0 a_1 a_2 t_3 \end{pmatrix},$$

where $a_0^2 = \lambda_{ij}$, $a_1^2 = \lambda_{ik}\lambda_{im}\lambda_{in}$ and $a_2^2 = \lambda_{jk}\lambda_{jm}\lambda_{jn}$. The other twelve singular points have now the following coordinates (i, j, k, m, n are all different),

$$\mathcal{E}_{0k} : (0 : -\lambda_{ik}a_2 : \lambda_{jk}a_1 : \lambda_{ik}\lambda_{jk}a_0)$$

$$\mathcal{E}_{ik} : (\lambda_{ik}a_2 : 0 : \lambda_{ik}\lambda_{jk}a_0 : \lambda_{jk}a_1)$$

$$\mathcal{E}_{jk} : (\lambda_{jk}a_1 : \lambda_{ik}\lambda_{jk}a_0 : 0 : \lambda_{ik}a_2)$$

$$\mathcal{E}_{mn} : (\lambda_{ik}\lambda_{jk}a_0 : \lambda_{jk}a_1 : -\lambda_{ik}a_2 : 0)$$

and the equation of the quartic takes the symmetric form

$$a_1^2(t_0^2 t_2^2 + t_1^2 t_3^2) + a_2^2(t_0^2 t_1^2 + t_2^2 t_3^2) + a_0^6(t_0^2 t_3^2 + t_1^2 t_2^2)$$
$$(25) \qquad + 2a_1 a_2(t_0 t_1 - t_2 t_3)(t_0 t_2 - t_1 t_3) + 2a_0^3 a_2(t_0 t_1 + t_2 t_3)(t_0 t_3 - t_1 t_2)$$
$$- 2a_0^3 a_1(t_0 t_2 + t_1 t_3)(t_0 t_3 + t_1 t_2) + 2\delta t_0 t_1 t_2 t_3 = 0,$$

where $\delta = -2\bar{\sigma}_{3,ij} + (\sigma_{1,ij}^2 - 6\sigma_{2,ij})\bar{\sigma}_{1,ij} + \sigma_{1,ij}(\bar{\sigma}_{2,ij} - 2\sigma_{1,ij}^2 + 9\sigma_{2,ij})$. Notice that this equation is precisely Equation (5) in [16, p. 301] which was found by using theta group techniques with an appropriate basis of sections of $[2\Theta]$, in which the action of the theta group related to this bundle is given by permutations and sign changes.

The equation with respect to a Göpel tetrad, such as $e_{00}, e_{0k}, e_{ij}, e_{mn}$ (all indices different) is found in the same way.

It is clear that in the case of $|2\Theta|$ no birational images of the Kummer surface are obtained by looking at sections which vanish at one or several half periods.

5.2. **The linear system $|3\Theta|$.** In the case $D = 3\Theta$ we will find several different projective images of the Kummer surface $\tilde{\mathcal{K}}_\Gamma$ of $\mathcal{J}_\Gamma$. Since D has odd multiplicity at the origin it is an odd section and the half periods $e_{00}, e_{01}, \ldots, e_{05}$ are even while the other ten half periods are odd. If follows from Lemma 5.1 that the linear system $|3\Theta|$ contains besides 3Θ another fifty totally reducible divisors:

$$\mathcal{D}_+ : \Theta_{0i} + \Theta_{0j} + \Theta_{ij} \qquad\qquad (0 < i < j \leq 5),$$
$$\mathcal{D}_+' : \Theta_{ij} + \Theta_{jk} + \Theta_{ki} \qquad\qquad (0 < i < j < k \leq 5),$$
$$\mathcal{D}_- : \Theta_{ij} + \Theta_{kl} + \Theta_{mn} \qquad\qquad (i, j, \ldots, n \text{ all } \neq),$$
$$\mathcal{D}_-' : \Theta + 2\Theta_{ij} \qquad\qquad (0 \leq i < j \leq 5).$$

The ten divisors $\mathcal{D}_+$ and the ten divisors $\mathcal{D}_+'$ are even since their multiplicity at the origin is two or zero, while the fifteen divisors $\mathcal{D}_-$ and the fifteen divisors $\mathcal{D}_-'$ are odd, their multiplicity at the origin being one or three. Here is a picture of a particular $\mathcal{D}_+$ and $\mathcal{D}_-$.
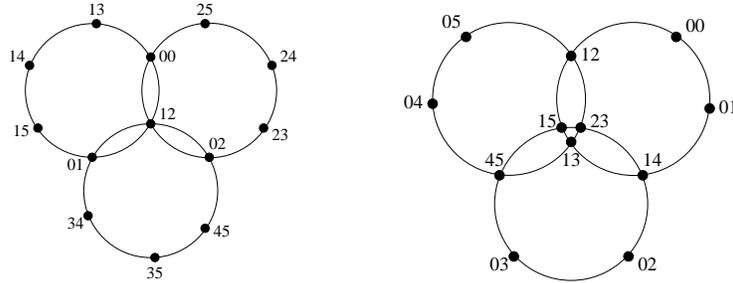
We denote their projections on $\tilde{\mathcal{K}}_\Gamma$ by C_+ and C_-. It follows from (4) that $h^0(C_+) = 4$ and $h^0(C_-) = 5$. This leads to two maps $\phi_{\mathcal{M}^+} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^3$ and $\phi_{\mathcal{M}^-} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^4$; we will investigate later in this paragraph the subsystem defined by odd sections that vanish at one of the odd half periods.

We first investigate the map $\phi_{\mathcal{M}^+}$. We find from Table 3 four independent even sections in $[3\Theta]$ and accordingly we define

(26)
$$\begin{aligned}
\theta_0 &= sv_1, \\
\theta_1 &= sv_2, \\
\theta_2 &= s(u_1 v_2 - u_2 v_1), \\
\theta_3 &= s((w_1 + u_1^2)v_2 - (w_2 + u_1 u_2)v_1),
\end{aligned}$$

where s denotes the section that cuts out 3Θ. The six half periods on Θ are even and the other ten half periods are odd.

Proposition 5.3. *The linear system $|3\Theta|^+$ has only the ten odd half periods as base points; however, it defines a regular map $\phi_{\mathcal{M}^+} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^3$. Its image is a quartic surface whose equation is given in terms of the coordinates θ_i, $i = 0, \dots, 3$ by*

(27)
$$\begin{aligned}
0 = {} &-\sigma_5(\sigma_1\theta_2 - \theta_3)\theta_0^3 + [(\sigma_1\sigma_4 + 3\sigma_5)\theta_1\theta_2 - \sigma_4(\theta_2^2 + \theta_1\theta_3)]\theta_0^2 \\
&+ [-2\sigma_5\theta_1^3 - (\sigma_1\sigma_3 + \sigma_4)\theta_1^2\theta_2 + 2\sigma_3\theta_1\theta_2^2 - \sigma_2\theta_2^3 + \sigma_3\theta_1^2\theta_3 + \sigma_1\theta_2^2\theta_3 - \theta_2\theta_3^2]\theta_0 \\
&+ \sigma_4\theta_1^4 + (\sigma_1\sigma_2\theta_2 - \sigma_3\theta_2 - \sigma_2\theta_3)\theta_1^3 - (\sigma_1^2\theta_2^2 - \theta_3^2)\theta_1^2 + (2\sigma_1\theta_2 - \theta_3)\theta_1\theta_2^2 - \theta_2^4.
\end{aligned}$$

$\phi_{\mathcal{M}^+}$ contracts the (-2)-curves B_i which correspond to the six even half periods and maps the ten other (-2)-curves B_i to lines. The image of Θ has degree three while the other theta curves map to lines. No other curves are contracted by $\phi_{\mathcal{M}^+}$.

Proof. If $X \in \mathcal{J}_\Gamma$ is a half period that does not belong to Θ then (13) implies that the corresponding polynomial $v(\lambda)$ is zero, so that all sections, given by (26) vanish and X belongs to the base locus of $|3\Theta|^+$. Suppose that we have another affine point X where all sections vanish, $X = [P + Q - 2\infty]$. If $\lambda(P) \neq \lambda(Q)$ then $v_1 = (\mu(P) - \mu(Q))/(\lambda(P) - \lambda(Q)) = 0$ implies $\mu(P) = \mu(Q)$. Further $v_2 = (\lambda(P)\mu(Q) - \lambda(Q)\mu(P))/(\lambda(P) - \lambda(Q)) = 0$ implies that $\mu(P) = \mu(Q) = 0$, which contradicts the fact that X is not a half period. If $\lambda(P) = \lambda(Q)$ then $v_1 = f'(\lambda(P))/(2\mu(P)) = 0$ implies $f'(\lambda(P)) = 0$ and $v_2 = \mu(P) - \lambda(P)f'(\lambda(P))/(2\mu(P)) = 0$ implies $\mu(P) = 0$, again a contradiction. In order to see what happens to the corresponding linear system $\mathcal{M}^+$ at B_{ij} we take a curve $X(t) = [P(t) + Q(t) - 2\infty]$, with $P(0) = \omega_i$ and $Q(0) = \omega_j$,

$$P(t) = \left(\lambda_i + t^2, c_i t + O(t^2)\right),$$
$$Q(t) = \left(\lambda_j + (\alpha t)^2, \alpha c_j t + O(t^2)\right),$$

where $c_i^2 = \prod_{j \neq i}(\lambda_i - \lambda_j) \neq 0$. The factor α was introduced here to represent the different directions at e_{ij}, which become points of the exceptional divisor E_{ij} and of B_{ij}. Computing (13) for these curves and taking the limit $t \to 0$ for their images in $\mathbb{P}^3$ we find

$$(\theta_0 : \theta_1 : \theta_2 : \theta_3) = (c_i - \alpha c_j : \alpha c_j \lambda_i - c_i \lambda_j : c_i \lambda_j^2 - \alpha c_j \lambda_i^2 : \star),$$

($\star$ is a finite number that is easily computed but whose value is not important for us) so that for any α there is at least one section which is non-zero. Notice that we don't need to consider the value $\alpha = \infty$ because of the symmetric role of P and Q. To show that no base point of $|3\Theta|^+$ lies on Θ, proceed as in the proof of Proposition 5.2: first consider $X(t) = [P + Q(t) - 2\infty]$ where $P \in \Gamma \setminus \{\infty\}$ and take $Q(t) = (\lambda(t), \mu(t)) \in \Gamma$ to be given by (10). If we evaluate the map $(v_1 : v_2 : u_1 v_2 - u_2 v_1 : (w_1 + u_1^2)v_2 - (w_2 + u_1 u_2)v_1)$ at $X(t)$ and take the limit for $t \to 0$ then we find $(1 : -\lambda(P) : \lambda(P)^2 : \star)$ (again the (finite) value of $\star$ is irrelevant). This shows that, besides possibly the origin, no point on the theta divisor is contained in the base locus of $|\mathcal{M}^+|$. Letting $P(t)$ as well as $Q(t)$ be given by (10) one check in a similar way that the origin is also not contained in the base locus, i.e., the base locus of $|\mathcal{M}^+|$ is empty and $\phi_{\mathcal{M}^+}$ is regular.

An equation for the image of $\phi_{\mathcal{M}^+}$ is computed as follows. Use the first three equations in (21) to eliminate all w_i linearly and use the first three equations of (26) to eliminate v_1, v_2 and u_2. From the remaining equations in (21) and (26) eliminate first s to obtain two equations in u_1 one of which is linear. Elimination of u_1 gives the announced equation for the quartic. It can be shown as in the proof of Proposition 5.2 that this quartic is not a complete square; this will be however most obvious after we have rewritten the equation in a more symmetric form, so we will not do this verification at this point.

We have that $\rho_i = 0$ at the six even half periods e_{oi} and $\rho_i = 1$ at the ten odd half periods so, using Proposition 3.2, we find that the image of $\phi_{\mathcal{M}^+}$ will have six singular points and will contain ten disjoint lines, coming from the (-2)-curves. Since Θ does not contain any of the odd half periods, Formula (6) implies that the image of Θ under $\phi_{\mathcal{M}^+}$ is a cubic curve which passes through the six singular points; the other theta curves all contain precisely four odd half periods so these curves map to fifteen lines.

Finally, suppose that some irreducible curve, which is different from the (-2)-curves, is contracted. Since it is different from the theta divisor it intersects the affine part $\mathcal{J}_\Gamma \setminus \Theta$ and can be written as $X(t) = [P(t) + Q(t) - 2\infty]$ where $P(t) = (\lambda_1(t), \mu_1(t))$ and $Q(t) = (\lambda_2(t), \mu_2(t))$. As in the proof of Proposition 5.2 we may assume that $\lambda_1(t) \neq \lambda_2(t)$ and $\mu_1(t) \neq \mu_2(t)$. Let us assume that the whole curve is mapped to the single point $(1 : c_1 : c_2 : c_3)$. Solving for λ_2 and μ_2 we find that

$$\lambda_2(t) = -\frac{c_1\lambda_1(t) + c_2}{\lambda_1(t) + c_1} \qquad \mu_2(t) = \frac{c_1^2 - c_2}{(\lambda_1(t) + c_1)^2}\mu_1(t).$$

Since $\mu_i^2(t) = f(\lambda_i(t))$ for $i = 1, 2$ we find that $\lambda_1(t)$ satisfies an algebraic equation of degree eight with leading term $(c_2 - c_1^2)\lambda_1^8(t)$. Then $c_2 = c_1^2$ because otherwise $\lambda_1(t)$ and hence also $\mu_1(t)$, $\lambda_2(t)$ and $\mu_2(t)$ are constant. However, if $c_2 = c_1^2$ then $\mu_2(t) = 0$ so that the curve corresponds to one of the theta curves. As we have seen, these theta curves map to lines, not to points. Therefore no such curve is contracted by $\phi_{\mathcal{M}^+}$. $\square$

We will now construct coordinates for $\mathbb{P}^3$ with respect to which the equation of the quartic takes a completely symmetric form. First we show that any four of the singular points $\mathcal{E}_{00}, \mathcal{E}_{01}, \ldots, \mathcal{E}_{05}$ can be taken as base point for $\mathbb{P}^3$. Since $\mathcal{T}_{00}$ is a cubic curve and passes through all six singular points it will be planar as soon as four of the singular points are coplanar. Then all six points singular are coplanar and hence also the fifteen lines $\mathcal{T}_{ij}$, $0 \leq i < j \leq 5$, which join these singular points.

Then these lines would have intersection points besides the singular points, which is impossible. We will take the points $\mathcal{E}_{01}, \ldots, \mathcal{E}_{04}$ as base points for $\mathbb{P}^3$, so we need to find the coordinates of these points. Notice that they are given by $\mathcal{E}_{0i} = \mathcal{T}_{ij} \cap \mathcal{T}_{oj}$. We do this by first computing the sections which cut out the divisors $\mathcal{D}_+$. If we express that a section

$$\alpha\theta_0 + \beta\theta_1 + \gamma\theta_2 + \delta\theta_3$$

vanishes on Θ_{0i} and Θ_{0j} (using the parametrization (18) of Θ_{0i}) then we find

$$\alpha = \lambda_i\beta - \lambda_i^2\gamma = \lambda_j\beta - \lambda_j^2\gamma, \quad \delta = 0,$$

and we obtain that

$$(28) \qquad\qquad f_{ij}^+ = \sigma_{2,ij}\theta_0 - \sigma_{1,ij}\theta_1 + \theta_2$$

is (up to a constant) the only even section that vanishes on Θ_{0i} and Θ_{0j}. Since we know that there exists an even section which vanishes in addition on Θ_{ij} this section must also vanish on Θ_{ij}. The latter fact can of course also be verified directly using (19) and (20). If we intersect the quartic surface (27) with the hyperplane $\theta_2 = \sigma_{1,ij}\theta_1 - \sigma_{2,ij}\theta_0$ then we find the equations for the four lines $\mathcal{T}_{0i}, \mathcal{T}_{0j}, \mathcal{T}_{ij}$ and $\mathcal{E}_{ij}$. On the one hand we get from it that a parametrization for $\mathcal{T}_{0i}$ ($1 \leq i \leq 5$) is given by ($t \in \mathbb{P}$)

$$\mathcal{T}_{0i} \;\; : \;\; (1 : -\lambda_i : \lambda_i^2 : t).$$

On the other hand we find two factors which give the following equations for for $\mathcal{T}_{ij}$ and $\mathcal{E}_{ij}$ (to see that it is the first one which corresponds to $\mathcal{T}_{ij}$ and not the second one, one can use e.g. (19) and (20)):

$$\mathcal{T}_{ij} \;\; : \;\; (1 : t - \lambda_i : \lambda_i^2 + \sigma_{1,ij}t : \lambda_i^2\bar{\sigma}_{1,i} + t(\sigma_{1,ij}\bar{\sigma}_{1,ij} + \sigma_{2,ij})).$$

$$\mathcal{E}_{ij} \;\; : \;\; (1 : t + \sigma_{1,ij} : \sigma_{1,ij}t + \sigma_{1,ij}^2 - \sigma_{2,ij} :$$
$$(\sigma_{1,ij}^2 + \bar{\sigma}_{2,ij})t + \sigma_{1,ij}^3 + (\bar{\sigma}_{2,ij} - \sigma_{2,ij})\sigma_{1,ij} - \bar{\sigma}_{3,ij}).$$

This leads to $\mathcal{E}_{0i} = \mathcal{T}_{0i} \cap \mathcal{T}_{ij} = (1 : -\lambda_i : \lambda_i^2 : \lambda_i^2\bar{\sigma}_{1,i})$ and $\mathcal{E}_{00} = \mathcal{T}_{0i} \cap \mathcal{T}_{0j} = (0 : 0 : 0 : 1)$. Now take the points $\mathcal{E}_{01}, \ldots, \mathcal{E}_{04}$ as base points and call the corresponding coordinates $t_1, \ldots, t_4$ then the quartic takes the following symmetric form

$$(29) \qquad\qquad \sum_{i=1}^4 \sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} \lambda_{ij}^2 \lambda_{jk}^2 \lambda_{ki}^2 \lambda_{im} \lambda_{in} t_i^2 t_j t_k = 0,$$

where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$. Since all terms in the new equation of the quartic are of the form $t_i^2 t_j t_k$ this equation can never be an exact square providing a simple proof for our earlier claim that the image of $\phi_{\mathcal{M}^+}$ is a quartic. With respect to the new basis for $\mathbb{P}^3$ the singular points $\mathcal{E}_{00}$ and $\mathcal{E}_{05}$ have the following coordinates:

$$\mathcal{E}_{00} : (\lambda_{23}\lambda_{34}\lambda_{42} : \lambda_{31}\lambda_{14}\lambda_{43} : \lambda_{41}\lambda_{12}\lambda_{24} : \lambda_{13}\lambda_{32}\lambda_{21}),$$

$$\mathcal{E}_{05} : \left(\frac{1}{\lambda_{12}\lambda_{13}\lambda_{14}\lambda_{15}} : \frac{1}{\lambda_{21}\lambda_{23}\lambda_{24}\lambda_{25}} : \frac{1}{\lambda_{31}\lambda_{32}\lambda_{34}\lambda_{35}} : \frac{1}{\lambda_{41}\lambda_{42}\lambda_{43}\lambda_{45}} \right).$$

Using the coordinates of $\mathcal{E}_{00}, \ldots, \mathcal{E}_{05}$ the new equations of the lines $\mathcal{T}_{ij}$ are immediately computed because $\mathcal{T}_{ij}$ passes through $\mathcal{E}_{0i}$ and $\mathcal{E}_{0j}$.

We now investigate the map ϕ_-. Table 3 gives us five independent sections of $[3\Theta]$. Still denoting by s the section that cuts out 3Θ, we define $\theta_0 = s$, $\theta_1 = su_1$, $\theta_2 = su_2$, $\theta_3 = s(u_1u_2 - w_2)$ and $\theta_4 = s(u_1w_2 + u_2w_1 + 2u_1u_2w_0)/2$.

Proposition 5.4. *The linear system* $|3\Theta|^-$ *is base-point-free, hence* $\phi_{\mathcal{M}^-} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^4$ *is a regular map. The image of this map is a complete intersection of a quadric and a cubic hypersurface whose equations are given, in terms of the coordinates* θ_i, $i = 0, \ldots, 4$ *by*

$$0 = 2\theta_0\theta_4 + \theta_1\theta_3 - (\sigma_2\theta_0 + \sigma_1\theta_1 - \theta_2)\theta_2, \tag{30}$$

and

$$
\begin{aligned}
0 = {}& (4\sigma_3\sigma_5 - \sigma_4^2)\theta_0^3 - 4(\sigma_2\sigma_5\theta_1 + \sigma_1\sigma_5\theta_2 - \sigma_5\theta_3 - \sigma_4\theta_4)\theta_0^2 \\
& + 4[\sigma_1\sigma_5\theta_1^2 + (\sigma_5 - \sigma_1\sigma_4)\theta_1\theta_2 + \sigma_3\theta_2\theta_3 - \theta_4^2]\theta_0 \\
& - 4(\sigma_5\theta_1^2 - \sigma_4\theta_1\theta_2 + \sigma_3\theta_2^2)\theta_1 + 4\theta_2(\sigma_1\theta_2 - \theta_3)(\sigma_2\theta_1 - \theta_3).
\end{aligned}
\tag{30}
$$

The theta divisor Θ *and the ten* (-2)-*curves* B_{ij}, $1 \le i < j \le 5$ *corresponding to the even half periods are the only divisors which are contracted by* $\phi_{\mathcal{M}^-}$, *while the other six exceptional divisors* B_{0i}, $0 \le i \le 5$, *map to lines and the other theta curves map to conics.*

Proof. The proof that the linear system $|2\Theta|^+$ is base-point-free applies verbatim to the present case because the sections $\theta_0, \ldots, \theta_3$ are defined in exactly the same way. The defining equation of θ_4 is easily rewritten in terms of the other θ_i giving the above equation of the quadric. Now obviously the quartic equation (22) holds between the sections, but that does not mean that the homogeneous ideal of the image is generated by the quadratic and the quartic polynomials. Indeed, if we add the quadratic polynomial in (30), multiplied by $-\sigma_1\theta_1\theta_2 + \theta_1\theta_3 + \theta_2^2 + 2\sigma_4\theta_0^2 - \sigma_2\theta_0\theta_2 - 2\theta_0\theta_4$, to the left hand side of the quartic (22) then the result is divisible by θ_0 and we are left with the cubic equation (30). Since the degree of the image is six the image is the complete intersection of the quadric and cubic hypersurface. Moreover, since $\phi_{[2\Theta]}$ does not contract any curves besides the curves B_i we can at least conclude that besides the B_i no curve *that intersects the affine part* $\mathcal{J}_\Gamma \setminus \Theta$ is contracted. In this case $\rho_{ij} = 0$ for $1 \le i < j \le 5$ (corresponding to the ten odd half periods) and $\rho_{0i} = 1$ for $0 \le i \le 5$ (corresponding to the other half periods). (6) shows that Θ is contracted by $\phi_{\mathcal{M}^-}$ and the same is true for the exceptional divisors B_{ij}, $1 \le i < j \le 5$. The remaining theta curves and exceptional divisors map to fifteen conics and six lines respectively. Notice that these six lines pass through a single singularity of the image. □

Again a more symmetric equation is obtained by choosing some of the singular points as base points, namely we choose $\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{34}, \mathcal{E}_{45}$ and $\mathcal{E}_{15}$ as base points. Using (14) we find that the old coordinates of $\mathcal{E}_{ij}$ are given by

$$\mathcal{E}_{ij} : (1 : \sigma_{1,ij} : \sigma_{2,ij} : \sigma_{1,ij}\sigma_{2,ij} - \bar{\sigma}_{3,ij} : (\sigma_{1,ij}\bar{\sigma}_{3,ij} + \sigma_{2,ij}\bar{\sigma}_{2,ij})/2 + \sigma_{2,ij}\sigma_{1,ij}\bar{\sigma}_{1,ij}).$$

If we define t_i to be the coordinates with respect to these five points then the equation of the quadric becomes

$$\lambda_{24}\lambda_{25}(\lambda_{13}^2 t_2 + \lambda_{14}\lambda_{15}t_4)t_1 + \text{ cycl } (1, 2, 3, 4, 5) = 0, \tag{31}$$

while the equation of the cubic becomes

(31)
$$\left(\frac{\lambda_{13}}{\lambda_{14}} t_2 + \frac{\lambda_{23}}{\lambda_{25}} t_3 + \frac{\lambda_{15}}{\lambda_{13}} t_4 + \frac{\lambda_{25}}{\lambda_{24}} t_5 \right) t_1^2 + (A t_3 + B t_4) t_1 t_2 + \text{ cycl } (1,2,3,4,5) = 0.$$

Since the constants A and B are quite complicated when expressed in terms of the λ_i we do not record their expressions here. Finally, let us compute the sections that cut out the image of the divisors $\mathcal{D}_-$. This is done as in the case of $\mathcal{D}_+$: such a section must be of the form

$$\alpha + \beta u_1 + \gamma u_2 + \delta(u_1 u_2 - w_2) + \epsilon(u_1 w_2 + u_2 w_1 + 2 u_1 u_2 w_0),$$

and it should vanish on $\mathcal{T}_{ij}$, $\mathcal{T}_{0k}$ and $\mathcal{T}_{mn}$, where $\{i, j, k, m, n\} = \{1, 2, 3, 4, 5\}$. If we normalize $\epsilon = 1$ then we get by using (19) and (20)

$$\begin{aligned}
\alpha &= 2\lambda_k^2 (\sigma_{2,ij} + \sigma_{2,mn}) - \lambda_k \bar{\sigma}_{3,k} - \bar{\sigma}_{4,k}, \\
\beta &= 2\lambda_k (\sigma_{2,ij} + \sigma_{2,mn}), \\
\gamma &= -2\bar{\sigma}_{1,ij} \bar{\sigma}_{1,mn}, \\
\delta &= -2\lambda_k, \\
\epsilon &= 1.
\end{aligned}$$

In the case of $|3\Theta|^-$ we can restrict ourselves to the sections with prescribed vanishing at the half periods. Every section of $|3\Theta|^-$ vanishes an odd number of times at the even half periods so that a prescribed vanishing at one of these half periods would imply that we consider $\mathcal{M}_\nu^-$ for $\nu = (0, \dots, 3, \dots, 0)$. Then formula (3.2) leads to $\dim |\mathcal{M}_\nu^-| = 9/2 + 1 - 14/4 = 2$, so the corresponding map can never be birational. Therefore we consider an odd half period e_{ij}, $1 \le i, j \le 5$ and define $\nu = (0, \dots, 2, \dots, 0)$ (the 2 being at position ij). Formula (3.2) now leads to $\dim |\mathcal{M}_\nu^-| = 4$, hence $\phi_{\mathcal{M}_\nu^-}$ maps to $\mathbb{P}^3$. Using the fact that $u_1 = -\lambda_i - \lambda_j = \sigma_{1,ij}$ and $u_2 = \lambda_i \lambda_j = \sigma_{2,ij}$ at e_{ij} we find from Table 3 that the following four independent sections vanish at e_{ij}.

$$\begin{aligned}
\theta_0 &= s(u_1 - \sigma_{1,ij}), \\
\theta_1 &= s(u_2 - \sigma_{2,ij}), \\
\theta_2 &= s(u_1 u_2 - w_2 - \sigma_{1,ij} \sigma_{2,ij} + \bar{\sigma}_{3,ij}), \\
\theta_3 &= s(u_1 w_2 + u_2 w_1 + 2 u_1 u_2 w_0 - \sigma_{1,ij} \bar{\sigma}_{3,ij} - \sigma_{2,ij} \bar{\sigma}_{2,ij} - 2\sigma_{1,ij} \sigma_{2,ij} \bar{\sigma}_{1,ij}).
\end{aligned}$$
(32)

We describe $\phi_{\mathcal{M}_\nu^-}$ and its image in the following proposition.

Proposition 5.5. *The linear system $|\mathcal{M}_\nu^-|$ is base-point-free, hence $\phi_{\mathcal{M}_\nu^-} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^3$ is regular. It contracts ten divisors, to wit the nine exceptional divisors B_{kl} corresponding to the even half periods, but not B_{ij}, and the theta divisor Θ. The image is a quartic which contains six lines $\mathcal{E}_{0i}$ ($0 \le i \le 5$) which are collinear at $\mathcal{T}$ and it contains sixteen conics, one of which is the image $\mathcal{E}$ of B_{00}.*

Proof. Using (11) for the half period e_{ij} it follows at once that the image of B_{ij} is a conic; alternatively this is seen from $\rho_{ij} = \nu_{ij} = 2$. Then (32) implies that the only possible base points coorespond to $s = 0$, the theta divisor. Using (10) we see that the theta divisor gets mapped to the single point $(0 : 0 : 0 : 1)$; using (11) for any of the other nine even half periods it follows that each gets contracted. The verification that the odd half periods map to lines and that the other theta

curves map to conics is similar. Let us assume that some other divisor D gets mapped to the point $(a : b : c : d) \in \mathbb{P}^3$ and let us show how this leads to a contradiction (assuming Γ smooth). We can parametrize D as $[P(t) + Q(t) - 2\infty]$ where $P(t) = (\lambda_1(t), \mu_1(t))$ and $Q(t) = (\lambda_2(t), \mu_2(t))$. Then we look for solutions of

$$a\lambda(t) = u_1(t) - \sigma_{1,ij},$$
$$b\lambda(t) = u_2(t) - \sigma_{2,ij},$$
(33) $\qquad c\lambda(t) = u_1(t)u_2(t) - w_2(t) - \sigma_{1,ij}\sigma_{2,ij} + \bar{\sigma}_{3,ij},$
$$d\lambda(t) = u_1(t)w_2(t) + u_2(t)w_1(t) + 2u_1(t)u_2(t)w_0(t)$$
$$- \sigma_{1,ij}\bar{\sigma}_{3,ij} - \sigma_{2,ij}\bar{\sigma}_{2,ij} - 2\sigma_{1,ij}\sigma_{2,ij}\bar{\sigma}_{1,ij}).$$

with $a, \ldots, d$ constant. The case $a = b = 0$ can be excluded at once because it does not correspond to a curve. Similarly when $\lambda_1(t) = \lambda_i$ and $a = 0$. If $\lambda_1(t) = \lambda_i$, so that $\mu_i(t) = 0$ then we may therefore assume that $a = 1$. Then the first two equations in (33) imply $b = \lambda_i$ and $\lambda(t) = \lambda_2(t) - \lambda_j$. The third equation in (33) then leads to $\lambda_2(t) = \lambda_i$ or $\lambda_2(t) = \lambda_j$, which are both unacceptable, or to the following linear equation in $\lambda_2(t)$:

$$\lambda_2(\lambda_i\bar{\sigma}_{1,i} - c) + \lambda_i(\lambda_i^2 + \sigma_{2,ij} + \bar{\sigma}_{2,ij} + c) + \bar{\sigma}_{3,ij} = 0.$$

If $\lambda_2(t)$ is not constant then it can only satisfy the above equation of both coefficients are zero, which happens for no value of c. By symmetry we can also exclude the case $\lambda_1(t) = l_j$. Also the case $b = -\lambda_1$ can be excluded because it leads to the previous cases $b = \lambda_i$ or $b = \lambda_j$. We treated these special cases because $\lambda_1(t) - \lambda_i$, $\lambda_1(t) - \lambda_j$ and $b + \lambda_1$ appear frequently as factors in the two cases considered next. The first one corresponds to the general case in which $a = 0$. Then we can take $b = 1$ and the first two equations lead to $\lambda_1 = -\lambda_2 + \lambda_i + \lambda_j$ and $\lambda = -(\lambda_2 - \lambda_i)(\lambda_2 - \lambda_j)$. If we substitute this in the third equation of (33), we find a monic equation in $\lambda_2(t)$ with constant coefficients, which leads to $\lambda_1(t)$ and $\lambda_2(t)$ being constant, a contradiction. The second case corresponds to the generic case $a \neq 0$. Taking $a = 1$ we find $\lambda(t) = -(\lambda_1(t) - \lambda_i)(\lambda_1(t) - \lambda_j)/(\lambda_1(t) + b)$ and $\lambda_2(t) = (\lambda_i + b)(\lambda_j + b)/(\lambda_1 + b) - b$. If we substitute this in the third equation of (33), we find an equation for $\lambda_2(t)$ which depends on b and c. As before one finds that if Γ is non-singular there are no values for b and c such that all coefficients of this polynomial are zero. A proof that the image is a quartic and an explicit equation for it will be given below. $\qquad\square$

Since the roots λ_i of f will appear explicitly in the equation of the quartic we will not write down the equations in terms of the θ_i but we pass at once to a set of natural coordinates, which will give the equation of the quartic a symmetric form. The conic $\mathcal{E}_{ij}$ intersects the lines $\mathcal{T}_{0i}$, $\mathcal{T}_{0j}$ and $\mathcal{T}_{ij}$ in three points (which are not collinear) and these points are independent of the image $\mathcal{T}$ of Θ (which is a singular point). We will take these points as basis points for $\mathbb{P}^3$. To do this we first need to find their coordinates, which is done in this case as follows. We use (18) to compute the images of Θ_{0i} and we take the limit for $\lambda \to \lambda_j$ $(i \neq j)$. This gives us the following coordinates:

$$\mathcal{E}_{ij} \cap \mathcal{T}_{0i} \; : \; (\lambda_{ij} : -\lambda_i\lambda_{ij} : \lambda_i(\lambda_i^2 + \lambda_i\lambda_j - \lambda_j^2) + \sigma_{2,ij}\bar{\sigma}_{1,ij} + \lambda_i\bar{\sigma}_{2,ij} + \bar{\sigma}_{3,ij}$$
$$: 2\lambda_i[\lambda_i^2\lambda_j - (\lambda_j^2 - \lambda_i\lambda_j - \lambda_i^2)\bar{\sigma}_{1,ij} + \lambda_j\bar{\sigma}_{2,ij} + \bar{\sigma}_{3,ij}]).$$

Using (19) and (20) we find the image of Θ_{ij} and the limit for $\lambda \to \infty$ gives the following intersection point:

$$\mathcal{E}_{ij} \cap \mathcal{T}_{ij} \; : \; (-\bar{\sigma}_{2,ij} + \sigma_{1,ij}(\bar{\sigma}_{1,ij} - \sigma_{1,ij}) + \sigma_{2,ij} : -\bar{\sigma}_{3,ij} + \sigma_{2,ij}(\bar{\sigma}_{1,ij} - \sigma_{1,ij})$$

$$: -\bar{\sigma}_{3,ij}\bar{\sigma}_{1,ij} + \sigma_{2,ij}(\bar{\sigma}_{1,ij}^2 - \bar{\sigma}_{2,ij} - \sigma_{1,ij}^2 + \sigma_{2,ij})$$

$$: 2[-\bar{\sigma}_{3,ij}(\bar{\sigma}_{2,ij} + \sigma_{1,ij}^2 - \sigma_{2,ij}) + \bar{\sigma}_{1,ij}\sigma_{1,ij}\sigma_{2,ij}(\bar{\sigma}_{1,ij} - \sigma_{1,ij})])$$

Also recall that Θ is mapped to $\mathcal{T} = (0 : 0 : 0 : 1)$. If we take the following points as base points for $\mathbb{P}^3$ (in that order)

$$\mathcal{T}, \mathcal{E}_{ij} \cap \mathcal{T}_{0i}, \mathcal{E}_{ij} \cap \mathcal{T}_{0j}, \mathcal{E}_{ij} \cap \mathcal{T}_{ij}$$

and we denote the corresponding coordinates (properly scaled) by $t_0, \ldots, t_3$ then we find the following equation for the quartic:

$$(t_1 t_2 + t_2 t_3 + t_3 t_1 - t_0^2)[\alpha_j t_1^2 - \alpha_i t_2^2 + \alpha t_3^2 + (\alpha_j - \alpha_i + \alpha)(t_1 t_2 + t_2 t_3 + t_3 t_1)]$$

(34)

$$+ t_0(t_1 t_2 + t_2 t_3 + t_3 t_1 + t_3^2)[\alpha(t_1 - t_2) + \beta(t_1 + t_2) + 2\gamma t_3]$$

$$- 2t_0 t_3[\alpha_j t_1^2 + \alpha_i t_2^2 + \gamma t_3^2)] = 0$$

where

$$\alpha = \lambda_{ij}^3, \qquad \alpha_l = \prod_{k \neq i,j} \lambda_{lk},$$

$$\beta = 4(\bar{\sigma}_{3,ij} + \sigma_{2,ij}\bar{\sigma}_{1,ij}) + \sigma_{1,ij}(\sigma_{1,ij}^2 - 2\bar{\sigma}_{2,ij} - 6\sigma_{2,ij}),$$

$$\gamma = \sigma_{1,ij}(\bar{\sigma}_{2,ij} - \sigma_{2,ij}) - 2\bar{\sigma}_{3,ij} + (\sigma_{1,ij}^2 - 2\sigma_{2,ij})(\sigma_{1,ij} - \bar{\sigma}_{1,ij}).$$

Now we can easily see that ϕ'_- is birational: if the equation of the quartic is a square then the coefficient in t_0 of degree zero is a square hence $\alpha_j t_1^2 - \alpha_i t_2^2 + \alpha t_3^2 + (\alpha_j - \alpha_i + \alpha)(t_1 t_2 + t_2 t_3 + t_3 t_1)$ and $t_1 t_2 + t_2 t_3 + t_3 t_1$ are proportional. In particular $\alpha = 0$ so that $\lambda_i = \lambda_j$ which is impossible since Γ is non-singular.

It was pointed out to us by the referee that a quartic Kummer surface in $\mathbb{P}^3$ with ten singular points was considered recently by J. E. Rosenberg (see [24]). Our surfaces are however different, because our surfaces correspond to blowing down 9 exceptional divisors E_i and the theta divisor, while the ones that appear in [24] correspond to blowing down 10 exceptional divisors E_i.

5.3. **The linear system** $|4\Theta|$. In this case all half periods are even and Lemma 5.1 implies that *up to a translation over a half period* the only totally reducible divisors in $|4\Theta|$ are, besides 4Θ, the following

$$\mathcal{D}_+ : \Theta + \Theta_{ij} + \Theta_{kl} + \Theta_{mn} \qquad\qquad (i, j, \ldots, n \text{ all } \neq),$$

$$\mathcal{D}'_+ : 2\Theta + 2\Theta_{ij} \qquad\qquad (0 \leq i < j \leq 5),$$

$$\mathcal{D}_- : \Theta + \Theta_{0i} + \Theta_{0j} + \Theta_{ij} \qquad\qquad (0 < i < j \leq 5),$$

$$\mathcal{D}'_- : \Theta + \Theta_{ij} + \Theta_{jk} + \Theta_{ki} \qquad\qquad (0 < i < j < k \leq 5).$$

Below we give a picture of particular divisors $\mathcal{D}_+$ and $\mathcal{D}_-$. One sees that the four curves in $\mathcal{D}_+$ intersect in twelve nodes while the curves in $\mathcal{D}_-$ intersect in four triple points. The divisors $\mathcal{D}_+$ and $\mathcal{D}'_+$ are even while the divisors $\mathcal{D}_-$ and $\mathcal{D}'_-$ are odd.
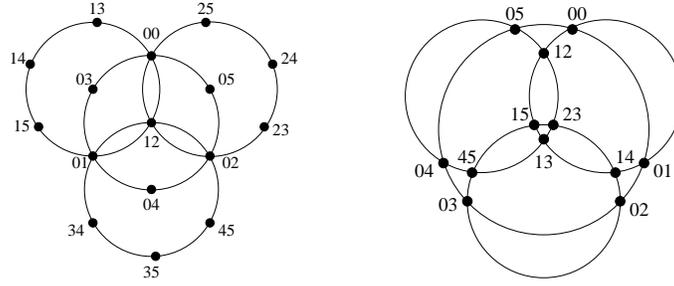
We denote by C_- the projection of $p^* \mathcal{D}_-$ on $\tilde{\mathcal{K}}_\Gamma$ and we write ϕ_- for $\phi_{[C_-]}$. By (3.2) we have that $h^0(\tilde{X}, C) = 6$. We denote by s the section that cuts out $[4\Theta]$. and we find from Table 3 that the following sections $\theta_0, \ldots, \theta_5$ provide a basis for the odd sections of 4Θ.

$$\theta_0 = s v_1,$$
$$\theta_1 = s v_2,$$
$$\theta_2 = s(u_1 v_2 - u_2 v_1),$$
$$\theta_3 = s((w_1 + u_1^2)v_2 - (w_2 + u_1 u_2)v_1),$$
$$\theta_4 = s(v_1 w_2 + u_1 w_0 v_2),$$
$$\theta_5 = s(u_1 u_2 v_2 - v_2 w_2 - u_2^2 v_1 - u_2 v_2 w_0).$$

In this case we easily find

Proposition 5.6. *The map $\phi_- : \tilde{X} \to \mathbb{P}^5$ is an isomorphism onto its image. The images $\mathcal{E}_{ij}$ and $\mathcal{T}_{ij}$ form two groups of sixteen disjoint lines, each line intersecting six lines of the other group.*

Proof. Comparing the section $\theta_0, \ldots, \theta_5$ to the sections that were used in the case of ϕ_+ for 3Θ we see that no affine point can be a base point. We will compute below an equation for the image of theta, which is a line since $\rho_i = 1$ for all i. So $|C|$ is basepoint-free. Moreover, since the only divisor which was contracted by ϕ_+ in the case of 3Θ was Θ, which is not contracted in this case, ϕ_- is an isomorphism. $\rho_i = 1$ for all i, hence all (-2)-curves are mapped to (disjoint) lines; their equations will be computed below. Also $C \cdot \Theta_{ij} = 4 - 6/2 = 1$ so all theta curves are mapped to sixteen disjoint lines. $\square$

We will find the relations between the θ_i by expressing the fact that the image contains a whole configuration of lines, coming from the theta curves Θ_{ij} and the (-2)-curves B_i. The lines $\mathcal{T}_{ij}$ (where i and j are not both zero) can be computed explicitly using the parametrizations for the divisors Θ_{ij}. For $\mathcal{T}_{0i}$ $(i \neq 0)$ we find by using (18) the following parametrization $(t \in \mathbb{P})$:

$$\left(1 : -\lambda_i : \lambda_i^2 : t : -t - \lambda_i(\sigma_2 + \lambda_i^2) : -\lambda_i t - \lambda_i^2(\lambda_i^2 + \bar{\sigma}_{2,i})\right).$$

For $\mathcal{T}_{ij}$ $(i, j \neq 0)$ use (19) and (20) to find

$$(t : 1 : \sigma_{1,ij} - \sigma_{2,ij}t : \sigma_{2,ij} + \sigma_{1,ij}\bar{\sigma}_{1,ij} - \bar{\sigma}_{1,ij}\sigma_{2,ij}t : \sigma_{1,ij}\bar{\sigma}_{1,ij} + \bar{\sigma}_{3,ij}t :$$
$$\sigma_{2,ij}\sigma_{1,ij} - \sigma_{2,ij}\bar{\sigma}_{1,ij} - \bar{\sigma}_{3,ij} - \sigma_{2,ij}^2 t)$$

Note that we cannot compute the lines $\mathcal{E}_{ij}$ in this way because all the functions v_i vanish at the half periods. However, any quadric which vanishes at the lines $\mathcal{T}_{ij}$ must also vanish at the lines $\mathcal{E}_{ij}$ because every line $\mathcal{E}_{ij}$ has six points lying on

the lines $\mathcal{T}_{ij}$. It is now easy to find and solve the (linear) conditions on α_{ij} for $\sum_{i \leq j} \alpha_{ij} v_i v_j$ to vanish on the lines $\mathcal{T}_{ij}$: we get the following set of independent quadrics:

$$\theta_2^2 - \sigma_1 \theta_1 \theta_2 + \theta_1 \theta_4 + \theta_0 \theta_5 = 0,$$

(35) $$\sigma_5 \theta_0^2 + \sigma_3 \theta_1^2 - \sigma_4 \theta_0 \theta_1 - \sigma_2 \theta_1 \theta_2 + \theta_1 \theta_5 + \theta_2 \theta_4 = 0,$$

$$\sigma_4(\theta_1^2 + \theta_0 \theta_2) + \sigma_2(\theta_2^2 - \theta_1 \theta_3) + \theta_3^2 - 2\sigma_5 \theta_0 \theta_1 +$$
$$(\sigma_1 \sigma_2 - 2\sigma_3)\theta_1 \theta_2 - \sigma_1 \theta_2(\theta_3 + \theta_4) + \theta_3 \theta_4 - \theta_2 \theta_5 = 0.$$

Note that again these equations do not involve the roots λ_i of $f(\lambda)$ explicitly. We will see that by using the λ_i explicitly we can make the equations much more symmetric. Before we can do this we need to compute the equations for the other lines ($\mathcal{T}_{00}$ and all $\mathcal{E}_{ij}$) and the ninety-six intersection points of the configuration. The following lemma provides an effective way to do this.

Lemma 5.7. *The hyperplane section which vanishes on the lines $\mathcal{T}_{00}$, $\mathcal{T}_{0i}$, $\mathcal{T}_{0j}$ and $\mathcal{T}_{ij}$ also vanishes on the lines $\mathcal{E}_{00}$, $\mathcal{E}_{0i}$, $\mathcal{E}_{0j}$ and $\mathcal{E}_{ij}$*

Proof. The points e_{00}, e_{0i}, e_{0j} and e_{ij} are triple points of the divisor $\Theta_{00} + \Theta_{0i} + \Theta_{0j} + \Theta_{ij}$ hence the lines $\mathcal{E}_{00}$, $\mathcal{E}_{0i}$, $\mathcal{E}_{0j}$ and $\mathcal{E}_{ij}$ have three points in common with the hyperplane that vanishes on $\mathcal{T}_{00}$, $\mathcal{T}_{0i}$, $\mathcal{T}_{0j}$ and $\mathcal{T}_{ij}$. $\square$

In fact, since the degree of $\phi_-(\tilde{\mathcal{K}}_\Gamma)$ is eight the hyperplane section must consist exactly of these eight lines. It is now easy to do the computation: since this hyperplane section is given as in (28) by $\sigma_{2,ij} \theta_0 - \sigma_{1,ij} \theta_1 + \theta_2 = 0$ it suffices to intersect the quadrics with the hyperplane

$$\theta_2 = \sigma_{1,ij} \theta_1 - \sigma_{2,ij} \theta_0$$

which amounts to solving the equations of the quadrics linearly for the remaining variables. Besides the lines $\mathcal{T}_{0i}$, $\mathcal{T}_{0j}$ and $\mathcal{T}_{ij}$ for which we gave the equations above we also find the following lines

$\mathcal{T}_{00} : (0 : 0 : 0 : 0 : 1 : t)$,

$\mathcal{E}_{00} : (0 : 0 : 0 : 1 : -1 : t)$,

$\mathcal{E}_{0i} : (1 : -\lambda_i : \lambda_i^2 : \lambda_i^2 \bar{\sigma}_{1,i} : t : \lambda_i t - \lambda_i^3 \bar{\sigma}_{1,i})$,

$\mathcal{E}_{0j} : (1 : -\lambda_j : \lambda_j^2 : \lambda_j^2 \bar{\sigma}_{1,i} : t : \lambda_j t - \lambda_j^3 \bar{\sigma}_{1,i})$,

$\mathcal{E}_{ij} : (t : 1 : \sigma_{1,ij} - \sigma_{2,ij} t : \sigma_{1,ij}^2 + \bar{\sigma}_{2,ij} - (\sigma_{1,ij} \sigma_{2,ij} + \bar{\sigma}_{3,ij}) t : \sigma_{1,ij} \bar{\sigma}_{1,ij} + \bar{\sigma}_{3,ij} t :$
$\quad \sigma_{2,ij}(\sigma_{1,ij} - \bar{\sigma}_{1,ij}) - \bar{\sigma}_{3,ij} - \sigma_{2,ij}^2 t)$.

We have added the right labels already: to identify which is which one may consider different values of i and/or j, identifying the last three lines; to distinguish $\mathcal{T}_{00}$ from $\mathcal{E}_{00}$ it suffices to check that $\mathcal{T}_{00}$ does not intersect any of the lines $\mathcal{T}_{ij}$.

Our next task is to find the coordinates of the ninety-six intersection points of the configuration. We will need them to simplify the equations of our quadrics. If we denote

$$p_{ij}^{mn} = \mathcal{E}_{ij} \cap \mathcal{T}_{mn}$$

then we find for any indices such that $\{i,j,k,m,n\} = \{1,2,3,4,5\}$

$$p_{00}^{00} = (0:0:0:0:0:1),$$

$$p_{0i}^{00} = (0:0:0:0:1:\lambda_i),$$

$$p_{00}^{0i} = (0:0:0:1:-1:-\lambda_i),$$

$$p_{0i}^{0i} = (1:-\lambda_i:\lambda_i^2:\lambda_i^2\bar{\sigma}_{1,i}:-\lambda_i(\lambda_i^2+\bar{\sigma}_{2,i}):-\lambda_i^2(\bar{\sigma}_{2,i}+\lambda_i\bar{\sigma}_{1,i}+\lambda_i^2)),$$

$$p_{0i}^{ij} = (1:-\lambda_i:\lambda_i^2:\lambda_i^2\bar{\sigma}_{1,i}:\bar{\sigma}_{3,ij}+\sigma_{2,ij}\bar{\sigma}_{1,ij}+\lambda_i^2\bar{\sigma}_{1,ij}:$$
$$\lambda_i(\bar{\sigma}_{3,ij}+\sigma_{2,ij}\lambda_i+\sigma_{2,ij}\bar{\sigma}_{1,ij})),$$

$$p_{ij}^{0i} = (1:-\lambda_i:\lambda_i^2:-\lambda_i^3-\lambda_i(\sigma_{2,ij}+\bar{\sigma}_{2,ij})-\bar{\sigma}_{3,ij}:\bar{\sigma}_{3,ij}+\sigma_{2,ij}\bar{\sigma}_{1,ij}+\lambda_i^2\bar{\sigma}_{1,ij}:$$
$$\lambda_i(\bar{\sigma}_{3,ij}+\sigma_{2,ij}\lambda_i+\sigma_{2,ij}\bar{\sigma}_{1,ij})),$$

$$p_{ij}^{ij} = (\sigma_{1,ij}\bar{\sigma}_{1,ij}-\bar{\sigma}_{2,ij}-\sigma_{1,ij}^2+\sigma_{2,ij}:-\bar{\sigma}_{3,ij}-\sigma_{1,ij}\sigma_{2,ij}+\bar{\sigma}_{1,ij}\sigma_{2,ij}:$$
$$-\sigma_{2,ij}^2+\sigma_{2,ij}\bar{\sigma}_{2,ij}-\sigma_{1,ij}\bar{\sigma}_{3,ij}:\sigma_{2,ij}\sigma_{1,km}\sigma_{1,kn}\sigma_{1,mn}-\sigma_{1,ij}\sigma_{2,ij}^2-$$
$$\bar{\sigma}_{3,ij}\sigma_{1,ij}\bar{\sigma}_{1,ij}:\bar{\sigma}_{3,ij}(\sigma_{2,ij}-\bar{\sigma}_{2,ij}-\sigma_{1,ij}^2)+\sigma_{1,ij}\sigma_{2,ij}\bar{\sigma}_{1,ij}(\bar{\sigma}_{1,ij}-\sigma_{1,ij}):$$
$$\bar{\sigma}_{3,ij}^2-\sigma_{2,ij}^2(\bar{\sigma}_{1,ij}^2-\bar{\sigma}_{2,ij}-\sigma_{1,ij}\bar{\sigma}_{1,ij}+\sigma_{2,ij})),$$

$$p_{ij}^{mn} = (\sigma_{1,mn}-\sigma_{1,ij}:\sigma_{2,mn}-\sigma_{2,ij}:\sigma_{1,ij}\sigma_{2,mn}-\sigma_{2,ij}\sigma_{1,mn}:\sigma_{2,mn}^2-$$
$$\sigma_{2,mn}(\sigma_{2,ij}-\sigma_{1,ij}^2)-\sigma_{1,ij}\sigma_{1,mn}\sigma_{2,ij}-\lambda_k(\sigma_{1,ij}\sigma_{2,mn}-\sigma_{2,ij}\sigma_{1,mn}):$$
$$(\sigma_{1,ij}\sigma_{2,ij}-\sigma_{1,mn}\sigma_{2,mn})\lambda_k+\sigma_{1,ij}\sigma_{1,mn}(\sigma_{2,mn}-\sigma_{2,ij}):$$
$$(\sigma_{2,mn}^2-\sigma_{2,ij}^2)\lambda_k+\sigma_{2,ij}\sigma_{2,mn}(\sigma_{1,ij}-\sigma_{1,mn})).$$

These points are used to compute the sixteen projective tranformations τ_{ij} which come from the sixteen translations on $\mathcal{J}_\Gamma$ over half periods; it actually suffices to compute the τ_{0i}. The transformation τ_{0i} should map the following seven points

$$p_{00}^{00}, p_{00}^{0i}, p_{0i}^{00}, p_{0i}^{0i}, p_{0j}^{0j}, p_{0m}^{0m}, p_{0n}^{0n}$$

to

$$p_{0i}^{0i}, p_{0i}^{00}, p_{00}^{0i}, p_{00}^{00}, p_{ij}^{ij}, p_{im}^{im}, p_{in}^{in}$$

(in that order), and similar for the other translations. If we introduce the following abbreviation

$$\chi_{ij}^k = \sum_{l=0}^k \bar{\sigma}_{j+l,i}\lambda_i^{k-l},$$

then we find that the matrix for τ_{0i} is given by $\begin{pmatrix} A_i & B_i \end{pmatrix}$, where

$$A_i = \begin{pmatrix}
\lambda_i^2\chi_{i0}^2 & \lambda_i\chi_{i1}^1 & -\lambda_i\chi_{i0}^1 \\
\lambda_i\chi_{i3}^1 & \lambda_i^4+\chi_{i3}^1 & \lambda_i^2\chi_{i0}^1 \\
-\lambda_i^2\chi_{i3}^1 & \lambda_i^3\chi_{i1}^1 & \chi_{i2}^2 \\
-\bar{\sigma}_{1,i}\lambda_i^2\chi_{i3}^1 & \chi_{i,1}^1\lambda_i^3\bar{\sigma}_{1,i}+\chi_{i,0}^4\sigma_2 & \sigma_1\chi_{i2}^2-\chi_{i0}^1\lambda_i^4 \\
\lambda_i\chi_{i3}^1(\lambda_i^2+\bar{\sigma}_{2,i}) & -\lambda_i^2\chi_{i1}^1(\lambda_i^2+\bar{\sigma}_{2,i}) & \lambda_i^2\chi_{i0}^1(\lambda_i^2+\bar{\sigma}_{2,i}) \\
-\left(\chi_{i3}^1\right)^2 & \lambda_i\chi_{i3}^1\chi_{i1}^1 & -\lambda_i\chi_{i3}^1\chi_{i0}^1
\end{pmatrix}$$

$$B_i = \begin{pmatrix} 0 & \lambda_i & -1 \\ 0 & -\lambda_i^2 & \lambda_i \\ 0 & \lambda_i^3 & -\lambda_i^2 \\ -\chi_{i0}^4 & -\lambda_i^4 - \chi_{i2}^2 & -\bar\sigma_{1,i}\lambda_i^2 \\ 0 & \bar\sigma_{1,i}\lambda_i^3 + \chi_{i3}^1 & \lambda_i(\lambda_i^2 + \bar\sigma_{2,i}) \\ 0 & \lambda_i\chi_{i3}^1 & \lambda_i^2\chi_{i0}^2 \end{pmatrix}.$$

The matrices τ_{ij} commute pairwise so they can be simultaneously diagonalized. The eigenvalues of τ_i are given by $\pm\varphi_i$ where $\varphi_i = \prod_{j\neq i} \lambda_{ij}$ and a complete set of common eigenvectors for all τ_{ij} is given by

$$W_0 = (0,0,0,1,0,0),$$
$$W_i = (1, -\lambda_i, \lambda_i^2, \bar\sigma_{1,i}\lambda_i^2, -\lambda_i(\lambda_i^2 + \bar\sigma_{2,i}), \bar\sigma_{4,i} + \lambda_i\bar\sigma_{3,i})$$

where $i = 1, \ldots, 5$. If we let W denote the matrix whose columns are the vectors W_i (properly normalized) and define $X = W^{-1}V$ then then equations of the three quadrics $V^T Q_i V = 0$ take the following symmetric form.

$$\varphi_1^{-1}t_1^2 + \varphi_2^{-1}t_2^2 + \varphi_3^{-1}t_3^2 + \varphi_4^{-1}t_4^2 + \varphi_5^{-1}t_5^2 = 0,$$
(36) $$\varphi_1^{-1}\lambda_1 t_1^2 + \varphi_2^{-1}\lambda_2 t_2^2 + \varphi_3^{-1}\lambda_3 t_3^2 + \varphi_4^{-1}\lambda_4 t_4^2 + \varphi_5^{-1}\lambda_5 t_5^2 = 0,$$
$$\varphi_1^{-1}\lambda_1^2 t_1^2 + \varphi_2^{-1}\lambda_2^2 t_2^2 + \varphi_3^{-1}\lambda_3^2 t_3^2 + \varphi_4^{-1}\lambda_4^2 t_4^2 + \varphi_5^{-1}\lambda_5^2 t_5^2 = t_0^2.$$

Of course one can get rid of all factors φ_i but we will not do this because it makes the coordinates of the ninety-six points more complex. It is easy to compute that these points have now the following coordinates.

$$(0 : \pm 1 : \pm 1 : \pm 1 : \pm 1 : \pm 1)$$
$$(1 : \pm(\lambda_1 - \lambda_i) : \pm(\lambda_2 - \lambda_i) : \pm(\lambda_3 - \lambda_i) : \pm(\lambda_4 - \lambda_i) : \pm(\lambda_5 - \lambda_i))$$

where the plus signs correspond to the origin resp. the points p_{00}^{0i}. For the other points p_{ij}^{kl} the i-th and j-th coordinates get a minus sign; notice that in this way all possible signs appear! The translation over a half period $\omega_i + \omega_j$ is now just given by flipping the sign of the i-th and j-th coordinates. This fact is useful in computing the new parametrizations of the thirty-two lines: one easily finds that $\mathcal{E}_{00}$ and Θ_{00} are given by

$$\mathcal{E}_{00} : (u : \lambda_1 u + r : \lambda_2 u + r : \lambda_3 u + r : \lambda_4 u + r : \lambda_5 u + r),$$
$$\Theta_{00} : (-u : \lambda_1 u + r : \lambda_2 u + r : \lambda_3 u + r : \lambda_4 u + r : \lambda_5 u + r),$$

and for the other lines $\mathcal{E}_{ij}$ and Θ_{ij} it suffices to add a minus sign in the i-th and j-th coordinates. In particular we have the following proposition:

Proposition 5.8. *The involution* $(t_0, t_1, t_2, t_3, t_4, t_5, t_6) \mapsto (-t_0, t_1, t_2, t_3, t_4, t_5, t_6)$ *restricts to an automorphism of the K-3 surface which interchanges the two families of sixteen lines.*

In the case of the odd sections of $[4\Theta]$ one can ask for higher vanishing at one of the half periods e_{ij} and find a quartic in $\mathbb{P}^3$. It is easy to very that in this case the image has six singular points which come from the six theta curves passing through that point. The exceptional divisor E_{ij} maps to a curve of degree three and all

the theta curves and exceptional divisors are mapped to lines. Compare this to the case of ϕ_+ for 3Θ: it is exactly the "dual". Computing the image one finds exactly the same image as in the latter case. The reason for this is that, as we have seen, the K-3 surface carries an automorphism which interchanges the two families of sixteen lines.

We finally consider the case of even sections of 4Θ. Since ϕ_+ leads to a Kummer surface in $\mathbb{P}^9$ we will restrict us here to a case in which we prescribe the vanishing (of order at least two since all half periods are even) at the points e_{00}, e_{0i}, e_{0j} and e_{ij}. Notice that these points form a group. The corresponding Kummer surface was extensively studied by Tomasz Szemberg (see [27]). We will give here the explicit calculation of the projective image, because the equations which were obtained by algebraic geometric methods in [27] are less explicit and its coefficients are not expressed in terms of the underlying curve. We define $\nu = (2, 2, 2, 2, 0, \ldots, \ldots, 0)$ (the first four half periods being e_{00}, e_{0i}, e_{0j} and e_{ij}). It is easy to verify that formula (3.2) leads to $\dim |\mathcal{M}_\nu^+| = 6$, hence $\phi_{\mathcal{M}_\nu^+}$ maps to $\mathbb{P}^5$.

Proposition 5.9. *The linear system $|\mathcal{M}_\nu^+|$ is base-point-free, hence $\phi_{\mathcal{M}_\nu^+} : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^5$ is regular. It contracts twelve divisors, to wit all exceptional divisors B_{kl} except B_{00}, B_{0i}, B_{0j} and B_{ij}, which are mapped to conics. The image is a complete intersection of three quadrics which can be taken as follows.*

$$
\begin{aligned}
0 &= \varphi(\lambda_i)u_1 v_1 - \varphi(\lambda_j)u_2 v_2, \\
0 &= \lambda_{ij}^3 u_2 v_2 + \varphi(\lambda_i)u_3 v_3, \\
0 &= (u_1 + u_2 + u_3 + v_1 + v_2 + v_3)^2 - 4\lambda_{ij}^{-2}\varphi'(\lambda_i)u_3 v_3 \\
&\quad - 4[(u_1 + v_2)u_2 + v_2(u_2 + v_1) + v_3(u_2 + v_2)],
\end{aligned}
$$

(37)

where $\varphi(\lambda) = (\lambda - \lambda_k)(\lambda - \lambda_m)(\lambda - \lambda_n)$.

Proof. We only indicate how (37) was computed and refer for the other statements to [27]. There is a natural choice for the six sections, namely for each pair of the curves $\Theta_{00}, \Theta_{0i}, \Theta_{0j}$ and Θ_{ij} we consider the even section of 4Θ which vanishes twice at this pair of curves. Since such a pair passes through the points e_{00}, e_{0i}, e_{0j} and e_{ij} these sections have the right vanishing properties. Notice that these sections can be constructed from the sections of $[2\Theta]$ by taking the tensor products $t_i \otimes t_j$, $0 \leq i < j \leq 3$, where t_0, t_1, t_2, t_3 are the sections that vanish twice at one of the four curves. If we define $\tilde{u}_i = t_0 \otimes t_i$ for $i = 1, 2, 3$ and $\tilde{v}_i = t_j \otimes t_k$ where $\{i, j, k\} = \{1, 2, 3\}$ then we have two obvious quadratic equations $\tilde{u}_1 \tilde{v}_1 = \tilde{u}_2 \tilde{v}_2 = \tilde{u}_3 \tilde{v}_3$. It remains to find a third quadric. The quickest way to do this is by using the equation of the Kummer surface (22), rewritten in terms of the basis $\{t_0, t_1, t_2, t_3\}$ for $H^0(\mathcal{J}_\Gamma, [2\Theta])$. It follows from (23) and (24) that these leads to the following change of coordinates:

$$
\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_i^2 & \lambda_i & 1 & 0 \\ \lambda_j^2 & \lambda_j & 1 & 0 \\ \sigma_{2,ij}\bar{\sigma}_{1,ij} + \bar{\sigma}_{3,ij} & -\sigma_{2,ij} & -\bar{\sigma}_{1,ij} & 1 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.
$$

If we substitute this in (25) and define $\varphi(\lambda) = (\lambda - \lambda_k)(\lambda - \lambda_m)(\lambda - \lambda_n)$ then we find the following equation for the classical Kummer quartic.

$$\begin{aligned}
&\left[\varphi(\lambda_j)t_0t_1 + \varphi(\lambda_i)t_0t_2 + \lambda_{ij}^2 t_0 t_3 + \lambda_{ij}t_1t_2 - t_2t_3 - t_3t_1\right]^2 \\
&- 4\varphi(\lambda_i)t_0t_1t_2(\varphi(\lambda_j)t_0 + \lambda_{ij}t_2 - 2t_3) \\
&+ 4t_1t_2t_3(\lambda_{ij}t_1 - t_3) - 4\lambda_{ij}\varphi'(\lambda_i)t_0t_1t_2t_3 = 0.
\end{aligned}$$

By defining u_i and v_i as appropriate multiples of $\tilde{u}_i$ and $\tilde{v}_i$ we get (37). $\quad\square$

6. Appendix

For comparison we give a more conceptual (but longer) proof of the fact that in the case $\mathcal{L} = 4\Theta$, $\nu = 0$ the map ϕ_- is an isomorphism and that its image is the complete intersection of three quadrics (see Section 5.3). This proof is based on Saint-Donat's theorem 2.2 and works only in the generic case. We will use the notation of Section 5.3.

Proposition 6.1. *If $A = \mathcal{J}_\Gamma$ is generic then the linear system $|C_-|$ has no base points and leads to a regular map $\phi_- : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^5$.*

Proof. It follows from (4) that $C_-^2 = 8$ so it suffices to show, according to Theorem 2.2, that $|C_-|$ has no fixed components. None of the curves B_i can belong to the base locus because, if we increase one of the ν_i to three then the number of sections drops. If some other divisor is a fixed component of $|C_-|$ then there is a symmetric divisor D on A such that every odd section of $H^0([\mathcal{D}_-])$ vanishes on D. Since D is actually totally symmetric it is linearly equivalent to $\Theta, 2\Theta, 3\Theta$ or 4Θ and we have a basis $\{ss_1, \ldots, ss_6\}$ of $H^0([\mathcal{D}_-])$, D being cut out by s. Then the sections $\{s_1, s_2, s_3, s_4, s_5, s_6\}$ represent a linearly independent set of sections with the same parity (either even or odd) in either $H^0(3\Theta)$, $H^0(2\Theta)$ or $H^0(\Theta)$. Which is impossible. $\quad\square$

Proposition 6.2. *If $A = \mathcal{J}_\Gamma$ is a generic Jacobi surface then the map $\phi_- : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^5$ is birational.*

Proof. We show that we are not in one of the exceptional cases of Saint-Donat's Theorem (Theorem 2.2). Let us first assume that $\tilde{\mathcal{K}}_\Gamma$ contains an irreducible curve C' for which $g(C') = 1$ and $C' \cdot C_- = 2$. Then there is a symmetric divisor D' on A such that

$$\tag{38} p^*D' = \pi^*C' + \sum_{i=1}^{16} \mu_i(D')E_i.$$

Since $C'^2/2 + 1 = g(C') = 1$ we get $C'^2 = 0$ and $D'^2 = \sum_{i=1}^{16} \mu_i(D')^2$. Then Formula (6) implies (for $D = \mathcal{D}_- \sim 4\Theta$) that the intersection $\Theta \cdot D'$ is given by

$$\tag{39} \Theta \cdot D' = 1 + \frac{1}{4}\sum_{i=1}^{16} \mu_i(D').$$

The Hodge inequality (see [11, p. 368]) $\Theta^2 D'^2 \leq (\Theta \cdot D')^2$ and the Cauchy-Schwarz inequality $(\sum_{i=1}^{16} \mu_i(D'))^2 \leq 16\sum_{i=1}^{16} \mu_i(D')^2$ then lead to

$$2\sum_{i=1}^{16} \mu_i(D')^2 = \Theta^2 D'^2 \leq 1 + \frac{1}{2}\sum_{i=1}^{16}\mu_i(D') + \sum_{i=1}^{16}\mu_i(D')^2,$$

an equality, which is easily rewritten as

$$(40) \qquad \sum_{i=1}^{16} \left(\mu_i(D') - \frac{1}{4} \right)^2 \le 2.$$

This means that each of the $\mu_i(D')$ must be either zero or one; if n of them are equal to one and the other ones are equal to zero then (40) reduces to $n \le 2$. Then $\Theta \cdot D'$ is only an integer for $n = 0$ in which case $\Theta \cdot D' = 1$, an impossibility if A is generic. This excludes the first case of the Saint-Donat Theorem.

We now assume that $\tilde{\mathcal{K}}_\Gamma$ contains a divisor C' such that $g(C') = 2$ and $C_- \sim 2C'$. Then $C_- \cdot C' = 2C'^2 = 4$. If we define D' as in (38) then we find as before

$$D'^2 = 4 + \sum_{i=1}^{16} \mu_i(D')^2, \qquad \Theta \cdot D' = 2 + \frac{1}{4} \sum_{i=1}^{16} \mu_i(D').$$

and proceed as in the first part of the proof: we apply the Hodge and Cauchy-Schwarz inequalities to get

$$2 \left(4 + \sum_{i=1}^{16} \mu_i(D')^2 \right) = \Theta^2 D'^2 \le 4 + \sum_{i=1}^{16} \mu_i(D') + \sum_{i=1}^{16} \mu_i(D')^2.$$

This inequality is easily rewritten as

$$\sum_{i=1}^{16} \left(\mu_i(D') - \frac{1}{2} \right)^2 \le 0,$$

which has no solution. Thus, both exceptional cases of Theorem 2.2 are excluded and ϕ_- is birational. $\qquad \square$

Proposition 6.3. *If $A = \mathcal{J}_\Gamma$ is generic then the birational map $\phi_- : \tilde{\mathcal{K}}_\Gamma \to \mathbb{P}^5$ is an embedding of the smooth Kummer surface $\tilde{\mathcal{K}}_\Gamma$ in $\mathbb{P}^5$.*

Proof. Since we know from the previous proposition that ϕ_- is birational it suffices to show that no curve is contracted. If $B_j = \pi_*(2E_j)$ were contracted then

$$0 = C_- \cdot B_j = \frac{1}{2} \left(p^* \mathcal{D}_- - \sum_{i=1}^{16} E_i \right) \cdot (2E_j) = 1,$$

a contradiction. Assume now that an irreducible divisor C' on $\tilde{\mathcal{K}}_\Gamma$, different from the curves B_j, is contracted. There exists a symmetric divisor D' on A such that $\pi^*(C') = p^*(D') - \sum_{i=1}^{16} \mu_i(D')E_i$. This leads to the following contradiction:

$$(41) \quad 2 \left(\sum_{i=1}^{16} \mu_i(D') \right)^2 \le 32 \sum_{i=1}^{16} \mu_i(D')^2 \le 32 D'^2 \le (\mathcal{D}_- \cdot D')^2 = \left(\sum_{i=1}^{16} \mu_i(D') \right)^2.$$

In the first inequality in (41) we used the Cauchy-Schwarz inequality and in the second one we used that $C'^2 \ge 0$. The third inequality follows from Hodge's inequality (cfr. supra) and the equality in (41) follows from

$$0 = C_- \cdot D = \frac{1}{2} \left(\mathcal{D}_- \cdot D' - \sum_{i=1}^{16} \mu_i(D') \right).$$

This shows that no curve is contracted hence ϕ_- is an isomorphism onto his image. $\qquad \square$

We finally show that the image is of ϕ_- is defined by quadratic equations.

Proposition 6.4. *If $A = \mathcal{J}_\Gamma$ is generic then the image of ϕ_- in $\mathbb{P}^5$ is given by an intersection of quadrics, in particular it is a complete intersection.*

Proof. We exclude the exceptional cases of Theorem 2.3. First, assume that there exists an irreducible curve C' such that $g(C') = 1$ and $C' \cdot C_- = 3$. There exists a symmetric divisor D' on A such that $\pi^*(C') = p^*(D') - \sum_{i=1}^{16} \mu_i(D')E_i$ and we find $C'^2 = 0$, $H^2 = \sum_{i=1}^{16} \sigma_i^2$ and $\Theta \cdot D' = \frac{3}{2} + \frac{1}{4}\sum_{i=1}^{16} \mu_i(D')$, leading to the following inequality for the $\mu_i(D')$

$$\tag{42} \sum_{i=1}^{16}\left(\mu_i(D') - \frac{3}{8}\right)^2 \leq \frac{9}{2}.$$

Since every term is at least equal to $9/64$ all $\mu_i(D')$ must be equal to 0 or 1. If we assume that n of them are equal to 1 and the others are zero then (42) reduces to $n \leq 9$ which gives only integer solution for $\Theta \cdot H$ when $n = 2$ or $n = 6$. If $n = 6$ then $\Theta \cdot D' = 3$ which is impossible on a generic Jacobian. If $n = 2$ then $\Theta \cdot D' = 2$ so that D' is algebraically equivalent to Θ, so D' is a translate of Θ. Since D' is symmetric it must be a translate of Θ over a half period. Now the equation $p^*H = \pi^*D + E_1 + E_2$ tells us that H has even multiplicity at all half periods except at two half periods, which is impossible, excluding the first exceptional case.

Second, let us assume that $\tilde{\mathcal{K}}_\Gamma$ contains two curves C' and C'' such that $g(C') = 2$, $g(C'') = 0$, $C' \cdot C'' = 1$ and $C_- \sim 2C' + C''$. Then

$$C_- \cdot C'' = (2C' + C'') \cdot C'' = 2 - 2 = 0$$

implying that C'' is a contracted curve for ϕ_-. We have seen however in Proposition 6.3 that no curve is contracted, excluding the second exceptional case. $\qquad\square$

For a generalization to higher genus we refer to [7].

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