HOCHSCHILD COHOMOLOGY OF POLYNOMIAL ALGEBRAS

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Abstract. In this paper we investigate the Hochschild cohomology groups $H^2(\mathbf{A})$ and $H^3(\mathbf{A})$ for an arbitrary polynomial algebra $\mathbf{A}$. We also show that the corresponding cohomology groups which are built from differential operators inject in $H^2(\mathbf{A})$ and $H^3(\mathbf{A})$ and we give an application to deformation theory. The results are not new, but the proofs given here are very elementary, depending only on simple properties of Hochschild cochains.

Contents

1. Introduction

The Hochschild cohomology groups $H^2(\mathbf{A})$ and $H^3(\mathbf{A})$ of a commutative algebra $\mathbf{A}$ over a field $\mathbb{F}$ play a fundamental role in the study of the rigidity of $\mathbf{A}$ as an associative algebra when $\mathrm{char}(\mathbb{F}) \neq 2$. Indeed, consider a formal deformation

$$\pi_\star(p,q) = \pi(p,q) + h\pi_1(p,q) + h^2\pi_2(p,q) + \cdots , \tag{1}$$

of the original product $\pi(p,q) = pq$ of elements $p, q \in \mathbf{A}$. It defines an associative product if and only if $[\pi_\star, \pi_\star] = 0$ ($[\cdot\,,\cdot]$ denotes the Gerstenhaber bracket), an equation which can be rewritten by using the Hochschild coboundary operator δ as

$$\delta\pi_n = \frac{1}{2}\sum_{i+j=n} [\pi_i, \pi_j]. \tag{2}$$

If the deformation (1) is associative up to order $n-1$ then the right hand side of (2) is a Hochschild 3-cocycle; the deformation extends to order n if and only this cocycle is a coboundary. Hence the obstruction lies in $H^3(\mathbf{A})$. Uniqueness is determined up to an element of $H^2(\mathbf{A})$ because on the one hand two solutions π_n and π'_n of (2) differ by a 2-cocycle, while on the other hand two such solutions which differ by a 2-coboundary define equivalent deformations.

In this paper we present an elementary proof that if $\mathbf{A}$ is a polynomial algebra then a 2-cocycle or 3-cocycle ϕ is a coboundary if and only if its skew-symmetrization vanishes. A complete proof of this fact appears already in [6], but

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our proof is more straightforward, relying on simple cohomological arguments and it allows us to show that a tridifferential operator ψ which is a coboundary is actually a coboundary of a bidifferential operator. As an application, we give a simple proof that any deformation (1) is equivalent to a deformation in which all π_i are bidifferential operators.

It should be noted that the homology of a polynomial algebra is well understood, and a more general result than ours is given in [4] (see Theorem 3.2.2). Our original purpose in proving these results was to apply them to understanding deformation quantization of polynomial Poisson algebras, which we did in [8]; we felt that the statement of these results in their present form and the simple proofs we give might be of some interest.

2. Hochschild cohomology

Let $\mathbf{A}$ denote any commutative algebra over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. For $n \geq 0$ the space of n-cochains is given by

$$C^n(\mathbf{A}) = \operatorname{Hom}(\mathbf{A}^n, \mathbf{A}).$$

The cochains form a complex $C^\bullet(\mathbf{A})$ for the Hochschild coboundary operator

$$\delta : C^n(\mathbf{A}) \to C^{n+1}(\mathbf{A})$$

which is defined by

$$\delta\varphi(p_1, \cdots, p_{n+1}) = p_1\varphi(p_2, \cdots, p_{n+1})$$
$$+\sum_{k=1}^{n}(-1)^k\varphi(p_1, \cdots, p_{k-1}, p_kp_{k+1}, p_{k+2}, \cdots, p_{n+1}) + (-1)^{n+1}\varphi(p_1, \cdots, p_n)p_{n+1},$$

for $\varphi \in C^n(\mathbf{A})$ (see [3]). The n-th cohomology group of this complex will be denoted by $H^n(\mathbf{A})$. Our aim in this section is to analyze $H^2(\mathbf{A})$ and $H^3(\mathbf{A})$ more thoroughly. We will give an explicit characterization in the case of polynomial algebras, in Theorems 2.1, 2.2 and 2.3. In the proofs below, for a polynomial algebra given by an ordered basis (free generating set) $\{x_i\}_{i\in\mathcal{I}}$, we will denote elements of the basis by the letters x and y, while arbitrary polynomials will be denoted by the letters $p, q, \ldots, v$. For a basis element x, the statement $x \leq p$ means that the basis elements appearing in the monomials in p have index greater than or equal to that of x, so that in particular $x \leq c$ for any constant c.

Any 2-cochain φ can be uniquely decomposed as the sum of a symmetric cochain φ^+ and a skew-symmetric cochain φ^-. Then φ is a cocycle precisely when both its symmetric and skew-symmetric parts are cocycles. To see this fact, suppose that φ is a 2-cocycle. Then $\delta\varphi(p, q, r) = p\varphi(q, r) - \varphi(pq, r) + \varphi(p, qr) - \varphi(p, q)r$. Let $\bar\varphi(p, q) = \varphi(q, p)$. Then

$$\delta\bar\varphi(p, q, r) = p\varphi(r, q) - \varphi(r, pq) + \varphi(qr, p) - \varphi(q, p)r = -\delta\varphi(r, q, p) = 0.$$

Since φ^+ and φ^- are linear combinations of φ and $\bar\varphi$, this shows the desired result. Furthermore, the coboundary of any 1-cochain is symmetric, which is immediate from the fact that if λ is a 1-cochain, then

$$\delta\lambda(p, q) = p\lambda(q) - \lambda(pq) + \lambda(p)q.$$

This implies that each skew-symmetric 2-cocycle determines a distinct cohomology class. Any biderivation is a 2-cocycle, since for a biderivation φ

$$\delta\varphi(a,b,c)=a\varphi(b,c)-\varphi(ab,c)+\varphi(a,bc)-\varphi(a,b)c=-\varphi(a,c)b+b\varphi(a,c)=0.$$

Furthermore, any skew-symmetric 2-cochain is a cocycle precisely when it is a biderivation. To see this, note that if φ is a skew-symmetric cocycle, then

$$\delta\varphi(a,b,c)-\delta\varphi(c,a,b)+\delta\varphi(b,c,a)=2(\varphi(a,bc)-b\varphi(a,c)-\varphi(a,b)c).$$

Since the left hand side vanishes, φ is a biderivation. These remarks hold for an arbitrary commutative algebra $\mathbf{A}$. When $\mathbf{A}$ is a polynomial algebra, we have a more complete characterization of $H^2(\mathbf{A})$.

Theorem 2.1. *Suppose that φ is a 2-cocycle on a polynomial algebra. Then φ is a coboundary precisely when it is symmetric. Furthermore, the cochain λ satisfying $\delta\lambda=\varphi$ can be chosen arbitrarily on basis elements. In particular, it can be chosen to satisfy $\lambda(x)=0$ when x is a basis element.*

Proof. For a symmetric 2-cocycle φ, we construct recursively a 1-cochain λ whose coboundary coincides with φ. Now

$$\delta\varphi(1,1,q)=\varphi(1,q)-\varphi(1,q)+\varphi(1,q)-\varphi(1,1)q=0,$$

so that $\varphi(1,q)=\varphi(1,1)q$. Let $\lambda(1)=\varphi(1,1)$, and define $\lambda(x)$ arbitrarily for all basis elements x. The property $\delta\lambda=\varphi$ holds precisely when

$$\lambda(pq)=p\lambda(q)+q\lambda(p)-\varphi(p,q). \tag{3}$$

When either p or q is constant, this equation holds by the preceding remarks; otherwise λ is evaluated at terms of lower degree on the right hand side, so the left hand side is defined recursively by this formula. But we need to check that if $pq=p'q'$ then the right hand sides of the decomposition above agree. We proceed by induction on the sum of the degrees of p and q. Thus we assume that λ is defined on polynomials of degree less than N and that it satisfies (3) for any p, q such that the degree of pq is less than N. For any p, q of degree less than N, the expression $\chi(p,q)=p\lambda(q)+q\lambda(p)-\varphi(p,q)$ is then well-defined. We need to show that if v has degree N and $v=pq=p'q'$ where the degrees of p, q, p' and q' are less than N, then $\chi(p,q)=\chi(p',q')$. Let us denote $r=gcd(p,p')$ and $s=gcd(q,q')$. By interchanging the roles of p' and q', if necessary, we may assume that r and s have nonzero degree. There are polynomials t and t' such that $p=rt$, $p'=rt'$, $q=st'$ and $q'=st$. Then

$$\begin{aligned}\chi(p,q)-\chi(p',q')&=\chi(rt,st')-\chi(rt',st)\\&=rt\lambda(st')+st'\lambda(rt)-\varphi(rt,st')-(t\leftrightarrow t')\\&=rt(s\lambda(t')+t'\lambda(s)-\varphi(s,t'))+st'(r\lambda(t)+t\lambda(r)-\varphi(r,t))\\&\qquad-\varphi(rt,st')-(t\leftrightarrow t')\\&=-\varphi(rt,st')-rt\varphi(s,t')-st'\varphi(r,t)-(t\leftrightarrow t')\\&=\delta\varphi(ts,r,t')-\delta\varphi(tr,s,t')+t'\delta\varphi(r,t,s),\end{aligned}$$

which vanishes because φ is a cocycle. □

Let us now turn our attention to the third Hochschild cohomology group. A 3-cochain ψ is called flip symmetric if it satisfies $\psi(p,q,r)=\psi(r,q,p)$, and flip skew-symmetric if $\psi(p,q,r)=-\psi(r,q,p)$. Every 3-cochain ψ can be uniquely decomposed as the sum of a flip symmetric cochain ψ^+ and a flip skew-symmetric cochain ψ^-. Moreover, ψ is a cocycle precisely when ψ^+ and ψ^- are cocycles. Furthermore, the coboundary of a symmetric 2-cochain is flip skew-symmetric, and the coboundary of a skew-symmetric 2-cochain is flip symmetric. The *Jacobi map* $J: C^3(\mathbf{A}) \to C^3(\mathbf{A})$ is given by

$$J\psi(p,q,r)=\psi(p,q,r)+\psi(q,r,p)+\psi(r,p,q).$$

If ψ is the coboundary of a symmetric cochain, it satisfies the *Jacobi identity* $J\psi=0$. For any 3-cocycle ψ, $\psi(1,1,1)=\delta\psi(1,1,1,1)=0$ and $\psi(1,p,1)=\delta\psi(1,1,p,1)=0$. Suppose that $\psi(p,1,1)=\psi(1,1,p)=0$ for all p. Then $\psi(1,p,q)=\delta\psi(1,1,p,q)$, $\psi(p,1,q)=\delta\psi(p,1,1,q)$, and $\psi(p,q,1)=\delta\psi(p,q,1,1)$, so these terms vanish for all p and q. These remarks are easy to check, and apply to any commutative algebra $\mathbf{A}$, not just a polynomial algebra.

Theorem 2.2. *Suppose that ψ is a 3-cocycle on a polynomial algebra. Then ψ is flip symmetric if and only if ψ is a Hochschild coboundary of a skew-symmetric cochain φ. Furthermore, the cochain $\varphi(x,y)$ can be chosen arbitrarily on basis elements $x<y$. In particular, it can be chosen so that $\varphi(x,y)=0$ when x and y are basis elements.*

Proof. By the above remarks we only need to verify that a flip symmetric cocycle ψ is a coboundary of a skew-symmetric cochain. If we define $\theta(p,1)=\psi(1,1,p)=-\theta(1,p)$, and extend θ in an arbitrary manner to a skew-symmetric cochain, then $\delta\theta(1,1,p)=-\psi(1,1,p)$. Replacing ψ by $\psi+\delta\theta$, we may assume that $\psi(1,1,p)=0$, so that ψ vanishes when any of its arguments is a constant. We define φ recursively, by first setting $\varphi(1,1)=\varphi(1,p)=\varphi(p,1)=0$. In addition, let us assume that $\varphi(x,y)$ is defined in an arbitrary manner for basis elements $x<y$, and extend it to all basis elements using the desired skew-symmetry property. Consider the following equalities which must be satisfied if $\psi=\delta\varphi$.

$$\begin{aligned}
\psi(p,q,u)&=p\varphi(q,u)-\varphi(pq,u)+\varphi(p,qu)-\varphi(p,q)u,\\
\psi(p,u,q)&=p\varphi(u,q)-\varphi(pu,q)+\varphi(p,uq)-\varphi(p,u)q,\\
\psi(u,p,q)&=u\varphi(p,q)-\varphi(up,q)+\varphi(u,pq)-\varphi(u,p)q.
\end{aligned}$$

Adding the first and third and subtracting the second of these equations, and using the desired skew-symmetry property for φ yields the following equation:

$$2\varphi(pq,u)=2p\varphi(q,u)+2q\varphi(p,u)-\psi(p,q,u)+\psi(p,u,q)-\psi(u,p,q), \tag{4}$$

The right hand side of this expression is evidently symmetric in p and q, and the equation holds when either p or q is constant. We wish to use it to define φ, as in the proof of Theorem 2.1. Thus we assume that $\varphi(r,u)$ is well-defined for any r of degree less than N and any u. For any u and for any p, q of degree less than N, the right hand side of (4) is well-defined; we will denote it by $2\chi(p,q,u)$. We need to show that if v has degree N and $v=pq=p'q'$ where the degrees of p, q, p' and q' are less than N, then $\chi(p,q,u)=\chi(p',q',u)$. We write $p=rt$, $p'=rt'$, $q=st'$

and $q' = st$, as in the proof of Theorem 2.1. Then

$$\begin{aligned}
&2(\chi(p,q,u) - \chi(p',q',u)) \\
&\quad = 2rt\varphi(st',u) + 2st'\varphi(rt,u) - \psi(rt,st',u) + \psi(rt,u,st') - \psi(u,rt,st') - (t \leftrightarrow t') \\
&\quad = -rt(\psi(s,t',u) - \psi(s,u,t') + \psi(u,s,t')) - st'(\psi(r,t,u) - \psi(r,u,t) + \psi(u,r,t)) \\
&\qquad\quad - \psi(rt,st',u) + \psi(rt,u,st') - \psi(u,rt,st') - (t \leftrightarrow t').
\end{aligned}$$

It is straightforward to check that the last line can be written as

$$\begin{aligned}
&\delta\psi(rt',t,s,u) - \delta\psi(rt',t,u,s) + \delta\psi(rt',u,t,s) - \delta\psi(u,rt',t,s) \\
&\qquad + s\delta\psi(t,r,t',u) - s\delta\psi(t,r,u,t')) - (t \leftrightarrow t'),
\end{aligned}$$

hence vanishes. To verify that φ is skew-symmetric, we compute $\varphi(pq,rs) + \varphi(rs,pq)$, using skew-symmetry of lower degree terms. We have

$$\begin{aligned}
2\varphi(pq,rs) &= 2p\varphi(q,rs) + 2q\varphi(p,rs) - \psi(p,q,rs) + \psi(p,rs,q) - \psi(rs,p,q) \\
&= p(2r\varphi(q,s) + 2s\varphi(q,r) + \psi(r,s,q) - \psi(r,q,s) + \psi(q,r,s)) + \\
&\qquad q(2r\varphi(p,s) + 2s\varphi(p,r) + \psi(r,s,p) - \psi(r,p,s) + \psi(p,r,s)) - \\
&\qquad \psi(p,q,rs) + \psi(p,rs,q) - \psi(rs,p,q).
\end{aligned}$$

From this we see that $2(\varphi(pq,rs) + \varphi(rs,pq))$ equals

$$\delta\psi(p,q,r,s) - \delta\psi(p,r,q,s) + \delta\psi(q,s,r,p) + ((p,q) \leftrightarrow (r,s))$$

and thus vanishes. It is easily checked that $\delta\varphi = \psi$. □

Theorem 2.3. *Suppose that ψ is a 3-cocycle on a polynomial algebra. Then ψ is a Hochschild coboundary of a symmetric cochain φ if and only if ψ is flip skew-symmetric and satisfies the Jacobi identity $J\psi = 0$. Moreover φ can be chosen to satisfy $\varphi(x,p) = 0$ whenever x is a basis element satisfying $x \leq p$.*

Proof. As in the previous theorem, we reduce to the case where $\psi(p,1,1) = 0$. Take an ordered basis of the algebra. Define φ by $\varphi(1,p) = \varphi(p,1) = 0$ for all p. We extend the definition recursively by setting

$$\varphi(xp,q) = x\varphi(p,q) + \psi(q,p,x) = \varphi(q,xp),$$

when $x \leq q$ and $x \leq p$. To show that φ is well defined and symmetric, we only need to show that if x is a basis element satisfying $x \leq p$ and $x \leq q$, then the expansion of $\varphi(xp,xq)$ yields the same result as the expansion of $\varphi(xq,xp)$. Now

$$\varphi(xp,xq) = x\varphi(p,xq) + \psi(xq,p,x) = x(x\varphi(q,p) + \psi(p,q,x)) + \psi(xq,p,x),$$

so that $\varphi(xp,xq) - \varphi(xq,xp) = \delta\psi(x,p,q,x) = 0$.

To show that $\delta\varphi = \psi$, we note that if any of p, q or r is constant, then both $\psi(p,q,r)$ and $\delta\varphi(p,q,r)$ vanish. We may proceed by induction on the sum of the degrees of p, q and r. If p can be factored as xp', where x satisfies $x \leq p'$, $x \leq q$

and $x \leq r$, then

$$\begin{aligned}
\delta\varphi(xp', q, r) &= xp'\varphi(q,r) - \varphi(xp'q, r) + \varphi(xp', qr) - \varphi(xp', q)r \\
&= xp'\varphi(q,r) - x\varphi(p'q, r) - \psi(r, p'q, x) \\
&\qquad + x\varphi(p', qr) + \psi(qr, p', x) - rx\varphi(p', q) - r\psi(q, p', x) \\
&= x(p'\varphi(q,r) - \varphi(p'q, r) + \varphi(p', qr) - \varphi(p', q)r) \\
&\qquad - \psi(r, p'q, x) + \psi(qr, p', x) - r\psi(q, p', x) \\
&= x\psi(p', q, r) - \psi(r, p'q, x) + \psi(qr, p', x) - r\psi(q, p', x) \\
&= \psi(xp', q, r).
\end{aligned}$$

On the other hand, if we can express $r = xr'$, where $x \leq p$, $x \leq q$ and $x \leq r$, then

$$\psi(p, q, xr') = -\psi(xr', q, p) = -\delta\varphi(xr', q, p) = \delta\varphi(p, q, xr'),$$

since ψ is flip skew-symmetric, and the coboundary of any symmetric cochain is also flip skew-symmetric. The only other possibility is that $q = xq'$, where $x \leq q'$, $x \leq p$ and $x \leq r$. But then we have

$$\begin{aligned}
\psi(p, xq', r) &= -\psi(xq', r, p) - \psi(r, p, xq') \\
&= -\delta\varphi(xq', r, p) - \delta\varphi(r, p, xq') \\
&= \delta\varphi(p, xq', r),
\end{aligned}$$

using the Jacobi identity $J\psi = 0$ and the fact that the coboundary of any symmetric cochain satisfies the Jacobi identity. Note that it is only at this last step that the Jacobi identity is used. □

In the proof above, the element φ we constructed satisfies $\varphi(x, p) = 0$ for any basis element x satisfying $x \leq p$. The following proposition implies that we could have assumed that $\varphi(x, p)$ is defined arbitrarily for $x \leq p$: we show that there exists a cocycle φ which takes any prescribed values $\varphi(x, p)$ for $x \leq p$.

Proposition 2.4. *On a polynomial algebra suppose that $\varphi(x, p)$ is any cochain defined for $x \leq p$, satisfying $\varphi(x, 1) = 0$. Then φ extends uniquely to a symmetric cocycle satisfying $\varphi(1, 1) = 0$.*

Proof. From the condition $\delta\varphi(x, p, q) = 0$ one derives the property

$$\varphi(xp, q) = x\varphi(p, q) + \varphi(x, pq) - \varphi(x, p)q.$$

If either p or q is constant, then the formula holds trivially. Otherwise, if $x \leq p$ and $x \leq q$, then the left hand side is defined recursively by the right hand side. The consistency and the symmetry condition $\varphi(xp, xq) = \varphi(xq, xp)$ follow from

$$\begin{aligned}
\varphi(xp, xq) &= x\varphi(p, xq) + \varphi(x, xpq) - \varphi(x, p)xq \\
&= x^2\varphi(q, p) + x\varphi(x, pq) - \varphi(x, q)xp + \varphi(x, xpq) - \varphi(x, p)xq.
\end{aligned}$$

If $\varphi(q, p) = \varphi(p, q)$, then the above formula is already symmetric in p and q, so the check of consistency and symmetry is trivial. To see that $\delta\varphi(p, q, r) = 0$, consider the case when $p = xp'$, where $x \le p'$, $x \le q$ and $x \le r$. Then

$$\begin{aligned}\delta\varphi(xp',q,r) &= xp'\varphi(q,r) - \varphi(xp'q,r) + \varphi(xp',qr) - \varphi(xp',q)r\\ &= xp'\varphi(q,r) - x\varphi(p'q,r) - \varphi(x,p'qr) + \varphi(x,p'q)r + x\varphi(p',qr) +\\ &\qquad \varphi(x,p'qr) - \varphi(x,p')qr - x\varphi(p',q)r - \varphi(x,p'q)r + \varphi(x,p')qr\\ &= x(p'\varphi(q,r) - \varphi(p'q,r) + \varphi(p',qr) - \varphi(p',q)r)\\ &= x\delta\varphi(p',q,r),\end{aligned}$$

which is zero by the induction hypothesis. The other cases follow from the flip skew-symmetry and the Jacobi identity $J(\delta\varphi) = 0$. □

Note that if $\psi \in C^3(\mathbf{A})$ is flip symmetric then its skew-symmetrization vanishes, while if ψ is flip skew-symmetric its skew-symmetrization coincides (up to a factor 2) with $J\psi$. Therefore, part of Theorems 2.2 and 2.3 can be reformulated by saying that a 3-cocycle ψ on a polynomial algebra is a coboundary if and only if its skew-symmetrization vanishes, as stated in the introduction.

3. Hochschild cohomology and differential operators

In this section we will assume that $\mathbf{A}$ is a polynomial algebra over a field $\mathbb{F}$ of characteristic 0 and we fix a basis $\{x_i\}_{i\in\mathcal{I}}$ for $\mathbf{A}$. We will give a characterization of Hochschild cochains in terms of (possibly infinite order) differential operators. First, let us establish some conventions on our terminology. For a basis element x_i of $\mathbf{A}$ we will denote the derivation $\partial/\partial x_i$ by ∂^i. For a multi-index $I = (i_1, \cdots, i_m)$, ∂^I will stand for the differential operator $\partial^{i_1}\dots\partial^{i_m}$, x_I will stand for the monomial $x_{i_1}\dots x_{i_m}$, and $|I| = m$ will be called its *order*. For a polynomial p, we will denote $\partial^I(x_I)$ by $I!$. Also, we shall write $I < I'$ to indicate that I is obtained by removing some of the indices in I'. By $\partial^{I_1}\otimes\cdots\otimes\partial^{I_n}$ we shall denote the n-differential operator of *order* $|I_1| + \cdots + |I_n|$ given by

$$\partial^{I_1}\otimes\cdots\otimes\partial^{I_n}(p_1,\cdots,p_n) = \partial^{I_1}(p_1)\dots\partial^{I_n}(p_n).$$

An expression of the form

$$\varphi = \sum_{I_1,\cdots,I_n} \varphi_{I_1,\cdots,I_n}\partial^{I_1}\otimes\cdots\otimes\partial^{I_n},$$

where $\varphi_{I_1,\cdots,I_n}$ are polynomials, and we sum over all multi-indices, gives a well-defined n-cochain on the polynomial algebra. When only finitely many non-zero terms appear then we say that φ is a *(finite order) differential operator,* otherwise such an expression is called a *formal differential operator.* The *order* of a differential operator φ is the largest m for which there is a nonzero term in φ of order m. Every n-cochain can be expressed as a formal differential operator, since we can solve for the polynomials $\varphi_{I_1,\cdots,I_n}$ above recursively by

$$I_1!\dots I_n!\varphi_{I_1,\cdots,I_n} = \varphi(x_{I_1},\cdots,x_{I_n}) - \sum_{(J_1,\cdots,J_n)<(I_1,\cdots,I_n)} \varphi_{J_1,\cdots,J_n}\partial^{J_1}(x_{I_1})\dots\partial^{J_n}(x_{I_n}).$$

In the following lemma, which characterizes when an n-cochain is a differential operator, if $k \in \mathbb{F}^{\mathcal{I}}$ and $I = (i_1,\dots,i_s)$ is a multi-index, then $(x-k)^I$ denotes the product $(x_1-k_1)^{i_1}\dots(x_s-k_s)^{i_s}$.

Lemma 3.1. *An n-cochain on a polynomial algebra is a differential operator precisely when there is some N such that for any $k \in \mathbb{F}^{\mathcal{I}}$,*

$$\varphi((x-k)^{I_1}, \cdots, (x-k)^{I_n})(k) = 0,$$

whenever $|I_1| + \cdots + |I_n| \geq N$.

Proof. If φ is a differential operator it suffices to take $N = 1 + \operatorname{ord}\varphi$. On the other hand, if the order of φ is infinite, we may find for any N a non-zero $\varphi_{I_1,\cdots,I_n}$, with $|I_1| + \cdots + |I_n| \geq N$, in particular this polynomial is non-zero at some point $(k_1, \cdots, , k_n)$. Then

$$\varphi((x-k)^{I_1}, \cdots, (x-k)^{I_n})(k) = I_1! \dots I_n! \varphi_{I_1,\cdots,I_n}(k) \neq 0.$$

□

In the proof of the above lemma, it was necessary to evaluate a polynomial at a point, so this argument cannot be extended to the ring of formal power series in the variables $\{x_i\}_i \in \mathcal{I}$, because evaluation at a point is not well defined. By examining the recursion formulas in Theorems 2.2 and 2.3 and applying Lemma 3.1, one obtains the following theorem.

Theorem 3.2. *Suppose that ψ is a tridifferential operator of order m on a polynomial algebra. If ψ is a Hochschild coboundary, then there exists a bidifferential operator φ of order m such that $\delta\varphi = \psi$.*

We conclude with an application to deformation theory.

Theorem 3.3. *Any deformation of a polynomial algebra is equivalent to a deformation whose cochains are bidifferential operators.*

Proof. Suppose that $\pi_\star = \pi + h\pi_1 + \dots$ is the given deformation, and that for some $n \in \mathbb{N}$ the cochains $\pi_1, \cdots, \pi_n$ are given by differential operators. Then we show that π_{n+1} can be replaced by a differential operator yielding an equivalent deformation (for the definition of equivalence of deformations we refer e.g. to [8]). It is easy to see that the coboundary of a differential operator, as well as the bracket of two differential operators is again a differential operator, so we can apply Theorem 3.2 to express $\delta(\pi_{n+1}) = \delta(C)$, for some differential operator C Thus $\delta(\pi_{n+1} - C) = 0$ and we can express $\pi_{n+1} - C = A + S$ where A is a skew-symmetric cocycle and S is a symmetric cocycle. Let $\pi'_{n+1} = C + A$. A is a differential operator because it is a biderivation, hence π'_{n+1} is a differential operator. π_{n+1} and π'_{n+1} differ by S, which is a coboundary, since it is a symmetric cocycle, so we can replace $\pi_\star$ by an equivalent deformation whose first $n+1$ terms are given by differential operators. □

Let us call a deformation $\pi_\star = \pi + h\pi_1 + \dots$ a deformation quantization when the cochains π_i are symmetric for even i and skew-symmetric for odd i. It follows from Theorem 3.3 that any deformation quantization of a polynomial algebra is equivalent to a deformation quantization whose cochains are bidifferential operators. Indeed, the bidifferential operator C for which $\delta\pi_{n+1} = \delta C$ has the same parity as π_{n+1} in view of Theorems 2.2 and 2.3.

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