A MINICOURSE ON COX RINGS OF SURFACES

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1. The Cox ring

The suggested references for this lecture are [CLS11] for the first two sections and [ADHL] for the last section. Throughout this lecture \( \mathbb{K} \) will be an algebraically closed field of characteristic zero.

1.1. Toric varieties. A toric variety is an irreducible normal variety \( X \) of dimension \( n \) which contains a torus \( T \cong (\mathbb{K}^*)^n \) as a Zariski open subset and such that the action of \( T \) on itself extends to an action of \( T \) on \( X \).

In this section we will recall the definition and the basic properties of the correspondence between toric varieties and fans. Given a toric variety \( X \) and \( u \in \mathbb{Z}^n \) we can consider the one parameter subgroup \( \lambda^u : \mathbb{K}^* \to (\mathbb{K}^*)^n, \ t \mapsto (t^{u_1}, \ldots, t^{u_n}) \) and look at \( \lim_{t \to 0} \lambda^u(t) \) in \( X \). This gives rise to a set of cones in \( \mathbb{R}^n \) such that, as \( u \) varies in the interior of each cone, the limit of \( \lambda^u \) exists and is constant. This collection of cones is the fan associated to \( X \). It can be easily proved that each orbit for the torus action contains a unique limit point, so that there is a one-to-one correspondence between the cones in the fan and the torus orbits in \( X \). For example, the cone \( \{0\} \) does correspond to the torus \( T \) and the cones of maximal dimension to the fixed points of \( T \).

Example 1.1.1. The projective plane \( \mathbb{P}^2 = \mathbb{K}^3 - \{0\}/\mathbb{K}^* \) is clearly a toric variety. In fact the map \( (\mathbb{K}^*)^2 \to \mathbb{P}^2, (t_1, t_2) \mapsto (1, t_1, t_2) \) identifies \( T \) with the open subset \( \{x_0x_1x_2 \neq 0\} \subset \mathbb{P}^2 \) and \( T \) acts on \( \mathbb{P}^2 \) by
\[
(t_1, t_2) \cdot (x_0, x_1, x_2) = (x_0, t_1x_1, t_2x_2).
\]

The fan of \( \mathbb{P}^2 \) is the following:

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For example one can easily check that \( \lim_{t \to 0} \lambda^{(a,b)}(t) = (1, 0, 0) \) if \( a, b > 0 \) and \( \lim_{t \to 0} \lambda^{(a,b)}(t) = (1, 0, 1) \) if \( a > 0, b = 0 \).

Conversely, a toric variety can be constructed from a fan in the following way. Let \( N \) be a lattice, i.e. a finitely generated and free abelian group, and \( M = \text{Hom}(N, \mathbb{Z}) \) be its dual. A rational polyhedral cone in \( N \) is a cone \( \sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R} \) generated by a finite set of vectors in \( N \):
\[
\sigma = \left\{ \sum_i \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0} \right\} \subset N_{\mathbb{R}}, \ v_i \in N.
\]

In particular \( \sigma \) is convex. Moreover, we will assume \( \sigma \) to be strongly convex, i.e. \( \sigma \cap (-\sigma) = \{0\} \). The dual cone \( \sigma^\vee \subset \mathbb{R}^n \) is
\[
\sigma^\vee = \{ m \in M_{\mathbb{R}} : (m, u) \geq 0 \text{ for all } u \in \sigma \},
\]
where \( (, ) \) is the natural bilinear pairing between \( M_{\mathbb{R}} \) and \( N_{\mathbb{R}} \). Given a cone \( \sigma \) one can define the semigroup
\[
S_\sigma = \sigma^\vee \cap M.
\]
By Gordan’s Lemma [CLS11, Proposition 1.2.17] \( S_\sigma \) is finitely generated, i.e. there are \( u_1, \ldots, u_l \in S_\sigma \) such that every element in \( S_\sigma \) is a linear combination of them with non-negative coefficients. This allows to define the map
\[
\varphi_\sigma : (\mathbb{K}^*)^n \to \mathbb{K}^l, \ t \mapsto (t^{u_1}, \ldots, t^{u_l}),
\]
where \( t^{u_1} = t_1^{u_{11}} \cdots t_n^{u_{1n}} \) is a Laurent monomial. The affine toric variety associated to \( \sigma \), denoted by \( X_\sigma \), is the Zariski closure of the image of \( \varphi_\sigma \). Equivalently
\[
X_\sigma = \text{Spec} \mathbb{K}[S_\sigma],
\]
where \( \mathbb{K}[S_\sigma] = \{ \sum_i a_i t^{u_i} : u \in S_\sigma, a_i \in \mathbb{K} \} \) is the semigroup algebra of \( S_\sigma \).

**Example 1.1.2.** Let \( \sigma \subset \mathbb{R}^2 \) be the cone generated by the vectors \( 2e_1 - e_2, e_2 \). Then \( \sigma_2^\circ \) is generated by \( e_1, e_1 + e_2, e_1 + 2e_2 \). The associated semigroup is \( S_\sigma = \langle e_1, e_1 + e_2, e_1 + 2e_2 \rangle \).

Thus \( \varphi_\sigma : (\mathbb{K}^*)^2 \to \mathbb{K}^3, (t_1, t_2) \mapsto (t_1, t_1 t_2, t_1^2 t_2) \) and the closure of its image is the quadratic cone \( X_\sigma = \{(x, y, z) \in \mathbb{R}^3 : xz = y^2 \} \).

Toric varieties are obtained by gluing affine toric varieties associated to the cones of a fan, i.e. a finite collection \( \Sigma \) of strongly convex rational polyhedral cones in \( \mathbb{N}_\mathbb{R} \) such that:

(i) if \( \tau \subset \sigma \) is a face of a cone \( \sigma \in \Sigma \), then \( \tau \in \Sigma \),

(ii) if \( \sigma_1, \sigma_2 \in \Sigma \), then \( \sigma_1 \cap \sigma_2 \) is a face of both \( \sigma_1 \) and \( \sigma_2 \).

Each cone \( \sigma \in \Sigma \) defines an affine toric variety \( X_\sigma \) and, if \( \tau \) is a face of \( \sigma \), then \( X_\tau \) can be identified with an open subset of \( X_\sigma \). This allows to construct a toric variety \( X_\Sigma \) by gluing two affine toric varieties \( X_{\sigma_1}, X_{\sigma_2} \) along \( X_{\sigma_1 \cap \sigma_2} \) for any \( \sigma_1, \sigma_2 \in \Sigma \).

**Example 1.1.3.** Let \( \sigma_0, \sigma_1, \sigma_2 \) be the maximal cones of the fan of \( \mathbb{P}^2 \). Then \( X_{\sigma_i} \cong \mathbb{P}^2 \) represent the usual affine charts. For example
\[
U_{\sigma_0} = \text{Spec} \mathbb{C}[S_{\sigma_0}] = \text{Spec} \mathbb{K}[x, y], \quad U_{\sigma_1} = \text{Spec} \mathbb{C}[S_{\sigma_1}] = \text{Spec} \mathbb{K}[x^{-1}, x^{-1}y]
\]
glue together along the open subset \( x \neq 0 \) by means of the isomorphism \( (x, y) \mapsto (x^{-1}, x^{-1}y) \). Taking \( x = x_1/x_0 \) and \( y = x_2/x_0 \) one easily recognizes \( U_{\sigma_i} \) as the standard affine subset \( U_i = \{ x_i \neq 0 \} \) of \( \mathbb{P}^2 \).

Let \( X_{\Sigma_1}, X_{\Sigma_2} \) be two toric varieties with fans \( \Sigma_1 \subset (N_1)_\mathbb{R} \) and \( \Sigma_2 \subset (N_2)_\mathbb{R} \) and with tori \( T_1 \) and \( T_2 \). A morphism \( F : X_{\Sigma_1} \to X_{\Sigma_2} \) is a toric morphism if \( F(T_1) \subset T_2 \) and \( F|_{T_1} \) is a group homomorphism. Any toric morphism is induced by a morphism of lattices compatible with the fans. In fact, let
\[
F : N_1 \to N_2
\]
be a \( \mathbb{Z} \)-linear map such that for any cone \( \sigma_1 \in \Sigma_1 \) there exists \( \sigma_2 \in \Sigma_2 \) such that \( F(\Sigma_1) \subset \sigma_2 \). This induces a homomorphism between the algebras \( \mathbb{K}[S_{\sigma_1}] \) and \( \mathbb{K}[S_{\sigma_2}] \) and thus a morphism \( f_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2} \). Such morphisms glue together to give a morphism
\[
f : X_{\Sigma_1} \to X_{\Sigma_2}.
\]
Observe that \( F(0) = 0 \), so that it induces a morphism \( f_0 : T_1 \to T_2 \).
Example 1.1.4. Let $\sigma = \text{Cone}(e_1, e_2) \subset \mathbb{R}^2$ and let
\[ F : \mathbb{Z}^2 \to \mathbb{Z}^2, \quad (x, y) \mapsto (2x, y - x). \]
Observe that $F_\mathbb{R}(\sigma) = \text{Cone}(2e_1 - e_2, e_2)$ is the cone in Example 1.1.2. Thus $F$ induces a morphism $f$ between $\mathbb{K}^2$ and the quadric cone. The corresponding map between the tori is $(\lambda, \mu) \mapsto (\lambda^2, \mu \lambda^{-1})$. This shows that $f$ is the quotient by the involution $(x, y) \mapsto (-x, -y)$.

1.2. The Cox construction. Let $\Sigma$ be a fan in $\mathbb{N}_\mathbb{R}$ and let $v_1, \ldots, v_r \in \mathbb{N}$ be the primitive vectors generating its one dimensional cones. In what follows we assume $\Sigma$ to be non-degenerate, that is the vectors $v_i$ generate $\mathbb{N}_\mathbb{R}$. Consider the $\mathbb{Z}$-linear map:
\[ P : \mathbb{Z}^r \to N, \quad e_i \mapsto v_i. \]
Let $\hat{\Sigma}$ be the fan in $\mathbb{Z}^r$ whose cones are the faces $\hat{\sigma}$ of the positive orthant $\Sigma_0$ such that $P_\mathbb{R}(\hat{\sigma}) \subset \sigma$ for some $\sigma \in \Sigma$. Thus $P$ defines a toric morphism:
\[ p : \hat{X} \to X = X_\Sigma, \]
where $\hat{X}$ is an open toric subvariety of $X_{\Sigma_0} \cong \mathbb{K}^n$ whose complement $\mathbb{K}^n \setminus \hat{X}$ is called irrelevant locus. We will now show that $p$ is the quotient by a group action. Let $P^\vee$ be the dual of $P$ and $K$ be its cokernel, so that we have an exact sequence:
\[ 0 \longrightarrow M \xrightarrow{P^\vee} \mathbb{Z}^r \xrightarrow{Q} K \longrightarrow 0, \]
where the injectivity of the first arrow follows from the fact that $\Sigma$ is non-degenerate. By means of the map $Q$, the algebra $\mathbb{K}[\mathbb{Z}^r \cap \Sigma_0^\vee] \cong \mathbb{K}[x_1, \ldots, x_r]$ acquires a grading by $K$:
\[ \mathbb{K}[x_1, \ldots, x_r]_w := \bigoplus_{w \in Q^{-1}(w) \cap \Sigma_0^\vee} \mathbb{K}x^w, \quad w \in K. \]

Definition 1.2.1. The Cox ring or total coordinate ring of $X_\Sigma$ is the $K$-graded polynomial ring
\[ R(X_\Sigma) = \mathbb{K}[x_1, \ldots, x_r]. \]

The grading of the Cox ring induces an action of the group $G := \text{Spec} \mathbb{K}[K]$ on $\mathbb{K}^r$ defined as follows:
\[ g \cdot (x_1, \ldots, x_r) := (\chi_{x_1}^w(g)x_1, \ldots, \chi_{x_r}^w(g)x_r), \quad \deg(x_i) = w_i. \]
We now show that the map $p$ is a good quotient by the action of $G$, which means that locally, on affine sets, it is of the form $\text{Spec}(A) \to \text{Spec}(A^G)$, where $A^G$ is the ring of invariants for the action of $G$.

Theorem 1.2.2. Let $X = X_\Sigma$ be the toric variety associated to a non-degenerate fan $\Sigma$. Then the morphism
\[ p : \hat{X} \to X \]
is a good quotient by the action of the group $\text{Spec}(\mathbb{K}[K])$.

Proof. We will prove the statement only for the affine spaces. Let $\hat{\sigma} \in \hat{\Sigma}$ and $\sigma \in \Sigma$ such that $P_\mathbb{R}(\hat{\sigma}) = \sigma$ and let $p : U_{\hat{\sigma}} \to U_\sigma$ be the associated morphism. The action of $G$ on $U_{\hat{\sigma}}$ is induced by the homomorphism:
\[ \mathbb{K}[S_{\hat{\sigma}}] \to \mathbb{K}[K] \otimes_{\mathbb{K}} \mathbb{K}[S_{\hat{\sigma}}], \quad x^u \mapsto \chi_Q(u) \otimes x^u. \]
Observe that the subalgebra of $G$-invariant polynomials is the degree 0 subalgebra with respect to the $K$-grading, thus:

\[ K[S_\sigma]^G = K[S_\sigma]_0 = K[S_\sigma]. \]

This concludes the proof. □

**Example 1.2.3.** In case $X = \mathbb{P}^2$ we have the exact sequence

\[ 0 \longrightarrow \mathbb{Z}^2 \overset{P^\vee}{\longrightarrow} \mathbb{Z}^3 \overset{Q}{\longrightarrow} \mathbb{Z} \longrightarrow 0 \]

where $P$ is given by the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $P^\vee$ by its transpose $A^T$ and $Q$ is the multiplication by $(1, 1, 1)$. The Cox ring is:

\[ \mathcal{R}(\mathbb{P}^2) = K[x_0, x_1, x_2], \quad \deg(x_i) = 1. \]

Observe that $\hat{\Sigma}$ is the boundary of the positive orthant, so that the Cox construction is the classical construction of the projective space as the quotient of $\mathbb{K}^3 - \{0\}$ by the diagonal action of $\mathbb{K}^*$.  

![Diagram](image)

**Example 1.2.4.** Let $Q$ be the affine quadric cone given in Example 1.1.2. The toric morphism described in Example 1.1.4 is the morphism $P$ of the Cox construction, so that $\hat{Q} = K^2$, $K = \mathbb{Z}/2\mathbb{Z}$ and $G = \mu_2$. The Cox ring is

\[ \mathcal{R}(Q) = K[x, y], \quad \deg(x) = \deg(y) = 1 \pmod{2}. \]

1.3. **The Cox ring of an algebraic variety.** We will now define the Cox ring for a wider class of varieties. Let $\text{WDiv}(X)$ be the group of Weil divisors of a normal algebraic variety $X$, i.e. the free abelian group generated by the irreducible hypersurfaces of $X$. A rational function $f \in K(X)^*$ defines a Weil divisor:

\[ \text{div}(f) = \sum_D \text{ord}_D(f)D, \]

where $\text{ord}_D(f)$ is the order of vanishing of $f$ along $D$. Such divisors are called principal divisors and they form a subgroup $\text{PDiv}(X)$ of $\text{WDiv}(X)$. The Class group of $X$ is the quotient group

\[ \text{Cl}(X) := \text{WDiv}(X)/\text{PDiv}(X). \]

In order to define the Cox ring, we will always assume $\text{Cl}(X)$ to be a finitely generated group. Moreover, for simplicity, we assume that $\text{Cl}(X)$ is a free group, i.e. $\text{Cl}(X) \cong \mathbb{Z}^r$ and we choose a subgroup $K$ of $\text{WDiv}(X)$ such that the class homomorphism

\[ c : K \rightarrow \text{Cl}(X), \quad D \mapsto [D] \]

is an isomorphism. We recall that, given a Weil divisor $D$, we can consider the Riemann-Roch space:

\[ \Gamma(X, \mathcal{O}_X(D)) = \{ f \in K(X)^* : \text{div}(f) + D \geq 0 \}. \]
**Definition 1.3.1.** Let $X$ be a normal algebraic variety such that $\text{Cl}(X)$ is finitely generated and $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$. The **Cox ring** of $X$ is defined to be the ring

$$R(X) := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)),$$

where the multiplication is the one in $K^*(X)$.

In case $\text{Cl}(X)$ has torsion, see [ADHL, §4.2] for the definition of the Cox ring.

We observe that $R(X)$ is a $K$-graded algebra over $K$, i.e. it has a direct sum decomposition into complex vector spaces $R(X)_D := \Gamma(X, \mathcal{O}_X(D))$, where $D \in K$, such that

$$f_1 \in R(X)_{D_1}, f_2 \in R(X)_{D_2} \Rightarrow f_1f_2 \in R(X)_{D_1 + D_2}.$$

Given a homogeneous element $f \in R(X)_D$ we define its **degree** to be $\deg(f) = |D|$.

It can be proved that two different choices $K_1, K_2$ for the grading group give isomorphic Cox rings (see Exercise 3).

In case $X = X_\Sigma$ is a toric variety it can be proved that the previous definition is equivalent to Definition 1.2.1. The condition $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$ is equivalent to ask that the fan $\Sigma$ is non-degenerate and the group $K$ is the Class group of $X_\Sigma$. By the previous discussion and [HK00, Corollary 2.10] the Cox ring $R(X)$ is a polynomial ring if and only if $X$ is a toric variety. In this sense, toric varieties are the simplest varieties in this theory.

**Exercises.**

1. The **Hirzebruch surface** $\mathbb{F}_n$ is the surface associated to the fan $\Sigma_n$, $n \geq 0$, in the following picture. Compute the Cox ring and work out the Cox construction of $\mathbb{F}_n$ (you can do this with the help of Magma, see the Appendix).

   ![Diagram](image)

2. Show that $\mathbb{F}_0$ is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$.

3. Show that if $K_1, K_2$ are two subgroups of $\text{WDiv}(X)$ such that the class homomorphism $c : K_1 \to \text{Cl}(X)$ is an isomorphism, then the associated Cox rings are isomorphic.

4. Let $X, Y$ be two algebraic varieties in the hypothesis of Definition 1.3.1 such that there exists an isomorphism $f : U_X \to U_Y$ where $U_X, U_Y$ are open subsets whose complements have codimension at least two ($f$ will be called a **small modification** in the next lecture). Show that $R(X)$ and $R(Y)$ are isomorphic.
2. Mori dream surfaces

The suggested references for this lecture are [ADHL] and [LV09b].

2.1. Varieties with finitely generated Cox ring. Let $X$ be a normal algebraic variety with finitely generated and free Class group and with $\Gamma(X, O_X^*) = \mathbb{K}^*$. In the previous lecture we defined the Cox ring of $X$ as follows:

$$R(X) = \bigoplus_{D \in K} \Gamma(X, O_X(D)),$$

where $K$ is a subgroup of $\text{WDiv}(X)$ isomorphic to $\text{Cl}(X)$ via the class map. As in the toric case, the $K$-grading of $R(X)$ induces an action of the group $G_X := \text{Spec } \mathbb{K}[K] \cong (\mathbb{K}^*)^r$ on $\text{Spec } R(X)$. Explicitly, if $D_1, \ldots, D_r$ is a basis of $K$ and $f \in R(X)_{\sum a_i D_i}$, then

$$(t_1, \ldots, t_r) \cdot f := t_1^{a_1} \cdots t_r^{a_r} f.$$ 

In case $R(X)$ is a finitely generated algebra, the analogous of the Cox construction holds. In case $X$ is projective the irrelevant ideal can be defined as follows. Let $D$ be an ample divisor on $X$ and $R_D := \bigoplus_{n \geq 0} R(X)_{nD}$. The irrelevant ideal of $X$ is

$$J_X := \sqrt{(R_D)},$$

where $(R_D)$ denotes the ideal generated by $R_D$.

**Theorem 2.1.1.** The variety $X$ is a good quotient of $\hat{X} = \text{Spec } R(X) - V(J_X)$ by $G_X$.

If $D$ is any movable divisor on $X$ (i.e. such that a positive multiple $|nD|$ has no components in its base locus) the same construction yields an irrelevant ideal $J_{X,D}$ and a variety

$$X' = (\text{Spec } R(X) - V(J_{X,D}))/G_X.$$ 

The varieties $X$ and $X'$ have isomorphic Cox rings and are related by a small modification, i.e. a birational map $f : X \rightarrow X'$ such that $f_U : U \rightarrow U'$ is an isomorphism, where $U, U'$ are open subsets whose complements have codimension two. As $D$ varies in the movable cone one obtains finitely many varieties $X'$ with the above property. This is a fundamental tool in the study of the birational geometry of $X$ and allowed to prove that the Mori program [link] can be carried out for any divisor on $X$ [HK00]. For this reason, the varieties with finitely generated Cox ring have been called Mori dream spaces.

A Mori dream space, in general, is not determined by its Cox ring (see [LV09b, Example 3.3] for an example of two smooth 3-folds which are not isomorphic and have isomorphic Cox rings). However, in the case of surfaces the following holds.

**Corollary 2.1.2.** Let $X, Y$ be two normal projective surfaces whose Cox rings are isomorphic as graded algebras and finitely generated, then $X$ and $Y$ are isomorphic.

In what follows we will concentrate on Mori dream surfaces and their classification.
2.2. **Mori dream surfaces.** In this section by *surface* we mean a smooth projective surface over the complex numbers, even if several of the stated results can be formulated more in general for normal surfaces. Moreover, we will always assume that the Class group of the surface is finitely generated. It follows from the exponential sequence [link] that the Class group is finitely generated if and only if the irregularity $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$ of the surface vanishes. For example, this holds for any rational surface.

We recall that the complete linear series $|D|$ of a divisor $D$ is the set of effective divisors linearly equivalent to $D$ and that it can be identified with the projectivization of $\Gamma(X, \mathcal{O}_X(D))$. Finally, we recall that on a smooth surfaces there exists an intersection product

$$\text{Cl}(X) \times \text{Cl}(X) \to \mathbb{Z}, \quad ([C], [D]) \mapsto C.D$$

which, in case $C, D$ are smooth curves intersecting transversally, is just the number of points in $C \cap D$ [link].

In order to give a characterization of Mori dream surfaces we first need to define certain geometric cones in $\text{Cl}(X)_\mathbb{Q} = \text{Cl}(X) \otimes \mathbb{Q}$, the rational Class group of $X$. The *effective cone* of $X$ is the cone in $\text{Cl}(X)_\mathbb{Q}$ generated by the classes of the irreducible curves on $X$:

$$\text{Eff}(X) := \left\{ \sum_i a_i [C_i] : a_i \in \mathbb{Q}_{\geq 0}, C_i \text{ irreducible curve} \right\}$$

The *nef cone* is the dual of the effective cone with respect to the intersection product:

$$\text{Nef}(X) := \{ [D] \in \text{Cl}(X)_\mathbb{Q} : D.C \geq 0 \text{ for any } [C] \in \text{Eff}(X) \}.$$ 

By the Kleiman criterion [link] the nef cone is the closure of the *ample cone*, i.e. the cone generated by classes of divisors $[D]$ such that a positive multiple $nD$ defines an embedding of the surface in projective space. Finally, a divisor $D$ is called *semiample* if its stable base locus is empty, i.e. there is no point of $X$ which belongs to the support of all the divisors $D' \in \lfloor nD \rfloor$ for all $n > 0$. The *semiample cone* $\text{SAample}(X)$ is the cone generated by the classes of semiample divisors in $\text{Cl}(X)_\mathbb{Q}$. The cones we have defined are related as follows:

**Lemma 2.2.1.** $\text{SAample}(X) \subseteq \text{Nef}(X) \subseteq \text{Eff}(X)$.

**Proof.** If $[D]$ is not nef, then $D.C < 0$ for some irreducible curve $C$. If $D'$ is a divisor linearly equivalent to a positive multiple of $D$ not containing $C$ in its support, then clearly $D'.C \geq 0$, so that $C$ is contained in the stable base locus of $[D]$. This proved the first inclusion. The second inclusion follows from the fact that the nef cone is the closure of the ample cone. \hfill $\Box$

**Example 2.2.2.** Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at one point $p$. This is the toric surface whose fan is obtained by adding the sum of two rays to the fan of $\mathbb{P}^2$. Its Class group is generated by the class $[H]$ of the pull-back of a line and by the class $[E]$ of the exceptional divisor.
Its Cox ring is
$$R(X) = \mathbb{C}[f_1, f_2, f_3, f_4],$$
where the $i$-th column of the matrix is the degree of $f_i$ in $\text{Cl}(X)$ with respect to the basis $[H], [E]$. The effective cone of $X$ is generated by $[H - E], [E]$ by Proposition 2.2.3. A class $[aH + bE]$ is nef if and only if
$$((aH + bE) \cdot E) = -b \geq 0, \quad ((aH + bE) \cdot (H - E)) = a + b \geq 0.$$ 
Thus the nef cone is generated by $[H]$ and $[H - E]$. Such classes are both semiample since $|H|$ contains the pull-back of the lines in $\mathbb{P}^2$ and $|H - E|$ the proper transforms of the lines through $p$, so that both of them have no base locus. Thus the semiample cone coincides with the nef cone.

A first necessary condition for a surface to be Mori dream is the following (which actually holds for any variety in the hypotheses of Definition 1.3.1). Let $D$ be an effective divisor on $X$ and $E \in K$ be such that $[E] = [D]$. In what follows we will denote by $f_D$ the unique (up to a constant) element in $R(X)_E$ such that $\text{div}(f_D) + E = D$.

**Proposition 2.2.3.** Let $X$ be a surface and let $\{f_i\}_{i \in I}$ be a homogeneous set of generators of $R(X)$. Then the effective cone of $X$ is generated by $\text{deg}(f_i), i \in I$. In particular, if $X$ is Mori dream, then its effective cone is polyhedral.

**Proof.** Let $D$ be an effective divisor of $X$ and $f_D \in R(X)$ be the corresponding element. Since $R(X)$ is generated by $f_1, \ldots, f_r$, then $f_D$ is a polynomial $P$ in $f_1, \ldots, f_r$. Let $f_1^{a_1} \cdots f_r^{a_r}$ be a monomial of $P$, then
$$[D] = \text{deg}(f_D) = \sum_{i=1}^{r} a_i \text{deg}(f_i),$$
giving the statement. 

Observe that if $D$ is a reduced, irreducible divisor with negative self-intersection, then $\dim \Gamma(X, \mathcal{O}_X(nD)) = 1$ for any positive integer $n$, since two distinct divisors in $|nD|$ would intersect non-negatively. This implies that $[D]$ is a generator of the effective cone. This remark allows us to give an example of a surface whose effective cone is not polyhedral.

**Example 2.2.4.** Let $X$ be the blow-up of $\mathbb{P}^2$ at nine points in the intersection of two general cubics. By a Theorem of Nagata [Nag60, Theorem 4a] $X$ contains infinitely many $(-1)$-curves, i.e. smooth rational curves with self-intersection $-1$. The classes of such curves span extremal rays of the effective cone, so that $\text{Eff}(X)$ is not polyhedral. By Proposition 2.2.3 $X$ is not a Mori dream surface.

The polyhedrality of the effective cone is not enough to give the finite generation of the Cox ring. We will show this in one example.
Example 2.2.5. Let $X$ be a very general quartic surface in $\mathbb{P}^3$ and let $\pi : \tilde{X} \to X$ be the blow-up of $X$ at a very general point $p \in X$. Let $H$ be the pull-back of a hyperplane section of $X$ and $E$ be the exceptional divisor of $\pi$. The Class group of $X$ is generated by $[H]$, $[E]$. The intersection of the tangent plane $T_pX$ with $X$ is a plane quartic curve with a node at $p$ and its proper transform in $\tilde{X}$ is a smooth curve $C$ of genus two such that $[C] = [H - 2E]$. The effective cone of $X$ clearly contains the cone generated by $[E]$ and $[C]$. Thus the nef cone is contained in its dual, which is generated by $[H]$ and $[C]$. Since both $[H]$ and $[C]$ are clearly nef (observe that $C$ is irreducible and $C^2 = 0$), then all the inclusions are equalities and we obtain:

$$\text{Eff}(X) = \langle [E], [C] \rangle, \quad \text{Nef}(X) = \langle [H], [C] \rangle.$$ 

In [AL11, Proposition 6.3] it is proved that $[C]$ is not semiample since, by the initial generality assumptions, $\dim H^0(X, \mathcal{O}_X(nC)) = 1$ for any positive integer $n$.

Assume that $\mathcal{R}(X)$ is finitely generated, so that there exists $[D] \in \text{Nef}(X)$ such that the degree of any generator either lives in the cone $\langle [D], [E] \rangle$ or is equal to $[C]$. A class $[D']$ as in the picture is ample, since it lives in the interior of the nef cone, so that there is $f \in \Gamma(X, \mathcal{O}_X(nD'))$ which is not divisible by $f_C$, for $n$ big enough. Thus $\deg(f)$ is a non-negative linear combination of the degrees of the generators of $\mathcal{R}(X)$ distinct from $f_C$. This means that $[nD'] = \deg(f)$ lives in the cone $\langle [D], [E] \rangle$, giving a contradiction.

The following theorem shows that the ones we have seen are the only obstructions for the finite generation of the Cox ring (see [AHL10, Corollary 2.6] and [GM05, Corollary 1]).

**Theorem 2.2.6.** Let $X$ be a smooth projective surface with $q(X) = 0$. Then the following are equivalent:

(i) $\mathcal{R}(X)$ is finitely generated.

(ii) $\text{Eff}(X)$ is polyhedral and $\text{Nef}(X) = \text{SAample}(X)$.

**Proof of (ii)$\Rightarrow$(i).** We start observing that a divisor $D$ of $X$ is semiample if and only if $[D]$ has no components in its stable base locus. In fact, if $[D]$ has stable base locus of dimension $< 1$, then $D$ is nef (see the proof of Lemma 2.2.1), so that it is semiample by hypothesis.

Let $w_1, \ldots, w_r$ be the primitive generators of the extremal rays of the effective cone such that $\dim \mathcal{R}(X)_{nw_i} \leq 1$ for any positive integer $n$ and fix $f_i \in \mathcal{R}(X)_{nw_i}$ with minimal $n_i$, so that

$$N_i(X) := \bigoplus_{n \geq 0} \mathcal{R}(X)_{nw_i} = \mathbb{C}[f_i].$$

By a Lemma of Zariski [HK00, Lemma 2.8] the following algebra is finitely generated:

$$\mathcal{S}(X) = \bigoplus_{w \in \text{SAample}(X)} \mathcal{R}(X)_w = \bigoplus_{w \in \text{Nef}(X)} \mathcal{R}(X)_w.$$
Given any non-zero \( f \in R(X) \), such that \( w \) is not semiample, we can write \( f = f^1 f_i \), where \( w(1) := \deg(f^1) = w - n_i w_i \), since the linear system associated to \( w \) has a curve in its stable base locus by the previous remark. If \( w(1) \) is not semiample, then we can write \( f = f^2 f_j \), with \( w(2) := \deg(f^2) \). This procedure stops in a finite number of steps since otherwise \( w(i) \) would leave the effective cone. Thus one obtains a semiample degree \( w(n) \) and \( f = f^n g \) with \( f^n \in S(X) \) and \( g \in \bigcup_i N_i(X) \). This shows that \( R(X) \) is generated by the \( f_i \)'s and a generating set of \( S(X) \), concluding the proof.

2.3. Towards a classification of Mori dream surfaces. We will now overview some known results about the classification of Mori dream surfaces according to their anticanonical Iitaka dimension:

\[
\kappa(-K_X) := \max\{\dim \varphi_{|-nK_X|(X)} : n \in \mathbb{N}\},
\]

where we denote by \( \varphi_{|-nK_X|} \) the rational map associated to the complete linear system \( |-nK_X| \). Unlike the Kodaira dimension of a surface, the anticanonical Iitaka dimension is not a birational invariant. Indeed, as Example 2.2.4 shows, having finitely generated Cox ring is not a birational invariant. The values of \( \kappa(-K_X) \) can be either \(-\infty, 0, 1 \) or \( 2 \). By the classification theory of surfaces, \( X \) is rational if \( \kappa(-K_X) \geq 1 \). We give examples of each type in the following table:

<table>
<thead>
<tr>
<th>( \kappa(-K_X) )</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>toric surfaces, generalized del Pezzo surfaces,</td>
</tr>
<tr>
<td></td>
<td>rational elliptic surfaces</td>
</tr>
<tr>
<td>1</td>
<td>K3 surfaces, Enriques surfaces</td>
</tr>
<tr>
<td>0</td>
<td>surfaces of general type</td>
</tr>
<tr>
<td>(-\infty)</td>
<td>surfaces of general type</td>
</tr>
</tbody>
</table>

We recall that a generalized del Pezzo surface is a rational surface whose anticanonical divisor is nef and with \( \kappa(-K_X) = 2 \). Examples of such surfaces are blow-ups of \( \mathbb{P}^2 \) at \( r \) points in general position with \( 0 \leq r \leq 8 \). For \( r = 6 \) one obtains smooth cubic surfaces in \( \mathbb{P}^3 \).

A rational elliptic surface is a rational surface carrying a surjective morphism \( \pi : X \to \mathbb{P}^1 \) whose general fiber is a smooth connected curve of genus one and such that it does not contain \((-1)\)-curves in its fibers.

A K3 surface is a surface with \( q(X) = 0 \) and \( K_X \sim 0 \). Finally, an Enriques surface is a surface with \( q(X) = 0 \) such that \( 2K_X \sim 0 \) and \( K_X \not\sim 0 \).

The following theorem, proved in [TVAV11] and [AL11], allows to give a complete classification of Mori dream surfaces with positive anticanonical Iitaka dimension.

**Theorem 2.3.1.** Let \( X \) be a smooth projective rational surface.

(i) If \( \kappa(-K_X) = 2 \), then \( X \) is a Mori dream surface;

(ii) If \( \kappa(-K_X) = 1 \) then \( X \) is a Mori dream surface if and only if \( \text{Eff}(X) \) is polyhedral, or equivalently if \( X \) contains finitely many \((-1)\)-curves.

By Theorem 2.2.6 proving that \( X \) is Mori dream is equivalent to prove that the effective cone of \( X \) is polyhedral and that any nef divisor is semiample. If \( \kappa(-K_X) \geq 1 \) then it can be proved that the nef cone always coincides with the semiample cone [AL11, Theorem 3.4]. Moreover, the effective cone of a surface with \( \kappa(-K_X) = 2 \) is always polyhedral by [Nak07, Proposition 3.3]. On the other
hand, there exist surfaces with \( \kappa(-K_X) = 1 \) whose effective cone has infinitely many extremal rays (see Example 2.2.4).

The condition for being Mori dream given in ii) can be stated more precisely, in particular if \(-K_X\) is nef it is equivalent to ask that the classes of the \((-2)\)-curves of \(X\) generate a subgroup of \(\text{Cl}(X)\) of rank 9.

If \(\kappa(-K_X) = 0\) and \(-K_X\) is nef, the \(X\) is either a K3 surface, an Enriques surface or a rational surface. If \(X\) is rational then \(-K_X\) is a non-trivial nef divisor which is not semiample, thus \(X\) is not Mori dream. Otherwise the following holds.

**Theorem 2.3.2.** Let \(X\) be a K3 surface or an Enriques surface. Then \(X\) is a Mori dream surface if and only if the automorphism group of \(X\) is finite.

K3 surfaces with this property have been classified in a series of papers by Piatetski-Shapiro and Shafarevich, Nikulin and Vinberg (see [AHL10] for references) and Enriques surfaces by Dolgachev and Kondo in [Dol84, Kon86].

In the case of rational surfaces with \(\kappa(-K_X) = 0\) and \(-K_X\) not nef, a necessary and sufficient condition such that the nef cone and the semiample cone coincide has been given recently in [LT]. In the same paper appears the first example of rational Mori dream surface with \(\kappa(-K_X) = 0\), which is a Coble surface (i.e. \(-K_X\) is not effective and \(-2K_X\) is effective) obtained as limit of a one dimensional family of Mori dream Enriques surfaces. The case \(\kappa(-K_X) = -\infty\) is still unexplored, except for some isolated examples.

**Exercises.**

1. Compute the effective cone, the nef cone and the semiample cone of the Hirzebruch surface \(F_2\) without using Theorem 2.2.6.

2. Show that if any element of the linear series \(|D|\) is irreducible, then any homogeneous set of generators of \(\mathcal{R}(X)\) contains a basis of \(\Gamma(X, \mathcal{O}_X(D))\).

3. Show that if \(\varphi: X \to \mathbb{P}^1\) is a fibration having at most one reducible fiber, then any homogeneous set of generators of \(\mathcal{R}(X)\) contains an element \(f_E\) which defines a smooth fiber \(E\) of \(\varphi\).

4. Let \(\pi: X \to \mathbb{P}^2\) be the blow-up of \(\mathbb{P}^2\) at \(r\) points, \(H\) be the pull-back of a line and \(E_1, \ldots, E_r\) be the exceptional divisors.
   
   (a) Prove that any curve \(D\) distinct from the \(E_i\)'s is linearly equivalent to \(dH - \sum a_i E_i\), for some integers \(d > 0\) and \(a_i \geq 0\).
   
   (b) Prove, by means of the adjunction formula [link], that a \((-1)\)-curve \(D\) satisfies \(D^2 = D.K_X = -1\), where \(K_X = -3H + \sum E_i\) is the canonical divisor of \(X\).
   
   (c) Show that \(X\) contains finitely many \((-1)\)-curves if \(r \leq 8\). (Hint: show that \(d\) is bounded).

5. Let \(X\) be the blow-up of \(\mathbb{P}^2\) at \(r\) collinear points. Show that the effective cone of \(X\) is generated by \([H - \sum E_i], [E_1], \ldots, [E_r]\), with the notation in the previous exercise. Is \(X\) a Mori dream surface?
### 3. Computing the Cox ring

Given a Mori dream space $X$, a natural problem is to find a presentation of its Cox ring, or at least to determine the degrees of its generators and relations. In [BP04] Batyrev and Popov identified the generators of the Cox ring of any del Pezzo surface, showing for example that it is generated by the elements defining the $(-1)$-curves of $X$ if the rank of the Class group is $4 \leq r \leq 8$. The ideal of relations of the Cox ring of del Pezzo surfaces has been computed in [STV07,SS07,LV09a,TVAV09]. In [CT06] the authors determined the generators of the Cox ring of the blow-up of $\mathbb{P}^n$ in any number of points that lie on a rational normal curve and in [AHL10] the Cox ring has been computed for certain K3 surfaces which double cover smooth rational surfaces.

This lecture deals with Cox rings of rational elliptic surfaces and is based on the results in [AGL].

#### 3.1. Rational elliptic surfaces

Let $X$ be a smooth rational surface carrying an *elliptic fibration*, i.e. a morphism $\pi : X \to \mathbb{P}^1$ whose general fiber is a smooth connected curve of genus one and such that there are no $(-1)$-curves contained in the fibers of $\pi$. We will also assume that the fibration has a section, i.e. a morphism $\sigma : \mathbb{P}^1 \to X$ such that $\pi \circ \sigma = \text{id}$. Any such morphisms is usually identified with its image in $X$, which is a $(-1)$-curve intersecting each fiber at one point.

The singular fibers of an elliptic fibration have been classified by Kodaira [link] according to the types $I_n$ ($n \geq 1$), $II$, $III$, $IV$, $II^*$, $III^*$, $IV^*$, $I_{n}^*$ ($n \geq 0$). The intersection graph of some of the reducible fibers is represented in the following figure, where each vertex represents a $(-2)$-curve and the number near to it is its multiplicity.

![Intersection Graph of Singular Fibers](image)

Fixed a section as an origin, the set of sections of an elliptic fibration has the structure of a group, called the *Mordell-Weil group* $\text{MW}(\pi)$. By Theorem 2.3.1 ii) a rational elliptic surface is Mori dream if and only if it contains a finite number of $(-1)$-curves or, equivalently, if the Mordell-Weil group is finite. The surfaces with this property have been classified by Miranda and Persson in [MP86]. The classification is reassumed in the following table, which describes 15 surfaces and a family depending of one parameter $a \in \mathbb{C} - \{0, 1\}$. 

<table>
<thead>
<tr>
<th>Type</th>
<th>$a$ Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$II^*(E_8)$</td>
<td>$a_8$</td>
<td>$\bullet$ $a_8$ $a_0$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$ $a_7$</td>
</tr>
<tr>
<td>$III^*(E_7)$</td>
<td>$a_7$</td>
<td>$\bullet$ $a_7$ $a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$</td>
</tr>
<tr>
<td>$IV^*(E_6)$</td>
<td>$a_6$</td>
<td>$\bullet$ $a_6$ $a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_7$</td>
</tr>
<tr>
<td>$I_n^*(D_{n+4})$</td>
<td>$a_n$</td>
<td>$\bullet$ $a_n$ $a_{n+1}$$a_{n+2}$ $a_{n+3}$ $a_{n+4}$</td>
</tr>
</tbody>
</table>
3.2. Generators of the Cox ring. In this section we will identify a set of generators for the Cox ring of any rational elliptic surface and we will explain, in one example, how to find the relations between them.

By Exercise 2 in the previous lecture any curve $D$ with negative self-intersection gives a generator $f_D$ of the Cox ring. In the case of a rational elliptic surface such curves are either $(-2)$-curves or $(-1)$-curves, which are components of fibers of $\pi$ and sections of $\pi$ respectively.

Moreover, by Exercise 3, if a fibration on $X$ has at most one reducible fiber, then the element $f_D$, where $D$ is a smooth fiber, is a generator of the Cox ring. This happens for the surfaces $X_{22}, X_{211}, X_{411}, X_{9111}$, where $\pi$ has a unique reducible fiber. Rational elliptic surfaces also carry rational fibrations, called conic bundles, which are the morphisms associated to divisors $D$ such that

$$D^2 = 0, \quad K_X.D = -2.$$ 

Since $-K_X.D = 2$, then a reducible fiber of a conic bundle either contains two $(-1)$-curves or a $(-1)$-curve with multiplicity two in its support. This implies [AGL, Proposition 2.4] that its support has the structure described by one of the following diagrams, where the white vertices represent $(-1)$-curves and the remaining ones represent $(-2)$-curves.

3. By 

$$\Gamma(X, \mathcal{O}_X(D)) = \varphi^*(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$$ 

is generated by elements of $\mathcal{O}(X)$ defining negative curves unless $\varphi$ contracts a chain of nine negative curves to one point $p$. In this case one of the generators of the Cox

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ring is \( f_D \), where \( D \) is the pull-back of a line in \( \mathbb{P}^2 \) not passing through \( p \). A divisor \( D \) with this property only exists for the surfaces \( X_{22}, X_{211} \) and \( X_{9111} \).

The main result in [AGL] states that the ones we have just described are the only generators of the Cox ring of a rational elliptic surface. More precisely, consider the following divisors:

(i) \((-1\)-curves or \((-2\)-curves,

(ii) conic bundles,

(iii) \(-K_X \)

(iv) nef divisors \( D \) with \( D^2 = 1, \ D \cdot (-K_X) = 3 \).

For each such divisor \( D \) choose a basis of \( \Gamma(X, \mathcal{O}_X(D)) \) and let \( \Omega \) be the union of all such bases. An element \( f \in \Omega \) will be called a distinguished section of \( \mathcal{R}(X) \).

**Theorem 3.2.2.** The Cox ring of an extremal rational elliptic surface is generated by distinguished sections.

The main ingredient of the proof is the following lemma. Let \( D \) be an effective divisor without components in its base locus and \( A, B \) be two smooth curves such that \( D \cdot A = 0 \). Then we have the following diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & \Gamma(X, \mathcal{O}_X(D-A)) & \longrightarrow \Gamma(X, \mathcal{O}_X(D)) & \longrightarrow & \Gamma(A, \mathcal{O}_A) \cong \mathbb{C} \\
\downarrow m_B & & \downarrow m_A & & \downarrow r_A & & \\
\end{array}
\]

where \( m_A, m_B \) are the multiplication maps by \( f_A \) and \( f_B \) respectively, \( r_A \) is the restriction to \( A \) and the horizontal sequence is exact.

**Lemma 3.2.3.** If \( \dim \Gamma(X, \mathcal{O}_X(D-B)) > 0 \) and \( A \) is not in the base locus of \( |D-B| \), then \( \Gamma(X, \mathcal{O}_X(D)) \) is generated by \( f_A, f_B, \Gamma(X, \mathcal{O}_X(D-A)) \) and \( \Gamma(X, \mathcal{O}_X(D-B)) \).

**Proof.** Observe that \( D - A \) is effective and \( r_A \) is surjective, since otherwise \( |D| \) would contain components in its base locus. Since \( A \) is not in the base locus of \( |D-B| \) then \( r_A \circ m_B \) is also surjective. Given \( f \in \Gamma(X, \mathcal{O}_X(D)) \), let \( r_A(f) = c \in \Gamma(A, \mathcal{O}_A) \) and let \( g \in \Gamma(X, \mathcal{O}_X(D-B)) \) such that \( r_A \circ m_B(g) = c \), then \( f - m_B(g) \in \Gamma(X, \mathcal{O}_X(D-A)) \), proving the statement.

Observe that, given an ample divisor \( H \) on \( X \), the intersections of \( D - A \) and \( D - B \) with \( H \) are strictly smaller than \( D \cdot H \), thus the previous Lemma allows to run an inductive procedure for identifying the generators of \( \Gamma(X, \mathcal{O}_X(D)) \).

**Example 3.2.4.** The elliptic surface \( X_{141} \) is obtained from the following pencil of plane cubics:

\[ x_2(x_0x_1 + x_0x_2 - x_1^2) + \lambda x_0x_1(x_0 - x_1) = 0, \]

by blowing up four times over the point 1, twice over the points 2, 3 and once over the point 4.

The fibration has two reducible fibers of type \( I_1^2 \) and \( I_4 \), which correspond to the cubics with \( \lambda = \infty \) and \( \lambda = 0 \) respectively. The elements of the Mordell-Weil group are the four \((-1\)-curves in the exceptional divisors over the points 1, 2, 3, 4. A basis for the Class group is given by the pull-back \( e_0 \) of the class of line in \( \mathbb{P}^2 \) and by the classes of the exceptional divisors \( e_1, \ldots, e_9 \) with \( e_i, e_j = -\delta_{ij} \), where \( e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 \) are the classes of the irreducible components of the
exceptional divisor over the point 1, $e_5 - e_6, e_6$ over the point 2, $e_7 - e_8, e_8$ over the point 3 and $e_9$ over the point 4. The Cox ring of $X_{141}$ is generated by the defining elements of the 14 curves with negative self-intersection, whose classes with respect to the basis $e_0, \ldots, e_9$ are given by the columns of the following matrix.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

3.3. Relations for the Cox ring of a blow-up. Let $f : X \to Y$ be a blow-up morphism between two Mori dream surfaces. There is a natural orthogonal decomposition:

$$\text{Cl}(X) \cong f^* \text{Cl}(Y) \oplus H,$$

where $H$ is freely generated by the classes of the integral components of the exceptional divisor $\text{Exc}(f)$. Let

$$\mathcal{R}(X) = \mathbb{C}[T, S]/I$$

be the Cox ring of $X$, where we denote by $S$ the set of elements $S_1, \ldots, S_n$ defining the integral components of $\text{Exc}(f)$ and by $T$ the set $T_1, \ldots, T_m$ of the remaining generators. We will now give a method for identifying the ideal $I$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathbb{C}[T, S] & \longrightarrow & \mathcal{R}(X) & \longrightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{f^*_R} & & \downarrow{f^*_z} & & \\
& & \mathbb{C}[T] & \longrightarrow & \mathcal{R}(Y) \\
\end{array}
$$

where

(i) $f^*_R$ is the natural injective pull-back homomorphism:

$$f^*_R : \mathcal{R}(Y) \to \mathcal{R}(X), \ h \mapsto h \circ f$$

which maps $\mathcal{R}(Y)_D$ to $\mathcal{R}(X)_{f^*D}$;

(ii) $\alpha : \mathbb{C}[T] \to \mathbb{C}[T, S], \ T_i \mapsto T_i m_i$, where $m_i$ is the unique monomial in the variables $S_j$ such that $\deg(T_i m_i) \in f^* \text{Cl}(Y)$;

(iii) $\beta : \mathbb{C}[T] \to \mathcal{R}(Y), \ T_i \mapsto s_i$, where $f^*_R(s_i) = T_i$.

Observe that the first row is exact and the square is commutative.

**Proposition 3.3.1.** Let $J$ be the saturation of the ideal generated by $\alpha(\ker(\beta))$. Then $J$ is prime and it is contained in $I$. In particular $J = I$ if $\dim J = \dim I$. 
Sketch of proof. The inclusion $\mathcal{J} \subset \mathcal{I}$ follows easily from the diagram and from the fact that $\mathcal{I}$ is a saturated ideal (in fact it is prime). The fact that $\mathcal{J}$ is prime follows from [BHKS, Proposition 3.3] since $\ker(\beta)$ is prime.

□

Example 3.3.2. Let $f : X_{141} \to \mathbb{P}^2$ be the blow-down morphism described in Example 3.2.4. The degrees of the generators $T_1, \ldots, T_5$ are the columns of the left block of the matrix in Example 3.2.4. The functions $s_i$ in this case are:

$$s_1 = x_0, \quad s_2 = x_1, \quad s_3 = x_0 - x_1, \quad s_4 = x_2, \quad s_5 = x_0 x_1 - x_1^2 + x_0 x_2.$$ 

A minimal basis of $\ker(\beta)$ is given by

$$T_1 - T_2 - T_3, \quad T_1 T_4 + T_2 T_3 - T_5,$$

and its image by $\alpha$ is generated by

$$T_1 S_1 S_2 S_3 S_4 S_9 - T_2 S_1 S_2 S_3 S_4 S_6 - T_3 S_1 S_2 S_4 S_7 S_8^2, \quad T_1 T_4 S_1^2 S_2^5 S_4 S_5 S_6 S_7 S_8 S_9^2 + T_2 T_3 S_1^2 S_2^5 S_4 S_5 S_6 S_7 S_8^2 - T_5 S_1^2 S_2^5 S_4 S_5 S_6 S_7 S_8.$$

A computation with Magma [BCP97] shows that its saturation is the ideal $\mathcal{J}$ generated by:

$$T_1 S_1 S_2 S_3 S_9 - T_2 S_5 S_6^2 - T_3 S_7 S_8^2, \quad T_1 T_4 S_2^3 + T_2 T_3 S_4 S_6 S_8 - T_5 S_1^2 S_2.$$

and that

$$\dim(\mathcal{J}) = \dim(\mathcal{I}) = \dim(\text{Spec } \mathcal{R}(X)) = 2 + \text{rk} \text{ Cl}(X) = 12.$$ 

Thus by Proposition 3.3.1 we have

$$\mathcal{R}(X_{141}) = \mathbb{C}[T_1, \ldots, T_5, S_1, \ldots, S_9] / (T_1 S_1 S_2 S_3 S_9 - T_2 S_5 S_6^2 - T_3 S_7 S_8^2, \quad T_1 T_4 S_2^3 + T_2 T_3 S_4 S_6 S_8 - T_5 S_1^2 S_2).$$

Observe that the pencil of lines through the point 1 contains three lines whose pull-backs in $X_{141}$ are monomials in the generators of the Cox ring, this gives the first relation of $\mathcal{R}(X_{141})$. The second relation is induced in the same way by the pencil of conics which pass through the points 1, 2, 3 and have tangent line $x_0 = 0$ at the point 1.

Exercises.

1. Show that the surface obtained blowing-up $\mathbb{P}^2$ at the base points of the Hesse pencil

$$x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0$$

has four fibers of type I_3, and thus is isomorphic to the surface $X_{3333}$.

2. Find a generating set for the Cox ring of the surface $X_{411}$ by means of Theorem 3.2.2.
Magma appendix

In this section we will give some Magma [BCP97] functions which allow to experiment on some topics in the notes. To run the code it is enough to copy it into the Magma calculator online (link).

Toric varieties and the Cox construction. The following Magma [BCP97] code constructs the projective plane starting from its fan

```magma
> rays := [ [0,1], [1,0], [-1,-1] ];
> cones := [ [1,2], [1,3], [2,3] ];
> F := Fan(rays,cones);
> X := ToricVariety(Rationals(),F);
> X;
```

and computes its Cox ring:

```magma
> CoxRing(X);
Cox ring with underlying Polynomial ring of rank 3 over Rational Field
Order: Lexicographical
Variables: $.1, $.2, $.3
The irrelevant ideal is:
($.3, $.2, $.1)
The grading is:
1, 1, 1
```

where the irrelevant ideal is the ideal in the Cox ring which defines the irrelevant locus. Conversely, one can construct a toric variety starting from a graded ring with a given irrelevant locus. The sequences \(Z\) and \(Q\) contain the free part and the torsion part of the degrees of the variables in \(R\). For example, the following code constructs the quadric surface from its Cox ring:

```magma
> R<u,v,x,y>:=PolynomialRing(Rationals(),4);
> irrel:=ideal<R|u,v> meet ideal<R|x,y>;
> B:=[irrel];
> Z:=\{[1,1,0,0],[0,0,1,1]\};
> Q:=\{\};
> C := CoxRing(R,B,Z,Q);
> X:=ToricVariety(C);
> X;
```

Toric variety of dimension 2
Variables: u, v, x, y
The components of the irrelevant ideal are:
(y, x), (v, u)
The 2 gradings are:
1, 1, 0, 0,
0, 0, 1, 1

A subscheme of \(X\) can be defined by means of the coordinates of the Cox ring. For example we can define the following curves in \(X\) and check how they intersect:

```magma
> X1:=Scheme(X,u);
> X2:=Scheme(X,x);
```
We can construct the map associated to a complete linear system, which gives as output the image variety $Y$ and the corresponding lattice morphism $f$. For example the following gives a projection of the quadric onto $\mathbb{P}^1$:

```
> Y,f:=Proj(Divisor(X,u));
> Y;
```

Toric variety of dimension 1
Variables: $.1, $.2
The irrelevant ideal is:
($$.2, $.1$$)
The grading is:
1, 1

```
> f;
```

Mapping from: 2-dimensional toric lattice $N$ to 1-dimensional
toric lattice $\text{quo}(N)$ given by a rule [no inverse]

Conversely, given a morphism between lattices, one can construct the associated map between toric varieties:

```
> F:=ToricVarietyMap(X,Y,f);
> F;
```

A map between toric varieties described by:
1,
$$(v) \cdot (u)^{-1}$$

**Blow-ups.** The blow-up of a toric variety at one point can be obtained as follows:

```
> P:=ProjectivePlane(Rationals());
> N:=Ambient(Fan(P));
> v:=[1,1];
> X:=Blowup(P,N!v);
```

Toric variety of dimension 2
Variables: $.1, $.2, $.3, $.4
The components of the irrelevant ideal are:
($$.4, $.3$$), ($$.2, $.1$$)
The 2 gradings are:
0, 0, 1, 1,
1, 1, 1, 0
A map between toric varieties described by:
1,
$$(v) \cdot (u)^{-1},
(v) \cdot (w)^{-1}$$

The blow-up of $\mathbb{P}^2$ at 4 general points is no longer a toric variety. We can define it by taking the linear system of cubics in $\mathbb{P}^2$ through the four points:

```
> P:=ProjectivePlane(Rationals());
> pts:=[P![[1,0,0], P![[0,1,0], P![[0,0,1], P![[1,1,1]]];
> L:=LinearSystem(P,3);
> L:=LinearSystem(L,pts);
```
and then taking the image of the morphism defined by such linear system:

```maple
> P5:=ProjectiveSpace(Rationals(),5);
> f:=map<P->P5|Sections(L)>;
> X:=Image(f);
> Degree(X);
```

**Rational elliptic surfaces.** A pencil of cubics in $\mathbb{P}^2$ can be identified with a curve over a function field in one variable:

```maple
> K<t>:=FunctionField(Rationals());
> P<x,y,z>:=ProjectivePlane(K);
> f:=x^2*y+y^2*z+z^2*x;
> g:=y^3-z^3-x*y*z;
> C:=Curve(P,f+t*g);
```

Moreover, it defines an elliptic curve once we fix a point $p \in K(t)$ as an origin:

```maple
> p:=Points(Scheme(P,[z,y]))[1];
> E:=EllipticCurve(C,p);
> E;
```

and we can compute the Kodaira types of the reducible fibers of the elliptic fibration associated to the pencil:

```maple
> KodairaSymbols(E);
```

**References**


[HK00] Yi Hu and Sean Keel, Mori dream spaces and GIT, Michigan Math. J. 48 (2000), 331–348. ↑6, 7, 10


[LV09b] ________, A survey on Cox rings, Geom. Dedicata 139 (2009), 269–287. ↑7


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