TRANSCENDENTAL LATTICES OF SOME K3-SURFACES

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Abstract. In a previous paper, [S2], we described six families of $K3$-surfaces (over $\mathbb{C}$) with Picard-number 19, and we identified surfaces with Picard-number 20. In these notes we classify some of the surfaces by computing their transcendental lattices. Moreover we show that the surfaces with Picard-number 19 are birational to a Kummer surface which is the quotient of a non-product type abelian surface by an involution.

0. Introduction

Given a K3-surface an important step toward its classification in view of the Torelli theorem is to compute the Picard lattice and the transcendental lattice. When the rank of the Picard lattice (i.e. the *Picard-number*, which we denote by ρ) of the K3-surface is 20, the maximal possible, the transcendental lattice has rank two. These $K3$-surfaces are called by Shioda and Inose *singular*. In [SI], Shioda and Inose classified such surfaces in terms of their transcendental lattice, more precisely they show the following:

Theorem 0.1. [SI, Theorem 4, §4] *There is a natural one-to-one correspondence from the set of singular $K3$-surfaces to the set of equivalence classes of positive-definite even integral binary quadratic forms with respect to $SL_2(\mathbb{Z})$.*

When the Picard-number is 19 the transcendental lattice has rank three and by results of Morrison, [M], and Nikulin, [N1], the embedding in the *K3-lattice* $\Lambda := -E_8 \oplus -E_8 \oplus U \oplus U \oplus U$ is unique, hence it identifies the moduli curve classifying the K3-surfaces. In general however it seems to be difficult to compute explicitly the transcendental lattice. In [S2] we describe six families of K3-surfaces with Picard-number 19 and we identify in each family four surfaces with Picard-number 20. The aim of these notes is to compute their transcendental lattice and to classify them. In [S2] we describe completely the Picard lattice of the general surface and of the special surfaces in two of the families and of six surfaces with Picard-number 20 in the other families. Here by using lattice-theory and results on quadratic forms we compute the transcendental lattices of these surfaces. The methods are similar to the methods used by Barth in [B] for describing the K3-surfaces of [BS].
By a result of Morrison, [M, Cor. 6.4], K3-surfaces with $\rho = 19$ and 20 have a Shioda-Inose structure, in particular this means that there is a birational map from the K3-surface to a Kummer surface. It is well known (cf. [SI]) that if $\rho = 20$, then the Kummer surface is the quotient of a product-type abelian variety by an involution. When $\rho = 19$ this is not always the case. In fact we use the transcendental lattices to show that in our cases the abelian variety is not a product of two elliptic curves. In this case we call the Shioda-Inose structure *simple*.
The paper is organized as follows: in section 1 we recall some basic facts about lattices and quadratic forms and the families of K3-surfaces from [BS] and [S2]. Then section 2

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is entirely devoted to the computations of the transcendental lattices of the K3-surfaces of [S2]. In section 3 we show that the Shioda-Inose structure of the surfaces with $\rho = 19$ is simple. In section 4 we compare our singular K3-surfaces with already known surfaces, more precisely with the Hessian surfaces which are described in [DvG]: we see that all our singular K3-surfaces are Hessian surfaces of some cubic surfaces and we see that some of them are extremal elliptic K3-surfaces in the meaning of [MP] and of [SZ]. Finally in section 5 we recall the rational curves generating the Neron-Severi group of the K3-surfaces.
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1. Notations and preliminaries

1.1. **Lattices and quadratic forms.** A *lattice* L is a free $\mathbb{Z}$-module of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form:

$$b : L \times L \longrightarrow \mathbb{Z}.$$

An isomorphism of lattices preserving the bilinear form is called an *isometry*, L is said to be *even* if the quadratic form associated to b takes only even values, otherwise it is called *odd*. The *discriminant* $d(L)$ of L is the determinant of the matrix of b, L is said to be *unimodular* if $d(L) = \pm 1$. If L is non-degenerate, i.e. $d(L) \neq 0$, then the *signature* of L is a pair (s_+, s_-) where $s_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L \otimes \mathbb{R}$, L is called *positive-definite (negative-definite)* if the quadratic form associated to b takes just positive (negative) values. We will denote by U the *hyperbolic plane* i.e. a free $\mathbb{Z}$-module of rank 2 with bilinear form with matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

moreover we denote by E_8 the unique even unimodular positive definite lattice of rank 8, with bilinear form with matrix:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $L^\vee = Hom_{\mathbb{Z}}(L, \mathbb{Z}) = \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid b(v, x) \in \mathbb{Z} \text{ for all } x \in L\}$ denote the dual of the lattice L, then there is a natural embedding of L in $L^\vee$ via $c \mapsto b(c, -)$, and we have:

Lemma 1.1. *(cf.* [BPV, Lemma 2.1, p. 12]*) If L is a non-degenerate lattice with bilinear form b. Then*
1. $[L^\vee : L] = |d(L)|$.
2. If M is a submodule of L with rank M =rank L, then

$$[L : M]^2 = d(M)d(L)^{-1}.$$

Let A be a finite abelian group. A quadratic form on A is a map:

$$q : A \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

together with a symmetric bilinear form:

$$b : A \times A \longrightarrow \mathbb{Q}/\mathbb{Z}$$

such that:
1. $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$
2. $q(a + a') - q(a) - q(a') \equiv 2b(a, a')\ (mod\ 2\mathbb{Z})$
Let L be a non-degenerate even lattice, the bilinear form b induces a $\mathbb{Q}$-valued bilinear form on $L^\vee$ and so a quadratic form

$$q_L : L^\vee/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

called *discriminant-form* of L. By a result of Nikulin [N1, Cor. 1.9.4], the signature and the discriminant form of an even lattice determines its *genus* (for the precise definition of genus see [Bu, p. 378]).
An embedding of lattices $M \hookrightarrow L$ is *primitive* if L/M is free.

Lemma 1.2. *(cf.* [N1, Prop. 1.6.1]*) Let $M \hookrightarrow L$ be a primitive embedding of non-degenerate even lattices and suppose L unimodular then:*
1. There is an isomorphism $M^\vee/M \cong (M^\perp)^\vee/M^\perp$.
2. $q_{M^\perp} = -q_M$.
(Here $M^\perp$ denotes the orthogonal complement of M in L).

Let now X be an algebraic $K3$-surface, the group $H^2(X, \mathbb{Z})$ with the intersection pairing has the structure of a lattice and by Poincaré duality it is unimodular. This is isometric to the *K3-lattice*:

$$\Lambda := -E_8 \oplus -E_8 \oplus U \oplus U \oplus U$$

(cf. [BPV, Prop.3.2, p. 241]). The *Neron-Severi group* $NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ and its orthogonal complement $T(X)$ in $H^2(X, \mathbb{Z})$ (the *transcendental lattice*) are primitive sublattice of $H^2(X, \mathbb{Z})$ and have signature $(1, \rho - 1)$ and $(2, 20 - \rho)$, $\rho =$rank$(NS(X))$. By the Lemma 1.2 we have

$$NS(X)^\vee/NS(X) \cong T(X)^\vee/T(X)$$

and the discriminat-forms differ just by their sign. Moreover by the Lemma 1.1 we have $|NS(X)^\vee/NS(X)| = |(T(X))^\vee/T(X)| = d(NS(X))$.

We recall some more facts about $K3$-surfaces X with $\rho = 20$ (*singular* K3-surfaces, cf. [SI, p. 128]). Denote by $\mathcal{Q}$ the set of 2×2 positive-definite even integral matrices:

$$Q := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix},\ a, b, c \in \mathbb{Z} \tag{1}$$

with determinant $d := 4ac - b^2 > 0$ and $a, c > 0$. We define an equivalent relation "$\sim$" by: $Q_1 \sim Q_2$ if and only if $Q_1 = {}^t\gamma Q_2 \gamma$ for some $\gamma \in SL_2(\mathbb{Z})$. Denote by $[Q]$ the equivalence class of Q and by $\mathcal{Q}/SL_2(\mathbb{Z})$ the set of equivalence classes. Then:

Theorem 1.1. *(cf.* [SI, Thm. 4]*). The map $X \mapsto [T(X)]$ establishes a bijective correspondence from the set of singular K3-surfaces onto $\mathcal{Q}/SL_2(\mathbb{Z})$.*

In particular $K3$-surfaces with $\rho = 20$ are classified in terms of their transcendental lattice. By [Bu, Thm. 2.3, p. 14], we choose a representative Q in *reduced* form, i.e. $-a \leq b \leq a \leq c$, and so $b^2 \leq ac \leq d/3$. Recall the following:

Theorem 1.2. *(*[Bu, Theorem 2.4, p. 15]*) With the exception of*

$$1.\begin{pmatrix} 2a & a \\ a & 2c \end{pmatrix} \sim \begin{pmatrix} 2a & -a \\ -a & 2c \end{pmatrix};\ 2.\begin{pmatrix} 2a & b \\ b & 2a \end{pmatrix} \sim \begin{pmatrix} 2a & -b \\ -b & 2a \end{pmatrix}$$

no distinct reduced quadratic forms are equivalent.

It is well known that the number of equivalence classes of forms of a given discriminant d, i.e. the *class number* of d, is finite. If there is only one class we say that d has class number one. In some other cases, even if we have more equivalence classes, we have however one class per genus. In [Bu, pp.81–82] with the assumption $g.c.d(a,c,b)=1$, i.e. we assume that the form is *primitive*, all the discriminants of class number one and of one class per genus are listed. If $g.c.d(a,c,b)\neq 1$ then the form is a multiple of a primitive form.

1.2. **Families of K3-surfaces.** Let $G\subset SO(3)$ denote the polyhedral group T, O or I, and let $\widetilde{G}\subset SU(2)$ be the corresponding binary groups. Let

$$\sigma : SU(2)\times SU(2)\to SO(4,\mathbb{R})$$

denote the classical $2:1$ covering. The images $\sigma(\widetilde{T}\times\widetilde{T}) := G_6$, $\sigma(\widetilde{O}\times\widetilde{O}) := G_8$ and $\sigma(\widetilde{I}\times\widetilde{I}) := G_{12}$ in $SO(4,\mathbb{R})$ are studied in [S1], where we show that there are 1-dimensional families in $\mathbb{P}_3(\mathbb{C})$ of G_n-invariant surfaces of degree n, which we denote by X^n_λ, λ a parameter in $\mathbb{P}_1$. In [BS] it is shown that the quotients $Y_{\lambda,G_n}=X^n_\lambda/G_n$, $n=6,8,12$ are families of $K3$-surfaces where the general surface has Picard-number 19 and there are four surfaces with Picard-number 20. Then in [S2] by taking special normal subgroups of G_n ($n=6,8$) and considering the quotient of X^6_λ resp. X^8_λ by these subgroups we find six more pencils of K3-surfaces. Using the notations there the subgroups are

$$\mathcal{G}:\ T\times V\quad (TT)'\quad V\times V\quad O\times T\quad (OO)''\quad T\times T$$

and the families of K3-surfaces are denoted by $Y_{\lambda,\mathcal{G}}$. Here V denotes the Klein four group in $SO(3,\mathbb{R})$ and the groups $(TT)', (OO)''$ are described in [S2], the others are the images in $SO(4,\mathbb{R})$ of the direct product of binary subgroups of $SU(2)$. Moreover $T\times V$, $(TT)'$ are subgroups of index 3 of G_6 and $V\times V$ has index 3 in $T\times V$ and in $(TT)'$; $O\times T$, $(OO)''$ are subgroups of index 2 of G_8 and $T\times T$ has index 2 in $O\times T$ and in $(OO)''$. In the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ the general surface has Picard-number 19 and we could identify four surfaces with Picard-number 20. We denote them by $Y^{(n,j)}_{\mathcal{G}}$, where $n=6$, $\mathcal{G}=T\times V$ and $j=1,2,3,4$ or $n=8$, $\mathcal{G}=O\times T$ and $j=1,2,3,4$. The general surface in the other families has Picard-Number 19 too and there are four special surfaces with Picard-number 20, but in [S2] we could not identify enough curves generating the Neron-Severi group. Only in some special case with $\rho=20$ we could describe explicitly the Picard lattice. Following the notations of [S2] the surfaces are

$$Y^{(6,1)}_{(TT)'},\ T^{(6,2)}_{(TT)'},\ Y^{(8,1)}_{(OO)''},\ Y^{(8,4)}_{(OO)''}, Y^{(8,1)}_{T\times T}, Y^{(8,4)}_{T\times T}.$$

2. Transcendental Lattices

In this section we identify first the transcendental lattice of the singular $K3$-surfaces, then of the surfaces with $\rho=19$. In each case we proceed as follows:
1. We determine generators for $NS^\vee/NS$ with the help of the intersection pairing $(-,-)$, which is defined on NS (recall that $NS^\vee=\{v\in NS\otimes_{\mathbb{Z}}\mathbb{Q}\mid (v,x)\in\mathbb{Z} \text{ for all } x\in NS\}$).
2. We determine the discriminant-form of NS.
3. We use Lemma 1.2 to determine the discriminant-form of T.

4. We list all the reduced quadratic forms which have the discriminant $d(T) = d(NS)$ (we will see that in each case the matrices have form 1 or 2 as in the Theorem 1.2).
5. We use the discriminant form to determine T, in fact we see that when the rank is two the discriminants have class number one or one class per genus. When the rank is three in our cases the discriminants are *small*, Def. 2.1, and these have one class per genus.
We denote the curves generating the Neron-Severi group by L_i, M_i, N_i, C_i, R_i, we recall their configuration in section 5. Moreover before discussing each case we recall from [S2] the divisible classes of NS, which together with the previous curves generate the Picard lattice. Finally, we denote by $\mathbb{Z}_m(\alpha)$ the cyclic group $\mathbb{Z}_m$ with the quadratic form taking the value $\alpha \in \mathbb{Q}/2\mathbb{Z}$ on the group's generator which we choose.

2.1. **The singular cases.** The family $Y_{\lambda,T\times V}$. We recall the following 3-divisible class of NS

$$\bar{L}' = L_1 - L_2 + L_4 - L_5 + L'_1 - L'_2 + L'_4 - L'_5 + L''_1 - L''_2 + L''_4 - L''_5$$

and the following 2-divisible classes of NS

$$\begin{aligned} h_1 &= L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_2, \\ h_2 &= L_1 + L_3 + L_5 + L''_1 + L''_3 + L''_5 + M_1 + M_3. \end{aligned}$$

The general $K3$-surface in the family has $\rho = 19$ and the family contains four singular $K3$-surfaces. The discriminant of the general $K3$-surface in the pencil, i.e. the determinant of the lattice NS, is $2 \cdot 3 \cdot 5$ which is the order of $NS^\vee/NS$ by the Lemma 1.1. We specify the following generators:

$$\begin{aligned} M &:= M_1 + M_2 + M_3/2, \\ N &:= L_1 - L_2 + L_4 - L_5 - L'_1 + L'_2 - L'_4 + L'_5/3, \\ L &:= (3L_0 - L_1 - L'_1 - L''_1 - 2L_2 - 2L'_2 - 2L''_2 - 3L_3 - 3L'_3 - 3L''_3 \\ &\quad -2L_4 - 2L'_4 - 2L''_4 - L_5 - L'_5 - L''_5)/5 \end{aligned}$$

where

$$\begin{aligned} M^2 &= -3/2 &&= 1/2 && \bmod 2\mathbb{Z}, \\ N^2 &= -8/3 &&= 4/3 && \bmod 2\mathbb{Z}, \\ L^2 &= -18/5 &&= 2/5 && \bmod 2\mathbb{Z}. \end{aligned}$$

Hence the discriminant form of the Picard lattice is

$$\mathbb{Z}_2(1/2) \oplus \mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) \cong \mathbb{Z}_{30}(7/30)$$

- *The singular case* $6, 1(6, 4)$. Here the discriminant is $-3 \cdot 5 = -15$ and the generators of $NS^\vee/NS$ are N and L. The discriminant form is
$$\mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) = \mathbb{Z}_{15}(26/15)$$
- *The singular case* $6, 2(6, 3)$. Here the discriminant is $-2^2 \cdot 3 \cdot 5 = -60$, and the generators are M, N, L and another class $M' = M_4/2$ with $M'^2 = -1/2 = 3/2 \mod 2\mathbb{Z}$. The discriminant form is
$$\mathbb{Z}_2(1/2) \oplus \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_3(4/3) \oplus \mathbb{Z}_5(2/5) \cong \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{30}(97/30).$$

The discriminant form of the transcendental lattice differs from the discriminant form of the Picard lattice just by the sign, as computed above, hence in the general case is

$$\mathbb{Z}_{30}(53/30)$$

and in the special cases is

$$\begin{aligned} 6, 1\ (6, 4): &\quad \mathbb{Z}_{15}(4/15), \\ 6, 2\ (6, 3): &\quad \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{30}(23/30). \end{aligned}$$

Here we identify the transcendental lattices of these four singular $K3$-surfaces, and in the next section of the general $K3$-surface.

- *The singular case* $6,1$ $(6,4)$. We classify all the reduced matrices with discriminant 15 (i.e. one class per genus, cf. [Bu, pp.19–20]). We have just the following possibilities

$$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix},\ A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

  By taking the generator $(4/15, -1/15)$ and the bilinear form defined by A, we find a lattice $\mathbb{Z}_{15}(4/15)$ which is exactly the lattice $T^\vee/T$, as recalled on page 3 the signature and the discriminant form determine the genus so $T = A$.

- *The singular case* $6,2$ $(6,3)$. We classify all the reduced matrices with discriminant 60 (cf. [Bu, pp.19–20]). We have just the following possibilities

$$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix},\ B := \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}.$$

  By taking the generators $(1/2, 0)$ and $(1/3, 1/10)$ and the quadratic form B we find a lattice $\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{30}(23/30)$ which is exactly the lattice $T^\vee/T$, hence $T = B$.

The family $Y_{\lambda,(TT)'}$. We recall the following 3-divisible class in NS:

$$\bar{L} = N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8 + N_9 - N_{10} + N_{11} - N_{12}.$$

Now we identify the transcendental lattice of $Y^{(6,1)}_{(TT)'}$ and of $Y^{(6,2)}_{(TT)'}$.

- *The singular case* $6,1$. In this case the discriminant is $-3 \cdot 5 = -15$ and we have the following generators of $NS^\vee/NS$:

$$\begin{array}{rcl} N & := & (N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8)/3, \\ L & := & (3L_3 - 3L_3')/5, \end{array}$$

  where

$$\begin{array}{rcccll} N^2 & = & -8/3 & = & 4/3 & \mod 2\mathbb{Z}, \\ L^2 & = & -18/5 & = & 2/5 & \mod 2\mathbb{Z}. \end{array}$$

  Hence the transcendental lattice is the same as in the case of $Y^{(6,1)}_{T\times V}$.

- *The singular case* $6,2$. Recall the following 2-divisible classes in NS:

$$\begin{array}{l} N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2, \\ N_1 + C_1 + N_4 + N_9 + C_3 + N_{12} + M_1 + M_3. \end{array}$$

  The discriminant is $-2^2 \cdot 3 \cdot 5 = -60$ and the classes

$$N,\ M = M_1 + M_2 + M_3/2,\ M' = N_5 + C_2 + N_8 + M_1 + M_3/2,\ L$$

  are generators for $NS^\vee/NS$. Where

$$\begin{array}{rcccll} N^2 & = & -8/3 & = & 4/3 & \mod 2\mathbb{Z}, \\ M^2 & = & 1/2 & & & \mod 2\mathbb{Z}, \\ M'^2 & = & 3/2 & & & \mod 2\mathbb{Z}, \\ L^2 & = & -18/5 & = & 2/5 & \mod 2\mathbb{Z}. \end{array}$$

Hence the transcendental lattice is the same as in the case of $Y^{(6,2)}_{T\times V}$.

The family $Y_{\lambda,O\times T}$. Recall the following 2-divisible class of NS:

$$\bar{L}' = L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_2,$$

and the following 3-divisible class of NS:

$$k_1 = L_1 - L_2 + L_4 - L_5 - L'_1 + L'_2 - L'_4 + L'_5 + N_1 - N_2 + N_3 - N_4.$$

The general surface in the pencil has $\rho = 19$ and we have four surfaces with $\rho = 20$. The discriminant of the general $K3$-surface in the pencil is $2^3\cdot 3\cdot 7 = 168$. We specify the following generators of $NS^\vee/NS$:

$$\begin{array}{lcl} M & := & L_1 + L_3 + L_5 + M_1/2, \\ M' & := & L_1 + L_3 + L_5 + M_2/2, \\ R & := & R_2/2, \\ N & := & N_1 - N_2 - N_3 + N_4/3, \\ L & := & (2L''_2 + 4L_0 - 2L_1 - 2L'_1 + 3L_2 + 3L'_2 - 3L_3 - 3L'_3 - 2L_4 - 2L'_4 - L_5 - L'_5)/7 \end{array}$$

where

$$\begin{array}{lllll} M^2 & = -2 & = 0 & \mod 2\mathbb{Z}, \\ M'^2 & = -2 & = 0 & \mod 2\mathbb{Z}, \\ R^2 & = -1/2 & = 3/2 & \mod 2\mathbb{Z}, \\ N^2 & = -4/3 & = 2/3 & \mod 2\mathbb{Z}, \\ L^2 & = -16/7 & = 12/7 & \mod 2\mathbb{Z}. \end{array}$$

Observe that the classes M, M' and L are not orthogonal to each other; in fact $M\cdot M' = 1/2 \mod 2\mathbb{Z}$ and $M\cdot L = M'\cdot L = 1 \mod 2\mathbb{Z}$. Hence the discriminant form of the Picard lattice is:

$$\mathbb{Z}_2(0)\oplus\mathbb{Z}_2(0)\oplus\mathbb{Z}_2(3/2)\oplus\mathbb{Z}_3(2/3)\oplus\mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(0)\oplus\mathbb{Z}_2(0)\oplus\mathbb{Z}_{42}(79/42).$$

- *The singular case* $8,1$. Here the discriminant is $-2^2\cdot 7 = -28$ and the generators for $NS^\vee/NS$ are M, M' and L. The discriminant form is

$$\mathbb{Z}_2(0)\oplus\mathbb{Z}_2(0)\oplus\mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(0)\oplus\mathbb{Z}_{14}(12/7).$$

- *The singular case* $8,2$. The discriminant is $-2^2\cdot 3\cdot 7 = -84$ and the generators for $NS^\vee/NS$ are $M+R$, $M'+R$, N and L. The discriminant form is

$$\mathbb{Z}_2(3/2)\oplus\mathbb{Z}_2(3/2)\oplus\mathbb{Z}_3(2/3))\oplus\mathbb{Z}_7(12/7 \cong \mathbb{Z}_2(3/2)\oplus\mathbb{Z}_{42}(163/42) = \mathbb{Z}_2(3/2)\oplus\mathbb{Z}_{42}(79/42).$$

- *The singular case* $8,3$. Here the discriminant is $-2^3\cdot 3\cdot 7 = -168$ and the generators for $NS^\vee/NS$ are R,

$$R' = M_1 + 2C + 3M_2/4,$$

  N and L, where $R'^2 = 1/4 \mod 2\mathbb{Z}$. The discriminant form is

$$\mathbb{Z}_2(3/2)\oplus\mathbb{Z}_4(1/4)\oplus\mathbb{Z}_3(2/3)\oplus\mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(3/2)\oplus\mathbb{Z}_{84}(221/84) = \mathbb{Z}_2(3/2)\oplus\mathbb{Z}_{84}(53/84).$$

- *The singular case* $8,4$. Recall the 2-divisible class in NS

$$L_1 + L_3 + L_5 + N_1 + C + N_4 + R_2 + M_1$$

  The discriminant is $-2^2\cdot 7 = -28$ and the generators for $NS^\vee/NS$ are $L'+R$,

$$M'' = M_1 + M_2 + R_2/2,$$

and L, where $M''^2 = 1/2 \mod 2\mathbb{Z}$.
The discriminant form is

$$\mathbb{Z}_2(3/2) \oplus \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_7(12/7)) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{14}(31/14) \cong \mathbb{Z}_2(3/2) \oplus \mathbb{Z}_{14}(3/14) \mod 2\mathbb{Z}.$$

The discriminant of the transcendental lattice differs from the discriminant form of the Picard lattice just by the sign, hence in the general case is

$$\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_{42}(5/42)$$

and in the special cases is

$$\begin{array}{ll} 8,1: & \mathbb{Z}_2(0) \oplus \mathbb{Z}_{14}(2/7), \\ 8,2: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{42}(5/42), \\ 8,3: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{84}(115/84), \\ 8,4: & \mathbb{Z}_2(1/2) \oplus \mathbb{Z}_{14}(25/14). \end{array}$$

Here we identify the transcendental lattice for this four singular cases, and in the next section for the general $K3$-surface.

- *The singular case* $8,1$. We classify all the reduced matrices with discriminant 28 ([Bu, pp.19–20]). We have just the following possibilities:

$$A := \begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}, \; B := \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}.$$

  Now take the form B and the generators $(0, 1/2)$ and $(3/14, 1/14)$. These span exactly the lattice we were looking for.

- *The singular case* $8,2$. We classify all the reduced matrices with discriminant 84 ([Bu, pp.19–20]). We have the following four cases:

$$\begin{pmatrix} 2 & 0 \\ 0 & 42 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 22 \end{pmatrix}, \; C := \begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix}.$$

  Now we take the form C and the generators $(1/2, 0)$ and $(8/21, -19/42)$ and we are done.

- *The singular case* $8,3$. We classify all the reduced matrices with discriminant 168 ([Bu, pp.19–20]). We have the following four cases

$$\begin{pmatrix} 2 & 0 \\ 0 & 84 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 28 \end{pmatrix}, \; E := \begin{pmatrix} 12 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 42 \end{pmatrix}.$$

  Now we take the form E and the generators $(1/2, 1/2)$ and $(1/12, 1/7)$. These span exactly the lattice we were looking for.

- *The singular case* $8,4$. The discriminant is 28 like in the case of $8,1$. Now by taking the form A and the generators $(1/2, 0)$ and $(0, 5/14)$ we are done.

The family $Y_{\lambda,(OO)''}$. Recall the following 2-divisible class of NS

$$\bar{L} = M_1 + M_2 + M_3 + M_4 + R_1 + R_3 + R_1' + R_3'$$

We identify the transcendental lattices of the surfaces $Y^{(8,1)}_{(OO)''}$ and $Y^{(8,4)}_{(OO)''}$.

- *The singular case* $8,1$. In this case the the discriminant is $-2^2 \cdot 7 = -28$ and we have the following generators in $NS^\vee/NS$

$$\begin{array}{lcl} L & := & 2L_2 + 4L_4 - 2L_2' - 4L_4'/7, \\ M & := & R_1 + R_3 + M_1 + M_3/2, \\ M' & := & R_1 + R_3 + M_1 + M_4/2, \end{array}$$

where

$$\begin{array}{lcccl} L^2 & = & 12/7 & & \mod 2\mathbb{Z}, \\ M^2 & = & M'^2 & = \ 0 & \mod 2\mathbb{Z}. \end{array}$$

Hence the transcendental lattice is the same as in the case $Y^{(8,1)}_{O\times T}$.

- *The singular case* $8,4$. Recall the following 4-divisible class in NS

$$W := R_1 + 2R_2 + 3R_3 + R_1' + 2R_2' + 3R_3' + 2N_1 + 2C_1 + 3M_1 + M_2 + 2N_3 + 2C_2 + 3M_3 + M_4.$$

Moreover specify the classes:

$$\begin{array}{lcl} v_1 & := & R_1 + 2R_2 + 3R_3/4, \\ v_2 & := & R_1' + 2R_2' + 3R_3'/4, \\ v_3 & := & 2N_1 + 2C_1 + 3M_1 + M_2/4, \\ v_4 & := & 2N_3 + 2C_2 + 3M_3 + M_4/4,. \end{array}$$

The discriminant is $-2^4 \cdot 7 = -112$ and the generators of $NS^\vee/NS$ are

$$v_1 + v_3/4,\ v_2 + v_4/4,\ L$$

with

$$(v_1 + v_3/4)^2 = (v_2 + v_4/4)^2 = 0 \mod 2\mathbb{Z}.$$

The discriminant form of the Picard lattice is

$$\mathbb{Z}_4(0) \oplus \mathbb{Z}_4(0) \oplus \mathbb{Z}_7(12/7) = \mathbb{Z}_4(0) \oplus \mathbb{Z}_{28}(12/7).$$

Hence the discriminant form of the transcendental lattice is

$$\mathbb{Z}_4(0) \oplus \mathbb{Z}_{28}(2/7).$$

We classify all the reduced matrices with discriminant 112, these are

$$\begin{pmatrix} 2 & 0 \\ 0 & 56 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 28 \end{pmatrix}, F := \begin{pmatrix} 8 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 8 & 4 \\ 4 & 16 \end{pmatrix}.$$

We take the matrix F and the generators $(1/4, 1/2)$ and $(1/4, 9/14)$, so we are done.

The family $Y_{\lambda,T\times T}$. A similar computation as before shows that in the singular case $8,1$, resp. $8,4$ the transcendental lattice has bilinear form with intersection matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \text{ resp. } \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}.$$

Remark 2.1. *Observe that if the reduced matrices had not been as in case 1 or 2 of Theorem 1.2, we would find two different isomorphism classes of K3-surfaces with the same discriminant and the same discriminant form (cf.* [SZ, p. 3]*).*

2.2. **The general cases.** Here we identify the transcendental lattice of the general surfaces, $\rho = 19$, in the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$. In the last section, we have identified the discriminant form of the transcendental lattice, we use it to determine T. We need the following:

Definition 2.1. *(cf.*[B, Def. 1.1]*) The discriminant $d = d_{NS} = -d_T$ is* small *if $4 \cdot d$ is not divisible by k^3 for any non square natural number k congruent to 0 or 1 modulo 4.*

Then if d_T is small , the lattice T is uniquely determined by its genus (cf. [CS, Thm. 21, p. 395] and [CS, p. 378] for the definition of genus when the rank is three), hence by signature and discriminant form ([N1, Cor. 1.9.4])

- *The family $Y_{\lambda,T\times V}$.* The candidate lattice is

$$T_0 := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

this has discriminant -30, and taking the generator

$$f_1 := \begin{pmatrix} 4/15 \\ -1/15 \\ 1/2 \end{pmatrix}$$

one computes $q_{T_0}(f_1) = -7/30 = 53/30 \mod 2\mathbb{Z}$, hence the discriminant form is $\mathbb{Z}_{30}(53/30)$. Since $d_T = -30$ is small, the transcendental lattice of the general K3-surface is T_0.

- *The family $Y_{\lambda,O\times T}$.* The candidate lattice is

$$T_1 := \begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

this has discriminant -168, and taking the generators

$$f_1 := \begin{pmatrix} 0 \\ 3/2 \\ 1/2 \end{pmatrix},\ f_2 := \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix},\ f_3 := \begin{pmatrix} 37/42 \\ -19/42 \\ -1/2 \end{pmatrix}$$

we find $q_{T_1}(f_i) = 0 \mod 2\mathbb{Z}$, $i = 1, 2$ and $q_{T_1}(f_3) = 5/42 \mod 2\mathbb{Z}$, hence the discriminant form is $\mathbb{Z}_2(0) \oplus \mathbb{Z}_2(0) \oplus \mathbb{Z}_{42}(5/42)$. Since $d_T = -168$ is small, we have $T = T_1$.

We collect the results in the table 1. In [S2] we are able to describe the Picard lattice only of some singular surfaces in the families $Y_{\lambda,(TT)'}$, $Y_{\lambda,(OO)''}$, $Y_{\lambda,T\times T}$ (none in the family $Y_{\lambda,V\times V}$), hence we compute here only the transcendental lattice of these surfaces. This explains the gaps in the table 1. We include also the results of [B] about the general surfaces of the families Y_{λ,G_6}, Y_{λ,G_8}, $Y_{\lambda,G_{12}}$ and also about the singular surfaces in these pencils, Barth computed the transcendental lattices of the singular surfaces too, but he did not published his result. In the table we give also the discriminants of the lattices.

2.3. **Moduli curve.** Let

$$\Omega = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega, \omega) = 0; (\omega, \bar{\omega}) > 0\},$$

where Λ is the $K3$-lattice, this is an open subset in a quadric in $\mathbb{P}^{21}$. If X is a K3-surface and $\omega_X \in H^{2,0}(X)$, then it is well known that $\omega_X \in \Omega$ and it is called a *period point*. Moreover also the converse is true: each point of Ω occurs as period point of some K3-surface, this is the so called *surjectivity of the period map* (cf. [BPV, Thm. 14.2]). Now let $M \subset \Lambda$ be a sublattice of signature $(1, \rho - 1)$ and define:

$$\Omega_M = \{[\omega] \in \Omega | (\omega, \mu) = 0 \text{ for all } \mu \in M\}.$$

This has dimension $20 - \rho = 20-\text{rank } M$. If rank M=19 then this space is a curve. Let X be a K3-surface with $\rho = 19$. Since in this case the embedding of $T(X)$ in Λ is unique up to isometry of Λ (cf. [M, Cor. 2.10]), $T(X)$ determines Ω_M, with $M = T(X)^{\perp} = NS(X)$ and so the moduli curve, which classify the K3-surfaces. Hence in our cases the transcendental lattices given in the table 1 identify the moduli curve of the K3-surfaces in the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ (in the case $\rho = 19$).

TABLE 1. Transcendental lattices

Family	general surface	singular surfaces			
Y_{λ,G_6}	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & -6 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 30 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$
d	-90	15	60	60	15
Y_{λ,G_8}	$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 28 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 28 \end{pmatrix}$
d	-336	28	84	168	112
$Y_{\lambda,G_{12}}$	$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 34 & 0 \\ 0 & 0 & -30 \end{pmatrix}$	$\begin{pmatrix} 12 & 6 \\ 6 & 58 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 220 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 132 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 34 \end{pmatrix}$
d	-3960	660	1320	792	132
$Y_{\lambda,T\times V}$	$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$
d	-30	15	60	60	15
$Y_{\lambda,O\times T}$	$\begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$	$\begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix}$	$\begin{pmatrix} 12 & 0 \\ 0 & 14 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$
d	-168	28	84	168	28
$Y_{\lambda,(TT)'}$	-	$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	-	-
d	-	15	60	-	-
$Y_{\lambda,(OO)''}$	-	$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$	-	-	$\begin{pmatrix} 8 & 0 \\ 0 & 14 \end{pmatrix}$
d	-	28	-	-	112
$Y_{\lambda,T\times T}$	-	$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	-	-	$\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$
d	-	7	-	-	28

3. Shioda-Inose structure

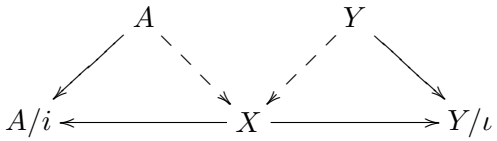
By a result of Morrison $K3$-surfaces with $\rho = 19$ or $\rho = 20$ admit a Shioda-Inose structure. Before discussing our cases we recall some definitions and facts.

Definition 3.1. *(cf.* [M, Def. 5.1]*) An involution ι on a $K3$-surface Y is a Nikulin involution if $\iota^*\omega = \omega$ for every $\omega \in H^{2,0}(Y)$.*

It is possible to show that every Nikulin involution has eight isolated fixed points and the minimal resolution of the quotient Y/ι is again a $K3$-surface ([N2, 11; Sect. 5]).

Definition 3.2. *(cf.* [M, Def. 6.1]*) A K3-surface Y admits a Shioda-Inose structure if there is a Nikulin Involution ι on Y with rational quotient map $\pi : Y - - \to X$ such that X is a Kummer surface, and π_* induces an Hodge isometry $T(Y)(2) \cong T(X)$ (where we denote by $T(Y)(2)$ the lattice $T(Y)$ with the bilinear form multiplied by 2).*

Hence we have the following diagram:



where A is the complex torus whose Kummer-surface is X, i is an involution on A with 16 fix-points and the rational maps from A to X and from Y to X are 2:1. By definition we have $T(Y)(2) \cong T(X)$ and by [M, Prop. 4.3], we have $T(A)(2) \cong T(X)$ hence the diagram induces an Hodge isometry $T(Y) \cong T(A)$.
In our cases the $K3$-surface which we consider are algebraic hence A is an abelian variety (cf. [M, Thm. 6.3, (ii)]). Moreover whenever Y is an algebraic $K3$-surface and $\rho(Y) = 19$ or 20 then Y admits always a Shioda-Inose structure (cf. [M, Cor. 6.4]). Whenever $\rho = 20$ Shioda and Inose show that $A = C_1 \times C_2$ where C_1 and C_2 are elliptic curves

$$C_i = \mathbb{C}/\mathbb{Z} + \mathbb{Z} \cdot \tau_i,\ i = 1, 2$$

with

$$\tau_1 = (-c + \sqrt{-d})/2a,\ \tau_2 = (c + \sqrt{-d})/2,\ (d = 4ab - c^2)$$

We show that in the case of the general $K3$-surfaces of the families $Y_{\lambda,T\times V}$ and of $Y_{\lambda,O\times T}$ the abelian surface $A(\lambda)$ is *simple*, i.e. it is not a product of elliptic curves, in this case we say that the Shioda-Inose structure is *simple*.
The transcendental lattice $T(A(\lambda))$ has rank 3 hence its orthogonal complement $NS(A(\lambda))$ in U^3 has rank 3 too and we have $NS(A(\lambda)) \cong T(Y(\lambda))(-1)$ because by [CS, Thm. 21, p. 395], the lattices are uniquely determined. We use this fact to show:

Theorem 3.1. *For general λ, $A = A(\lambda)$ is not a product of elliptic curves.*

Proof. (cf. [B, Thm. 5.1]) We show that A does not contain any elliptic curve C, i.e. a curve with $C^2 = 0$, by using Fermat's infinite descent applied to some Diophantine equation.
The general surface in $Y_{\lambda,T\times V}$: We have intersection form on the transcendental lattice with matrix

$$T_0 := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

hence the form on $NS(A)$ is

$$\begin{pmatrix} -4 & -1 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The associated quadratic form is

$$-4x^2 - 2xy - 4y^2 + 2z^2,\ x, y, z \in \mathbb{Z}.$$

If A contains an elliptic curve, then there are $x, y, z \in \mathbb{Z}$ with

$$2z^2 = 4x^2 + 2xy + 4y^2$$

hence

$$8z^2 = 16x^2 + 8xy + 16y^2.$$

Put $u = 4x + y$, then

$$8z^2 = u^2 + 15y^2. \tag{2}$$

Hence we have $u^2 = 3z^2 \mod 5\mathbb{Z}$, since 3 is not a square modulo 5 we have $u = z = 0 \mod 5\mathbb{Z}$, hence $u = 5u_1$, $z = 5z_1$, so

$$3y^2 = 5(8z_1^2 - u_1^2) \tag{3}$$

hence $y = 5y_1$ and substituting in (3) and dividing by 5 we find

$$15y_1^2 = 8z_1^2 - u_1^2,$$

which is the same as (2).

The general surface in $Y_{\lambda, O\times T}$: We have intersection form on the transcendental lattice with matrix:

$$T_1 := \begin{pmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Hence the form on $NS(A)$ is

$$\begin{pmatrix} -10 & -4 & 0 \\ -4 & -10 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The quadratic form is

$$-10x^2 - 8xy - 10y^2 = 2z^2,\ x, y, z \in \mathbb{Z}.$$

If A contains an elliptic curve, then there are $x, y, z \in \mathbb{Z}$ with

$$2z^2 = 10x^2 + 8xy + 10y^2$$

hence dividing by 2 and multiplying by 5 we find

$$5z^2 = 25x^2 + 20xy + 25y^2 = (5x + 2y)^2 + 21y^2.$$

Put $u = 5x + 2y$, then

$$5z^2 = u^2 + 21y^2. \tag{4}$$

Hence we have $u^2 = 5z^2 \mod 7\mathbb{Z}$. Since 5 is not a square modulo 7 we have $u = 7u_1$, $z = 7z_1$, so we obtain

$$3y^2 = 7(5z_1^2 - u_1^2) \tag{5}$$

hence $3y^2 = 0 \mod 7\mathbb{Z}$ and so $y = 7y_1$. Substituting in (5) and dividing by 7 we find

$$21y_1^2 = 5z_1^2 - u_1^2,$$

which is again (4). □

4. Hessians and extremal elliptic K3-surfaces

Many of the singular K3-surfaces of this article appear already in other realizations.
In [DvG] Dardanelli and van Geemen give a criteria to establish if a singular K3-surface is the desingularization of the Hessian of a cubic surface:

Proposition 4.1. *(cf.* [DvG, Prop. 2.4.1]*) Let T be an even lattice of rank 2,*

$$T = \begin{pmatrix} 2n & a \\ a & 2m \end{pmatrix}.$$

There is a primitive embedding $T \hookrightarrow T_{Hess}$ if and only if at least one among a, n and m is even. In this case T embeds in $U \oplus U(2)$.

Here $T_{Hess} = U \oplus U(2) \oplus A_2(-2)$. If we look in table 1 we see that all our singular $K3$-surfaces are desingularizations of Hessian surfaces of cubic surfaces. In particular Dardanelli and van Geemen study explicitly the singular K3-surfaces with

$$T = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

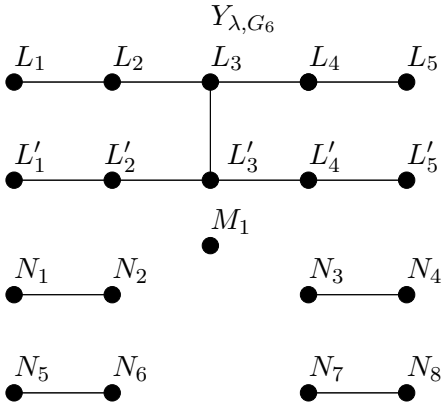
They call the surface X_{10} and show that it is the desingularization of the Hessian surface of the cubic surface with 10 Eckardt points. The latter has e.g. the following equation in $\mathbb{P}_4$ (this is the Clebsch diagonal surface):

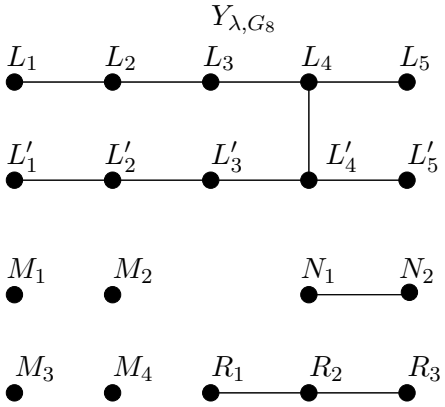
$$\sum_{i=0}^{4} x_i^3 = 0, \ \sum_{i=0}^{4} x_i = 0.$$

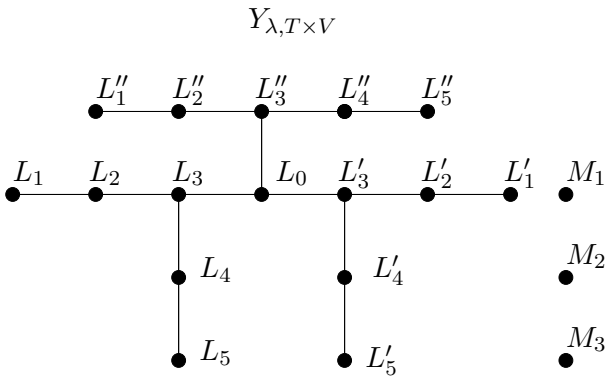
Finally observe that the singular surfaces of the families Y_{λ,G_6}, $Y_{\lambda,T\times V}$ and $Y_{\lambda,(TT)'}$ have an *extremal elliptic fibration*, in the sense of Miranda and Persson (cf. [MP]) and of Shimada and Zhang (cf. [SZ]), in fact these are the numbers: 322, 173, 102, 148, 276 in the list [SZ, Table 2, pp. 15-24].

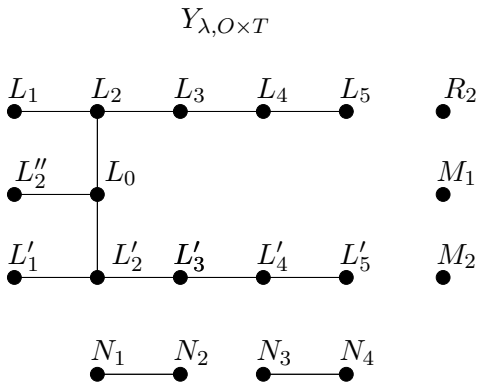
5. Figures: Configurations of rational curves

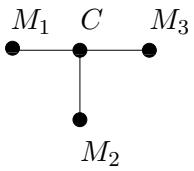
In this section we recall from [S2] the configurations of (-2)-rational curves generating the Neron-Severi group. In the case of the families $Y_{\lambda,T\times V}$ and $Y_{O\times T}$ the curves L_i, L_i' and L_i'' on the general K3-surface are also contained in the Neron-Severi group of the singular K3-surfaces, but we do not draw their configuration again. Moreover since the singular surfaces $Y_{T\times V}^{(6,1)}$ and $Y_{T\times V}^{(6,4)}$, as well as the surfaces $Y_{T\times V}^{(6,2)}$ and $Y_{T\times V}^{(6,3)}$ have the same graph, we draw just one picture.


Y_{λ,G_6}
L_1 L_2 L_3 L_4 L_5
L'_1 L'_2 L'_3 L'_4 L'_5
M_1
N_1 N_2 N_3 N_4
N_5 N_6 N_7 N_8

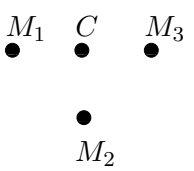

Y_{λ,G_8}
L_1 L_2 L_3 L_4 L_5
L'_1 L'_2 L'_3 L'_4 L'_5
M_1 M_2 N_1 N_2
M_3 M_4 R_1 R_2 R_3


$Y_{\lambda,T\times V}$
L''_1 L''_2 L''_3 L''_4 L''_5
L_1 L_2 L_3 L_0 L'_3 L'_2 L'_1 M_1
L_4 L'_4 M_2
L_5 L'_5 M_3

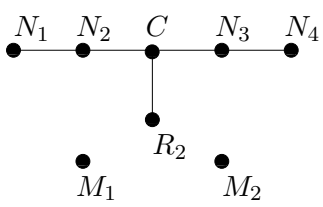

$Y_{\lambda,O\times T}$
L_1 L_2 L_3 L_4 L_5 R_2
L''_2 L_0 M_1
L'_1 L'_2 L'_3 L'_4 L'_5 M_2
N_1 N_2 N_3 N_4



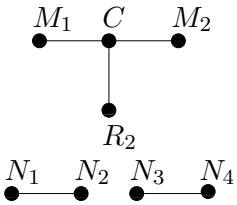
$Y^{(6,1)}_{\lambda,T\times V}(Y^{(6,4)}_{\lambda,T\times V})$



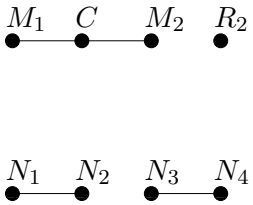
$Y^{(6,2)}_{\lambda,T\times V}(Y^{(6,3)}_{\lambda,T\times V})$



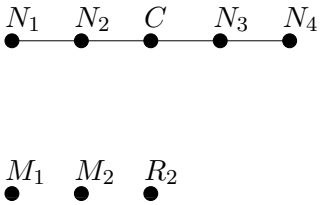
$Y^{(8,1)}_{\lambda,O\times T}$



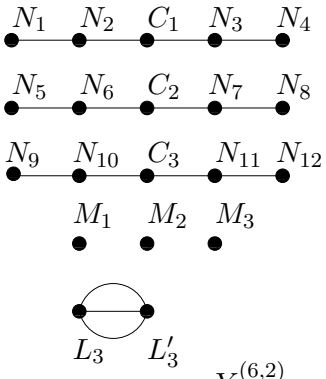
$Y^{(8,2)}_{\lambda,O\times T}$



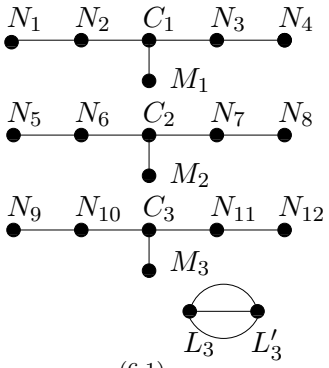
$Y^{(8,3)}_{\lambda,O\times T}$



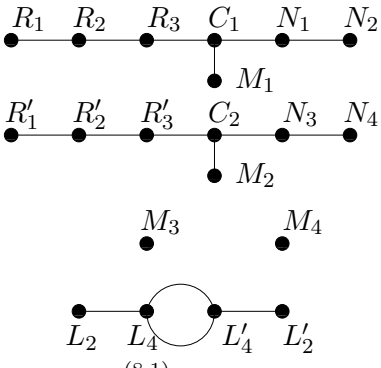
$Y^{(8,4)}_{\lambda,O\times T}$



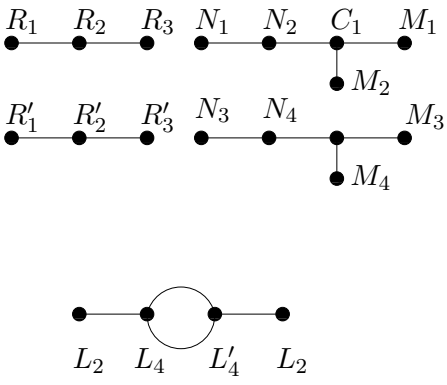
$Y^{(6,2)}_{\lambda,(TT)'}$



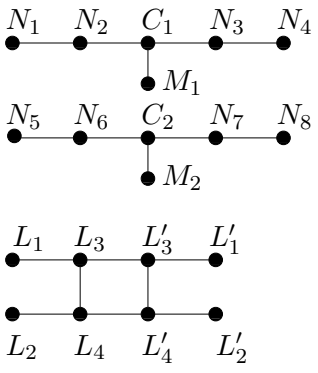
$Y^{(6,1)}_{\lambda,(TT)'}$



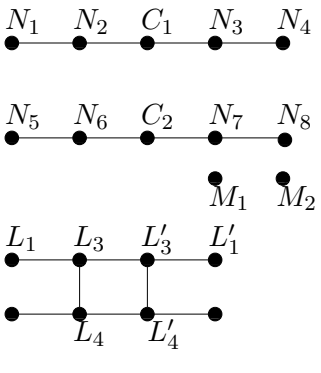
$Y^{(8,1)}_{\lambda,(OO)''}$



$Y^{(8,4)}_{\lambda,(OO)''}$



$Y^{(8,1)}_{\lambda,T\times T}$



$Y^{(8,4)}_{\lambda,T\times T}$

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