The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces

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Motivation

- Paper of the physicists Berglund and Hübsch (1992): they describe a special mirror construction for Calabi-Yau threefolds.
- Recent generalization by Chiodo and Ruan (2010).
- Recent results by Krawitz, Borisov, Ebeling,...
- Analyze the construction for K3 surfaces and look for relation with the mirror symmetry for lattice polarized K3 surfaces decribed by Dolgachev–Voisin.

The construction

Let

$$W = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{a_{ij}}$$

be a *potential*, we assume that W is

- *non-degenerate*: the only critical point is the origin.
- *invertible*: the matrix $A := (a_{ij})$ is an $N \times N$-invertible matrix (over $\mathbb{Q}$).

Let $A^{-1} := (a^{ij})$ be the inverse matrix and $q_i := \sum_{j=1}^{N} a^{ij}$ be the *charges*. Then:

- $\{W = 0\} \subset \mathbb{P}(w_1, \ldots, w_N)$ is an hypersurface in a w.p.s, where $w_i = dq_i$, d the least positive integer ($d \neq 0$) such that $dq_i \in \mathbb{Z}$, $\forall i$, d is the *total degree* of W.

Definition

A compact, complex manifold W with at most canonical singularities, trivial canonical bundle and $H^0(X, \Omega^l_X) = 0$ for $0 < l < \dim W$ is a *Calabi Yau* (singular) manifold.

We have $W \subset \mathbb{P}(w_1, \ldots, w_N)$ is CY iff $\sum q_i = 1$ and the w_i satisfy some numerical conditions (W is *well formed*):

- $\gcd(w_1, \ldots, \hat{w}_i, \ldots, w_N) = 1$.
- $\gcd(w_1, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_N)$ divides d.

Up to now we will always assume that W is CY.

Diagonal Automorphisms

Let

$$\mathrm{Aut}(W) = \{\gamma = (\exp(2\pi i\beta_1), \ldots, \exp(2\pi i\beta_N)) \in (\mathbb{C}^*)^N \,|\, W(\gamma x) = W(x)\}$$

denote the **diagonal** automorphisms group of W. Then:

- $\mathrm{Aut}(W)$ is finite abelian.
- $\mathrm{Aut}(W)$ is generated by $\rho_j = (\exp(2\pi i a^{1j}), \ldots, \exp(2\pi i a^{Nj})$.
- $\langle \rho_1 \cdot \ldots \cdot \rho_N \rangle = \langle (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_N)) \rangle = J_W$ is the *trivial subgroup* (acts trivially on the w.p.s. $\mathbb{P}(w_1, \ldots, w_N)$).
- $J_W \subset \mathrm{Aut}(W) \cap \mathrm{SL}_N(\mathbb{C}) := \mathrm{SL}(W)$, since $\sum q_i = 1$.

The BHCR mirror symmetry

The transposed matrix $A^T = (a_{ji})$ defines a *transposed potential* W^T,

- W is CY iff W^T is CY.

Let $J_W \subset G_W \subset \mathrm{SL}(W)$ we can associate to G_W a *transposed group* G_W^T (Krawitz 2009) with:

- $(G_W^T)^T = G_W$,
- $(J_W)^T = \mathrm{SL}(W^T)$, $(\mathrm{SL}(W))^T = J_{W^T}$,
- $J_{W^T} \subset G_W^T \subset \mathrm{SL}(W^T)$.

Theorem (Chiodo-Ruan 2010)

Let W be a non-degenerate, invertible potential defining a CY manifold. Denote $\widetilde{G_W} := G_W/J_W$ and $\widetilde{G_W^T} := (G_W)^T/J_{W^T}$ then the orbifolds $[W/\widetilde{G_W}]$ and $[W^T/\widetilde{G_W^T}]$ are mirror of each other, in the following sense

$$H_{CR}^{p,q}([W/\widetilde{G_W}], \mathbb{C}) \cong H_{CR}^{(N-2)-p,q}([W^T/\widetilde{G_W^T}], \mathbb{C})$$

The BHCR for K3 surfaces

- If $\dim(W) = 2$ the theorem gives no information.
- Apply the construction to K3 surfaces with non-symplectic involution.
- Compare it with the mirror construction by Dolgachev-Voisin.

Consider a K3 surface W with non-symplectic involution $\iota : x \mapsto -x$ and equation given by a non-degenerate, invertible potential:

$$x^2 = f(y, z, w) \subset \mathbb{P}(w_1, w_2, w_3, w_4).$$

Let $J_W \subset G_W \subset \mathrm{SL}(W)$, $J_{W^T} \subset G_W^T \subset \mathrm{SL}(W^T)$ and $\widetilde{G_W} := G_W / J_W$, $\widetilde{G_W^T} := (G_W)^T / J_{W^T}$.

Let $\widetilde{W} \longrightarrow W/\widetilde{G_W}$ and $\widetilde{W^T} \longrightarrow W^T/\widetilde{G_W^T}$ be minimal resolutions.

Remark: The groups $\widetilde{G_W}$ and $\widetilde{G_W^T}$ act symplectically.

Theorem (Artebani, Boissière, Sarti, 2011)

The BHCR-mirror couples $\widetilde{W}$ and $\widetilde{W^T}$ belong to the mirror families described by Dolgachev and Voisin.

Example

Consider a K3 surface:

$$W : \quad x^2 = z^7y + y^3w + w^{10} \subset \mathbb{P}(5,3,1,1), \quad \deg(W) = 10$$

Let $G_W = J_W$ which has order 10, the total degree of W. One computes $(r, a, \delta) = (3, 1, 1)$.
Consider the transposed K3 surface:

$$W^T : \quad x^2 = z^7 + zy^3 + yw^{10} \subset \mathbb{P}(7,4,2,1), \quad \deg(W^T) = 14,$$

one computes $\widetilde{J_W^T} = \mathrm{SL}(W^T)/J_{W^T} \cong \mathbb{Z}/3\mathbb{Z}$. For the minimal resolution of $W^T/\widetilde{G_W^T}$ one has $(r, a, \delta) = (17, 1, 1)$.
Methods: Classification of non-degenerate invertible potentials (Kreuzer–Skarke 1992), Reid's list of K3 surfaces in w.p.s., study of singularities of hypersurfaces in w.p.s., combinatorics to compute $\mathrm{SL}(W)$ and $\mathrm{SL}(W^T)$,...