Non-symplectic automorphisms of prime order on K3 surfaces

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K3 surfaces

A K3 surface X is a smooth, compact, complex, simply connected surface such that

$$K_X \sim 0.$$

In particular

$$H^0(X, \mathcal{O}_X(K_X)) = \mathbb{C} \cdot \omega_X$$

where ω_X is a nowhere vanishing holomorphic 2-form.

We study automorphisms of X, i.e. biholomorphic maps

$$\sigma : X \longrightarrow X$$

An easy example

Consider the K3 surface

$$X : \underbrace{x_0^4 + x_1^4 + x_0 x_2^3 + x_1 x_3^3}_{:=f} = 0$$

in $\mathbb{P}_3$ and the automorphism

$$\begin{array}{rccc} \sigma : & \mathbb{P}_3 & \longrightarrow & \mathbb{P}_3 \\ & (x_0 : x_1 : x_2 : x_3) & \mapsto & (x_0 : x_1 : \zeta_3 x_2 : x_3) \end{array}$$

where $\zeta_3 = e^{\frac{2i\pi}{3}}$. In the chart $x_0 \neq 0$, $\frac{\partial f}{\partial x_3} \neq 0$ the holomorphic 2-form on X can be written as

$$\omega := \frac{dx_1 \wedge dx_2}{\frac{\partial f}{\partial x_3}} = \frac{dx_1 \wedge dx_2}{3 x_3^2 x_1}$$

the action is

$$\sigma(\omega) = \zeta_3 \omega$$

The fixed locus

$$X : \; x_0^4 + x_1^4 + x_0 x_2^3 + x_1 x_3^3 = 0$$

$$\begin{array}{cccc} \sigma : & \mathbb{P}_3 & \longrightarrow & \mathbb{P}_3 \\ & (x_0 : x_1 : x_2 : x_3) & \mapsto & (x_0 : x_1 : \zeta_3 x_2 : x_3) \end{array}$$

The fixed locus is the intersection of X with the plane $\{x_2 = 0\}$ and the point $(0 : 0 : 1 : 0)$ hence

$$X^\sigma = \{\text{Plane curve of genus 3}\} \cup \{\text{pt}\}$$

Automorphisms operating non trivially on the holomorphic 2-form, like σ, are called *non-symplectic*

Non-symplectic automorphisms

An automorphism σ of X is called *purely non-symplectic* if

$$\sigma^*(\omega_X) = \zeta_I \omega_X, \quad \zeta_I = e^{\frac{2i\pi}{I}}$$

In this case the order of σ is finite equal to I.

- Nikulin '80: list of possibilities for non symplectic automorphisms on K3 surfaces.
- Possible prime orders are: $I = 2, 3, 5, 7, 11, 13, 17, 19$
- A first problem: determine the fixed locus.

Involutions

Let

$$i : X \longrightarrow X, \quad i^2 = id, \quad i^*(\omega_X) = -\omega_X$$

Nikulin ('80) computed the fixed locus of i:

- X^i can be empty. In this case the quotients are well known and are exactly the Enriques surfaces.
- X^i is the disjoint union of smooth curves (Nikulin gives the complete list of possibilities)

The orders 11, 13, 17, 19

If the order is $I = 13$, 17, 19 there is only one couple (X_I, σ_I). The examples are produced by using elliptic fibrations ($t \in \mathbb{C}$).

I	X_I	σ_I	fixed points	fixed curves
13	$y^2 = x^3 + t^5 x + t$	$(\zeta_{13}^5 x, \zeta_{13} y, \zeta_{13}^2 t)$	9	1 rational
17	$y^2 = x^3 + t^7 x + t^2$	$(\zeta_{17}^7 x, \zeta_{17}^2 y, \zeta_{17}^2 t)$	7	–
19	$y^2 = x^3 + t^7 x + t$	$(\zeta_{19}^7 x, \zeta_{19}^2 y, \zeta_{19}^2 t)$	4	–

If $I = 11$ there are 3 possibilities for the fixed locus.

(Kondo 1989, Oguiso and Zhang 1999/ Artebani, S., Taki 2009)

The orders 3, 5, 7

At a fixed point $x \in X^\sigma$ the action can be linearized as follows

$$\begin{pmatrix} \zeta_I^{t+1} & 0 \\ 0 & \zeta_I^{I-t} \end{pmatrix}, \quad t = 0, \ldots, I-2$$

- the fixed locus is non empty and contains smooth disjoint curves ($t = 0$) or isolated fixed points
- if the fixed locus contains a fixed curve of genus $g \geq 1$, then this is the only fixed curve of genus $g \geq 1$, the other are rational.
- we can write

$$X^\sigma = C_g \cup R_1 \cup \ldots \cup R_k \cup \{p_1, \ldots, p_n\}$$

where C_g is a smooth curve of genus $g \geq 0$, R_i is a smooth rational curve and p_j is an isolated fixed point.

Two important sublattices

Recall that for K3 surfaces

$$H^2(X,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}, \quad U = \{\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$$

The action of σ on X induces an action on cohomology

$$\sigma^* : H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z})$$

We have two important primitive sublattices

$$S(\sigma) = \{x \in H^2(X,\mathbb{Z}) \,|\, \sigma^*(x) = x\}, \quad T(\sigma) = S(\sigma)^{\perp}$$

Properties of the sublattices

- $S(\sigma) \subset S_X$ (the Picard group of X) and (the transcendental lattice of X) $T_X = S_X^\perp \subset T(\sigma)$
- T_X and $T(\sigma)$ are free modules over $\mathbb{Z}[\zeta_I]$ via the action of σ^*
- $\mathrm{rank}(T(\sigma))$ is even equal to $(I-1)m$
- $|\det S(\sigma)| = |\det T(\sigma)| = I^a$ for some $a \neq 0$
- (Rudakov-Shafarevich '89) the lattice $S(\sigma)$ is uniquely determined by its rank and a.

The classification

Main tools:

- Lefschetz formulas, topological and holomorphic
- properties of the lattices $S(\sigma)$ and $T(\sigma)$
- Smith exact sequences

One obtain relations

$$g = \frac{m-a}{2}, \quad n = 22 - m(I-2) - \frac{24}{I-1}, \quad k = \frac{12}{I-1} - \frac{m+a}{2}$$

and a complete description of the fixed locus and a list of the lattices $T(\sigma)$ and $S(\sigma)$.

Recall that:
$X^\sigma = C_g \cup R_1 \cup \ldots \cup R_k \cup \{p_1, \ldots, p_n\}$, $\quad \mathrm{rank}\,(T(\sigma)) = (I-1)m$,
$|\det T(\sigma)| = I^a$.

The order five

The local actions at a fixed point are

$$A_{5,0} = \begin{pmatrix} \zeta_5 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{5,2} = \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & \zeta_5^3 \end{pmatrix}, \quad A_{5,1} = \begin{pmatrix} \zeta_5^2 & 0 \\ 0 & \zeta_5^4 \end{pmatrix}.$$

and we obtain a table

n_1	n_2	k_0	k_1	k_2	$T(\sigma)$	$S(\sigma)$
1	0	0	0	1	$H_5 \oplus U \oplus E_8 \oplus E_8$	H_5
3	1	0	1	0	$H_5 \oplus U \oplus E_8 \oplus A_4$	$H_5 \oplus A_4$
3	1	0	0	0	$H_5 \oplus U(5) \oplus E_8 \oplus A_4$	$H_5 \oplus A_4^*(5)$
5	2	1	1	0	$U \oplus H_5 \oplus E_8$	$H_5 \oplus E_8$
5	2	1	0	0	$U \oplus H_5 \oplus A_4^2$	$H_5 \oplus A_4^2$
7	3	2	0	0	$U \oplus H_5 \oplus A_4$	$H_5 \oplus A_4 \oplus E_8$
9	4	3	0	0	$U \oplus H_5$	$H_5 \oplus E_8 \oplus E_8$

where

$$H_5 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_4^*(5) = \begin{pmatrix} -4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & -4 \end{pmatrix}$$

The families of K3 surfaces

The generic K3 surface in each family has Picard group $S_X = S(\sigma)$.

One can describe projective models of each family.

Example: $o(\sigma) = 5$, $S_X = S(\sigma) = H_5$,

$$t^2 = x_0(x_0 - x_1)\prod_{i=1}^{4}(x_0 - \lambda_i x_1) + x_2^5 x_1 \subset \mathbb{P}(3,1,1,1), \quad \lambda_i \in \mathbb{C}$$

The transformation

$$\sigma(t, x_0, x_1, x_2) = (t, x_0, x_1, \zeta_5 x_2), \quad \zeta_5 = e^{\frac{2i\pi}{5}}$$

is a non-symplectic automorphism of order 5,

$$X^\sigma = \{\mathrm{pt}\} \cup \{\text{curve of genus } 2\}$$