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Non-geometric Calabi-Yau Backgrounds and K3 automorphisms

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Abstract: We consider compactifications of type IIA superstring theory on mirror-folds obtained as K3 fibrations over two-tori with non-geometric monodromies involving mirror symmetries. At special points in the moduli space these are asymmetric Gepner models. The compactifications are constructed from non-geometric automorphisms that arise from the diagonal action of an automorphism of the K3 surface and of an automorphism of the mirror surface. We identify the corresponding gaugings of $\mathcal{N}=4$ supergravity in four dimensions, and show that the minima of the potential describe the same four-dimensional low-energy physics as the worldsheet formulation in terms of asymmetric Gepner models. In this way, we obtain a class of Minkowski vacua of type II string theory which preserve $\mathcal{N}=2$ supersymmetry. The massless sector consists of $\mathcal{N}=2$ supergravity coupled to 3 vector multiplets, giving the STU model. In some cases there are additional massless hypermultiplets.

Contents

1 Introduction

Geometric compactifications constitute only a subset of string backgrounds and have interesting generalisations to non-geometric backgrounds. Examples arise from spaces with local fibrations that have transition functions that include stringy duality symmetries. Spaces with torus fibrations and T-duality or U-duality transition functions are T-folds or U-folds [1], while those with Calabi-Yau fibrations and mirror symmetry transition functions are mirror-folds [1]. Such non-geometric spaces often have fewer moduli than their geometric counterparts, and the non-geometry typically breaks some of the symmetries, including supersymmetries, and provide an interesting tool for probing quantum geometry. Solvable worldsheet conformal field theories (CFT's) such as asymmetric orbifolds can arise at special points in the moduli space of a non-geometric background [2], allowing a complete analysis and important checks on general arguments.

Our focus here will be on mirror-folds of the type IIA superstring constructed from K3 bundles over T^2 with transition functions involving the mirror involution of the K3 surface. Previously Kawai et Sugawara have considered in [3] K3 mirror-folds with monodromies that, at least when the fiber is compact, break all supersymmetry; in the present work, we consider in contrast monodromies that preserve 8 supersymmetries, i.e. which preserve a quarter of the 32 supersymmetries of the type IIA string, or a half of the 16 supersymmetries of type IIA compactified on $K3$. As a result, we find interesting mirror-folds which give $D = 4$, $\mathcal{N} = 2$ Minkowski vacua of type IIA superstring theory. As we shall see, particular examples give precisely the STU model of [4] at low energies.

Our constructions can be viewed as particular cases of reductions with a duality twist [2]. In such a construction, a theory in D dimensions with discrete duality symmetry $G(\mathbb{Z})$ (e.g. T-duality or U-duality) is compactified on a d-torus with a $G(\mathbb{Z})$ monodromy around each of the d circles, giving a string-theory generalization of Scherk-Schwarz reduction [5]. In many cases, the theory in D dimensions has an action of the continuous group G which is a symmetry of the low energy physics, but which is broken to a discrete subgroup in the full string theory. For a field ϕ transforming under G as $\phi \mapsto g\phi$ the ansatz is of the form

$$\phi(x^\mu, y^i) = g(y)\hat{\phi}(x) \tag{1.1}$$

where y^i, $i = 1, \ldots, d$, are coordinates on T^d and x^μ, $\mu = 0, \ldots, D - d - 1$, are the remaining coordinates. With periodicities $y^i \sim y^i + 2\pi R_i$, the mondromies $g(y_i)^{-1}g(y_i + 2\pi R_i)$ must be in $G(\mathbb{Z})$ for each i.

Of particular interest are the special cases in which there are points in the moduli space in D dimensions that happen to be fixed under the action of the monodromies. For example, consider a theory where the moduli space in D dimensions contains the moduli space for a 2-torus, *i.e.* $SL(2, \mathbb{R})/U(1)$, identified under the action of the discrete group $SL(2, \mathbb{Z})$. There are special points in the moduli space which are invariant under finite subgroups of $SL(2, \mathbb{Z})$ isomorphic to $\mathbb{Z}_r$ for $r = 2, 3, 4, 6$. If one considers reductions on a circle where the monodromy is in one of these $\mathbb{Z}_r$ subgroups, then at any point in the moduli space invariant under the action of the monodromy, the reduction with a duality twist can be viewed as a $\mathbb{Z}_r$ orbifold by a $\mathbb{Z}_r$ twist together with a shift around the circle by $2\pi R/r$ [2]. From the effective field theory point of view, the moduli give scalar fields in $D - 1$ dimensions and the reduction gives a potential for these fields. At each fixed point in moduli space (i.e. at each point that is preserved by the monodromy) the potential has a minimum at which it vanishes, giving a Minkowski compactification [2]. In these cases, at the special points in moduli space there is both a stringy orbifold construction and a supergravity construction, giving complementary pictures of the same reduction. This extends to a reduction on T^d to $D-d$ dimensions with a duality twist on each of the d circles. The question of finding reductions giving Minkowski space in $D-d$ dimensions becomes related to that of finding fixed points in moduli space preserved by a subgroup of $G(\mathbb{Z})$.

Our starting point will be the theory in six dimensions obtained from compactifying IIA string theory on a $K3$ surface. The moduli space of metrics and B-fields on $K3$ gives the moduli space of $K3$ CFT's and is given by [6, 7]

$$\frac{O(4,20)}{O(4)\times O(20)}\,,$$

identified under the discrete subgroup $O(\Gamma_{4,20})\subset O(4,20)$ preserving the lattice $\Gamma_{4,20}$, which is the lattice of total cohomology of the K3 surface as well as the lattice of D-brane charges in type IIA. $O(\Gamma_{4,20})$ is the perturbative duality group of two-dimensional conformal field theories with $K3$ target spaces, which includes mirror symmetries. We then reduce to four dimensions on T^2 with an $O(\Gamma_{4,20})$ twist around each circle. Generically, such reductions break all supersymmetry; we will focus here on a class of monodromies admitting $\mathcal{N}=2$ vacua in four dimensions. From the string theory point of view, we will find them by considering points in the moduli space of sigma-models with K3 target spaces that are preserved by finite subgroups of $O(\Gamma_{4,20})$, and taking orbifolds by such automorphisms combined with shifts on the circles.

The particular automorphisms of K3 CFTs that we will use are inspired by a worldsheet construction of asymmetric Gepner models presented in [8, 9], following earlier works [10, 11] (see [12, 13] for later generalizations). They preserve only space-time supercharges from the worldsheet left-movers, and the asymmetry means that they are non-geometric in general.

From the supergravity point of view, the conventional reduction on T^2 without twists gives $\mathcal{N}=4$ supergravity coupled to twenty-two $\mathcal{N}=4$ abelian vector multiplets. The twisted reduction gives a gauged version of this supergravity with a non-abelian gauge group and a scalar potential. Vacua arise from minima of the potential, and of particular interest are theories with non-negative potentials and minima that give zero vacuum energy and a Minkowski vacuum. Our construction gives string theory compactifications whose supergravity limits are of this type, and moreover preserve $\mathcal{N}=2$ supersymmetry.

The relation between the asymmetric Gepner models that underlie these compactifications and gauged supergravities was suggested by one of the authors in [8] and explored by Blumenhagen et al. in [12] for a related but distinct class of models. In that work they considered quotients of Calabi-Yau compactifications by non-geometric automorphisms, rather than the freely-acting quotients involving torus shifts giving rise to fibrations over tori that we consider here. In the freely-acting quotient, the scale of (spontaneous) space-time supersymmetry breaking can be made arbitrarily small rather than being tied to the string scale as it would be for quotients with fixed points. This means that for our constructions, the gauged supergravity approach gives a good description of the low-energy physics. We will identify the gauging directly from geometric considerations (rather than from identifying the massless spectra of the four-dimensional theories) and analyse the worldsheet constructions of $K3$ mirror-folds with $\mathcal{N} = 2$ supersymmetry from an algebraic geometry viewpoint. This approach will provide a powerful mathematical framework – that applies to Calabi-Yau three-folds as well – and give an explicit construction of the low-energy four-dimensional gauged supergravity.

We consider algebraic K3 surfaces defined as (the minimal resolution of) the zero-loci of quasi-homogeneous polynomials in weighted projective spaces $\mathbb{P}_{w_1,\ldots,w_n}$. These surfaces are characterized in particular by their *Picard number* ρ, the rank of their Picard lattice $S\,(X) = H^2(X, \mathbb{Z}) \cap H^{(1,1)}(X)$ where $1 \leqslant \rho \leqslant 20$. For the algebraic K3 surfaces of fixed ρ, the moduli space of CFTs factorizes into

$$\frac{O(2, 20-\rho)}{O(2)\times O(20-\rho)} \times \frac{O(2,\rho)}{O(2)\times O(\rho)} \tag{1.2}$$

identified under a discrete subgroup, as we will review in section 2. The first factor is interpreted as the complex structure moduli space of the algebraic surface and the second as the complex Kähler moduli space.

The definition of mirror symmetry for K3 surfaces is more subtle than for Calabi-Yau three-folds, as $K3$ is a hyperkähler manifold. For algebraic K3 surfaces, the notion of *lattice-polarized mirror symmetry* (LP-mirror symmetry) was introduced around 40 years ago by Pinkham [14] and independently by Dolgachev and Nikulin [15–17]; see the article by Dolgachev [18] for an introduction. In the special case considered by Aspinwall and Morrison [7], this amounts to an $O(\Gamma_{4,20})$ transformation that maps a K3 surface of Picard number ρ to one with Picard number $20-\rho$, interchanging the two factors in eq. (1.2). Another notion of mirror symmetry, perhaps more familiar to physicists, is the Berglund-Hübsch construction [19] (BH mirror symmetry), generalizing the Greene-Plesser construction of mirror Gepner models [20] to generic Landau-Ginzburg models with non-degenerate invertible polynomials. These two constructions (BH and LP mirror symmetry) overlap but do not always agree.

A key ingredient to reconcile these two approaches to $K3$ mirror symmetry, which will also play a central role in the present study of non-geometric automorphisms, is to consider *non-symplectic automorphisms* [21], which are automorphisms of the surface acting on the holomorphic two-form ω as $\sigma_p^\star\,:\,\omega \mapsto \zeta_p\,\omega$, where *e.g.* $\zeta_p = \exp 2i\pi/p$. As we will review in section 2, at least when p is a prime number, the lattice-polarized mirror symmetry w.r.t. the invariant sublattice associated with the action of σ_p, coincides with the Berglund-Hübsch mirror symmetry [22, 23]. An important corollary that we will obtain is that the automorphism σ_p of the surface on the one-hand and the corresponding automorphism σ_p^T of the (Berglund-Hübsch) mirror surface on the other hand act on sub-lattices of $\Gamma_{4,20}$ that are orthogonal to each other, denoted respectively $T(\sigma_p)$ and $T(\sigma_p^T)$. It is expected that a similar statement is true for automorphisms of non prime order, and we expect the physics to work out in a similar fashion for such cases as well.

The non-geometric automorphisms of $K3$ CFTs that we study in this work correspond each to an $O(\Gamma_{4,20})$ transformation induced by a block-diagonal isometry in $O(T(\sigma_p) \oplus T(\sigma_p))$, the first block giving the isometry associated with the action of the automorphism σ_p on the K3 surface, and the second block giving the isometry associated with the action of the automorphism σ_p^T on the mirror K3 surface. These isometries of the lattice $\Gamma_{4,20}$ can be thought of as *mirrored automorphisms* of $K3$ CFTs of order p. They can be decomposed as follows:

$$\hat{\sigma}_p := \mu^{-1} \circ \sigma_p^T \circ \mu \circ \sigma_p \,, \tag{1.3}$$

where μ denotes the Berglund-Hübsch/lattice mirror involution. Here σ_p is an order p large diffeomorphism of $K3$, μ maps the $K3$ to its mirror, σ_p^T is an order p large diffeomorphism of the mirror $K3$, and μ^{-1} maps the mirror $K3$ back to the original one.

Taking the quotient by two such automorphisms combined with shifts on the two one-cycles of a two-torus, at special points in the K3 moduli space fixed under the two automorphisms, gives the asymmetric Gepner models of [8]. We extend this construction to all points in moduli space using a reduction with duality twists. It is in general a difficult problem to find $O(\Gamma_{4,20})$ transformations that have fixed points in the K3 moduli space and so can lead to Minkowski vacua. We will show in section 3 that the fixed points of the monodromies that we consider correspond indeed precisely to the Gepner model construction of [8] on the worldsheet (which are Landau-Ginzburg points in the moduli space of $K3$ CFTs with enhanced discrete symmetry) and lead to four-dimensional theories with $\mathcal{N} = 2$ Minkowski vacua.

The type IIA string theory compactified on K3 is non-perturbatively dual to the heterotic string compactified on T^4 [24]. The duality symmetry group $O(\Gamma_{4,20})$ acts on the heterotic side through isometries of the Narain lattice, containing T-duality transformations as well as diffeomorphisms and shifts of the B-field, and is often referred to as the heterotic T-duality group. The reduction from 6-dimensions on T^2 with $O(\Gamma_{4,20})$ monodromies round each circle provides twisted reductions of precisely the type introduced and studied in [2]. These reductions can be regarded in general as T-fold reductions of the heterotic string, with transition functions involving the T-duality group $O(\Gamma_{4,20})$ [1]. At the special points of the moduli space that are fixed-points of the twists, the construction reduces to a reduction of asymmetric orbifold type, with a quotient by elements of the T-duality group $O(\Gamma_{4,20})$ combined with shifts on T^2 [2]. Then the type IIA K3 mirrorfolds are dual to heterotic T-folds, and at special points in the moduli space these become type IIA Gepner-type models and heterotic models of asymmetric orbifold type. Finding automorphisms with interesting fixed points is in general a difficult problem; the novelty here is that algebraic geometry leads us to a very interesting class of automorphisms that, when used in either the type IIA or the heterotic description, gives a rich class of models with $\mathcal{N}=2$ supersymmetry.

This work is organized as follows. The first half of the paper, up to subsection 3.2, is more algebraic-geometry oriented while the rest of the article deals with the physical aspects. In detail, section 2 provides the necessary mathematical background about $K3$ surfaces, mirror symmetry and $K3$ automorphisms and section 3 presents the mirrored automorphisms of mirror pairs of $K3$ surfaces and their relation with asymmetric Gepner model constructions. In section 4, we will consider the Scherk-Schwarz compactification of the 6-dimensional supergravity corresponding to type IIA compactified on $K3$ to obtain a four-dimensional gauged $\mathcal{N}=4$ supergravity. We show that for suitable choices of the twists, two gravitini become massive and two remain massless, giving vacua preserving $\mathcal{N}=2$ supersymmetry. In the final section, we will translate the $O(\Gamma_{4,20})$ monodromies defined in section 3 into the gauged supergravity framework, and obtain the moduli space of the low-energy theory.

2 Mathematical background

The moduli space of two-dimensional conformal field theories defined by quantizing non-linear sigma models on K3 surfaces is given by [6, 7]:

$$\mathcal{M}_\Sigma \cong O(\Gamma_{4,20})\backslash O(4,20)/O(4)\times O(20)\,, \tag{2.1}$$

where $O(\Gamma_{4,20})$ is the isometry group of the even and unimodular (*i.e.* self-dual) lattice $\Gamma_{4,20}$ of signature $(4,20)$:

$$\Gamma_{4,20} \cong \Gamma_{3,19}\oplus U\,, \tag{2.2}$$

and $\Gamma_{3,19}$ is the K3 lattice

$$\Gamma_{3,19} \cong E_8\oplus E_8\oplus U\oplus U\oplus U\,, \tag{2.3}$$

which is isometric to the second cohomology group $H^2(X,\mathbb{Z})$ of a K3 surface X endowed with its cup product. Here U is the even unimodular lattice of signature $(1,1)$ and E_8 is the even unimodular lattice of signature $(0,8)$ associated with the Dynkin diagram E_8.

Automorphisms of $K3$ CFT's correspond to isometries of the $\Gamma_{4,20}$ lattice. The geometric automorphisms of the surface form a subgroup $O(\Gamma_{3,19}) \ltimes \mathbb{Z}_{3,19} \subset O(\Gamma_{4,20})$, generated by large diffeomorphisms of the surface in $O(\Gamma_{3,19}) \subset O(\Gamma_{4,20})$ and transformations $\mathbb{Z}_{3,19}$ corresponding to shifts of the B-field by representatives of integral cohomolgy classes. Isometries that are not in $O(\Gamma_{3,19}) \ltimes \mathbb{Z}_{3,19}$ are 'non-geometric'.

An important sublattice of the K3 lattice $\Gamma_{3,19}$ of a K3 surface X is the *Picard lattice*[1] which is defined to be:

$$S\,(X) = H^2(X, \mathbb{Z}) \cap H^{(1,1)}(X)\,. \tag{2.4}$$

The rank of this lattice (*i.e.* the rank of the corresponding Abelian group) $\rho(X)$, or Picard number, is at least one for any algebraic K3 surface, and its signature is $(1, \rho - 1)$. The *transcendental lattice* $T(X)$ of an algebraic K3 surface X is defined to be the sub-lattice of $H^2(X, \mathbb{Z}) \cong \Gamma_{3,19}$ orthogonal to the Picard lattice:

$$T(X) = H^2(X, \mathbb{Z}) \cap S\,(X)^{\perp} \hookrightarrow \Gamma_{3,19}\,. \tag{2.5}$$

This lattice has signature $(2, 20 - \rho)$.

For sigma-model CFTs on algebraic K3 surfaces, the Picard lattice can be enlarged to the 'quantum Picard lattice':

$$S^{\mathfrak{Q}}(X) \cong S\,(X) \oplus U\,. \tag{2.6}$$

with signature $(2, \rho)$. The transcendental lattice can also be viewed as the orthogonal complement of the quantum Picard lattice, i.e. $T(X) = \Gamma_{4,20} \cap S^{\mathfrak{Q}}(X)^{\perp}$. The moduli space of non-linear sigma-model CFTs on algebraic K3 surfaces with a Picard lattice of Picard number ρ then factorizes as [7]:

$$\mathcal{M}^{\rho}_{\Sigma} \cong O(T(X))\backslash O(2, 20-\rho)/(O(2)\times O(20-\rho)) \times O(S^{\mathfrak{Q}}(X))\backslash O(2, \rho)/(O(2)\times O(\rho)). \tag{2.7}$$

One may think of the first factor as corresponding to the complex structure moduli space and the second one to the complex Kähler moduli space.

Mirror symmetry of Calabi-Yau three-folds exchanges their Hodge numbers $h^{2,1}$ and $h^{1,1}$, and exchanges the complex structure and complex Kähler moduli spaces. For $K3$ surfaces the situation is different since (i) all $K3$ surfaces are diffeomorphic to each other and so have the same Hodge numbers and topology, and (ii) as these manifolds are hyperkähler, the complex structure and complex Kähler moduli are not unambiguously defined. For algebraic $K3$ surfaces there are two different notions of mirror symmetry that we will review in turn below, and these will both play a role in the construction of the non-geometric automorphisms we use in this paper.

[1] The Picard group or Picard lattice is generated by algebraic curves of the surface, *i.e.* curves that are holomorphically embedded in X.

2.1 Lattice-polarized mirror symmetry

A first notion of mirror symmetry of algebraic $K3$ surfaces is the *lattice-polarized* (LP) mirror symmetry discussed by Dolgachev in [18] and introduced by Pinkham [14], Nikulin and Dolgachev [15–17]. This construction uses the embedding of a given lattice M as a sub-lattice of the Picard lattice, which should be primitive. A primitive embedding of a lattice M into a lattice N, $\iota\,:\, M \hookrightarrow N$, is such that, viewing N as an Abelian group and $\iota(M)$ as a subgroup, the quotient $N/\iota(M)$ is a torsion-free Abelian group.[2]

Definition 1. *Let M be an even lattice of rank $t+1$ and signature $(1,t)$, with $t \leqslant 18$, admitting a primitive embedding in the K3 lattice, $\iota\,:\, M \hookrightarrow \Gamma_{3,19}$. Assuming that its orthogonal complement $\iota(M)^\perp \subset \Gamma_{3,19}$ admits a primitive embedding $\iota'\,:\, U \hookrightarrow \iota(M)^\perp$, the mirror lattice $M^\vee$ to M of rank $19-t$ and signature $(1,18-t)$ is defined through the decomposition*

$$\iota(M)^\perp = \iota'(U) \oplus M^\vee\,. \tag{2.8}$$

If there exists a primitive embedding $\jmath\,:\, M \hookrightarrow S(X)$ of M in the Picard lattice of an algebraic $K3$ surface X, then we say that X is an M-polarized K3 surface. Then two K3 surfaces X and $X^\vee$ form a lattice mirror pair if X is M-polarized and $X^\vee$ is $M^\vee$-polarized with $M^\vee$ the mirror lattice to M.

In other words, the LP mirror construction associates each M-polarized K3 surface with an $M^\vee$-polarized K3 surface, with $\mathrm{rk}\,(M^\vee) = 20 - \mathrm{rk}\,(M)$. The moduli space of the family of M-polarized K3 surfaces and the moduli space of the family of $M^\vee$-polarized K3 surfaces are called *LP mirror moduli spaces*.

Example 1. The Aspinwall-Morrison construction of mirror symmetry [7] is a particular instance of LP mirror symmetry. The authors considered an algebraic $K3$ surface X polarized by the whole Picard lattice, *i.e.* with M embedded primitively as $\iota(M) = S(X)$. One has then

$$S(X)^\perp = T(X) = \iota'(U) \oplus M^\vee\,, \tag{2.9}$$

and $M^\vee$ is the Picard lattice of the mirror surface. In other words, mirror symmetry is defined as the map

$$X \mapsto\; X^\vee\,, \tag{2.10}$$

where

$$T(X) = S^{\mathfrak{Q}}(X^\vee), \qquad S^{\mathfrak{Q}}(X) = T(X^\vee)\,, \tag{2.11}$$

which exchanges the quantum Picard lattice and the transcendental lattice. Hence both factors in (2.7) are exchanged under this involution, which can be viewed as exchanging the complex structure and the complex Kähler moduli spaces of sigma-model CFTs on the surface.

[2] As a counter-example, if (e_1, e_2) is a basis of N, and if $\iota(M)$ is spanned by $\{e_1+e_2, e_1-e_2\}$, the embedding $\iota\,:\, M \hookrightarrow N$ is not primitive as $N/\iota(M) \cong \mathbb{Z}_2$.

2.2 Berglund-Hübsch mirror symmetry

The second notion of mirror symmetry, *Berglund-Hübsch (BH) mirror symmetry*, is not specific to K3 surfaces. It follows from the Greene-Plesser construction [20] of mirror Gepner models [25] discovered in physics, exchanging the vector and axial R-currents of the worldsheet $(2,2)$ superconformal field theories. It was generalized by Berglund and Hübsch [19] and Krawitz [26]; later Chiodo and Ruan proved in [27] that it coincides with cohomological mirror symmetry.

Let us consider a K3 surface realized as the minimal resolution of a hypersurface in a weighted projective space $\mathbb{P}_{[w_1 w_2 w_3 w_4]}$ with $\gcd(w_1, \ldots, w_4) = 1$. A polynomial $W : \mathbb{C}^4 \to \mathbb{C}$ is quasi-homogeneous of degree d if

$$\forall \lambda \in \mathbb{C}^* \ , \quad W(\lambda^{w_1} x_1, \ldots, \lambda^{w_4} x_4) = \lambda^d W(x_1, \ldots,\ x_4) \, . \tag{2.12}$$

It is non-degenerate if the origin $x_1 = \cdots = x_4 = 0$ is the only critical point and if the fractional weights $w_1/d, \ldots, w_4/d$ are uniquely determined by W. If furthermore the number of monomials equals the number of variables the polynomial is said to be *invertible*. By rescaling one can then put an invertible polynomial W in the form

$$W = \sum_{i=1}^{4} \prod_{j=1}^{4} x_j^{\ a^i_j} \, , \tag{2.13}$$

where the square matrix $A_W := (a^i_j)$ is invertible. If $\sum_{\ell=1}^4 w_\ell = d$, the hypersurface $\{W = 0\}$ in $\mathbb{P}_{[w_1 w_2 w_3 w_4]}$ admits a minimal resolution X_W which is a smooth K3 surface.

We denote by G_W the Abelian group of all diagonal scaling transformations preserving the polynomial W:

$$G_W = \{(\mu_1, \ldots, \mu_4) \in (\mathbb{C}^\star)^4 \, | \ W(\mu_1 x_1, \ldots \mu_4 x_4) = W(x_1, \ldots x_4)\} \, , \tag{2.14}$$

and SL_W its subgroup containing elements of the form

$$(\mu_1, \ldots, \mu_4) = \left(e^{2i\pi g_1}, \ldots, e^{2i\pi g_4}\right) \ , \quad \sum_{\ell=1}^{4} g_\ell \in \mathbb{Z} \, , \tag{2.15}$$

which corresponds in physics to the group of supersymmetry-preserving symmetries. The group G_W always contains the element j_W with $(g_1 = w_1/d, \ldots, g_4 = w_4/d)$ generating a cyclic group J_W of order d.

Let us consider a subgroup $G \subset G_W$ such that $J_W \subseteq G \subseteq SL_W$, and the quotient group $\tilde{G} = G/J_W$. The minimal resolution of the orbifold $X_W/\tilde{G}$, denoted $X_{W,G}$, is also a K3 surface, that we associate with the pair (W, G). In physics, an orbifold of a $K3$ sigma-model (or Landau-Ginzburg model) by a discrete group G satisfying the condition $J_W \subseteq G \subseteq SL_W$ preserves all space-time supersymmetry.

We now introduce the Berglund-Hübsch mirror symmetry, which follows in the physical context from the isomorphism between the superconformal field theories associated with a pair of Landau-Ginzburg orbifolds, generalizing the original Greene-Plesser construction of mirror Gepner model orbifolds.

Definition 2. *Let (W, G) be associated with the minimal resolution of $X_W/\tilde{G}$, a smooth K3 surface. The pair (W^T, G^T) is obtained as follows:*

- *W^T is specified by the matrix $A_{W^T} := (A_W)^T$.*
- *$G^T = \{g \in G_{W^T}\,,\ gA_W h^T \in \mathbb{Z}\,,\ \forall h \in G\}$.*

The Berglund-Hübsch mirror surface of $X_{W,G}$ is then given by X_{W^T,G^T}, the minimal resolution of the orbifold $X_{W^T}/\widetilde{G^T}$.

Here X_{W^T} is the surface $W^T(\tilde{x}^1, \ldots \tilde{x}^4) = 0$ and $g = (g_1, \ldots, g_4)$ is an automorphism of X_{W^T} acting on the coordinates $\tilde{x}^\ell$ as $\tilde{x}^\ell \mapsto \exp(2\pi i\, g_\ell)\, \tilde{x}^\ell$ for $\ell = 1, 2, 3, 4$ while $h = (h_1, \ldots, h_4)$ specifies an automorphism of X_W under which the coordinates scale as $x^\ell \mapsto \exp(2\pi i\, h_\ell)\, x^\ell$. It is straightforward to check that, if the pair (W, G) is associated with a smooth $K3$ surface obtained as the minimal resolution of $X_W/\tilde{G}$, then the mirror pair (W^T, G^T) is associated with a smooth $K3$ surface obtained as the minimal resolution of $X_{W^T}/\widetilde{G^T}$, as $J_{W^T} \subseteq G^T \subseteq SL_{W^T}$.

Example 2. A Fermat-type K3 surface in a weighted projective space is defined by the polynomial

$$W = x_1^{d/w_1} + x_2^{d/w_2} + x_3^{d/w_3} + x_4^{d/w_4}\,, \tag{2.16}$$

where $d = \mathrm{lcm}\,(w_1, \ldots, w_4)$. This polynomial is preserved by the symmetries under which any of the coordinates scales as $x^\ell \mapsto \exp(2\pi i\, w_\ell/d)x^\ell$ and the other coordinates are invariant. They generate the group of diagonal symmetries $G_W \cong \mathbb{Z}_{d/w_1} \times \mathbb{Z}_{d/w_2} \times \mathbb{Z}_{d/w_3} \times \mathbb{Z}_{d/w_4}$. Consider the case in which we choose the group G to be $J_W = \langle (w_1/d, \ldots, w_4/d) \rangle$. Then $\tilde{G}$ is the trivial group and $X_{W,G} = X_W$. The mirror surface is characterized by the same polynomial as $W^T = W$, and the dual group G^T is given by the elements

$$g = (g_1, \ldots, g_4) \in G_{W^T} = G_W$$

satisfying the condition

$$r \sum_{\ell=1}^{4} g_\ell \in \mathbb{Z}\,, \forall r \in \{0, \ldots, d-1\}\,. \tag{2.17}$$

As $g_\ell = n_\ell \frac{w_\ell}{d}$ for some integer n_ℓ, we get the condition

$$\sum_{\ell=1}^{4} \frac{w_\ell}{d}\, n_\ell \in \mathbb{Z}\,. \tag{2.18}$$

This means that $G^T = SL_W$ in this case.

2.3 Non-symplectic automorphisms of prime order

The two notions of mirror symmetry of algebraic $K3$ surfaces do not necessarily agree. In particular, as was shown in [28], the LP mirror of a surface polarized by its whole Picard lattice is not always identical to the BH mirror of the same surface. There exists nevertheless a class of lattice-polarized mirror symmetries, in which the surface is polarized by a sub-lattice of the Picard lattice, that gives the same results as the Berglund-Hübsch construction, and that will be instrumental in our construction of non-geometric automorphisms.

Let us first define a non-symplectic automorphism of order p of a $K3$ surface X as a diffeomorphism $\sigma_p : X \to X$ of the surface acting on the holomorphic two-form $\omega(X)$ as

$$\sigma_p^\star \, : \, \omega(X) \mapsto \zeta_p \, \omega(X) \, , \tag{2.19}$$

where ζ_p is a primitive p-th root of unity, *i.e.* such that ζ_p generates a cyclic group isomorphic to $\mathbb{Z}/p\mathbb{Z}$, *e.g.* $\zeta_p = \exp(2i\pi/p)$.[3] If p is a prime number then it is straightforward to see that $2 \leq p \leq 19$ (see [21, Theorem 0.1]).

The automorphism $\sigma_p : X \to X$ acts on 2-forms through $\sigma_p^\star$. Let $S(\sigma_p)$ be the sub-lattice of $H^2(X, \mathbb{Z})$ invariant under the action of the isometry $\sigma_p^\star$ and $T(\sigma_p)$ its orthogonal complement. The rank of the lattice $S(\sigma_p)$ will be denoted by ρ_p. As was shown by Nikulin [21], the invariant sublattice $S(\sigma_p)$ is a subset of the Picard lattice,

$$S(\sigma_p) \subseteq S(X) \tag{2.20}$$

and both $S(\sigma_p)$ and $T(\sigma_p)$ are primitive sub-lattices of the $K3$ lattice. The following lemma was also proved in [21]:

Lemma 1. *Let σ_p be an order p non-symplectic automorphism of a K3 surface X. There exists a positive integer q such that the action $\sigma_p^\star$ on the vector space $T(\sigma_p) \otimes \mathbb{C}$ can be diagonalized as*

$$\begin{pmatrix} \zeta_p \mathbb{I}_q & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & & \zeta_p^n \mathbb{I}_q & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta_p^{p-1} \mathbb{I}_q \end{pmatrix} \tag{2.21}$$

where $\mathbb{I}_q$ is the identity matrix in q dimensions and all integers $n \in \{1, p-1\}$ with $\gcd(n,p) = 1$ *appear once, i.e. all primitive p-roots of unity are eigenvalues and the corresponding eigenspaces are all of dimension q.*

[3] These autormorphisms are often called in the literature *purely non-symplectic* but for simplicity well call them just non-symplectic. When the order p is a primer number, as it is the case in the present work, each non-symplectic automorphism is purely non-symplectic.

Remark 1. From [29, Proposition 9.3] and [21] the action of the non-symplectic automorphism on the K3 lattice is unique up to conjugation with isometries.

We will consider a particular type of hypersurface in weighted projective space admitting a non-symplectic automorphism of prime order p, whose non-degenerate invertible polynomial is of the form

$$W = x_1^p + f(x_2, x_3, x_4)\,. \tag{2.22}$$

We will call such surfaces *p-cyclic*, following [23]. They admit the obvious order p non-symplectic automorphism $\sigma_p \,:\, x_1 \mapsto \zeta_p\, x_1$. By construction, the BH mirror of a p-cyclic surface has its defining polynomial of the form $W^T = \tilde{x}_1^p + \tilde{f}(\tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$, therefore it also admits an order p automorphism $\sigma_p^T \,:\, \tilde{x} \mapsto \zeta_p\, \tilde{x}$.

The following theorem was proved for $p = 2$ by Artebani et al. [22] and for $p \in \{3, 5, 7, 13\}$ by Comparin et al. [23].[4]

Theorem 1. *Let $X_{W,G}$ be a p-cyclic $S(\sigma_p)$-polarized K3 surface, where $S(\sigma_p) \subseteq S(X)$ is the sub-lattice of the Picard lattice invariant under the action of the non-symplectic automorphism σ_p of prime order, with $p \in \{2, 3, 5, 7, 13\}$. Let X_{W^T,G^T} be its Berglund-Hübsch mirror, polarized by the invariant sublattice $S(\sigma_p^T)$ associated with the non-symplectic automorphism σ_p^T. Then $X_{W,G}$ and X_{W^T,G^T} belong to mirror families of $K3$ surfaces in the sense of lattice-polarized mirror symmetry.*

We obtain from this theorem a simple corollary which will play an important role in the construction of non-geometric compactifications. We first define the quantum invariant sublattice as the orthogonal complement of $T(\sigma_p)$ in the $\Gamma_{4,20}$ lattice, namely

$$S^{\Omega}(\sigma_p) \cong S(\sigma_p) \oplus U\,, \tag{2.23}$$

which has signature $(2, \rho_p)$ with $\rho_p \leqslant \rho(X)$. For a lattice L we denote by $L^{\mathbb{R}}$ the real vector space $L \otimes \mathbb{R}$ generated by its basis vectors.

Corollary 1. *Let $X_{W,G}$ be a p-cyclic $S(\sigma_p)$-polarized K3 surface, where $S(\sigma_p)$ is the invariant sublattice under the σ_p action, with $p \in \{2, 3, 5, 7, 13\}$. and X_{W^T,G^T} its LP mirror, regarded as an $S(\sigma_p^T)$-polarized K3 surface, which is also its Berglund-Hübsch mirror following Theorem 1.*

From the theorem we have $T(\sigma_p) = S(\sigma_p)^\perp \cap \Gamma_{3,19} = U \oplus S(\sigma_p^T)$ and similarly $T(\sigma_p^T) = S(\sigma_p^T)^\perp \cap \Gamma_{3,19} = U \oplus S(\sigma_p)$. Hence $T(\sigma_p^T)$ is the orthogonal complement[5] of $T(\sigma_p)$ in $\Gamma_{4,20}$:

$$T(\sigma_p^T) \cong T(\sigma_p)^\perp \cap \Gamma_{4,20}\,. \tag{2.24}$$

[4] As shown by Nikulin [21], K3 surfaces admitting non-symplectic automorphisms of prime order up to $p = 19$ exist; however for the values $p \in \{11, 17, 19\}$ one cannot present the surface in a p-cyclic form, as was noticed in [23].

[5] The embedding of $T(\sigma_p)$ may not be unique in $\Gamma_{4,20}$ but we tacitly choose the embedding such that $T(\sigma_p^T)$ is the orthogonal complement.

We obtain then the orthogonal decomposition over $\mathbb{R}$:

$$\Gamma^{\mathbb{R}}_{4,20} \cong T(\sigma_p^T)^{\mathbb{R}} \oplus T(\sigma_p)^{\mathbb{R}} . \tag{2.25}$$

Example 3. In order to illustrate this general construction, we consider first the self-mirror K3 surface X given by the hypersurface

$$w^2 + x^3 + y^7 + z^{42} = 0 \tag{2.26}$$

in the weighted projective space $\mathbb{P}_{[21,14,6,1]}$. It inherits the orbifold singularities of the ambient space, and these should be minimally resolved in order to obtain a smooth K3 surface. This K3 surface admits a non–symplectic automorphism of order 42, acting on z by

$$z \mapsto e^{2i\pi/42} z$$

and leaving the other coordinates invariant. Then this implies, by [21, Theorem 0.1], that the rank of the transcendental lattice is a multiple of the Euler function of 42, which is 12. Since the rank is necessarily less than 22, which is the rank of the K3 lattice, the rank is necessarily equal to 12 and the rank of the Picard lattice is then 10. Interestingly, by [28] the generic K3 surface in the weighted projective space $\mathbb{P}_{[21,14,6,1]}$ has Picard lattice of rank 10, which is the same as the rank of the Picard lattice of the surface (2.26) of Fermat type. The Picard lattice $S(X)$ is isometric to a self dual lattice of signature $(1, 9)$, which is

$$S(X) \cong E_8 \oplus U . \tag{2.27}$$

Thus this surface is its own mirror in the sense of Aspinwall-Morrison, as we have

$$S^{\mathfrak{Q}}(X) \cong T(X) \cong E_8 \oplus U \oplus U . \tag{2.28}$$

The moduli space of complex structures associated with this surface corresponds to the set of space-like two-planes in $\mathbb{R}^{3,19}$ orthogonal to the basis vectors of $S(X)$, quotiented by $O(T(X))$, the group of isometries of the transcendental lattice:

$$\mathcal{M}_{\text{cs}} \cong O(T(X))\backslash O(2, 10)/O(2) \times O(10) \tag{2.29}$$

of real dimension 20.

The hypersurface (2.26) admits several non-symplectic automorphisms of prime order. The hypersurface is p-cyclic for $p = 2, 3, 7$ and the corresponding automorphisms $\sigma_2, \sigma_3, \sigma_7$ are of order 2, 3 and 7. Their action is

$$\begin{aligned} \sigma_2 &: w \mapsto -w , \\ \sigma_3 &: x \mapsto e^{2i\pi/3} x , \\ \sigma_7 &: y \mapsto e^{2i\pi/7} y . \end{aligned} \tag{2.30}$$

In all cases, the invariant sublattice $S(\sigma_p) \subset \Gamma_{3,19}$ is identified with the Picard lattice $S(X)$ (see [23, 29]), and $S^{\mathfrak{Q}}(\sigma_p) \cong S^{\mathfrak{Q}}(X)$. Its orthogonal complement in $\Gamma_{4,20}$ is naturally the transcendental lattice of the surface, hence $T(\sigma_p) \cong T(X)$.

The action of σ_3 on the transcendental lattice (2.28) of the surface (2.26) is given, in the appropriate basis over $\mathbb{C}$, by six copies of the companion matrix of the cyclotomic polynomial $\Phi_3 = \prod_{n=1}^{2}(z - e^{2i\pi n/3})$, using Lemma 1.[6]

Recall that by Remark 1 the action can be given in a unique way on the transcendental lattice. It splits into an action onto the $U \oplus U$ lattice and onto the E_8 lattice. For $U \oplus U$ we choose a lattice basis in which the lattice metric (or Gram matrix) is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The action of the isometry σ_3 on $U \oplus U$ is then given by the following matrix in $O(U \oplus U)$:

$$M_3^{U\oplus U} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 3 \\ -3 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{2.31}$$

For the E_8 lattice, choosing a lattice basis in which the lattice metric is

$$E_8^{st} := \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \tag{2.32}$$

[6] The companion matrix of a polynomial of the form $P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$ is the matrix $\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$, whose characteristic polynomial is P.

the action of σ_3 on E_8 is given by the following matrix in $O(E_8)$:

$$M_3^{E_8} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 2 & -2 \\ 2 & -1 & 1 & -1 & 0 & -1 & 2 & -3 \\ 2 & -1 & 2 & -1 & -1 & -1 & 3 & -4 \\ 3 & -2 & 2 & 0 & -2 & -1 & 4 & -5 \\ 2 & -1 & 1 & 0 & -1 & -1 & 2 & -2 \\ 2 & -2 & 2 & 0 & -1 & -1 & 2 & -3 \\ 1 & -1 & 1 & 0 & 0 & -1 & 1 & -2 \end{pmatrix}. \tag{2.33}$$

Likewise, the action of σ_7 is given in the appropriate basis over $\mathbb{C}$ by two copies of the companion matrix of the cyclotomic polynomial $\Phi_7 = \prod_{n=1}^{6}(z - e^{2i\pi n/7})$, and finally the action of σ_2 on $T(X)$ is simply given by minus the identity matrix in twelve dimensions.

For this surface $|SL_W/J_W| = 1$ hence $J_{W^T} = J_W$. Therefore, using either of the non-symplectic automorphisms of prime order, one finds that the surface X_{W,J_W} is its own Berglund-Hübsch mirror and, polarized by $S(\sigma_p) \cong S(X)$, is also its own mirror in the sense of LP mirror symmetry.

Example 4. Let us consider the hypersurface

$$w^2 + x^3 + y^8 + z^{24} = 0 \tag{2.34}$$

in the weighted projective space $\mathbb{P}_{[12,8,3,1]}$. In this case $|SL_W/J_W| = 2$ and there are two choices of G with $J_W \subseteq G \subseteq SL_W$, either $G = J_W$ or $G = SL_W$. Berglund-Hübsch mirror symmetry provides then the mirror pair (W, J_W) and (W, SL_W).

The surface defined by eq. (2.34) admits a non-symplectic automorphism of order 3 acting as $\sigma_3 : x \mapsto e^{2i\pi/3}x$, while keeping the other variables fixed. As was shown in [23], the surface X_{W,J_W} is polarized by the invariant lattice $S(\sigma_3) = E_6 \oplus U$ and the transcendental lattice is contained in the lattice $T(\sigma_3) = E_8 \oplus A_2 \oplus U \oplus U$. Then $\operatorname{rank} S(X) \geq 8$. On the other hand, the surface also admits a non–symplectic automorphism of order 24 acting by $z \mapsto e^{2i\pi/24}z$ that gives (see Lemma 1) $\operatorname{rank} T(X) = 8$ or 16. The second case contradicts the previous inequality so that $\rho_X = \operatorname{rank} S(X) = 14$.

It was shown in [23] that the $S(\sigma_3)$-polarized X_{W,J_W} and the $S(\sigma_3^T)$-polarized surface X_{W,SL_W} form a LP mirror pair. Indeed we have:

- For (W, J_W), the invariant sublattice is $S(\sigma_3) = E_6 \oplus U$ while $T(\sigma_3) = E_8 \oplus A_2 \oplus U \oplus U$.
- For (W, SL_W), the invariant sublattice is $S(\sigma_3^T) = E_8 \oplus A_2 \oplus U$ and $T(\sigma_3^T) = E_6 \oplus U \oplus U$.

First, for the surface (W, J_W), the action of σ_3 on $T(\sigma_3)$, is given as follows (by Remark 1 the action is unique up to conjugation by isometries). On the A_2 lattice, by taking the lattice metric (Gram matrix)

$$A_2^{st} := \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \tag{2.35}$$

one gets

$$M_3^{A_2} := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \tag{2.36}$$

while on $E_8 \oplus U \oplus U$ it is the same as for the previous surface, see equation (2.31). Second, for the mirror surface (W, SL_W), the action of σ_3^T on $T(\sigma_3^T)$ is given as follows. On $U \oplus U$, it is given by the matrix $M_3^{U\oplus U}$ defined in (2.31) and on E_6 the action is given by (see the Appendix)

$$M_3^{E_6} := \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix} \tag{2.37}$$

in a lattice basis in which the lattice metric is

$$E_6^{st} := \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

The surface (2.34) admits also a non-symplectic automorphism of order 2, acting as $\sigma_2 : w \mapsto -w$. Following [22], the invariant lattice of σ_2 is of rank six and isometric to $S(\sigma_2) \cong D_4 \oplus U \subset T(X)$, hence $T(\sigma_2) \cong E_8 \oplus D_4 \oplus U \oplus U$. The previous theorem indicates that the $S(\sigma_2)$-polarized surface X_{W,J_W} and the $S(\sigma_2^T)$-polarized surface X_{W,SL_W} form a LP mirror pair. One can check that (W, SL_W) admits an order two non-symplectic automorphism σ_2^T of the invariant lattice $S(\sigma_2^T) \cong E_8 \oplus D_4 \oplus U$.

3 Non-geometric automorphisms of K3 sigma-models

In this section we define *mirrored automorphisms* of sigma-model CFTs with K3 target spaces, combining the action of non-symplectic automorphisms of a surface and of its mirror, and study the corresponding isometries of the lattice $\Gamma_{4,20}$. This construction is inspired by the non-geometric string theory compactifications that were obtained in [8] as asymmetric orbifolds of Gepner models.

3.1 Mirrored automorphisms and isometries of the $\Gamma_{4,20}$ lattice

We consider a p-cyclic K3-surface X, associated with a given non-symplectic automorphism σ_p of prime order. The BH mirror of this surface admits also a non-symplectic automorphism σ_p^T of the same order; by the theorem 1, these two surfaces, polarized by the invariant sublattices with respect to σ_p and σ_p^T respectively, are also LP mirrors. The automorphism σ_p has an action on the vector space $T(\sigma_p)^{\mathbb{R}}$ while the automorphism σ_p^T acts on the vector space $T(\sigma_p^T)^{\mathbb{R}}$. By the corollary 1 these vector spaces are orthogonal to each other in $\Gamma_{4,20}^{\mathbb{R}}$.

Inspired by physical considerations that will be illustrated in the next subsection, we will consider a mirrored automorphism that combines σ_p and σ_p^T into a non-geometrical automorphism $\hat{\sigma}_p$ of the CFT defined by quantizing the non-linear sigma model with a p-cyclic K3 surface target space. To prove that its action on the lattice $\Gamma_{4,20}$ is well-defined we will need the following proposition:

Proposition 1. *Let $X_{W,G}$ be a p-cyclic $S(\sigma_p)$-polarized K3 surface, where $S(\sigma_p)$ is the invariant sublattice under the σ_p action, with p prime, and X_{W^T,G^T} its LP mirror, regarded as an $S(\sigma_p^T)$-polarized K3 surface.*

Observe that $\Gamma_{4,20}$ is an over-lattice of finite index[7] *of the lattice $T(\sigma_p^T) \oplus T(\sigma_p)$, with index given by $|\det T(\sigma_p)| = |\det T(\sigma_p^T)|$, where for a lattice L, $\det L$ is the determinant of the lattice metric.*

The equality of the determinants follows from the fact that the lattice $\Gamma_{4,20}$ is unimodular (see [30, Corollary 2.6]). By properties of K3 surfaces (see [21, Theorem 0.1] and [30, Lemma 2.5]) the automorphisms σ_p and σ_p^T act trivially on the discriminant groups of $T(\sigma_p^T)$ and of $T(\sigma_p)$ so that we can extend the diagonal action by σ_p and σ_p^T to the whole lattice $\Gamma_{4,20}$.

Corollary 1 and Proposition 1 allow us to define 'mirrored automorphisms' of CFTs with p-cyclic K3 surfaces target spaces in the following way:

Definition 3. *Let $(X_{W,G}, \jmath)$ be a p-cyclic K3 surface polarized by the invariant sublattice $S(\sigma_p)$ and $(X_{W^T,G^T}, \jmath^T)$, polarized by $S(\sigma_p^T)$, its LP mirror, with $p \in \{2, 3, 5, 7, 13\}$.*

By Corollary 1 and Proposition 1 the diagonal action by (σ_p, σ_p^T) on the lattice $T(\sigma_p^T) \oplus T(\sigma_p)$ can be extended to an isometry of the lattice $\Gamma_{4,20}$, that we associate with the action of a CFT automorphism denoted $\hat{\sigma}_p$, that we name mirrored automorphism.

[7] A lattice L is an over-lattice of a lattice M if M is a sub-lattice of L and if M has finite index $[L : M]$ in L (viewing M as a sub-group of the Abelian group L), such that both lattices have the same rank. The dual lattice $M^\vee$ of a lattice M is a free $\mathbb{Z}$-module that contains M. The quotient $M^\vee/M$ is called the discriminant group of the lattice.

The action of the mirrored automorphism $\hat{\sigma}_p$ on the vector space $\Gamma_{4,20}^{\mathbb{R}} \cong T(\sigma_p)^{\mathbb{R}} \oplus T(\sigma_p^T)^{\mathbb{R}}$, is then given by

$$\hat{\sigma}_p^{\star}\big|_{T(\sigma_p)^{\mathbb{R}}} = \sigma_p^{\star} \tag{3.1a}$$

$$\hat{\sigma}_p^{\star}\big|_{T(\sigma_p^T)^{\mathbb{R}}} = (\sigma_p^T)^{\star} \tag{3.1b}$$

For other values of p, including the non-prime cases, the physical construction suggests that similar non-geometric automorphisms acting as in eq. (3.1) can be defined. However the mathematical classification of these automorphisms is not yet complete (see [31]).

The isometry of the lattice $\Gamma_{4,20}$ induced by $\hat{\sigma}_p$ is not in the geometric group $O(\Gamma_{3,19}) \ltimes \mathbb{Z}_{3,19}$ and so is non-geometrical. The automorphism $\hat{\sigma}_p$ acts as

$$\hat{\sigma}_p := \mu^{-1} \circ \sigma_p^T \circ \mu \circ \sigma_p \,, \tag{3.2}$$

where μ denotes the BH/LP mirror involution which maps the $K3$ to its mirror; μ^{-1} maps the mirror $K3$ back to the original one, σ_p is a diffeomporphism of the original $K3$ and σ_p^T is a diffeomorphism of its mirror.

Due to Proposition 1 the lattice isometry induced by $\hat{\sigma}_p$ generates the order p isometry subgroup

$$O(\hat{\sigma}_p) := \langle \hat{\sigma}_p^{\star} \rangle \subset O(\Gamma_{4,20}) \,. \tag{3.3}$$

Being of finite order, it is conjugate to a subgroup of the maximal compact subgroup

$$\big[O(2) \times O(20-\rho_p)\big] \times \big[O(2) \times O(\rho_p)\big] \subset O(2, 20-\rho_p) \times O(2, \rho_p)$$

Explicitly, the action of the non-geometric automorphism $\hat{\sigma}_p$ on $\Gamma_{4,20}$, hence on the CFT with a $K3$ target space, is obtained by considering the geometrical action of σ_p on $T(\sigma_p)$ and, for the BH mirror surface, the action of σ_p^T on $T(\sigma_p^T)$. The lattice $\Gamma_{4,20}$ is an over-lattice of index $p^k := |\det T(\sigma_p)|$, k a non-negative integer, of the sum $T(\sigma_p) \oplus T(\sigma_p^T)$ (recall that $T(\sigma_p)$ and $T(\sigma_p^T)$ are p-elementary lattices i.e. the discriminant groups are sums $(\mathbb{Z}/p\mathbb{Z})^{\oplus k}$, k as before). Then to construct $\Gamma_{4,20}$ one should add to the generators of the lattice $T(\sigma_p) \oplus T(\sigma_p^T)$ exactly k classes of the form $(a+b)/p$ with a/p in the discriminant group of $T(\sigma_p)$ and b/p in the discriminant group $T(\sigma_p^T)$, and we ask also that $((a+b)/k)^2 \in \mathbb{Z}$. The action of the isometry (σ_p, σ_p^T) on these classes is then obtained by linearity (over the rationals). We get in this way a set of generators of $\Gamma_{4,20}$ on which we have an isometry $\hat{\sigma}_p$ of order p which is induced by the isometry (σ_p, σ_p^T) on $T(\sigma_p) \oplus T(\sigma_p^T)$, i.e. the restriction of $\hat{\sigma}_p$ to that lattice is equal to (σ_p, σ_p^T)[8].

[8]To make the situation more clear we consider a concrete example. This is in fact a more general situation in lattice theory. We consider the hyperbolic lattice U with generators e, f such that $e^2 = f^2 = 0$ and $ef = 1$. One can primitively embed the lattice $\langle 2 \rangle \oplus \langle -2 \rangle$ into U by sending

Given that, for any given p-cyclic K3 surface, all the relevant sublattices $T(\sigma_p)$ and $T(\sigma_p^T)$ have been tabulated in [22, 23], explicit forms of the $\Gamma_{4,20}$ isometries can be determined from lattice theory for any given example, see *e.g.* [32]. The corresponding matrices can be diagonalized on $\mathbb{C}$ according to lemma 1 and are characterized respectively by a set of rank$(T(\sigma_p))$ angles and a set of rank$(T(\sigma_p^T))$ angles; these angles will be discussed further in the following sections.

Example 5. We consider the self-mirror surface (2.26) already discussed in example 3. The action of the geometrical automorphism σ_3 on the K3 lattice $\Gamma_{3,19}$ has been described there, see eqs. (2.31,2.33). The lattice $\Gamma_{4,20}$ admits an orthogonal decomposition into the invariant quantum lattices $S^{\mathfrak{Q}}(\sigma_p)$ and $S^{\mathfrak{Q}}(\sigma_p^T)$ – or equivalently into $T(\sigma_p^T)$ and $T(\sigma_p)$ – corresponding respectively to the quantum Picard lattice $S^{\mathfrak{Q}}(X)$ and the transcendental lattice $T(X)$ of this surface.

As the surface and its mirror are isomorphic to each other, it is straightforward to define the action of the mirrored CFT automorphism $\hat{\sigma}_3$. It has a diagonal action on

$$\Gamma_{4,20} \cong \Big(E_8 \oplus U \oplus U\Big) \oplus \Big(E_8 \oplus U \oplus U\Big)\,, \tag{3.4}$$

duplicating the action of σ_3 on the transcendental lattice that was studied in subsection 2.3. The action of $\hat{\sigma}_3$ on the sigma-model CFT associated with the surface (2.26) is therefore given by the block-diagonal 24×24 integer matrix:

$$\hat{M}_3 := \begin{pmatrix} M_3^{E_8} & 0 & 0 & 0 \\ 0 & M_3^{U\oplus U} & 0 & 0 \\ 0 & 0 & M_3^{E_8} & 0 \\ 0 & 0 & 0 & M_3^{U\oplus U} \end{pmatrix} \in O(\Gamma_{4,20})\,, \tag{3.5}$$

where $M_3^{E_8}$ is given by eq. (2.33) and $M_3^{U\oplus U}$ by eq. (2.31).

Example 6. By considering the K3 surface given by equation (2.34) and its mirror we have the orthogonal decomposition of the vector space $\Gamma_{4,20}^{\mathbb{R}}$:

$$\Gamma_{4,20}^{\mathbb{R}} \cong \Big(U \oplus U \oplus E_6\Big)^{\mathbb{R}} \oplus \Big(U \oplus U \oplus A_2 \oplus E_8\Big)^{\mathbb{R}}\,. \tag{3.6}$$

the two generators to $a := e+f$ and $b := e-f$, in this way the lattice $\langle 2\rangle \oplus \langle -2\rangle$ has index two in U. One takes now the isometric involution on $\langle 2\rangle \oplus \langle -2\rangle$ which acts as $\iota := (\mathrm{id}, -\mathrm{id})$. The discriminant group of $\langle 2\rangle$, resp. $\langle -2\rangle$, is generated by $a/2$, resp. $b/2$. One considers now the class $w := (a+b)/2$, which has square $w^2 = 0 \in \mathbb{Z}$; the lattice generated by a and $(a+b)/2$ has determinant 1 and it is in fact isometric to U. To see this one takes the generators w and $v := (a-(a+b))/2 = (a-b)/2$ and the induced involution $\hat{\iota}$ acts exchanging v and w; in particular it cannot be put in a diagonal form over $\mathbb{Z}$.

The action of $\hat{\sigma}_3$ is then induced from the block-diagonal 24×24 integer matrix

$$\begin{pmatrix} M_3^{U\oplus U} & 0 & 0 & 0 & 0 \\ 0 & M_3^{E_6} & 0 & 0 & 0 \\ 0 & 0 & M_3^{U\oplus U} & 0 & 0 \\ 0 & 0 & 0 & M_3^{A_2} & 0 \\ 0 & 0 & 0 & 0 & M_3^{E_8} \end{pmatrix} \in O\big(T(\sigma_p) \oplus T(\sigma_p^T)\big) . \tag{3.7}$$

where the various matrices $M_3^{U\oplus U}$, $M_3^{E_6}$, $M_3^{U\oplus U}$, $M_3^{A_2}$ and $M_3^{E_8}$ are given respectively in eqs. (2.31,2.33,2.35,2.37).

This matrix is an isometry of the lattice $T(\sigma_p) \oplus T(\sigma_p^T) \cong U \oplus U \oplus E_6 \oplus U \oplus U \oplus A_2 \oplus E_8$. The latter is a sublattice of index 3 of $\Gamma_{4,20}$ (see Proposition 1) and more precisely the lattice $E_6 \oplus A_2$ is a sublattice of index 3 of E_8. Now the isometries of order three $\sigma_3^\star$ and $(\sigma_3^T)^\star$ have no fixed vectors on A_2, respectively E_6 and act trivially on the discriminant groups. They can be then combined (see Definition 3) to give an isometry of order three on E_8 without fixed vectors, but up to isometry there is only one such isometry on E_8 which is given by equation (2.33). So the action of $\hat{\sigma}_3$ on the sigma-model CFT associated with this surface is given by the matrix $\hat{M}_3$ in eq. (3.5), as for the previous example.

3.2 Symmetries of Landau-Ginzburg mirror pairs

The Gepner models arise at special points in the moduli space (2.1) of sigma-model CFTs on K3 surfaces. These Gepner points play a special role in the present context as some of them are fixed under the action of the mirrored automorphisms defined in the previous subsection.

A Gepner model for a K3 surface [25] is a $(4,4)$ superconformal field theory obtained as the infrared fixed point of a (2,2) Landau-Ginzburg orbifold [11, 33] with Fermat type superpotential

$$W = Z_1^{k_1} + Z_2^{k_2} + Z_3^{k_3} + Z_4^{k_4} \,, \tag{3.8}$$

quotiented by the order $K = \gcd(k_1, \ldots, k_3)$ diagonal $\mathbb{Z}_K$ symmetry j_W, acting on the chiral superfields as

$$j_W \ : \ Z_\ell \mapsto e^{2i\pi/k_\ell} Z_\ell \tag{3.9}$$

and realizing the projection onto integral R-charges. We will refer to this orbifold as the diagonal $\mathbb{Z}_K$ orbifold. The twisted sectors of this orbifold are labelled by $\gamma \in \{0, \ldots, K-1\}$ and will be referred to as γ-twisted sectors.

The Landau-Ginzburg orbifold/Gepner model with superpotential (3.8) has a quantum Abelian symmetry [34] which is not present in the large-volume limit of the sigma-model. In the diagonal $\mathbb{Z}_K$ orbifold theory, the quantum symmetry acts on a field in the γ-twisted sector of the model as:

$$\sigma^{\mathrm{Q}} \ : \ \phi_\gamma \ \mapsto e^{\frac{2i\pi}{42}\gamma} \phi_\gamma \,. \tag{3.10}$$

At the infrared fixed point, the superconformal field theory obtained from this model is an orbifold of a product of $\mathcal{N}=2$ minimal model CFTs, as every single-field Landau-Ginzburg model with superpotential $W = X^{k_\ell}$ flows to a super-coset CFT $SU(2)_{k_\ell}/U(1)_{k_\ell}$ [35]. These Gepner models lead to IIA superstring theory compactifications in six dimensions with $\mathcal{N}=(1,1)$ supersymmetry, or, compactifying further on a two-torus, to $\mathcal{N}=4$ supersymmetry in four-dimensions.

Asymmetric orbifolds

We will now explain the relation between the mirrored automorphisms introduced in subsection 3.1 and the non-geometric orbifolds of Gepner models presented in [8, 9], following earlier works [10, 11].

We consider the Gepner model corresponding to the Landau-Ginzburg orbifold of superpotential (3.8) and assume that $p := k_1$ is a prime number. The theory admits the order p symmetry

$$\sigma_p \;:\; Z_1 \mapsto e^{2i\pi/p} Z_1 \,. \tag{3.11}$$

Quotienting the Gepner model by this automorphism alone would break all space-time supersymmetry. Indeed one can see that in the corresponding orbifold theory (see [8] for details):

- all the worldsheet operators corresponding to space-time supercharges are charged under the $\mathbb{Z}_p$ symmetry hence are projected out of the spectrum,
- the b-twisted sectors of the $\mathbb{Z}_p$ orbifold by the symmetry (3.11) contain states with non-integer left and right $U(1)_R$-charges whenever $b \neq 0$.

One observes that one can define a subgroup of the quantum symmetry group of the model (3.8), isomorphic to $\mathbb{Z}_p$, generated by:

$$\sigma_p^{\mathrm{Q}} := (\sigma^{\mathrm{Q}})^{K/p} \,. \tag{3.12}$$

One can then modify the orbifold of the LG orbifold/Gepner model (3.8) by the symmetry (3.11) that we described above by adding a specific *discrete torsion* keeping the space-time supercharges coming from the left-moving sector in the spectrum.

Starting from the Gepner model, one defines the $\mathbb{Z}_p$ orbifold projection by assigning to every state in the theory a charge

$$\hat{Q}_p \equiv Q_p + Q_p^{\mathrm{Q}} \mod 1 \equiv Q_p + \frac{\gamma}{p} \mod 1 \,, \tag{3.13}$$

where Q_p is the charge of the given state under the action of σ_p and Q_p^{Q} is the charge under the quantum symmetry σ_3^{Q}, and by projecting onto states with $\hat{Q}_p \in \mathbb{Z}$. This discrete torsion has also an effect in the twisted sectors $b \neq 0$ of the new $\mathbb{Z}_p$ orbifold.

In those sectors the diagonal $\mathbb{Z}_K$ orbifold projection is modified, as one projects onto states with $\hat{Q}_K \in \mathbb{Z}$, where

$$\hat{Q}_K \equiv Q_K - \frac{b}{p} \mod 1\,. \tag{3.14}$$

The charge assignments (3.13) and (3.14) are related to each other by modular invariance.

One can check, by inspecting the one-loop partition function, that the $\mathbb{Z}_p$ orbifold projection w.r.t. the charge $\hat{Q}_p$ keeps all space-time supercharges from the left-movers, while none of the space-time supercharges from the right-movers is invariant. Furthermore the diagonal $\mathbb{Z}_K$ projection w.r.t. the charge $\hat{Q}_K$ keeps only states with integer left R-charge. Hence space-time supersymmetry from the left-movers on the worldsheet is preserved by this orbifold with discrete torsion. Notice that one could have used the charge $Q_p - Q_p^{\mathrm{Q}}$ instead, in which case the invariant space-time supercharges come from the right-movers.

Under mirror symmetry, the right R-charges in every $\mathcal{N} = 2$ minimal model are mapped to minus themselves. As a consequence, mirror symmetry exchanges the geometrical automorphism σ_p with its quantum counterpart σ_p^{Q}. In view of the discussion in section 2, a generator σ_p^{Q} of an order p subgroup of the quantum symmetry of a Landau-Ginzburg orbifold superconformal field theory is identified with a non-symplectic automorphism σ_p^T of the corresponding BH mirror K3 surface. Hence, the mirrored automorphisms introduced in subsection 3.1 correspond precisely, at the Gepner points in the moduli space, to the orbifolds with discrete torsion described here.

The two-fold choice of discrete torsion in the definition of the Landau-Ginzburg model symmetries $\hat{Q}_p := Q_p \pm Q_p^{\mathrm{Q}}$ that we have noticed above corresponds, in the language of subsection 3.1, to the possibility of pairing the action of σ_p either with the action of σ_p^T or of its inverse.

Fractional mirror symmetry

We have described in the previous subsection orbifolds of Gepner models with discrete torsion, that preserve all space-time supersymmetry from the left-movers, and none from the right-movers at first sight. As discussed in [9], they belong to a more general family of quotients of Gepner models by non-symplectic automorphisms of the corresponding K3 surfaces with discrete torsion. This construction leads generically to non-geometric compactifications, *i.e.* that do not belong to the moduli space of compactifications on smooth manifolds, preserving $\mathcal{N} = 2$ supersymmetry in four dimensions (after further compactification on T^2). Similar constructions exist for Calabi-Yau three-folds, leading to $\mathcal{N} = 1$ in four dimensions.

However, in the specific case of an orbifold of a p-cyclic K3 surface (or more generically of a p-cyclic CY manifold) by a non-symplectic automorphism of order

p as considered in the present work, the twisted sectors $b \neq 0$ contain right-moving operators that, despite having non-integral right R-charge, have the properties of generators of space-time supersymmetry. Interestingly, the worldsheet model for the non-geometric compactification is actually isomorphic as a $(2,2)$ superconformal field theory to the original Calabi-Yau model in this case.

This isomorphism implies the existence of quantum symmetries, called *fractional mirror symmetries* in [9], between geometric and non-geometric compactifications in the Landau-Ginzburg regime. Outside of the Gepner point such symmetry extends to a map between a $(2,2)$ non-linear sigma-model on a Calabi-Yau manifold and a non-geometric worldsheet model. A linear-sigma model description [36] was proposed in [9], generalizing the ideas of [37].

In the following, we will focus on freely-acting orbifolds combining this type of $K3$ non-geometrical orbifold with a shift along a circle; in this case the accidental isomorphism and corresponding restoration of $\mathcal{N}=4$ space-time supersymmetry do not play a role, as the b-twisted sectors with $b \neq 0$ will only contain massive states from the space-time point of view.

3.3 Worldsheet construction of non-geometric backgrounds: summary

The mirrored automorphisms described in subsections 3.1 for the geometry and 3.2 for the field theory are the building blocks of non-geometric compactifications of type IIA superstring theory, whose vacua correspond to the non-geometric freely-acting orbifolds of [8] that we will now summarize briefly.

The starting point is a Gepner model for a K3 surface as described in subsection 3.2. Consider the tensor product of this superconformal field theory with the free $c=3$ superconformal theory with a two-torus target space of coordinates Y_1, Y_2. We consider a freely-acting supersymmetry-breaking $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ orbifold of this $K3 \times T^2$ superconformal model generated by

$$g_1 : \ Z_1 \mapsto e^{2i\pi/k_1} Z_1 \ , \ Y_1 \mapsto Y_1 + 2\pi/k_1 \tag{3.15a}$$

$$g_2 : \ Z_2 \mapsto e^{2i\pi/k_1} Z_2 \ , \ Y_2 \mapsto Y_2 + 2\pi/k_2 \tag{3.15b}$$

As it is, this orbifold breaks all space-time supersymmetry for the reasons given in the previous subsection.

We add to each of these two freely-acting orbifold actions a *discrete torsion* of the same type as in eqs. (3.13,3.14) above. As described there the discrete torsion is such that all these models have integral left-moving R-charges but non-integral right-moving ones, hence a type IIA superstring theory built upon one of these models will have a four-dimensional Minkowski vacuum with $\mathcal{N}=2$ space-time supersymmetry, all space-time supercharges being obtained from the left-moving Ramond ground states.

The two gravitini obtained from the right-moving Ramond sector are indeed massive in the $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ orbifold theory. Because of the freely-acting nature of this orbifold, no massless states could possibly arise from the corresponding twisted sectors. If one chooses an orthogonal two-torus of radii R_1 and R_2, the masses squared of the two massive gravitini of broken $\mathcal{N}=4$ supersymmetry are [8]:

$$m^2 = \left(\frac{1}{k_1 R_1}\right)^2 + \left(\frac{1}{k_2 R_2}\right)^2 . \tag{3.16}$$

The massless spectra of all these models were computed in [9].[9] It was found there that the massless states are identified with a subset of the chiral rings of the $K3$ SCFT, containing states built out of the identity operator in the $SU(2)_{k_1}/U(1)_{k_1}$ and $SU(2)_{k_2}/U(1)_{k_2}$ minimal models; we will consider the corresponding moduli spaces in subsection 5.2 from another perspective.

In about half of the possible constructions, this subset is empty and hence all the moduli of the original $K3$ SCFT have become massive; the only remaining massless moduli are the T and U moduli of the two-torus and the axio-dilaton modulus S, that are part of space-time vector multiplets. It gives the $\mathcal{N}=2$ four-dimensional STU supergravity model at low energies (compared to the inverse size of the torus). In the remaining constructions, some of the $K3$ moduli survive and appear in the low energy theory in massless hypermultiplets. We now turn to the second part of this article, where we analyse these constructions from the low-energy four-dimensional viewpoint.

4 $\mathcal{N}=4$ gauged supergravity from duality twists

In this section we study the supergravity dimensional reduction that corresponds to the stringy construction considered in the previous sections. We have considered type IIA superstring theory compactified on $K3 \times T^2$ identified under certain automorphisms. This requires being at a point in the $K3$ moduli space which is a fixed point under the automorphisms. (These fixed points were found in the last section from Landau-Ginzburg orbifolds.) This construction is extended to general points in moduli space by a compactification with duality twists [2]. We will here discuss the supergravity limit of this, which is a dimensional reduction of Scherk-Schwarz type [5].

We consider then type IIA superstring theory compactified on $K3$ to 6 dimensions and then further compactified on T^2 with duality twists with non-geometric monodromy. In the supergravity limit, compactifying IIA supergravity on $K3$ gives $\mathcal{N}=(1,1)$ supergravity in six dimensions coupled to 20 vector multiplets, and this has a duality symmetry $O(4,20)\times\mathbb{R}$. Then further compactifying on T^2 with an

[9] In [9] all pair of purely non-symplectic automorphisms were considered, for prime and non-prime order p.

$O(4,20)$ monodromy round each circle gives a Scherk-Schwarz reduction of the supergravity, resulting in a gauged $\mathcal{N}=4$ supergravity in four dimensions. This construction has been discussed extensively in the supergravity literature; see e.g. [5, 38–40] and references therein. For our string theory constructions, the monodromies are required to be in the duality group $O(\Gamma_{4,20})$, *i.e.* the isometry group of the lattice of total cohomology of the $K3$ surface as was discussed in section 2; in the physics literature it is often refered to as $O(4,20;\mathbb{Z})$.

We will focus here on the case with monodromies that are in the $O(4)\times O(20)$ subgroup of $O(4,20)$ as it is for these compact monodromies that fixed points in the moduli space corresponding to Minkowski minima of the $D=4$ supergravity scalar potential are possible. As we shall see, some interesting features arise for these special cases, and will give some vacua that break the $\mathcal{N}=4$ supersymmetry in four dimensions to $\mathcal{N}=2$. We will summarize the supergravity results here, and give more details elsewhere.

In this section we will consider the supergravity reduction with monodromies in the continuous group $O(4,20)$ and in the following section we will consider the consistent type IIA superstring theory compactifications that arise from the discrete monodromies constructed in section 3.

4.1 Twisted reduction on T^2

The starting point in six dimensions is $\mathcal{N}=(1,1)$ supergravity coupled to 20 vector multiplets, and this has a rigid duality symmetry $G=O(4,20)$ (and a further rigid symmetry consisting of constant shifts of the dilaton). There is also a local symmetry which in the bosonic sector is $H=O(4)\times O(20)$. In extending to the fermionic sector, the local symmetry is actually a double cover of this, $H_s=Pin(4)\times O(20)$. The 24 vector fields A^I_m $(I=1,\ldots,24)$ transform as the **24** of $G=O(4,20)$ and are invariant under H. The fermions are invariant under $G=O(4,20)$ but transform under H_s. The scalars consist of a dilaton ϕ and scalars taking values in the coset G/H and can be represented by a G-valued vielbein field $\hat{\mathcal{V}}$ transforming as $\hat{\mathcal{V}}\to g\hat{\mathcal{V}}h^{-1}$ under the action of $h\in H, g\in G$. The vielbein $\hat{\mathcal{V}}$ represents $\dim(\mathrm{G})=276+1$ degrees of freedom, but $\dim(\mathrm{H})=196$ of these can be removed by local H transformations.

We now turn to dimensional reduction on T^2 with twists in G, giving rise to a gauged supergravity in four dimensions. Consider first the untwisted case. Simple dimensional reduction (with no twists) on T^2 gives rise to $\mathcal{N}=4$ supergravity coupled to 22 vector multiplets in four dimensions. The massless Abelian theory has $G=SL(2)\times O(6,22)$ global symmetry, and a local symmetry $Pin(6)\times O(22)$, acting on the bosonic sector through $O(6)\times O(22)$. The $\mathcal{N}=4$ supergravity multiplet contains the vielbein, four gravitini ψ^i_μ, six graviphotons A^m_μ, four spin-half fermions χ^i and a complex scalar τ, which takes value in $SL(2)/SO(2)$. The $SL(2)$ acts as usual through fractional linear transformations $\tau\mapsto(a\tau+b)/(c\tau+d)$.

The 22 vector multiplets in four dimensions each contain a vector A^a, four gaugini λ^{ai} and six real scalars. 132 scalars parameterize the coset space $O(6,22)/O(6)\times O(22)$ while the remaining two parameterize the coset space $SL(2)/U(1)$. The scalars in $O(6,22)/O(6)\times O(22)$ can be conveniently expressed in terms of a vielbein $\mathcal{V}\in O(6,22)$, such that they transform under global $G=O(6,22)$ and local $O(6)\times O(22)$ as $\mathcal{V}\mapsto h^{-1}\mathcal{V}g$. From the vielbein $\mathcal{V}$, one can construct a metric $\mathcal{M}$ on the coset space given by $\mathcal{M}=\mathcal{V}^t\mathcal{V}$ which is invariant under H and transforms tensorially under G: $\mathcal{M}\mapsto g^t\mathcal{M}g$. The group $O(6,22)$ preserves a metric η_{MN} of signature $(6,22)$ and for the supergravity theory we can choose a basis in which this is the diagonal metric $(\mathbb{I}_6,-\mathbb{I}_{22})$. However, in the next section when we apply the supergravity analysis to string theory, we will take η_{MN} to be a lattice metric on $\Gamma_{4,20}\oplus U\oplus U$.

A gauged version of this supergravity (with electric gauge group) can be obtained by choosing a subgroup K of the rigid $G=O(6,22)$ symmetry (of dimension 28 at most) and promoting it to a local symmetry, using a minimal coupling to the 28 vector fields already in the theory.[10] For this to work, the vector representation of $O(6,22)$ must be the adjoint representation of $K\subset O(6,22)$. The gauging of the four-dimensional supergravity is completely specified by the structure constants ${t_{MN}}^P$ of the gauge group K $(M,N=1,\dots 28)$ which satisfy the Jacobi identity and the constraint that K preserves the $O(6,22)$ invariant metric η_{MN} which is the condition that

$$t_{MNP}=\eta_{MQ}{t_{NP}}^Q$$

is completely antisymmetric. Supersymmetry then requires the addition of a scalar potential V, together with fermion mass terms given by

$$e^{-1}\mathcal{L}_{3/2}=\tfrac{1}{3}A_1^{ij}\bar{\psi}_{\mu i}\Gamma^{\mu\nu}\psi_{\nu j}+\tfrac{1}{3}A_2^{ij}\bar{\psi}_{\mu i}\Gamma^{\mu}\chi_j-{A_{2ai}}^j\bar{\psi}^i_\mu\Gamma^\mu\lambda^a_j+h.c. \tag{4.1}$$

and

$$e^{-1}\mathcal{L}_{1/2}=-{A_{2ai}}^j\bar{\chi}^i(\lambda^a)_j+\tfrac{1}{2}A_2^{ij}\bar{\lambda}^a_i\lambda_{aj}+{A_{ab}}^{ij}\bar{\lambda}^a_i\lambda^b_j+h.c. \tag{4.2}$$

in terms of certain scalar-dependent tensors $A_1^{ij},A_2^{ij},{A_{2ai}}^j,{A_{ab}}^{ij}$. Here $i,j=1,\dots 4$ are $SU(4)$ indices. Supersymmetry and gauge invariance put strong restrictions on the subgroups K that can be gauged, and fixes the form of the scalar potential and the tensors $A_1^{ij},A_2^{ij},{A_{2ai}}^j,{A_{ab}}^{ij}$; see [41–43] and references therein. In particular, the scalar potential is

$$V=\frac{1}{48\Im(\tau)}t_{MNP}t_{QRS}\left(\mathcal{M}^{MQ}\mathcal{M}^{NR}\mathcal{M}^{PS}-3\mathcal{M}^{MQ}L^{NR}L^{PS}\right)+\frac{1}{24\Im(\tau)}t_{MNP}t^{MNP} \tag{4.3}$$

where $\Im(\tau)$ denotes the imaginary part of τ, the scalar in $SL(2)/U(1)$ and $\mathcal{M}^{MQ}$ is the metric on $O(6,22)/O(6)\times O(22)$ discussed above.

[10]Note that this is not the most general gauging of the supergravity, but we will restrict ourselves to this class of gaugings here.

We now turn to the twisted reduction of the six-dimensional supergravity on T^2 to obtain a gauged $\mathcal{N} = 4$ supergravity. We consider a rectangular torus for simplicity, with coordinates y^i, $i = 1, 2$, with $y^1 \sim y^1 + 2\pi R_1$ and $y^2 \sim y^2 + 2\pi R_2$. Then the complex Kähler modulus is $T = iR_1R_2$ and complex structure modulus is $U = iR_2/R_1$.

We introduce twists around each of the two circles as described in the introduction. A field $\psi(x^\mu, y^i)$ (where y^i are coordinates on T^2 and x^μ, $\mu = 0, \ldots, 3$, are the coordinates of the four-dimensional space-time) is taken to depend on y through G transformations. Specifically, suppose $\psi(x^\mu, y^i)$ transforms in a representation of G, $\psi \mapsto R[g]\psi$ under a rigid transformation $h \in H_R$. Then the Scherk-Schwarz ansatz is

$$\psi(x^\mu, y^i) = R[g_1(y^1)]\, R[g_2(y^2)]\, \psi_0(x)\,, \tag{4.4}$$

giving the $D = 6$ field $\psi(x^\mu, y^i)$ as the transformation of a $D = 4$ field $\psi_0(x)$ under a y-dependent G transformation $g_1(y^1)$ around the first cycle and a y-dependent G transformation $g_2(y^2)$ around the second cycle. The two G transformations are required to commute,

$$g_1(y^1)g_2(y^2) = g_2(y^2)g_1(y^1)\,,$$

and the y-dependence is taken to be exponential, so that

$$g_1(y^1) = e^{N_1 y^1}\ , \quad g_2(y^2) = e^{N_2 y^2}\,. \tag{4.5}$$

Then the monodromies are

$$(g_1(0))^{-1}g_1(2\pi R_1) = e^{2\pi R_1\, N_1}, \qquad (g_2(0))^{-1}g_2(2\pi R_2) = e^{2\pi R_2\, N_2} \tag{4.6}$$

for two commuting elements N_1, N_2 of the Lie algebra of G, $[N_1, N_2] = 0$. In the six-dimensional supergravity, the only fields transforming under G are the vector fields A and the scalar fields, represented by the vielbein $\hat{\mathcal{V}}$. These then get non-trivial y dependence, while the fermions and graviton do not, as they are singlets under G. This picture depends on using the formalism in which the local H symmetry is not fixed. Choosing a physical gauge for the local H symmetry would mean that a G transformation must be accompanied by a compensating H transformation that act on the fermions through an H_s transformation, so that in this gauge the fermions also get y dependence, and this requires the choice of a lift of the twist in H to one in the double cover H_s.

Full details of the reduction for the bosonic sector are given in [44], and here we will just quote the results needed, mostly following the notation of [44]. The $O(4, 20)$ invariant metric is η_{IJ} where $I, J = 1, \ldots 24$; for the supergravity theory we can choose a basis in which this is the diagonal metric $(\mathbb{I}_4, -\mathbb{I}_{20})$. However, in the next section when we apply the supergravity analysis to string theory, we will take

η_{IJ} to be the metric on the lattice $\Gamma_{4,20}$. The generators of the gauge group K can be combined into a $O(6,22)$ vector T_M as

$$T_M = \begin{pmatrix} Z_i \\ X^i \\ T_I \end{pmatrix} . \tag{4.7}$$

The Lie algebra of K is then [44]

$$[T_M, T_N] = t_{MN}{}^P T_P \,,$$

where the structure constants of the gauge group are

$$t_{iI}{}^J = N_{iI}{}^J, \qquad N_{IJi} = L_{JK} f_{iI}{}^K, \tag{4.8}$$

and all other structure constants are zero. Then the only non-vanishing commutators are

$$[Z_i, T_I] = N_{iI}{}^J T_J, \qquad [T_I, T_J] = N_{IJi} X^i \,. \tag{4.9}$$

Suppose now that there is a point $\hat{\mathcal{V}} = \hat{\mathcal{V}}_0$ in the moduli space $O(4,20)/O(4) \times O(20)$ that is fixed under both the monodromies. From the arguments of [2], the fixed point gives a minimum of the scalar potential where the potential vanishes, giving a Minkowski vacuum. Then we can perform an $O(4,20)$ transformation with $g = (\hat{\mathcal{V}}_0)^{-1}$ to bring the fixed point to the origin, $\hat{\mathcal{V}}_0 = 1$. The subgroup of $O(4,20)$ preserving $\hat{\mathcal{V}}_0 = 1$ is $O(4) \times O(20)$ so both monodromies (4.6) must be in this subgroup.

The supergravity reduction is then specified by two commuting monodromies N_1 and N_2 in the Lie algebra of $O(4) \times O(20)$. These are then in a Cartan subalgebra $SO(2)^{12}$ and we can choose a basis (by $O(4) \times O(20)$ conjugation) in which they are both diagonal and each specified by 12 angles:

$$\begin{aligned} 2\pi R_1 N_1 &= \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \theta_2 \\ -\theta_2 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} , \\ 2\pi R_2 N_2 &= \begin{pmatrix} 0 & \tilde{\theta}_1 \\ -\tilde{\theta}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \tilde{\theta}_2 \\ -\tilde{\theta}_2 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & \tilde{\theta}_{12} \\ -\tilde{\theta}_{12} & 0 \end{pmatrix} . \end{aligned} \tag{4.10}$$

The monodromies in $O(4) \times O(20)$ are then

$$\begin{aligned} e^{2\pi R_1 N_1} &= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} , \\ e^{2\pi R_2 N_2} &= \begin{pmatrix} \cos\tilde{\theta}_1 & \sin\tilde{\theta}_1 \\ -\sin\tilde{\theta}_1 & \cos\tilde{\theta}_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \cos\tilde{\theta}_{12} & \sin\tilde{\theta}_{12} \\ -\sin\tilde{\theta}_{12} & \cos\tilde{\theta}_{12} \end{pmatrix} . \end{aligned} \tag{4.11}$$

We choose a basis in which the angles θ_1, θ_2 and $\tilde{\theta}_1, \tilde{\theta}_2$ specify monodromies in the $O(4)$ factor and the remaining angles specify monodromies in $O(20)$.

These twists will in general give masses to fields that are charged under $U(1)^{12} \subset O(4) \times O(20)$. For a state with charges q_i $(i, j = 1, \dots 12)$ under $U(1)^{12}$, the mass m will be given by

$$m^2 = \left(\sum_{i=1}^{12} \frac{q_i \theta_i}{2\pi R_1} \right)^2 + \left(\sum_{i=1}^{12} \frac{q_i \tilde{\theta}_i}{2\pi R_2} \right)^2 . \tag{4.12}$$

Using this formula, the masses of all fields can be found by finding the charges q_i.

The 28 vector fields are in the $\mathbf{28}$ of $O(6,22)$ and this decomposes into $(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{24})$ under $O(2,2) \times O(4,20)$. A twist with all angles non-zero makes the vectors in the $(\mathbf{1},\mathbf{24})$ representation massive and leaves the $(\mathbf{4},\mathbf{1})$ vectors massless. The 24 vector fields in the $(\mathbf{1},\mathbf{24})$ representation can be written as 12 complex vector fields A_i, where A_i has charge $q_i = 1$ and $q_j = 0$ for $j \neq i$. Then A_i has mass m given by

$$m^2 = \left(\frac{q_i \theta_i}{2\pi R_1} \right)^2 + \left(\frac{q_i \tilde{\theta}_i}{2\pi R_2} \right)^2 . \tag{4.13}$$

If the angles $\theta_i, \tilde{\theta}_i$ are both zero for some i, then the vector A_i will remain massless.

The scalars in $O(6,22)/O(6) \times O(22)$ can be parameterized by fields transforming as the $(\mathbf{6},\mathbf{22})$ under $O(6) \times O(22) \subset O(6,22)$. Under $O(2) \times O(2) \times O(4) \times O(20)$ these decompose as[11]

$$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{20}) + (\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{20}) . \tag{4.14}$$

Of these, only those in the $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ representation are singlets under $O(4) \times O(20)$ and hence invariant under the $U(1)^{12}$ twist. These scalars parameterize

$$\frac{O(2,2)}{O(2) \times O(2)} \subset \frac{O(6,22)}{O(4) \times O(22)} .$$

The axion and dilaton in $SL(2)/U(1)$ are also uncharged and remain massless, so the scalars in

$$\frac{SL(2)}{U(1)} \times \frac{O(2,2)}{O(2) \times O(2)} \sim \left[\frac{SL(2)}{U(1)} \right]^3$$

remain massless.

All other scalars become generically massive, with masses given by eq (4.12). This formula indicates also that, for a given set of charges $\{q_i, i = 1, \dots, 12\}$, some values of the angles θ_i, $\tilde{\theta}_i$ can lead to accidentally massless scalars.

It will be useful to decompose the indices $i = 1, \dots 12 = (M, A)$ into indices $M = 1, 2$ labeling the Cartan subalgebra of $O(4)$ and indices $A = 3, \dots 12$ labeling the Cartan subalgebra of $O(20)$, so that the charges are $q_i = (q_M, q_A)$. Then, for example, the 80 real scalars in the $(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{20})$ representation take values in the coset

[11] Here $O(2) \times O(4) \times \subset O(6)$ and $O(2) \times O(20) \subset O(22)$.

$O(4,20)/O(4)\times O(20)$ and can be written in terms of complex scalars ϕ_{MA}, ρ_{MA} where the scalar ϕ_{NB} has charges $q_i = (q_M, q_A)$ where $q_M = \delta_{MN}$ and $q_A = \delta_{AB}$ while ρ_{NB} has charges $q_i = (q_M, q_A)$ where $q_M = \delta_{MN}$ and $q_A = -\delta_{AB}$. Then ϕ_{MA} has mass squared

$$m^2 = \left(\frac{\theta_M + \theta_A}{2\pi R_1}\right)^2 + \left(\frac{\tilde{\theta}_M + \tilde{\theta}_A}{2\pi R_2}\right)^2 , \tag{4.15}$$

and ρ_{MA} has mass squared

$$m^2 = \left(\frac{\theta_M - \theta_A}{2\pi R_1}\right)^2 + \left(\frac{\tilde{\theta}_M - \tilde{\theta}_A}{2\pi R_2}\right)^2 . \tag{4.16}$$

Then for generic twists with all angles non-zero, the massless bosonic fields consist of the graviton, 4 vector fields in the **4** of $O(2,2)$ and 6 scalars in the coset space $[SL(2)/U(1)]^3$; this is precisely the bosonic sector of the STU model [4].

We now turn to the fermion mass terms (4.1),(4.2). At the origin, $\mathcal{V} = \mathbb{I}_{28}$, the mass matrices of the model simplify considerably to give [45]

$$\begin{aligned} &A_1^{ij} = A_2^{ij} = \tfrac{1}{8\sqrt{\tau_2}}([G_m]_{ik})^\star [G_n]_{kl}([G_p]_{lj})^\star t_{mnp} , \quad A_{2ai}{}^{j} = -\tfrac{1}{4\sqrt{\tau_2}}[G_m]_{ik}([G_n]_{kj})^\star t_{amn} , \\ &A_{ab}{}^{ij} = -\tfrac{1}{2\sqrt{\tau_2}}[G_m]^{ij} t_{abm} \end{aligned} \tag{4.17}$$

where G_m are the 't Hooft matrices used to convert an $SO(6)$ vector index to an antisymmetric pair of $Spin(6) = SU(4)$ indices. The first matrix A_1 gives direct access to the fraction of supersymmetry preserved by the vacuum, as it provides the mass term for the gravitini given by

$$\tfrac{1}{3}A_1^{ij}\bar{\psi}_{\mu i}\Gamma^{\mu\nu}\psi_{\nu j} + h.c. \tag{4.18}$$

where A_1^{ij} is a complex symmetric matrix. The mass matrix for $\psi_{\mu i}$ is

$$(M^2)_i{}^j = A_{1ik}A_1^{kj} \tag{4.19}$$

where $A_{1ij} = (A_1^{ij})^*$. This is a Hermitian matrix whose eigenvalues are (after a calculation using formulæ from [45]) $(m_1)^2$ and $(m_2)^2$, both with degeneracy two, where

$$(m_1)^2 = \left(\frac{\theta_1 - \theta_2}{4\pi R_1}\right)^2 + \left(\frac{\tilde{\theta}_1 - \tilde{\theta}_2}{4\pi R_2}\right)^2 \tag{4.20}$$

and

$$(m_2)^2 = \left(\frac{\theta_1 + \theta_2}{4\pi R_1}\right)^2 + \left(\frac{\tilde{\theta}_1 + \tilde{\theta}_2}{4\pi R_2}\right)^2 . \tag{4.21}$$

These formulæ can be understood as follows. The $D = 6$ supergravity has a local $H_s = Pin(4) \times O(20)$ symmetry and a global $O(4,20)$ symmetry. The fermions transform under $Spin(4) \times O(20) = SU(2) \times SU(2) \times O(20) \subset H_s$, but do not transform under the global $O(4,20)$ symmetry. If the local H_s symmetry is fixed, $O(4,20)$ transformations must be accompanied by compensating H_s transformations. As a result, reductions with $O(4,20)$ twists result in twists of the fermions by compensating H_s transformations. The gravitini transform as $(\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1})$ under $SU(2) \times SU(2) \times O(20)$, and as a result they become, after gauge fixing, twisted under the $U(1)^2 \subset O(4)$ but not under the $U(1)^{10} \subset O(20)$. The charges of the gravitini in the $(\mathbf{2},\mathbf{1},\mathbf{1})$ representation under $U(1)^2 \subset O(4)$ are $(q_1, q_2) = (1/2, 1/2)$ while those of the gravitini in the $(\mathbf{1},\mathbf{2},\mathbf{1})$ representation under $U(1)^2 \subset O(4)$ are $(q_1, q_2) = (1/2, -1/2)$. This then results in the mass formulæ (4.20,4.21) on using (4.12).

Similarly, the masses of the spin-1/2 fields can be found by calculating the tensors appearing in the mass formulæ (4.2), or by finding the twists in the gauge-fixed theory. Here we do the latter. Under $SU(2) \times SU(2) \times O(20)$, the spin-1/2 fields transform as

$$3 \times (\mathbf{2},\mathbf{1},\mathbf{1}) + 3 \times (\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{20}) + (\mathbf{1},\mathbf{2},\mathbf{20})\,.$$

The fermions in the $3 \times (\mathbf{2},\mathbf{1},\mathbf{1})$ representation will all get mass m_1 given by (4.20) while those in the $3 \times (\mathbf{1},\mathbf{2},\mathbf{1})$ representation will all get mass m_2 given by (4.21). The remaining fermions will all be massive for generic angles.

We see that something special happens if $\theta_1 = \theta_2$ and $\tilde{\theta}_1 = \tilde{\theta}_2$ so that m_2 is zero or $\theta_1 = -\theta_2$ and $\tilde{\theta}_1 = -\tilde{\theta}_2$ so that m_1 is zero. In either case, there are 2 massless gravitini and six massless spin-half fields. In this case the vacuum breaks the $\mathcal{N} = 4$ local supersymmetry to $\mathcal{N} = 2$ supersymmetry, and the massless fields fit into the $\mathcal{N} = 2$ supergravity multiplet with three massless vector multiplets, which is just the spectrum of the STU model. Unlike most occurrences of the STU model in string theory, in the present case it does not occur as a truncation of a richer theory, but describes the whole low-energy sector of the theory.

For generic angles, both m_1 and m_2 are non-zero therefore all the fermions become massive and all supersymmetry is broken.

For the ungauged $\mathcal{N} = 4$ theory, there is a local H_s symmetry and a global $O(6,22)$ symmetry. A monodromy in $O(4) \times O(20) \subset O(6,22)$ will break the global $O(6,22)$ to $O(2,2)$. However, $O(4)$ has a subgroup $SO(3)_1 \times SO(3)_2$, and if the $O(4)$ monodromy is restricted to be in $SO(3)_2$, then $SO(3)_1$ survives as a symmetry in the gauged supergravity. This corresponds to the case $m_1 = 0$ above. Similarly, $m_2 = 0$ corresponds to a monodromy in $SO(3)_1$ with $SO(3)_2$ surviving as a symmetry. In the formalism with the local H_s fixed, there is an $SU(4) \times O(22)$ global symmetry, of which $SU(4)$ is an R-symmetry. The monodromies are in $SU(2)_1 \times SU(2)_2 \times O(20) \subset SU(4) \times O(20)$. If $m_1 = 0$, then the monodromies lie in $SU(2)_2 \times SO(20) \subset SU(2)_1 \times SU(2)_2 \times SO(20)$ and hence $SU(2)_1$ survives as an R-symmetry in the low-energy theory, and similarly if $m_2 = 0$ then the monodromies lie in $SU(2)_1 \times SO(20) \subset$

$SU(2)_1 \times SU(2)_2 \times SO(20)$ and hence $SU(2)_2$ survives as an R-symmetry in the low-energy theory. The surviving $SU(2)$ is the R-symmetry for the unbroken $\mathcal{N} = 2$ supersymmetry.

4.2 Aspects of the low-energy $\mathcal{N} = 2$ theory

In this subsection we give further details of how the $\mathcal{N} = 4$ multiplets of the $D = 4$ supergravity decompose into massless and massive multiplets of $\mathcal{N} = 2$ supersymmetry for the cases in which m_1 is zero or m_2 is zero, and study aspects of the effective $\mathcal{N} = 2$ theory valid at energies much less than the supersymmetry breaking scale set by the gravitini masses.

4.2.1 Massless multiplets

The fields in the $D = 4$ supergravity theory fit into an $\mathcal{N} = 4$ supergravity supermultiplet and 22 $\mathcal{N} = 4$ vector supermultiplets. Once the local $SU(4) \times O(22)$ symmetry has been fixed, all fields transform under rigid $SU(4) \times SO(22)$ transformations. We now give the representations of the component fields under this global $SU(4) \times SO(22)$, and the decomposition of these representations into $SU(2)_1 \times SU(2)_2 \times SO(20) \subset SU(4) \times SO(2))$ representations, which will be useful for studying the $\mathcal{N} = 2$ multiplet structure. The $\mathcal{N} = 4$ supergravity multiplet is

	$SU(4) \times SO(22)$	$SU(2)_1 \times SU(2)_2 \times SO(20)$
2	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$
3/2	$(\mathbf{4}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1})$
1	$(\mathbf{6}, \mathbf{1})$	$2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{2}, \mathbf{1})$
1/2	$(\mathbf{4}', \mathbf{1})$	$(\mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1})$
0	$2 \times (\mathbf{1}, \mathbf{1})$	$2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})$

(4.22)

while for the 22 vector multiplets we have

	$SU(4) \times SO(22)$	$SU(2)_1 \times SU(2)_2 \times SO(20)$
1	$(\mathbf{1}, \mathbf{22})$	$(\mathbf{1}, \mathbf{1}, \mathbf{20}) + 2 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})$
1/2	$(\mathbf{4}, \mathbf{22})$	$(\mathbf{2}, \mathbf{1}, \mathbf{20}) + (\mathbf{1}, \mathbf{2}, \mathbf{20}) + 2 \times (\mathbf{2}, \mathbf{1}, \mathbf{1}) + 2 \times (\mathbf{1}, \mathbf{2}, \mathbf{1})$
0	$(\mathbf{6}, \mathbf{22})$	$(\mathbf{2}, \mathbf{2}, \mathbf{20}) + 2 \times (\mathbf{1}, \mathbf{1}, \mathbf{20}) + 2 \times (\mathbf{2}, \mathbf{2}, \mathbf{1}) + 4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})$

(4.23)

For the gravitini and spin-1/2 fields, the representation $\mathbf{4}$ corresponds to left-handed fermions transforming in the $\mathbf{4}$ and right handed ones transforming in the $\bar{\mathbf{4}}$, i.e. $\mathbf{4} \sim \bar{\mathbf{4}}_R + \mathbf{4}_L$. Similarly, $\mathbf{4}' \sim \bar{\mathbf{4}}_L + \mathbf{4}_R$.

If $m_1 = 0$, then the monodromies lie in $SU(2)_2 \times SO(20) \subset SU(2)_1 \times SU(2)_2 \times SO(20)$ and $SU(2)_1$ survives as an R-symmetry in the low-energy theory. The massless states are the ones that are singlets under $SU(2)_2 \times SO(20)$, i.e.

$$\begin{array}{|c|c|} 2 & (\mathbf{1},\mathbf{1},\mathbf{1}) \\ 3/2 & (\mathbf{2},\mathbf{1},\mathbf{1}) \\ 1 & 4\times(\mathbf{1},\mathbf{1},\mathbf{1}) \\ 1/2 & 3\times(\mathbf{2},\mathbf{1},\mathbf{1}) \\ 0 & 6\times(\mathbf{1},\mathbf{1},\mathbf{1}) \end{array} \tag{4.24}$$

This gives $\mathcal{N}=2$ supergravity with three massless $\mathcal{N}=2$ vector multiplets, which is precisely the content of the STU model.

4.2.2 Massive multiplets

In generic models, the remaining states now organize themselves in massive multiplets. The massive states that are singlets under SO(20) are:

$$\begin{array}{|c|c|} 3/2 & (\mathbf{1},\mathbf{2},\mathbf{1}) \\ 1 & (\mathbf{2},\mathbf{2},\mathbf{1}) \\ 1/2 & 3\times(\mathbf{1},\mathbf{2},\mathbf{1}) \\ 0 & 2\times(\mathbf{2},\mathbf{2},\mathbf{1}) \end{array} \tag{4.25}$$

This gives a BPS gravitino multiplet and two BPS hypermultiplets.

The remaining fields from the original $\mathcal{N}=4$ ungauged theory are all in a $\mathbf{20}$ of $SO(20)$, namely:

$$\begin{array}{|c|c|} 1 & (\mathbf{1},\mathbf{1},\mathbf{20}) \\ 1/2 & (\mathbf{2},\mathbf{1},\mathbf{20})+(\mathbf{1},\mathbf{2},\mathbf{20}) \\ 0 & (\mathbf{2},\mathbf{2},\mathbf{20})+2\times(\mathbf{1},\mathbf{1},\mathbf{20}) \end{array} \tag{4.26}$$

States in the $(\mathbf{2},\mathbf{20})$ of $SU(2)_2 \times SO(20)$ form a BPS hypermultiplet and states in the $(\mathbf{1},\mathbf{20})$ give a BPS massive vector multiplet (one scalar gets eaten by the vector).

4.2.3 Accidental massless multiplets

In certain models, a fraction of the BPS hypermultiplets are neutral under the monodromy and are therefore massless. From the mass formula (4.12), a supergravity field with charges q_i $(i,j=1,\ldots 12)$ under $U(1)^{12}$ will be massless if

$$\sum_{i=1}^{12} q_i\theta_i = 0 \tag{4.27}$$

and

$$\sum_{i=1}^{12} q_i\tilde{\theta}_i = 0\,. \tag{4.28}$$

For example, for the scalars with mass (4.15) this will be the case if $\theta_M=-\theta_A$ and $\tilde{\theta}_M=-\tilde{\theta}_A$.

4.2.4 Further accidental massless multiplets from KK modes

There can be further accidental massless multiplets from Kaluza-Klein modes [2]. For a trivial reduction without monodromy on the two circles with coordinates y_1, y_2, each field has a mode expansion of the form

$$\phi(x^\mu, y^1, y^2) = \sum_{n_1,n_2} e^{in_1 y^1/R_1 + in_2 y^2/R_2} \phi_{n_1,n_2}(x) \tag{4.29}$$

with a sum over integers n_1, n_2. The mode $\phi_{n_1,n_2}(x)$ then has mass m with

$$m^2 = \left(\frac{n_1}{R_1}\right)^2 + \left(\frac{n_2}{R_2}\right)^2 . \tag{4.30}$$

For a reduction with duality twists of the type discussed above, this formula is modified for fields that are charged under $U(1)^{12} \subset O(4) \times O(20)$. For a mode $\phi_{n_1,n_2}(x)$ with charges q_i $(i, j = 1, \ldots 12)$ under $U(1)^{12}$, the mass m will be given by the following modification of (4.12):

$$m^2 = \left(\frac{2\pi n_1 + \sum_{i=1}^{12} q_i \theta_i}{2\pi R_1}\right)^2 + \left(\frac{2\pi n_2 + \sum_{i=1}^{12} q_i \tilde{\theta}_i}{2\pi R_2}\right)^2 . \tag{4.31}$$

In the truncated supergravity theory, the condition for massless states were (4.27) and (4.28). Now we see that the condition that the full Kaluza-Klein spectrum contains massless modes is that

$$\sum_{i=1}^{12} q_i \theta_i = 0 \mod 2\pi \tag{4.32}$$

and

$$\sum_{i=1}^{12} q_i \tilde{\theta}_i = 0 \mod 2\pi . \tag{4.33}$$

For the hypermultiplets ϕ_{MA}, the condition that there be a massless KK mode is that

$$\theta_M + \theta_A = 0 \mod 2\pi \tag{4.34}$$

and

$$\tilde{\theta}_M + \tilde{\theta}_A = 0 \mod 2\pi , \tag{4.35}$$

while for ρ_{MA} the condition is

$$\theta_M - \theta_A = 0 \mod 2\pi \tag{4.36}$$

and

$$\tilde{\theta}_M - \tilde{\theta}_A = 0 \mod 2\pi . \tag{4.37}$$

For the gravitini, there is a similar modification of the mass formulæ. The gravitini KK modes will include massless spin-3/2 fields if

$$\theta_1 + \theta_2 = 0 \mod 4\pi$$

and

$$\tilde{\theta}_1 + \tilde{\theta}_2 = 0 \mod 4\pi\,,$$

or

$$\theta_1 - \theta_2 = 0 \mod 4\pi$$

and

$$\tilde{\theta}_1 - \tilde{\theta}_2 = 0 \mod 4\pi\,.$$

These are for gravitini modes that are periodic in y^1, y^2; the conditions for anti-periodic ones would be slightly different.

5 Compactifications with non-geometric monodromies

We will now apply the supergravity framework developed in the last section to the non-geometric compactifications analysed in sections 2 and 3 from the algebraic geometry and string theory viewpoints. In string theory, the non-compact symmetry groups $O(4,20)$ and $O(6,22)$ are broken to the discrete subgroups preserving the charge lattice [2]. In particular, $O(4,20)$ is broken to the group $O(\Gamma_{4,20})$ preserving the lattice $\Gamma_{4,20}$, and we choose the natural basis in which the metric η_{IJ} is the metric on the lattice $\Gamma_{4,20}$ given in section 2. As a result, the mass parameters introduced in the twisted reduction now take discrete values. Our aim here is to find the non-geometric type IIA compactifications, consisting of K3 fibrations over two-tori with non-geometric twists, in the sense of [2], that at fixed points of the twist reproduce the orbifold constructions of [8] summarized in section 3.

We start with a $(p_1,p_2)-$*cyclic* K3 surface, *i.e.* a hypersurface in a weighted projective space defined by a polynomial of the form

$$W = x_1^{p_1} + x_2^{p_2} + f(x_3, x_4)\,, \tag{5.1}$$

where p_1 and p_2 are prime numbers. As we have seen, such surface admits two non-symplectic automorphisms σ_{p_1} and σ_{p_2} generating an automorphism group isomorphic to $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. Using Definition 3 one can associate to them non-geometric automorphisms $\hat{\sigma}_{p_1}, \hat{\sigma}_{p_2}$ generating a subgroup $O(\hat{\sigma}_{p_1}) \times O(\hat{\sigma}_{p_2})$ of the duality group $O(\Gamma_{4,20})$, isomorphic to $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$; see eqs. (3.1,3.3).

In subsection 3.3 we defined orbifold compactifications consisting of identifying the IIA superstring theory on $K3 \times T^2$ under the action of $\hat{\sigma}_{p_1}$ combined with a shift of $2\pi R_1/p_1$ for the first one-cycle of the torus, and $\hat{\sigma}_{p_2}$ combined with a shift of $2\pi R_2/p_2$ for the second one-cycle of the torus. This is all defined at a particular point in the moduli space of CFTs on $K3$ that is a fixed point under these transformations, corresponding to the (p_1, p_2)-*cyclic* K3 surface at a Landau-Ginzburg point.

The reduction with a duality twist construction gives a way to extend this to all points in moduli space, and then the supergravity analysis of section 4 gives the resulting low energy effective field theory. The twisted reduction gives a fibration of K3 surfaces over T^2 with two non-geometric monodromies in $O(\Gamma_{4,20})$ associated with the one-cycles of the torus:

- An order p_1 monodromy belonging to $O(\hat{\sigma}_{p_1})$, associated with the non-geometric automorphism $\hat{\sigma}_{p_1}$, for the first one-cycle of the torus.
- An order p_2 monodromy belonging to $O(\hat{\sigma}_{p_2})$, associated with the non-geometric automorphism $\hat{\sigma}_{p_2}$, for the second one-cycle of the torus.

The action of $\hat{\sigma}_p$ is deduced from the action of the geometrical automorphism σ_p on the vector space generated by $T(\sigma_p)$ and from the action of the automorphism σ_p^T of the mirror surface on the vector space generated by $T(\sigma_p^T)$, see eqs. (3.1). These give the monodromies and hence the structure constants of the associated gauged supergravity.

Example 7. In example 3 we constructed an explicit example of an order three non-geometric automorphism that leaves invariant the self-mirror K3 surface (2.26) at the Gepner point. From (3.5), the corresponding $O(\Gamma_{4,20})$ element is given by the 24×24 matrix $\hat{M}_3$

$$\hat{M}_3 := \begin{pmatrix} M_3^{E_8} & 0 & 0 & 0 \\ 0 & M_3^{U\oplus U} & 0 & 0 \\ 0 & 0 & M_3^{E_8} & 0 \\ 0 & 0 & 0 & M_3^{U\oplus U} \end{pmatrix}, \tag{5.2}$$

where $M_3^{E_8}$ is given by eq. (2.33) and $M_3^{U\oplus U}$ by eq. (2.31). The matrix $M_3^{U\oplus U}$ has eigenvalues $\exp\frac{2i\pi}{3}$ and $\exp\frac{4i\pi}{3}$ with degeneracy two for each, and the matrix $M_3^{E_8}$ has the eigenvalues $\exp\frac{2i\pi}{3}$ and $\exp\frac{4i\pi}{3}$ each with degeneracy four.

The corresponding twisted reduction on $K3\times T^2$ is obtained using the formalism presented in subsection 4.1. Suppose we reduce on the y_1 circle with monodromy

$$e^{2\pi R_1 N_1} = \hat{M}_3 \tag{5.3}$$

This can be put in the form (4.11) by a change of basis, with the twelve angles θ_i given by $2\pi/3$ with degeneracy 16 and $4\pi/3$ with degeneracy 8. From this, one can find N_1 which in this basis takes the form (4.10). Then from (4.8) the structure constants ${t_{iI}}^J$ are given by

$${t_{iI}}^J = {N_{iI}}^J\,. \tag{5.4}$$

Similarly, the order seven non-geometric automorphism would give a monodromy matrix $\hat{M}_7$ which can be brought to the diagonal form (4.11) with the twelve angles θ_i given by $\exp(2ir\pi/7)$, for $r = 1, ..., 6$, each with degeneracy 4.

To specify the reduction, we choose the monodromy $e^{2\pi N_2}$ for the other circle from another automorphism, *e.g.* that resulting from σ_3 or σ_7, and this gives the structure constants $t_{2I}{}^J$.

In full generality, using Lemma 1 in section 2, the $GL(24;\mathbb{Z})$ matrices associated with the non-geometric automorphisms can be diagonalized over $\mathbb{C}$, or equivalently can be written as elements of $O(4,20)$, the group preserving the Minkowski metric diag $(1_4, -1_{20})$, by a change of basis. The monodromies in $\mathbb{Z}_{p_1} \subset O(4) \times O(20) \subset O(4,20)$ can be brought to the standard form (4.11) specified by 12 angles θ_i which satisfy $\exp(i\theta_i p_1) = 1$.[12] In the same way the monodromies in $\mathbb{Z}_{p_2} \subset O(4) \times O(20) \subset O(4,20)$ are specified by 12 angles $\tilde{\theta}_i$ which satisfy $\exp(i\tilde{\theta}_i p_2) = 1$.

From these angles, the full structure of the effective supergravity theory can be read off, as seen in section 4. The scalar potential of the supergravity admits minima at the fixed points of the automorphisms that reproduce the four-dimensional physics obtained from the asymmetric Gepner models considered in section 3.

The stringy compactifications discussed in this work have Minkowski vacua that preserve $\mathcal{N} = 2$ supersymmetry as we will show below, and have three massless vector multiplets S,T and U associated respectively to the axion-dilaton and to the T^2 moduli. About half of the corresponding asymmetric Gepner models, for instance the self-mirror surface (2.26) with $\hat{\sigma}_3$ and $\hat{\sigma}_7$ monodromies, give just $\mathcal{N} = 2$ STU supergravity at low energies, while in the other cases the low-energy theory contains some additional massless hypermultiplets, depending of the choice of K3 surfaces and of automorphisms; the associated moduli space will be discussed in subsection 5.2.

5.1 Gravitini masses and supersymmetry

As we have seen, the isometry induced by the action of the non-geometrical automorphism $\hat{\sigma}_p$ generates a finite order subgroup conjugate to a subgroup of $[O(2) \times O(20 - \rho_p)] \times [O(2) \times O(\rho_p)]$. As far as gravitini masses are concerned, only the space-like subgroup $O(2) \times O(2) \subset O(4) \subset O(4,20)$ plays a role, where the first $O(2)$ factor acts as an order p rotation in the space-like two-plane in the vector space generated by $T(\sigma_p)$ while the second $O(2)$ factor acts as an order p rotation in the space-like two-plane in the vector space generated by $T(\sigma_p^T)$.[13]

[12]More explicitly, there exists a positive integer q such that the complex numbers $\{\exp i\theta_i, i = 1, \ldots, 12\}$ are given by the primitive p roots of unity, q times each.

[13]For instance, for the order three automorphism studied in example 3, each $O(2)$ generator comes from the $O(2,2;\mathbb{Z})$ generator given by eq. (2.31) after $O(2,2;\mathbb{R})$ conjugation.

The parts of the monodromies in $O(2) \times O(2) \subset O(4)$ transformations are specified by the angles θ_1, θ_2 and $\tilde{\theta}_1, \tilde{\theta}_2$. These are then

$$\theta_1 = \frac{2\pi}{p_1}, \qquad \theta_2 = \varepsilon_1 \frac{2\pi}{p_1} \tag{5.5}$$

and

$$\tilde{\theta}_1 = \frac{2\pi}{p_2}, \qquad \tilde{\theta}_2 = \varepsilon_2 \frac{2\pi}{p_2}\,. \tag{5.6}$$

Here $\varepsilon_i = 0$ if there is no discrete torsion. If there is discrete torsion, then $\varepsilon_i \in \{-1, 1\}$ corresponding to the two possible choices of discrete torsion for each cycle, as seen from the corresponding worldsheet description in subsection 3.2.

As explained in section 4, $\mathcal{N} = 2$ supersymmetry is preserved only if $\theta_1 = \theta_2$ and $\tilde{\theta}_1 = \tilde{\theta}_2$ or $\theta_1 = -\theta_2$ and $\tilde{\theta}_1 = -\tilde{\theta}_2$. This requires $\varepsilon_1 = \varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2 = -1$. Otherwise all supersymmetry is broken. Note that accidental supersymmetry from KK modes cannot arise here if $p_1 > 2$ or $p_2 > 2$.

We then draw the following conclusions which are in accord with the Gepner model description of the vacua that was obtained in [8] (see in particular around eq. (4.3) in [8]):

- For 'geometric' non-symplectic automorphisms of K3 surfaces, which have vanishing discrete torsion $\varepsilon_i = 0$, all gravitini become massive and so all the spacetime supersymmetry is broken.

- The two non-geometric twists preserve the same $\mathcal{N} = 2 \subset \mathcal{N} = 4$ space-time supersymmetry if $\varepsilon_1 = \varepsilon_2 = \pm 1$. Then two of the gravitini remain massless, while the other two acquire an equal mass. In the worldsheet description, $\varepsilon_1 = \varepsilon_2$ means that the same choice of discrete torsion was made for both of the corresponding Gepner model freely acting orbifolds, so that they both preserve space-time supercharges from either the left-movers or the right-movers.

- Whenever $\varepsilon_1 = \varepsilon_2 = \pm 1$ the isometry preserves $SO(3) \subset SO(4)$ and hence an $SU(2)_R \subset SU(4)_R$ of the space-time R-symmetry is preserved.

Then only the non-geometric twists with discrete torsion that pair the non-symplectic automorphisms with the corresponding automorphisms acting on the mirror K3 surfaces can be compatible with $\mathcal{N} = 2$ vacua in four dimensions.

To conclude, there is a perfect agreement between the gauged $\mathcal{N} = 4$ supergravity and the worldsheet construction. Note finally that the mass scale of the spontaneous supersymmetry breaking $\mathcal{N} = 4 \to \mathcal{N} = 2$ is set by the (inverse of the) volume of the two-torus [8] and can be taken to be much smaller than the string mass scale. Therefore it makes sense to analyse the model within the four-dimensional supergravity framework (while ten-dimensional supergravity would be inappropriate for these non-geometric constructions).

5.2 Moduli space

The mathematical formulation of the non-geometric automorphisms that we have given in section 3 provides precise predictions for the scalar manifolds arising from the moduli space of our models, parametrized by the vacuum expectation values of the accidental massless hypermultiplets discussed in subsection 4.2 from a supergravity viewpoint.

From the general construction of compactifications with duality twists [2], the minima of the effective four-dimensional potential (4.3) correspond to the intersection of the fixed-point loci of the two monodromies used in the reduction. Hence the remaining massless hypermultiplets, if any, correspond to K3 moduli that are invariant under both automorphisms $\hat{\sigma}_p$ used in a particular compactification.

For a given non-geometric automorphism $\hat{\sigma}_p$, these moduli arise first as deformations of the algebraic surface that preserve its p-cyclic form (2.22), *i.e.* such that the surface admits the action of the non-symplectic automorphism σ_p. The global structure of these moduli spaces was studied in [46] and can be summarized briefly as follows (for the details see [46]). The complex vector space $T(\sigma_p)^{\mathbb{C}} := T(\sigma_p) \otimes \mathbb{C}$, generated by the orthogonal complement of the invariant lattice $S(\sigma_p)$, admits a decomposition in terms of the eigenspaces of $\sigma_p^\star$, see lemma 1. One has from (2.21)

$$T(\sigma_p)^{\mathbb{C}} = T_{\zeta_p} \oplus T_{\zeta_p^2} \cdots \oplus T_{\zeta_p^{p-1}} \,. \tag{5.7}$$

Following [46], let us define

$$\mathcal{B}_p = \{ z \in \mathbb{P}(T_{\zeta_p}) \,,\; (z,z) = 0, (z,\bar{z}) > 0 \} \,, \tag{5.8}$$

where $\mathbb{P}(T_{\zeta_p})$ denotes the projective space associated with the complex vector space T_{ζ_p} and $(-,-)$ denotes the bilinear form on the K3 lattice $\Gamma_{3,19}$ that induces in a natural way a bilinear form on $T(\sigma_p)$.[14] We define also

$$\Gamma_p = \{ \gamma \in O(T(\sigma_p)) \,,\; \gamma \circ \sigma_p^\star = \sigma_p^\star \circ \gamma \} \,, \tag{5.9}$$

the subgroup of the isometry group $O(T(\sigma_p))$ commuting with the action of σ_p.

Then the K3 surface with non-symplectic automorphism σ_p has period (i.e. holomorphic two form $\omega(X)$) lying in the following space:

$$\mathcal{M}^p_{\text{CS},\text{Fix}} \cong \Gamma_p \backslash \mathcal{B}_p \,. \tag{5.10}$$

[14] Recall that given a K3 surface X, a *marking* is the choice of an isometry $\phi : H^2(X,\mathbb{Z}) \longrightarrow \Gamma_{3,19}$ and this extends in a natural way to $\phi_{\mathbb{C}} : H^2(X,\mathbb{C}) \longrightarrow \Gamma_{3,19} \otimes \mathbb{C}$. Then if $\omega(X)$ is the holomorphic 2-form we have $H^{2,0}(X) = \mathbb{C}\omega(X)$ and $\phi_{\mathbb{C}}(H^{2,0}(X)$ is a point of

$$\Omega_{K3} = \{ [\sigma] \in \mathbb{P}(\Gamma_{3,19} \otimes \mathbb{C}) | (\sigma,\sigma) = 0, (\sigma,\bar{\sigma}) > 0 \}$$

which is an open (analytic) subset in a 20-dimensional quadric of the 21-dimensional projective space $\mathbb{P}(\Gamma_{3,19} \otimes \mathbb{C})$. The point $\phi_{\mathbb{C}}(H^{2,0}(X)$ is called *period point of the marked K3 surface* and the moduli space of marked K3 surfaces is a quotient of Ω_{K3}.

For $p > 2$, $\mathcal{B}_p$ is of complex dimension $\text{rank}(T(\sigma_p))/(p-1) - 1$ and is isomorphic to a complex ball.[15] For $p = 2$ one gets a Hermitian symmetric space of complex dimension $\text{rank}(T(\sigma_p)) - 2$.

We now consider the non-geometric automorphism $\hat{\sigma}_p$ constructed from σ_p as defined in section 3. To understand its action on the CFT moduli space one has to look also at the mirror surface X_{W^T,G^T} which admits an action of the non-symplectic automorphism σ_p^T. In the same way as before, we define[16]

$$T(\sigma_p^T) \otimes \mathbb{C} = T^T_{\zeta_p} \oplus T^T_{\zeta_p^2} \cdots \oplus T^T_{\zeta_p^{p-1}}\,, \tag{5.11}$$

$$\mathcal{B}_p^T = \{z \in \mathbb{P}(T^T_{\zeta_p})\,,\ (z,z) = 0, (z,\bar{z}) > 0\}\,, \tag{5.12}$$

and

$$\Gamma_p^T = \{\gamma \in O(T(\sigma_p^T))\,,\ \gamma \circ (\sigma_p^T)^\star = (\sigma_p^T)^\star \circ \gamma\}\,. \tag{5.13}$$

The moduli space of K3 surfaces with non-symplectic action by σ_p^T is then given by $\Gamma_p^T \backslash \mathcal{B}_p^T$.

To summarize, by using the description of the period map for K3 surfaces, the K3 surface with non-symplectic automorphism σ_p has period in $\Gamma_p \backslash B_p$, and the mirror K3 surface has period in $\Gamma_p^T \backslash B_p^T$. By using the definition of $\hat{\sigma}_p$ (see Definition 3), we expect that the hypermultiplet moduli space of CFTs invariant under the action of the non-geometric automorphism $\hat{\sigma}_p$ is obtained by the direct product of these two spaces:

$$\hat{\mathcal{M}}^p_{\Sigma,\text{Fix}} \cong \Gamma_p \backslash \mathcal{B}_p \ \times\ \Gamma_p^T \backslash \mathcal{B}_p^T\,, \tag{5.14}$$

and is of complex dimension (recall that we are considering here $p \in \{3,5,7,13\}$):

$$\dim_{\mathbb{C}}\left(\hat{\mathcal{M}}^p_{\Sigma,\text{Fix}}\right) = \begin{cases} \frac{24}{p-1} - 2 & , \quad p > 2\,, \\ 20 & , \quad p = 2\,. \end{cases} \tag{5.15}$$

Interestingly, this dimension is the same for every automorphism σ_p of a given prime order p, regardless of the rank of the corresponding invariant lattice $S(\sigma_p)$. We have checked this result against some of the string theory spectra computed in [8] and found agreement.

With two monodromy twists associated with the two one-cycles of the two-torus, one should consider the intersection of the corresponding moduli spaces, which is easier to do case by case. For instance, for the self-mirror K3 surface (2.26) twisted

[15] For the self-mirror surface, for instance, local coordinates on the moduli space $\Gamma_p \backslash \mathcal{B}_p$ are given by the monomial deformations of (2.26) that are invariant under the action of σ_p. For the automorphism σ_3 one gets the monomials $\{y^n z^{42-6n}, n = 1,\ldots,5\}$, generating a space of complex dimension five.

[16] Recall that $S^{\Omega}(\sigma_p) \cong T(\sigma_p^T)$.

by the non-geometric monodromies $\hat{\sigma}_3$ and $\hat{\sigma}_7$, this intersection is just a point[17] and so there are no massless hypermultiplets in the low energy supergravity and we obtain just the $\mathcal{N} = 2$ STU supergravity model (provided that $\varepsilon_1 = \varepsilon_2$ so that the two automorphisms preserve the same half of the supersymmetry).

6 Conclusion

In this work we have constructed a new class of $\mathcal{N} = 2$ four-dimensional non-geometric compactifications of type IIA superstring theories, that consist of K3 fibrations over two-tori with non-geometric monodromies which lead in most cases to pure $\mathcal{N} = 2$ STU supergravity with no hypermultiplets at low energies.

The monodromies correspond to non-geometric automorphisms that we have obtained by pairing a non-symplectic automorphism of a K3 surface with a non-symplectic automorphism of the mirror surface. We have demonstrated that the action of such an automorphism can be lifted to an isometry of the lattice $\Gamma_{4,20}$, *i.e.* an element of the duality group $O(\Gamma_{4,20})$ of CFTs on K3 surfaces, and hence leads to a well-defined string theory compactification. We have shown that the fixed loci of these automorphisms are given by asymmetric Gepner model orbifolds, considered recently in [8]. The new understanding of these non-geometric backgrounds in terms of mirrored automorphisms should apply to non-geometric automorphisms of Calabi-Yau three-folds as well (except naturally the lattice-related aspects).

We have analysed the models from the four-dimensional $\mathcal{N} = 4$ gauged supergravity perspective valid at low energies. The matrices corresponding to the $\Gamma_{4,20}$ isometries that we have constructed provide directly the structure constants which parametrise the gauged supergravities obtained from twisted reductions of $K3 \times T^2$, and we have showed that the minima of the superpotential preserve $\mathcal{N} = 2$ supersymmetry in four dimensions, as expected from the string theory constructions of such vacua. In some cases, hypermultiplets remain massless in the four dimensional theories; we have obtained, using results from the mathematical literature, the corresponding hypermultiplet moduli space, whose dimension agrees with the string theory predictions.

We plan to provide more details on the four-dimensional gauged supergravity construction in a companion paper. In particular we will analyse the scalar manifold of the low-energy theory in order to show explicitly that the hypermultiplet moduli space predictions from algebraic geometry are verified, and check that all the consistency conditions of gauged supergravity are met for these particular gaugings.

[17] In fact the moduli space of K3 surfaces with an action by σ_7 is one-dimensional, and the K3 surfaces of the family carry an elliptic fibration [29, Example 6.1, 1)]. One can check that only one K3 surface of the family admits also a non-symplectic automorphism of order three, this is then the K3 surface in example (2.26). This explains why the intersection is only one point.

The duality between the type IIA string theory compactified on K3 and the heterotic string compactified on T^4 [24] gives a heterotic dual to our constructions consisting of a toroidal reduction of the heterotic string with monodromy twists, that gives an asymmetric orbifold construction at fixed points; such models were introduced in [2]. Particular examples of heterotic asymmetric orbifolds are given by CHL compactifications [47]; the latter correspond, on the type IIA side, to symplectic automorphisms of $K3$ surfaces. Here, algebraic geometry leads us to a particularly interesting class of constructions that correspond to non-symplectic $K3$ automorphisms on the type IIA side and preserve $\mathcal{N} = 2$ supersymmetry. The corresponding heterotic orbifolds will be discussed further in a forthcoming publication.

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7 Appendix: Explicit lattice computations

To get the action of the isometry on the lattice E_8, respectively E_6, with the standard bilinear form E_i^{st}, $i = 6, 8$ as given in Examples 3, 4 we use the fact that thanks to the automorphism σ_3 of order three these two lattices have the structure of a $\mathbb{Z}[\zeta]$-module, where ζ is a primitive third root of unity (recall that the ring $\mathbb{Z}[\zeta]$ is called the *ring of Eisenstein integers*). Lattices with this property are very much investigated in number theory, see [48] for a precise introduction of the basic tools needed in this section and a general introduction on the subject.

Recall that the multiplication of an element $a + b\zeta \in \mathbb{Z}[\zeta]$ with an element x in the lattice, is defined as

$$(a + \zeta b) \cdot x := ax + b\sigma_3^\star(x).$$

Since $\sigma_3^\star$ by construction does not fix any vector on E_8, resp. E_6, and $\mathbb{Z}[\zeta]$ is a principal ideal domain then these two lattices are free over $\mathbb{Z}[\zeta]$ of rank 4, respectively 3.

Let e_1, e_2, e_3, e_4 be a set of generators of the $\mathbb{Z}[\zeta]$-module E_8 so that $E_8 = \mathbb{Z}[\zeta]e_1 \oplus \mathbb{Z}[\zeta]e_2 \oplus \mathbb{Z}[\zeta]e_3 \oplus \mathbb{Z}[\zeta]e_4$ and clearly $\mathbb{B}_8 = \{e_1, e_2, e_3, e_4, \zeta e_1, \zeta e_2, \zeta e_3, \zeta e_4\}$ is a set of generators of E_8 as an integral lattice. One can consider a similar set of generators for E_6 as $\mathbb{Z}[\zeta]$-module and a set of generators $\mathbb{B}_6$ for E_6 as a lattice over the integers. Following [49, Chapter 1], consider the hermitian forms on E_8, respectively E_6 (as $\mathbb{Z}[\zeta]$-lattices)

$$h_{E_8} = \begin{pmatrix} 3 & \theta & 0 & 0 \\ \bar\theta & 3 & \theta & 0 \\ 0 & \bar\theta & 3 & \theta \\ 0 & 0 & \bar\theta & 3 \end{pmatrix}, \qquad h_{E_6} = \begin{pmatrix} 3 & \theta & 0 \\ \bar\theta & 3 & \theta \\ 0 & \bar\theta & 3 \end{pmatrix}$$

where $\theta = \zeta - \bar\zeta$. One can then define a bilinear form

$$b_{E_i}(\alpha, \beta) := -\frac{1}{3}(h_{E_i}(\alpha, \beta) + \rho(h_{E_i}(\alpha, \beta))),\ (\alpha, \beta) \in E_i \times E_i$$

on the lattices E_i with the set of generators $\mathbb{B}_i$, $i = 6, 8$, where ρ denotes the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\zeta)$ that sends ζ to $\bar\zeta$ (see [32]). Observe that the element (α, β) is considered on the left hand side as an element of the rank i integral lattice and on the right hand side as an element of the rank $i/2$ module over $\mathbb{Z}[\zeta]$.

With the help of MAGMA one can find a base change matrix T_{E_i} with integer coefficients such that

$$E_i^{st} = T_{E_i}^t b_{E_i} T_{E_i}, \qquad i = 6, 8.$$

where recall that E_i^{st} is the standard lattice metric as given in the Examples 3, 4 and by abuse of notation we identify the bilinear form b_{E_i} with its symmetric $i \times i$ matrix. The action of the isometry in the above given set of generators $\mathbb{B}_i$ of E_i, $i = 6, 8$ (as an integral lattice) is easy to write, since this is a block matrix with 4, respectively 3, blocks of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

We call these two matrices H_{E_i}, $i = 6, 8$. Then the action of the isometry in the set of generators with the standard lattice metric is given by

$$J_3^{E_i} = T_{E_i}^{-1} H_{E_i} T_{E_i}, \qquad i = 6, 8.$$

These are the matrices given in the equations (2.33), (2.37).

In a similar way one can compute the action of the automorphism $\sigma_3^\star$ on A_2 and on $U \oplus U$. In this case the matrices of the hermitian forms have a very easy form

$$h_{A_2} = (3) \qquad h_{U\oplus U} = \begin{pmatrix} 0 & \theta \\ \bar\theta & 0 \end{pmatrix}.$$

With the same notation as above we find that in these two cases $T_{A_2} = T_{U\oplus U} = id$, which is not necessarily the case in general (e.g. one can check in the previous computation that $T_{E_i} \neq id$ since $b_{E_i} \neq E_i^{st}$, for $i = 6, 8$).

As an example we do the explicit computations for A_2. Here recall

$$b_{A_2}(\alpha,\beta) := -\frac{1}{3}(h_{A_2}(\alpha,\beta) + \rho(h_{A_2}(\alpha,\beta))),\ (\alpha,\beta) \in A_2 \times A_2.$$

As a $\mathbb{Z}[\zeta]$-lattice we have that $A_2 = \mathbb{Z}[\zeta]e_1$ for a generator e_1 so that $\{e_1, \zeta e_1\}$ is a basis of A_2 as an integral lattice. Now with $h_{A_2} = (3)$ we have

$$\begin{aligned} b_{A_2}(e_1,e_1) = -\frac{1}{3}(h_{A_2}(e_1,e_1) + \rho(h_{A_2}(e_1,e_1))) = -\frac{1}{3}(3+3) = -2,\\ b_{A_2}(\zeta e_1, \zeta e_1) = -\frac{1}{3}(\zeta 3\bar{\zeta} + \bar{\zeta}3\zeta) = -2,\\ b_{A_2}(e_1,\zeta e_1) = b_{A_2}(\zeta e_1, e_1) = -\frac{1}{3}(3\zeta + 3\bar{\zeta}) = 1, \end{aligned}$$

and we get $b_{A_2} = A_2^{st}$. Now the action of the automorphism $\sigma_3^\star$ on the basis $e := e_1$, $f := \zeta e_1$ of the integral lattice A_2 corresponds by definition of the structure of $\mathbb{Z}[\zeta]$-module to the multiplication by ζ. So we have that the automorphism sends e to f and since $\zeta^2 e_1 = -e_1 - \zeta e_1$ we get that the image of f is $-e-f$ as given in Example 4. In the case of E_i, $i = 6, 8$ we determine with MAGMA a matrix T_{E_i} that changes the basis $\mathbb{B}_i$ to the basis for the standard action and we use then this matrix to get the action of the automorphism on the basis of the standard action.

Observe that one could use a similar method to determine the action of the automorphism $\sigma_7^\star$ of Example 3 on the lattice $U \oplus U \oplus E_8$. This is a $\mathbb{Z}[\zeta_7]$-module of rank 2 (ζ_7 denotes a primitive seventh root of unity) but we do not know the explicit matrix of the hermitian form $h_{U\oplus U\oplus E_8}$ which is a 2×2-hermitian matrix (to determine such forms is in general a difficult problem; see [48]). As seen above this would allow us to find the matrix of the base change $T_{U\oplus U\oplus E_8}$ that can then be used to give the action of $\sigma_7^\star$ on $U \oplus U \oplus E_8$ with the standard bilinear form.

References

[1] C. M. Hull, *A Geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065, [hep-th/0406102].

[2] A. Dabholkar and C. Hull, *Duality twists, orbifolds, and fluxes*, *JHEP* **09** (2003) 054, [hep-th/0210209].

[3] S. Kawai and Y. Sugawara, *Mirrorfolds with K3 fibrations*, *JHEP* **02** (2008) 065, [arXiv:0711.1045].

[4] M. J. Duff, J. T. Liu, and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys.* **B459** (1996) 125–159, [hep-th/9508094].

[5] J. Scherk and J. H. Schwarz, *How to Get Masses from Extra Dimensions*, *Nucl. Phys.* **B153** (1979) 61–88.

[6] N. Seiberg, *Observations on the Moduli Space of Superconformal Field Theories*, *Nucl. Phys.* **B303** (1988) 286–304.

[7] P. S. Aspinwall and D. R. Morrison, *String theory on K3 surfaces*, hep-th/9404151.

[8] D. Israël and V. Thiéry, *Asymmetric Gepner models in type II*, *JHEP* **02** (2014) 011, [arXiv:1310.4116].

[9] D. Israël, *Nongeometric Calabi-Yau compactifications and fractional mirror symmetry*, *Phys. Rev.* **D91** (2015) 066005, [arXiv:1503.0155]. [Erratum: Phys. Rev.D91,no.12,129902(2015)].

[10] A. N. Schellekens and S. Yankielowicz, *New Modular Invariants for $N = 2$ Tensor Products and Four-dimensional Strings*, *Nucl. Phys.* **B330** (1990) 103–123.

[11] K. A. Intriligator and C. Vafa, *Landau-Ginzburg orbifolds*, *Nucl. Phys.* **B339** (1990) 95–120.

[12] R. Blumenhagen, M. Fuchs, and E. Plauschinn, *Partial SUSY Breaking for Asymmetric Gepner Models and Non-geometric Flux Vacua*, *JHEP* **01** (2017) 105, [arXiv:1608.0059].

[13] R. Blumenhagen, M. Fuchs, and E. Plauschinn, *The Asymmetric CFT Landscape in D=4,6,8 with Extended Supersymmetry*, *Fortsch. Phys.* **65** (2017), no. 3-4 1700006, [arXiv:1611.0461].

[14] H. Pinkham, *Singularités exceptionnelles, la dualité étrange d'Arnold et les surfaces $K-3$*, *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), no. 11 A615–A618.

[15] I. Dolgachev and V. V. Nikulin, *Exceptional singularities of v. i. arnold and k3 surfaces*, *Proc. USSR Topological Conference in Minsk, 1977.*

[16] I. Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, in *Bourbaki seminar, Vol. 1982/83*, vol. 105 of *Astérisque*, pp. 251–278. Soc. Math. France, Paris, 1983.

[17] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, *Math. USSR Izv.* **14** (1980) 103–167.

[18] I. V. Dolgachev, *Mirror symmetry for lattice polarized k3 surfaces*, *Journal of Mathematical Sciences* **81** (1996), no. 3 2599–2630, [alg-geom/9502005].

[19] P. Berglund and T. Hubsch, *A Generalized construction of mirror manifolds*, *Nucl. Phys.* **B393** (1993) 377–391, [hep-th/9201014]. [AMS/IP Stud. Adv. Math.9,327(1998)].

[20] B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau Moduli Space*, *Nucl. Phys.* **B338** (1990) 15–37.

[21] V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces.*, *Trans. Mosc. Math. Soc.* **2** (1980) 71–135.

[22] M. Artebani, S. Boissière, and A. Sarti, *The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces*, *Journal de Mathematiques Pures et Appliquees* **102** (2014), no. 4 758 – 781.

[23] P. Comparin, C. Lyons, N. Priddis, and R. Suggs, *The mirror symmetry of K3 surfaces with non-symplectic automorphisms of prime order*, *Adv. Theor. Math. Phys.* **18** (12, 2014) 1335–1368.

[24] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B438** (1995) 109–137, [hep-th/9410167].

[25] D. Gepner, *Space-Time Supersymmetry in Compactified String Theory and Superconformal Models*, *Nucl. Phys.* **B296** (1988) 757.

[26] M. Krawitz, N. Priddis, P. Acosta, N. Bergin, and H. Rathnakumara, *FJRW-Rings and Mirror Symmetry*, *Communications in Mathematical Physics* **296** (2010), no. 1 145–174.

[27] A. Chiodo and Y. Ruan, *LG/CY correspondence: The state space isomorphism*, *Advances in Mathematics* **227** (2011), no. 6 2157 – 2188.

[28] S.-M. Belcastro, *Picard lattices of families of k3 surfaces*, *Communications in Algebra* **30** (2002), no. 1 61–82, [http://dx.doi.org/10.1081/AGB-120006479].

[29] M. Artebani, A. Sarti, and S. Taki, *$K3$ surfaces with non-symplectic automorphisms of prime order*, *Math. Z.* **268** (2011), no. 1-2 507–533. With an appendix by Shigeyuki Kondō.

[30] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, vol. 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[31] P. Comparin and N. Priddis, *Equivalence of mirror constructions for $K3$ surfaces with non-symplectic automorphism*, preprint 2017, arXiv:1704.00354.

[32] J. H. Conway, N. J. A. Sloane, and E. Bannai, *Sphere-packings, Lattices, and Groups*. Springer-Verlag New York, Inc., New York, NY, USA, 1987.

[33] C. Vafa, *String Vacua and Orbifoldized L-G Models*, *Mod. Phys. Lett.* **A4** (1989) 1169.

[34] C. Vafa, *Quantum Symmetries of String Vacua*, *Mod. Phys. Lett.* **A4** (1989) 1615.

[35] E. Witten, *On the Landau-Ginzburg description of N=2 minimal models*, *Int. J. Mod. Phys.* **A9** (1994) 4783–4800, [hep-th/9304026].

[36] E. Witten, *Phases of N=2 theories in two-dimensions*, *Nucl. Phys.* **B403** (1993) 159–222, [hep-th/9301042]. [AMS/IP Stud. Adv. Math.1,143(1996)].

[37] K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

[38] C. M. Hull and R. A. Reid-Edwards, *Flux compactifications of string theory on twisted tori*, *Fortsch. Phys.* **57** (2009) 862–894, [hep-th/0503114].

[39] N. Kaloper and R. C. Myers, *The Odd story of massive supergravity*, *JHEP* **05** (1999) 010, [hep-th/9901045].

[40] M. Porrati and F. Zwirner, *Supersymmetry Breaking in String Derived Supergravities*, *Nucl. Phys.* **B326** (1989) 162–184.

[41] M. Weidner, *Gauged supergravities in various spacetime dimensions*, *Fortsch. Phys.* **55** (2007) 843–945, [hep-th/0702084].

[42] J. Schon and M. Weidner, *Gauged N=4 supergravities*, *JHEP* **05** (2006) 034, [hep-th/0602024].

[43] B. de Wit, H. Samtleben, and M. Trigiante, *The Maximal D=4 supergravities*, *JHEP* **06** (2007) 049, [arXiv:0705.2101].

[44] R. A. Reid-Edwards and B. Spanjaard, *N=4 Gauged Supergravity from Duality-Twist Compactifications of String Theory*, *JHEP* **12** (2008) 052, [arXiv:0810.4699].

[45] C. Horst, J. Louis, and P. Smyth, *Electrically gauged N=4 supergravities in D=4 with N=2 vacua*, *JHEP* **03** (2013) 144, [arXiv:1212.4707].

[46] I. V. Dolgachev and S. Kondō, *Moduli of K3 Surfaces and Complex Ball Quotients*, pp. 43–100. Birkhäuser Basel, Basel, 2007.

[47] S. Chaudhuri, G. Hockney, and J. D. Lykken, *Maximally supersymmetric string theories in D ¡ 10*, *Phys. Rev. Lett.* **75** (1995) 2264–2267, [hep-th/9505054].

[48] E. Bayer-Fluckiger, *Lattices and number fields*, in *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, vol. 241 of *Contemp. Math.*, pp. 69–84. Amer. Math. Soc., Providence, RI, 1999.

[49] D. Allcock, J. A. Carlson, and D. Toledo, *The moduli space of cubic threefolds as a ball quotient*, *Mem. Amer. Math. Soc.* **209** (2011), no. 985 xii+70.