Non-symplectic automorphisms of K3 Surfaces

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- Intro
- K3 surfaces determined by non-symplectic automorphisms, esp. order 16
- Invariant lattices of purely non-symplectic automorphisms

K3 Surfaces

Definition

A K3 surface is a compact, complex surface S such that

- The irregularity $q(S) = h^0(\Omega_S^1) = 0$,
- There exists a unique nonvanishing global 2-form ω_S, so that $H^0(\Omega_S^2) = \mathbb{C}\omega_S$ (so geometric genus $p_g(S) = h^0(\Omega_S^2) = 1$).

Automorphisms

An automorphism is a biholomorphic map $\sigma : S \to S$

Definition

An automorphism $\sigma : S \to S$ is

- symplectic if $\sigma^* \omega_S = \omega_S$
- non-symplectic if $\sigma^* \omega_S = \lambda \omega_S$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$
- purely nonsymplectic if σ has order n and $\sigma^* \omega_S = \zeta_n \omega_S$, where ζ_n is a primitive n-th root of unity.

Lattices

Definition

A lattice L is a free abelian group of finite rank, together with a non–degenerate symmetric bilinear form

$$B : L \times L \to \mathbb{Z}.$$

A lattice is even if $B(x, x) \in 2\mathbb{Z}$ for each $x \in L$. The signature (t_+, t_-) of a lattice is the signature of B.
An embedding $M \hookrightarrow L$ is primitive if L/M is a free group.

Definition

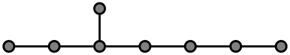
The discriminant group $A_L := Hom(L, \mathbb{Z})/L$ is a finite group.
L is unimodular if A_L is trivial.

The K3 lattice

For K3 surfaces, $H^2(S,\mathbb{Z}) \cong U^3 \oplus (E_8)^2$ via the cup product (an even unimodular lattice of signature $(3, 19)$.

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

E_8 is the unique negative definite lattice of rank 8 associated to the root lattice:



Sublattices

There two important sublattices:

1. The invariant lattice $S(\sigma) = \{x \in H^2(S, \mathbb{Z}) \mid \sigma^* x = x\}$.
   If σ is purely non-symplectic then $S(\sigma) \subseteq \mathrm{NS}(S)$.

2. The transcendental lattice $T_S = \mathrm{NS}(S)^\perp$.
   If σ is purely non-symplectic then $T_S \subseteq S(\sigma)^\perp$

Here $^\perp$ refers to the orthogonal complement in $H^2(S, \mathbb{Z})$.

Brandhorst results

Fact: If σ is purely non-symplectic of order n, then $\varphi(n) \mid \operatorname{rk} T_S$.

Theorem (Brandhorst)

Let S be a K3 surface and σ a purely nonsymplectic automorphism of order n such that $\operatorname{rk} T_S = \varphi(n)$ and ζ_n is not an eigenvalue of $\sigma|_{\mathrm{NS}\otimes\mathbb{C}}$. Set $d = |\det \mathrm{NS}|$, then S is determined up to isomorphism by the pair (n, d).

If $\varphi(n) \geq 12$, then σ can also be uniquely determined:

n	$\det T$	S
$13, 26$	13	$y^2 = x^3 + t^5 x + t$
26	13	$y^2 = x^3 + t^7 x + t^4$
$21, 42$	1	$y^2 = x^3 + t^5(t^7 - 1)$
$21, 42$	7^2	$y^2 = x^3 + 4t^4(t^7 - 1)$
$21, 42$	7^2	$y^2 = x^3 + t^3(t^7 + 1)$
21	7^2	$x_0^3 x_1 + x_1^3 x_2 + x_0 x_2^3 - x_0 x_3^3$
28	1	$y^2 = x^3 + x + t^7$
	2^6	$y^2 = x^3 + (t^7 + 1)x$
	2^6	$y^2 = x^3 + (t^7 + 1)x$
$17, 34$	17	$y^2 = x^3 + t^7 x + t^2$
34	17	$x_0 x_1^5 + x_0^5 x_2 + x_1^2 x_2^4 = y^2$
32	2^2	$y^2 = x^3 + t^2 x + t^{11}$
	2^4	$y^2 = x_0(x_1^5 + x_0^4 x_2 + x_1 x_2^4)$
36	1	$y^2 = x^3 - t^5(t^6 - 1)$
	3^4	$y^2 = x^3 + x + t^9$
	$2^6 3^2$	$x_0 x_3^3 + x_0^3 x_1 + x_1^4 + x_2^4$
40	2^4	$z^2 = x_0(x_0^4 x_2 + x_1^5 - x_2^5)$
48	2^2	$y^2 = x^3 + t(t^8 - 1)$
$19, 38$	19	$y^2 = x^3 + t^7 x + t$
38	19	$y^2 = x_0^5 x_1 + x_0 x_1^4 x_2 + x_2^6$
$27, 54$	3	$y^2 = x^3 + t(t^9 - 1)$
27	3^3	$x_0 x_3^3 + x_0^3 x_1 + x_2(x_1^3 - x_2^3)$
$25, 50$	5	$z^2 = (x_0^6 + x_0 x_1^5 + x_1 x_2^5)$
$33, 66$	1	$y^2 = x^3 + t(t^{11} - 1)$
44	1	$y^2 = x^3 + x + t^{11}$

Brandhorst results

If $\varphi(n) \leq 10$, then σ is not necessarily determined.

n	$\det T$	X
$3, 6$	3	$y^2 = x^3 - t^5(t-1)^5(t+1)^2$
4	2^2	$y^2 = x^3 + 3t^4x + t^5(t^2-1)$
$5, 10$	5	$y^2 = x^3 + t^3x + t^7$
8	2^2	$y^2 = x^3 + tx^2 + t^7$
	2^4	$t^4 = (x_0^2 - x_1^2)(x_0^2 + x_1^2 + x_2^2)$
12	1	$y^2 = x^3 + t^5(t^2-1)$
	$2^2 3^2$	$y^2 = x^3 + t^5(t^2-1)^2$
	2^4	$y^2 = x^3 + t^5(t^2-1)^3$
$7, 14$	7	$y^2 = x^3 + t^3x + t^8$
$9, 18$	3	$y^2 = x^3 + t^5(t^3-1)$
	3^3	$y^2 = x^3 + t^5(t^3-1)^2$
16	2^2	$y^2 = x^3 + t^2x + t^7$
	2^4	$y^2 = x^3 + t^3(t^4-1)x$
	2^6	$y^2 = x^3 + x + t^8$
20	2^4	$y^2 = x^3 + (t^5-1)x$
	$2^4 5^2$	$y^2 = x^3 + 4t^2(t^5+1)x$
24	2^2	$y^2 = x^3 + t^5(t^4+1)$
	2^6	$y^2 = x^3 + (t^8+1)$
	$2^2 3^4$	$y^2 = x^3 + t^3(t^4+1)^2$
	$2^6 3^4$	$y^2 = x^3 + x + t^{12}$
$15, 30$	5^2	$y^2 = x^3 + 4t^5(t^5+1)$
	3^4	$y^2 = x^3 + t^5x + 1$
$11, 22$	11	$y^2 = x^3 + t^5x + t^2$

For $(n,d) = (16, 2^2), (24, 2^2)$ we have $\mathrm{Aut}(S) \cong \mathbb{Z}/n\mathbb{Z}$.
For $(n,d) = (12, 1), (16, 2^4), (20, 2^4)$ we have $\mathrm{Aut}(S) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.
All others have infinite automorphism group.

Order 16 surfaces

For order 16, these were also classified by al Tabbaa and Sarti.
$r = \operatorname{rk} S(\sigma)$, N_σ: number of isolated fixed points,
k_σ: number of fixed rational curves.

Theorem (al Tabbaa, Sarti)

Let S be a K3 surface and σ be a purely non-symplectic automorphism of order 16 on S. Assume that $\mathrm{NS} = S(\sigma^8)$ has rank 14 (i.e. $\operatorname{rk} T_S = \varphi(16)$). Then one of the following distinct cases holds:

- *There is a curve C of genus $g(C) = 2$ or 3 in the fixed locus of σ^8.*
- *There is an elliptic curve C in the fixed locus of σ^4.*

r	N_σ	k_σ	$g(C)$	NS
13	12	1	3	$U \oplus D_4 \oplus E_8$
11	10	1	2	$U(2) \oplus D_4 \oplus E_8$
7	4	0	2	$U(2) \oplus D_4 \oplus E_8$
9	8	1	1	$(U \oplus D_4^3)$
7	6	0	1	$(U \oplus D_4^3)$

An elliptic fibration

Definition

An elliptic fibration on S is a morphism $\pi : S \to \mathbb{P}^1$ such that for generic $t \in \mathbb{P}^1$, the fiber $\pi^{-1}(t)$ is a smooth elliptic curve.
An elliptic fibration $\pi : S \to \mathbb{P}^1$ is Jacobian if it admits a section $s : \mathbb{P}^1 \to S$.

Proposition

If $\pi : S \to \mathbb{P}^1$ is an elliptic K3 surface, then π has a positive finite number of singular fibers.

Fact: The volume form of S is $\omega_S = \frac{dt \wedge dx}{2y}$.

Order 16 surfaces

There are 3 surfaces S with:

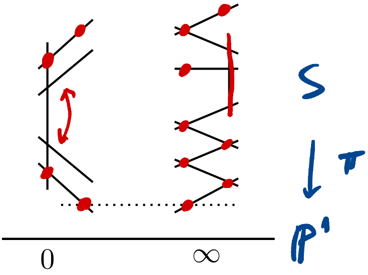
- S has a purely non-symplectic automorphism of order 16.
- $\operatorname{rk} T_S = 8$
- $\det|\,\mathrm{NS}\,| = |A_{N_S}|$ is one of $2^2, 2^4, 2^6$.

r	N_σ	k_σ	$g(C)$	NS
13	12	1	3	$U \oplus D_4 \oplus E_8$
11	10	1	2	$U(2) \oplus D_4 \oplus E_8$
7	4	0	2	$U(2) \oplus D_4 \oplus E_8$
9	8	1	1	$(U \oplus D_4^3)$
7	6	0	1	$(U \oplus D_4^3)$

16	2^2	$y^2 = x^3 + t^2x + t^7$	$(\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^{10} t)$
	2^4	$y^2 = x^3 + t^3(t^4 - 1)x$	$(\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$
	2^6	$y^2 = x^3 + x + t^8$	$(-x, iy, \zeta_{16} t)$

Order 16 surfaces

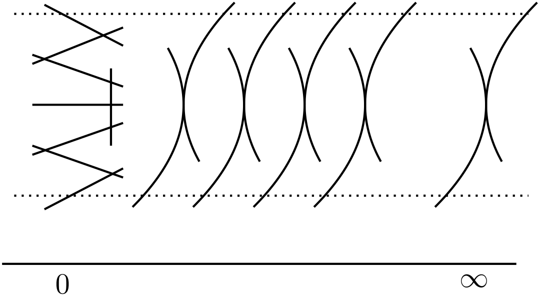
1. Equation: $y^2 = x^3 + t^2x + t^7$
2. Singular fibers: Type I_0^* over $t = 0$ and type II^* fiber over $t = \infty$.



3. $\mathrm{Aut}(S) \cong \mathbb{Z}/16\mathbb{Z}$.
4. $\sigma : (x, y, t) \to (\xi_{16}^2 x, \xi_{16}^{11} y, \xi_{16}^{10} t)$.

Order 16 surfaces

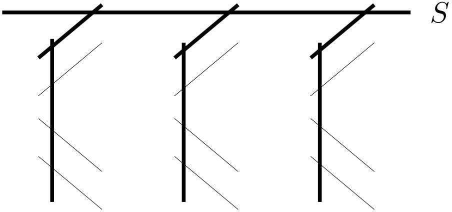
1. Equation: $y^2 = x^3 + t^3(t^4 - 1)x$
2. Singular fibers: Type III^* over $t = 0$ and five fibers of type III, one of them over $t = \infty$.



3. $\mathrm{Aut}(S) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$.
4. $\sigma_1 : (x, y, t) \to (\xi_{16}^6 x, \xi_{16}^9 y, \xi_{16}^4 t)$
   $\sigma_2 : (x, y, t) \to (\xi_{16}^6 \frac{y^2 - x^3}{x^2}, \xi_{16}^9 \frac{x^3 y - y^3}{x^3}, \xi_{16}^4 t)$

Order 16 surfaces

1. Equation: $y^2 = x^3 + x + t^8$
2. Singular fibers: Type IV^* over $t = \infty$ and 16 fibers of type I_0 (nodal rational curve)



α

3. $\mathrm{Aut}(S)$ is infinite
4. $\sigma_1 : (x, y, t) \to (-x, iy, \xi_{16}^{13} t)$
   $\sigma_2 : (x, y, t) \to (-x, -iy, \xi_{16}^{5} t)$.

What about the last surface?

Let S denote this last surface.
$\sigma_1 : (x, y, t) \to (-x, iy, \xi_{16}^{13} t)$
$\sigma_2 : (x, y, t) \to (-x, -iy, \xi_{16}^{5} t)$.

Lemma (Karayayla, Comparin, Priddis, Sarti)

$\mathrm{Aut}(S) \cong \mathrm{MW}(S) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z})$.

Definition

We say that (S, σ) is isomorphic to (X, γ) if there exists an isomorphism $f : S \longrightarrow X$ such that $\sigma = f^{-1}\gamma f$.

Theorem (Comparin, Priddis, Sarti)

If $\sigma \in \mathrm{Aut}(S)$ is purely non-symplectic of order 16, then σ is conjugate to some power of σ_1 or σ_2.

Mirror Symmetry

Why invariant lattices?
Among other things, invariant lattices seem to be very important for mirror symmetry for K3 surfaces.

Mirror symmetry for K3 surfaces

Definition

Let M be a lattice of signature $(1,t)$, $t \leq 18$. If there exists a primitive embedding $j : M \hookrightarrow \mathrm{NS}(S)$, we call S an M-polarizable K3 surface

Definition

Let M be a primitive sublattice of L_{K3} of signature $(1,t)$ with $t \leq 18$ such that $(M)^{\perp}_{L_{K3}} \cong U \oplus M^{\vee}$. We define $M^{\vee}$ (up to isometry) to be the *mirror lattice* to M. Given an M-polarizable K3 surface X and an M'-polarizable K3 surface X' with $M' = M^{\vee}$ (or equivalently $M = (M')^{\vee}$), we say X and X' are *LPK3 mirror K3 surfaces*.

Mirror symmetry for K3 surfaces

We can construct a K3 surface $X_{W,G}$ from a pair (W, G), where W is a quasihomogeneous polynomial with respect to one of 95 K3 weight systems, and G is a group of symmetries.

Theorem

Let W be an invertible polynomial of the form $W = x_0^n + f(x_1, x_2, x_3)$ quasihomogeneous with respect to one of Reid and Yonemura's 95 weight systems, and G a group of symmetries of W satisfying $J_W \leq G \leq \mathrm{SL}_W$. Then the K3 surface $X_{W,G}$, polarized by $S(\sigma_n)$, and its BHK mirror $X_{W^\vee,G^\vee}$ polarized by $S(\sigma_n^\vee)$, form an LPK3 mirror pair.

Order 16 surfaces

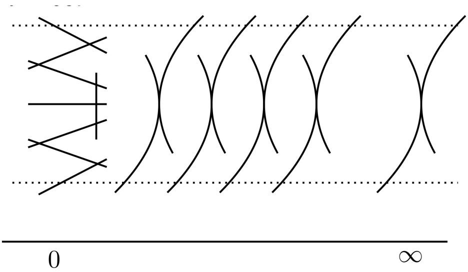
r	N_σ	k_σ	$g(C)$	NS
13	12	1	3	$U \oplus D_4 \oplus E_8$
11	10	1	2	$U(2) \oplus D_4 \oplus E_8$
7	4	0	2	$U(2) \oplus D_4 \oplus E_8$
9	8	1	1	$U \oplus D_4^3$
7	6	0	1	$U \oplus D_4^3$

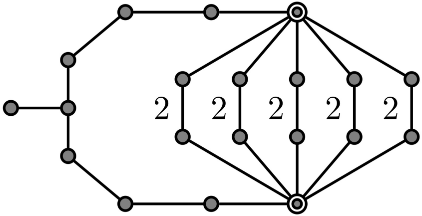
r	$S(\gamma)$
13	$U \oplus E_8 \oplus A_3$
11	$T_{2,5,6}$
7	$U(2) \oplus D_4 \oplus \langle -8 \rangle$

r	$S(\gamma)$
9	$T_{3,4,4}$
7	$U \oplus D_4 \oplus \langle -8 \rangle$

Example

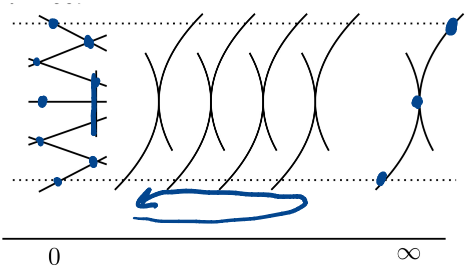
Equation: $y^2 = x^3 + t^3(t^4 - 1)x$

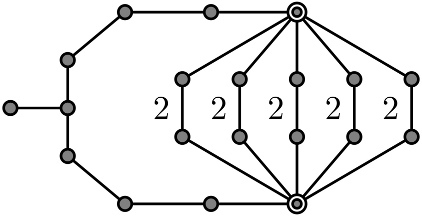


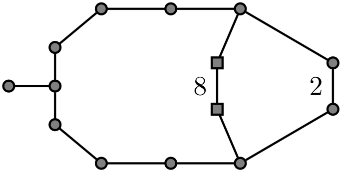


Example: $\sigma_1 : (x, y, t) \to (\xi_{16}^6 x, \xi_{16}^9 y, \xi_{16}^4 t)$

r	N_σ	k_σ	$g(C)$	NS
11	10	1	2	$U(2) \oplus D_4 \oplus E_8$

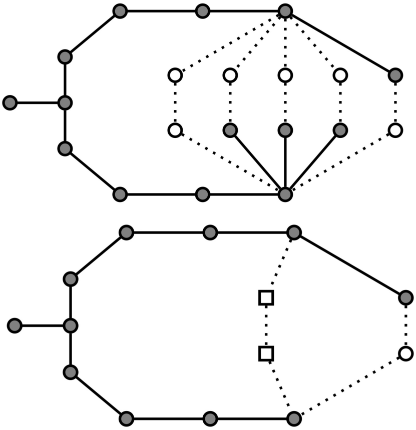






Example $\sigma_1 : (x, y, t) \to (\xi_{16}^6 x, \xi_{16}^9 y, \xi_{16}^4 t)$

r	N_σ	k_σ	$g(C)$	NS
11	10	1	2	$U(2) \oplus D_4 \oplus E_8$



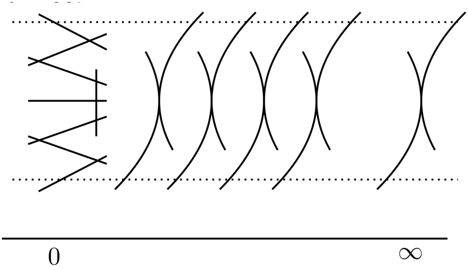
$T_{2,5,6}$

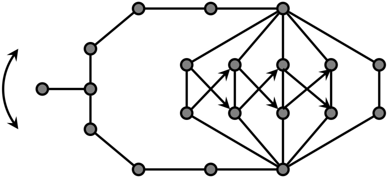
An embedding $M \hookrightarrow L$ is primitive if L/M is a free group.

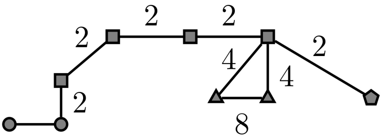
Example

$\sigma_2 : (x,y,t) \to (\xi_{16}^6 \frac{y^2-x^3}{x^2}, \xi_{16}^9 \frac{x^3y-y^3}{x^3}, \xi_{16}^4 t)$

r	N_σ	k_σ	$g(C)$	NS
7	4	0	2	$U(2) \oplus D_4 \oplus E_8$







■ = self-intersection -4

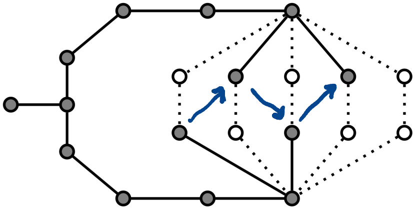
▲ = self-intersection -8

⬟ = self-intersection 0

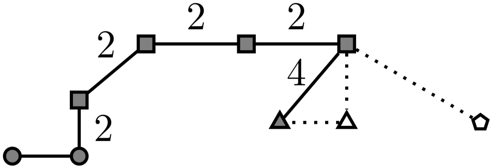
Example

$$\sigma_2 : (x, y, t) \to (\xi_{16}^6 \frac{y^2 - x^3}{x^2}, \xi_{16}^9 \frac{x^3 y - y^3}{x^3}, \xi_{16}^4 t)$$

r	N_σ	k_σ	$g(C)$	NS
7	4	0	2	$U(2) \oplus D_4 \oplus E_8$



$U(2) \oplus D_4 \oplus \langle -8 \rangle$.



Thank you for your attention!