A cone conjecture for log Calabi-Yau Surfaces

Jennifer Li

Second ECOS-ANID workshop in Algebraic Geometry

July 19, 2021

A Cone Conjecture for log Calabi-Yau Surfaces

Morrison '93 (Conjecture)

X: Calabi-Yau 3-fold $\begin{cases} X : \mathbb{C}, \text{ proj. var. of dim } 3 \\ K_X = 0 \\ \pi_1(X) = 0 \end{cases}$

$\mathrm{Nef}(X) \subset H^2(X, \mathbb{R})$

Then

$\mathrm{Aut}(X) \curvearrowright \mathrm{Nef}(X)$ with a rational polyhedral fundamental domain.
RPFD

* In particular: $\mathrm{Aut}(X)$ acts with finitely many orbits on faces of $\mathrm{Nef}(X)$.

Motivation : Mirror symmetry

Assuming Morrison's conjecture

By a construction of Looijenga $\rightsquigarrow$ $\underbrace{\text{a complex analytic space}}$

$\| \quad$ Conjecture

a neighborhood of a boundary point of the moduli space of the mirror Calabi-Yau 3-fold.

Thesis

We prove a version of Morrison's conjecture for log Calabi-Yau surfaces.

A log Calabi-Yau surface is a pair (Y, D):

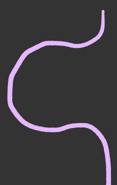
$\begin{cases} Y: \text{smooth, complex, projective surface} \\ D \subset Y: \text{normal crossing divisor} \\ K_Y + D = 0 \end{cases}$

* Always assume

$D \neq \emptyset$

D : singular

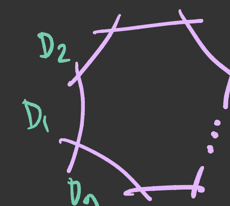
(Y, D) log CY surface $\overset{\text{A.F.}}{\Longrightarrow}$ every connected component of D is either

①  or

smooth curve of genus 1

②  or

rational nodal curve

③ 

$D = D_1 + \cdots + D_n$

n = length of D

cycle of $\mathbb{P}^1$'s

$\Longrightarrow$ Y is rational and D is one connected component of type ② or ③.

D is singular + classification of surfaces

Two examples

Ex 1 Y: smooth projective toric surface w/ toric boundary D

$D \in |-K_Y|$

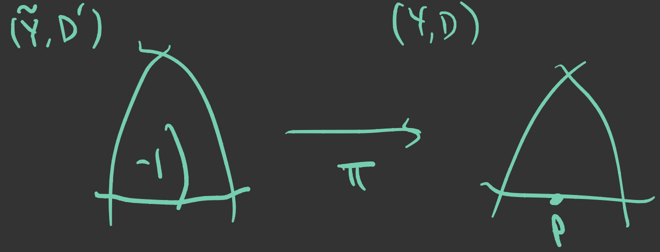
Then (Y, D) is a log CY surface.

Ex 2 (Y, D): log CY surface

$p \in D$: smooth point

$\tilde{Y} \xrightarrow[\pi]{} Y$: blowup of $p \in Y$

$D' \subset \tilde{Y}$: strict transform of D



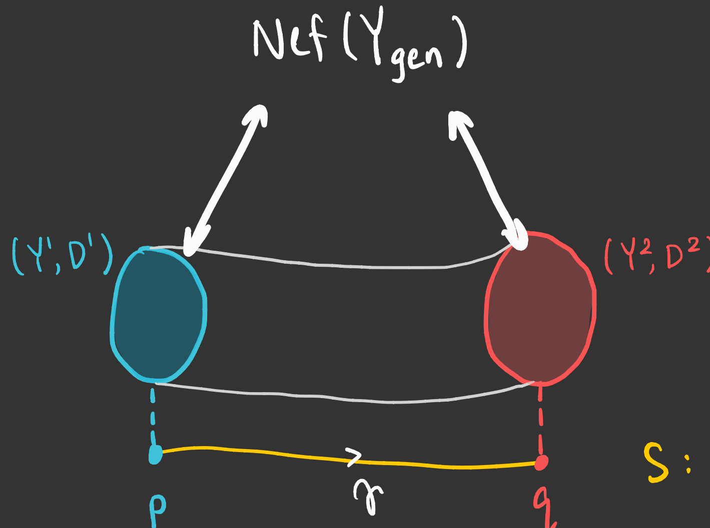
Then $(\tilde{Y}, D')$ is a log CY surface.

(Y, D) is generic if there are no $\underbrace{(-2)\text{-curves}}_{C \cong \mathbb{P}^1 \text{ and } C^2 = -2}$ $C \subset Y \setminus D$.

Fact If (Y^1, D^1) and (Y^2, D^2) are two generic pairs in some connected component of the moduli space, then $\mathrm{Nef}(Y^1) \cong \mathrm{Nef}(Y^2)$.

$$\begin{array}{ccc} \mathrm{Nef}(Y^1) & & \mathrm{Nef}(Y^2) \\ \cap & & \cap \\ H^2(Y^1, \mathbb{R}) & \xrightarrow{\sim} & H^2(Y^2, \mathbb{R}) \end{array}$$

from parallel transport along path γ



S : connected component

Need for Thm 1 Statement

1. $\underset{\substack{\| \text{ Gross-Hacking-Keel} \\ \text{the monodromy group}}}{\overset{\text{admissible group}}{\mathrm{Adm}}} := \{ \theta \in \mathrm{Aut}\,(\underbrace{\mathrm{Pic}\,Y, \odot}_{\substack{\| \\ (H^2(Y,\mathbb{Z}),\, \cup)}}) \mid \theta(\mathrm{Nef}(Y_{gen})) = \mathrm{Nef}(Y_{gen})$ and $\theta([D_i]) = [D_i]$ for $i = 1, \ldots, n \}$

dot product

2. $\mathrm{Nef}^{e}(Y) = \mathrm{Nef}(Y) \cap \mathrm{Eff}(Y)$ where

$$\mathrm{Eff}(Y) = \{ \textstyle\sum a_i [C_i] \mid a_i \in \mathbb{R}_{\geq 0} \text{ and } C_i \subset Y \text{ curve} \} = \mathrm{Curv}(Y)$$

$$\mathrm{Nef}(Y) = \{ L \in \mathrm{Pic}(Y) \otimes \mathbb{R} \mid L \cdot C \geq 0 \text{ for all } \overset{\text{curve}}{C} \subset Y \} = \overline{\mathrm{Nef}^{e}(Y)}$$

$$\underset{\text{Ample cone (open)}}{\mathrm{Int}\,(\mathrm{Nef}(Y))} \subseteq \underset{\text{interior + some rays on the boundary}}{\mathrm{Nef}^{e}(Y)} \subseteq \underset{\text{closed}}{\mathrm{Nef}(Y)}$$

Thm 1 Adm $\curvearrowright$ $\mathrm{Nef}^e(Y_{gen})$ with a RPFD.

Pf (sketch)

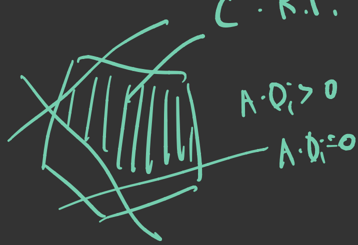
1. $\mathrm{Eff}(Y_{gen})$ is covered by cones

$$C(E_1, \dots, E_k) := \langle D_1, \dots, D_n, \underbrace{E_1, \dots, E_k}_{\text{disjoint, interior } (-1)\text{-curves}} \rangle_{\mathbb{R}\geq 0}$$

Proved by Engel-Friedman for $\mathbb{Z}$-coefficients

$\Rightarrow$ $\mathrm{Nef}^e(Y_{gen})$ is covered by

$$\underbrace{C'(E_1, \dots, E_k)}_{\text{R.P.}} := \underbrace{C(E_1, \dots, E_k)}_{\text{R.P.}} \cap \mathrm{Nef}(Y_{gen})$$



Thm 1 Pf (cont.ed)

2. Adm $\circlearrowright$ all collections $\{E_1, \ldots, E_k\}$ with finitely many orbits.

From Friedman:

$$\text{orbits} \longleftrightarrow \underbrace{\text{deformation types of log CY surfaces obtained from } (Y,D) \text{ by contracting } (-1)\text{-curves}}_{\text{finitely}}$$

$1+2 \Rightarrow \mathrm{Nef}^e(Y) = \bigcup C'(E_1, \ldots, E_k)$

Adm $\circlearrowright$ $\{C'(E_1, \ldots, E_k)\}$ w/ finitely many orbits.

$\Longrightarrow$ Adm $\circlearrowright$ $\mathrm{Nef}^e(Y_{gen})$ with a RPFD.

Looijenga (arXiv: 0908.0165v2)

Need for Thm 2 Statement

1. In each deformation type, there exists a unique log CY surface (Y_e, D_e) s.t.

the Deligne mixed Hodge structure on $H_2(Y_e \setminus D_e, \mathbb{Z})$ is split

$$\Updownarrow$$

Friedman (arXiv:1502.02560v2)

If $L \in \mathrm{Pic}(Y)$ and $L \cdot D_i = 0$ for all i, then $L|_D \simeq \mathcal{O}_D$

2. $\mathrm{Aut}(Y, D) = \{\theta \in \mathrm{Aut}(Y) \mid \theta(D_i) = D_i \text{ for } i = 1, \dots, n\}$.

Thm 2 $\mathrm{Aut}(Y_e, D_e) \circlearrowright \mathrm{Nef}^e(Y_e)$ with a R.P.F.D.

Remark The claim

$$\mathrm{Aut}(Y, D) \circlearrowright \mathrm{Nef}^e(Y) \text{ with a R.P.F.D.}$$

is usually false for (Y, D), a log CY.

e.g. $(\bar{Y} = \mathbb{P}^2,\ \bar{D} = $ rational nodal cubic$)$

$Y = Bl^9\, \mathbb{P}^2$ (blowup at 9 general points on $\bar{D}$)

$D = \bar{D}'$ (strict transform of $\bar{D}$)

Then $\mathrm{Aut}(Y, D)$ is trivial but $\mathrm{Nef}(Y)$ is not R.P.

Similar to an example of Nagata (smooth cubic).

Thm 2 $\mathrm{Aut}(Y_e, D_e) \circlearrowright \mathrm{Nef}^e(Y_e)$ with a R.P.F.D.

Pf (Idea)

(Thm 1) $\mathrm{Adm} \circlearrowright \mathrm{Nef}^e(Y_{gen})$ with a R.P.F.D.

$\Downarrow$ use a proof similar to Sterk for K3 surfaces

Thm 2

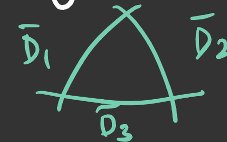
Thm 3 (Y_e, D_e): log CY surface with negative definite or negative semidefinite boundary.

If $n \le 6$, then $\mathrm{Nef}(Y_e)$ is R.P. and $\underbrace{\underset{\mathrm{Adm}/W}{\underset{\parallel}{\mathrm{Aut}(Y_e, D_e)}} = \{e\}}_{\text{Looijenga: } n \le 5}$ for $n = 6$.

Ex 1 ($n=3$; Y_e)

Thm (Looijenga) (Y,D) w/ neg. def or neg. semidef boundary

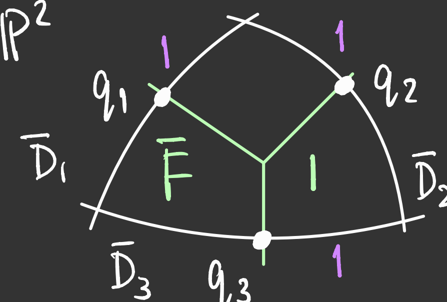
Then (Y,D) is the b.u. of smooth pts of the bdry of $(\bar{Y},\bar{D})$, where $\bar{Y}=\mathbb{P}^2$ and $\bar{D}=\bar{D}_1+\bar{D}_2+\bar{D}_3$



Our case: choose $\underbrace{q_1,q_2,q_3}_{\text{collinear}}$ and blow up those pts some number of times

$(\bar{Y},\bar{D})$

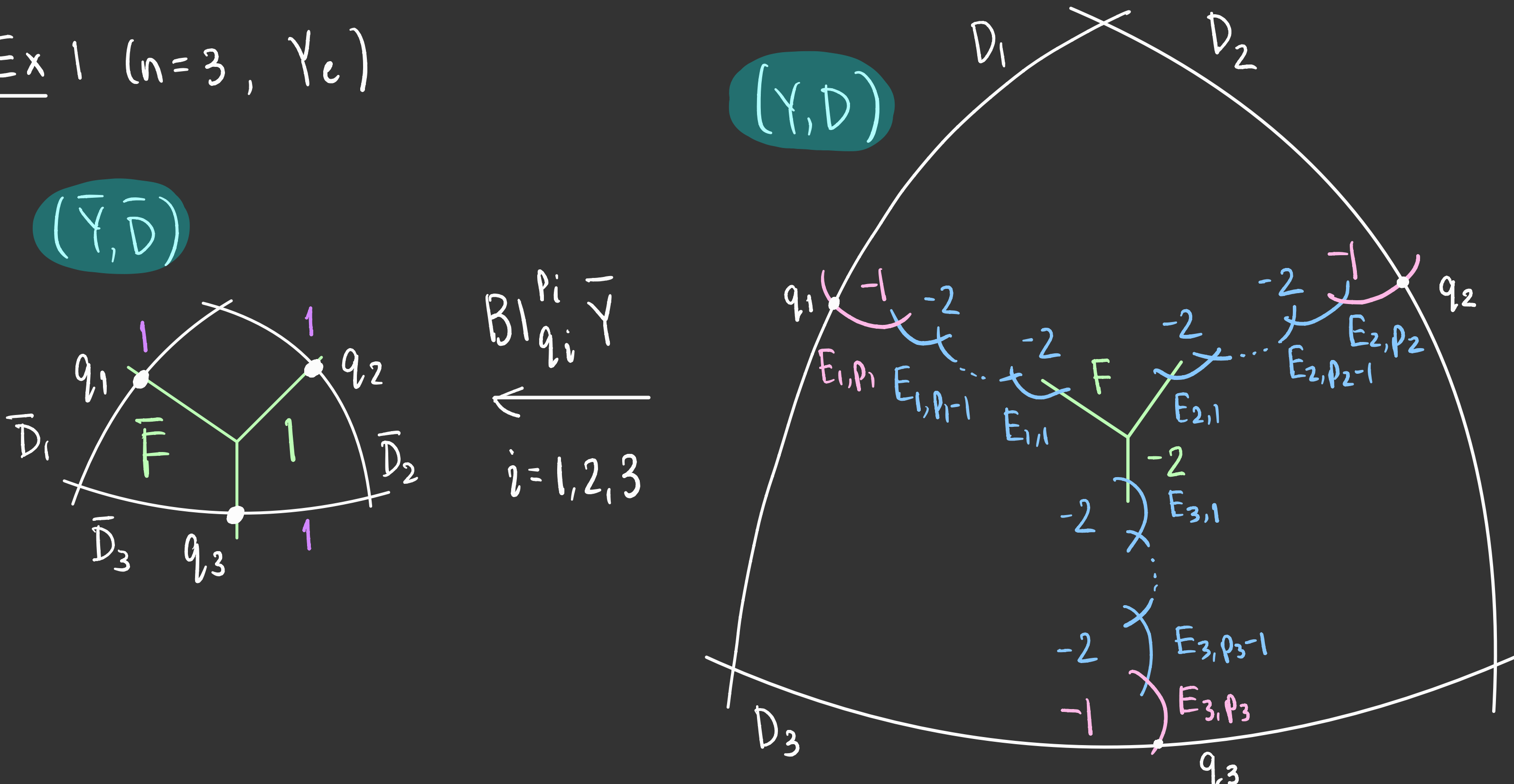
$\bar{Y}=\mathbb{P}^2$



$\bar{F}$: line through q_i

↳ drawn as tropicial curve

Ex 1 ($n=3$, Y_e)

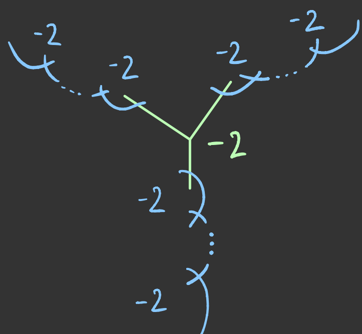


Prop. $\mathrm{Curv}(Y_e) = \langle D_i, E_{i,j}, F \mid i = 1,2,3 \text{ and } 1 \le j \le p_i \rangle_{\mathbb{R}_{\ge 0}}$

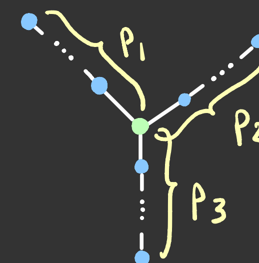
$\Rightarrow \mathrm{Nef}(Y_e) = (\mathrm{Curv}(Y_e))^*$ is also R.P.

Ex 2 ($n=3$, Y_{gen})

[Looijenga ($n \le 5$) $\mathrm{Adm} = W$]. W is associated to the root system



dual graph $\rightsquigarrow$



$W \curvearrowright \mathrm{Nef}^e(Y_{gen})$ with fundamental domain $\underbrace{\mathrm{Nef}(Y_e)}_{\text{R.P.}}$.

Ex 3 $(n=7)$

Y : sm. proj. surface

section σ_0 $\uparrow$ $\downarrow f$ minimal elliptic fibration

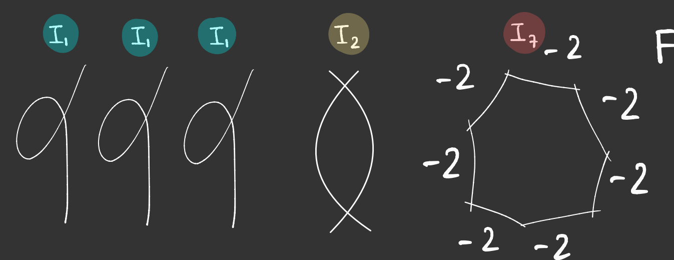
C : sm. proj. curve

$$MW(Y \xrightarrow{f} C) = \left(\{ \text{sections of fibration } Y \xrightarrow{f} C \},\ \text{group law} \right)$$

There exists (Y_e, D_e)

an elliptic fibration $\downarrow$

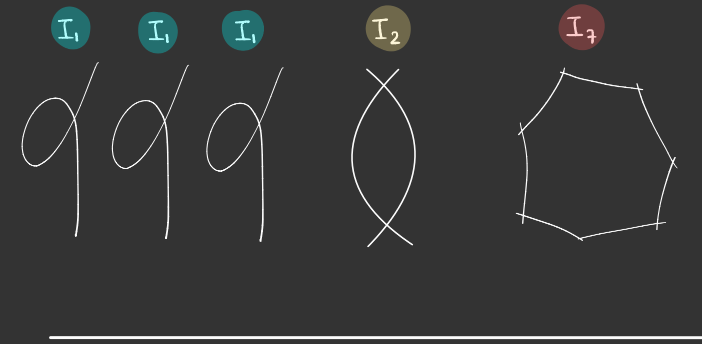
$\mathbb{P}^1$



$$MW(Y \xrightarrow{f} \mathbb{P}^1) = F^{\perp} / \langle F,\ \Gamma \text{ s.t. } f(\Gamma) = pt \rangle_{\mathbb{Z}}$$

Ex 3 $(n=7)$

$$\begin{array}{c}(Y_e, D_e)\\ \downarrow \\ \mathbb{P}^1\end{array}$$



In this case, $|MW| = \infty$.

(in fact, $MW = \mathbb{Z}$)

* In particular: $|\mathrm{Aut}(Y_e, D_e)| = \infty$

$\Rightarrow$ $\mathrm{Nef}(Y_e)$ is not R.P.

Ex 3 $(n=7)$

$MW \circlearrowright \{(-1)\text{-curves}\}$ transitively

and $\mathrm{Aut}(Y_e, D_e) \leq MW$ with finite index.

$\Longrightarrow \mathrm{Aut}(Y_e, D_e) \circlearrowright \{(-1)\text{-curves}\}$ with finitely many orbits.

(Totaro) $\left\{ \begin{array}{l} \mathrm{Aut}(Y_e, D_e) \circlearrowright \mathrm{Nef}^e(Y_e) \text{ with a RPFD.} \\ \text{In fact, } \mathrm{Aut}(Y, D) \circlearrowright \mathrm{Nef}^e(Y) \text{ with a RPFD} \\ \text{whenever } Y \text{ is a rational elliptic surface.} \end{array} \right.$

Deformations of cusps

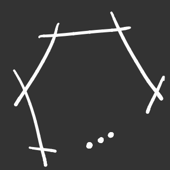
Mirror symmetry interpretation

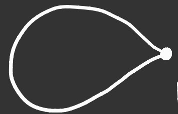
(Y, D) : log Calabi-Yau surface

D is negative definite

Then contract : Gravert

$$(Y, D) \longrightarrow (Y', p)$$





$$\mathrm{Nef}(Y') = \mathrm{Nef}(Y) \cap \langle D_1, \dots, D_n \rangle_{\mathbb{R}}^{\perp}$$ a face of $\mathrm{Nef}(Y)$

Thm 4 $\mathrm{Adm} \circlearrowright \mathrm{Nef}^e(Y'_{gen})$ with a RPFD.

Tools: Thm 1 + Looijenga's work

Deformations of cusps (cont.ed)

Thm 4 $\mathrm{Adm} \circlearrowright \mathrm{Nef}^e(Y'_{gen})$ with a RPFD.

A construction of Looijenga gives a complex analytic germ $(0 \in S)$ from the action of Adm on $\mathrm{Nef}^e(Y'_{gen})$

* Cusp singularities come in dual pairs.

Conjecture The germ $(0 \in S)$ is a smoothing component of the deformation space of the dual cusp singularity $(q \in X)$ to $(p \in Y')$. Moreover, the general fiber over this component S is mirror to $Y \setminus D$.

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Mirror symmetry: Calabi-Yau varieties come in pairs X, Y

complex geometry of X is related to symplectic geometry of Y

Our cone conjecture is the analogue of the Morrison conjecture for CY 3-folds,

except: Automorphism group $\rightsquigarrow$ Monodromy group

replaced by

Thanks!