ELLIPTIC FIBRATIONS AND SYMPLECTIC AUTOMORPHISMS ON K3 SURFACES

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Abstract. Nikulin has classified all finite abelian groups acting symplectically on a K3 surface and he has shown that the induced action on the K3 lattice $U^3 \oplus E_8(-1)^2$ depends only on the group but not on the K3 surface. For all the groups in the list of Nikulin we compute the invariant sublattice and its orthogonal complement by using some special elliptic K3 surfaces.

0. Introduction

An automorphism of a K3 surface is called symplectic if the induced action on the holomorphic 2-form is trivial. The finite groups acting symplectically on a K3-surface are classified by [Nik1], [Mu], [X], where also their fixed locus is described. In [Nik1] Nikulin shows that the action of a finite abelian group of symplectic automorphisms on the K3 lattice $U^3 \oplus E_8(-1)^2$ is unique (up to isometries of the lattice), i.e. it depends only on the group and not on the K3 surface. Hence one can consider a special K3 surface, compute the action, then, up to isometry, this is the same for any other K3 surface with the same finite abelian group of symplectic automorphisms. It turns out that elliptic K3 surfaces are good candidate to compute this action, in fact one can produce symplectic automorphisms by using sections of finite order.

The finite abelian group in the list of Nikulin are the following fourteen groups:

$$\mathbb{Z}/n\mathbb{Z},\ 2 \leq n \leq 8,\ (\mathbb{Z}/m\mathbb{Z})^2,\ m = 2,3,4,$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})^i,\ i = 3,4.$$

For all the groups G but $(\mathbb{Z}/2\mathbb{Z})^i,\ i = 3,4$, there exists an elliptic K3 surface with symplectic group of automorphisms G generated by sections of finite order (cf. [Shim]). In general it is difficult to describe explicitly the action on the K3 lattice. An important step toward this identification is to determine the invariant sublattice, its orthogonal complement and the action of G on this orthogonal complement. In [Nik1] Nikulin gives only rank and discriminant of these lattices, however the discriminant are wrong if the group is not cyclic (cf. also [G2]). In the case of $G = \mathbb{Z}/2\mathbb{Z}$ the action on the K3 lattice was computed by Morrison (cf. [Mo]), and in the cases of $G = \mathbb{Z}/p\mathbb{Z}$, $p = 3,5,7$ we computed in [GS] the invariant sublattice and its orthogonal in the K3 lattice. In this paper we conclude the description of these lattices for all the fourteen groups, in particular we can compute their discriminant which are not always the same as those given in [Nik1]. Its is interesting that some of the orthogonal to the invariant lattices are very well known lattices. We denote by $\Omega_G := (H^2(X,\mathbb{Z})^G)^\perp$ then

G	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/3\mathbb{Z})^2$
Ω_G	$K_{12}(-2)$	$\Lambda_{12}(-1)$	$\Lambda_{14.3}(-1)$	$\Lambda_{15}(-1)$	$K_{16.3}(-1)$

where $K_{12}(-2)$ is the Coxeter-Todd lattice, the lattices Λ_n are laminated lattices and the lattice $K_{16.3}$ is a special sublattice of the Leech-lattice (cf. [CS1] and [PP] for a description). All these give lattice packings which are very dense.

In each case one can compute the invariant lattice and its orthogonal complement by using an elliptic K3 surface. In the case of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z}$ we use always the same elliptic fibration, with six fibers of type I_4, with symplectic automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated

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by sections, then we consider the subgroups. The K3 surface admitting this elliptic fibration is the Kummer surface $Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$ and the group of its automorphisms is described in [KK]. This fibration admits also symplectic automorphisms group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^i$, $i = 3, 4$, in this case some of the automorphisms come from automorphisms of the base of the fibration, i.e. of $\mathbb{P}^1$. Hence also for these two groups we are able to compute the invariant sublattice and its orthogonal complement.

Once we know the lattices Ω_G we can describe all the possible Nèron Severi groups of algebraic K3 surfaces with minimal Picard number and group of symplectic automorphisms G. Moreover we can prove that if there exists a K3 surface with one of those lattices as Néron Severi group, then it admits a certain finite abelian group G as group of symplectic automorphisms. These two facts are important for the classification of K3 surfaces with symplectic automorphism group (cf. [vGS], [GS]), in fact they give information on the coarse moduli space of algebraic K3 surface with symplectic automorphisms. For example an algebraic K3 surface X has G as a symplectic automorphism group if and only if $\Omega_G \subset NS(X)$ (cf. [Nik1, Theorem 4.15]) and moreover $\rho(X) \geq 1 + rk(\Omega_G)$.

The paper is organized as follows: in the sections 1 and 2 we recall some basic results on K3 surfaces and elliptic fibrations, in section 3 we show how to find elliptic K3 surfaces with symplectic automorphism group $(\mathbb{Z}/2\mathbb{Z})^i$, $i = 3, 4$. The section 4 recalls some facts on the elliptic fibration described by Keum and Kondo in [KK] and contains a description of the lattices $H^2(X,\mathbb{Z})^G$ and $(H^2(X,\mathbb{Z})^G)^\perp$ in the cases $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^i$, $i = 2, 3, 4$. In the section 5 we give the equations of the elliptic fibrations for the remaining G and compute the invariant lattice and its orthogonal. Finally the section 6 describe the Nèron Severi group of K3 surfaces with finite abelian symplectic automorphism group and deal with the moduli spaces. In Appendix we describe briefly the elliptic fibrations which can be used to compute the lattices Ω_G for the group G which are not analyzed in Section 4 and we give a basis for these lattices. The proofs of these results are essentially the same as the proof of Proposition 4.1 in Section 4.

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1. Basic results

Let X be a K3 surface. The second cohomology group of X with integer coefficients, $H^2(X,\mathbb{Z})$, with the pairing induced by the cup product is a lattice isometric to $\Lambda_{K3} := E_8(-1)^2 \oplus U^3$ (the K3 lattice), where U is the lattice with pairing $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_8(-1)$ is the lattice associated to the Dynkin diagram E_8 (cf. [BPV]). Let g be an automorphism of X. It induces an action g^*, on $H^2(X,\mathbb{Z})$. This isometry induces an isometry on $H^2(X,\mathbb{C})$ which preserve the Hodge decomposition. In particular $g^*(H^{2,0}(X)) = H^{2,0}(X)$.

Definition 1.1. *An automorphism $g \in Aut(X)$ is **symplectic** if and only if $g^*_{|H^{2,0}(X)} = Id_{|H^{2,0}(X)}$ (i.e. $g^*(\omega_X) = \omega_X$ with $H^{2,0}(X) = \mathbb{C}\omega_X$). We will say that a group of automorphisms **acts symplectically** on X if all the elements of the group are symplectic automorphisms.*

Remark 1.1. [Nik1, Theorem 3.1 b)] An automorphism g of X is symplectic if and only if $g^*_{|T_X} = Id_{T_X}$. □

In [Nik1] the finite abelian groups acting symplectically on a K3 surface are listed and many properties of this action are given. Here we recall the most important. Let G be a finite abelian group acting symplectically on a K3 surface X, then

- G is one of the following fourteen groups

$$(1) \quad \begin{array}{l} \mathbb{Z}/n\mathbb{Z},\ 2 \leq n \leq 8,\ (\mathbb{Z}/m\mathbb{Z})^2,\ m = 2,3,4, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})^i,\ i = 3,4; \end{array}$$

- the desingularization of the quotient X/G is a K3 surface;

- the action induced by the automorphisms of G on $H^2(X,\mathbb{Z})$ is unique up to isometries. In particular the lattices $H^2(X,\mathbb{Z})^G$ and $(H^2(X,\mathbb{Z})^G)^\perp$ depend on G but they do not depend on X, up to isometry.

The last property implies that we can consider a particular K3 surface X admitting a finite abelian group G of symplectic automorphisms to analyze the isometries induced on $H^2(X,\mathbb{Z})\simeq\Lambda_{K3}$ and to describe the lattices $H^2(X,\mathbb{Z})^G$ and $\Omega_G:=(H^2(X,\mathbb{Z})^G)^\perp$. In particular since T_X is invariant under the action of G (by Remark 1.1),

$$H^2(X,\mathbb{Z})^G \hookleftarrow NS(X)^G\oplus T_X$$

and the inclusion has finite index, so considering the orthogonal lattices, we obtain

$$(H^2(X,\mathbb{Z})^G)^\perp=(NS(X)^G)^\perp.$$

For a systematical approach to the problem how to construct K3 surfaces admitting certain symplectic automorphisms we will consider K3 surfaces admitting an elliptic fibration. We recall here some basic facts.

Let X be a K3 surface admitting an elliptic fibration, i.e. there exists a morphism $X\rightarrow\mathbb{P}^1$ such that the generic fiber is a non singular genus one curve and such that there exists a section $s:\mathbb{P}^1\rightarrow X$, which we call the zero section. There are finitely many singular fibers, which can also be reducible. In the following we will consider only singular fibers of type I_n, $n\geq 0$, $n\in\mathbb{N}$. The fibers of type I_1 are curves with a node, the fibers of type I_2 are reducible fibers made up of two rational curves meeting in two distinct points, the fibers of type I_n, $n>2$ are made up of n rational curves meeting as a polygon with n edges. We will call C_0 the irreducible component of a reducible fiber which meets the zero section. The irreducible components of a fiber of type I_n are called C_i where $C_i\cdot C_{i+1}=1$ and $i\in\mathbb{Z}/n\mathbb{Z}$. Under the assumption $C_0\cdot s=1$, these conditions identify the components completely once the component C_1 is chosen, so these conditions identify the components up to the transformation $C_i\leftrightarrow C_{-i}$ for each $i\in\mathbb{Z}/n\mathbb{Z}$. All the components of a reducible fiber of type I_n have multiplicity one, so a section can intersect a fiber of type I_n in any component.

The set of the sections of an elliptic fibration form a group (the Mordell Weil group), with the group law which is induced by the one on the fibers.
Let Red be the set $Red=\{v\in\mathbb{P}^1\,|\,F_v \text{ is reducible}\}$. Let r be the rank of the Mordell Weil group (recall that if there are no sections of infinite order then $r=0$) and let $\rho=\rho(X)$ denote the Picard number of the surface X. Then

$$\rho(X)=rkNS(X)=r+2+\sum_{v\in Red}(m_v-1)$$

(cfr. [Shio, Section 7]) where m_v is the number of irreducible components of the fiber F_v.

Definition 1.2. *The* **trivial lattice** *Tr_X (or Tr) of an elliptic fibration on a surface is the lattice generated by the class of the fiber, the class of the zero section and the classes of the irreducible components of the reducible fibers which do not intersect the zero section.*

The lattice Tr admits U as sublattice and its rank is $rk(Tr)=2+\sum_{v\in Red}(m_v-1)$. Recall that $NS(X)\otimes\mathbb{Q}$ is generated by Tr and the sections of infinite order.

Theorem 1.1. [Shio, Theorem 1.3] *The Mordell Weil group of the elliptic fibration on the surface X is isomorphic to the quotient $NS(X)/Tr:=E(K)$.*

In Section 8 of [Shio] a pairing on $E(K)$ is defined. The value of this pairing on a section P depends only on the intersection between the section P and the reducible fibers and between P and the zero section. Now we recall the definition and the properties of this pairing.
Let $E(K)_{tor}$ be the set of the torsion elements in the group $E(K)$.

Lemma 1.1. [Shio, Lemma 8.1, Lemma 8.2] *For any $P \in E(K)$ there exists a unique element $\phi(P)$ in $NS(X) \otimes \mathbb{Q}$ such that:*
i) $\phi(P) \equiv (P)$ mod $Tr \otimes \mathbb{Q}$ (where (P) is the class of P modulo $Tr \otimes \mathbb{Q}$)
ii) $\phi(P) \perp Tr$.
The map $\phi : E(K) \to NS(X) \otimes \mathbb{Q}$ defined above is a group homomorphism such that $Ker(\phi) = E(K)_{tor}$.

Theorem 1.2. [Shio, Theorem 8.4] *For any $P, Q \in E(K)$ let $\langle P, Q\rangle = -\phi(P) \cdot \phi(Q)$ (where $\cdot$ is induced on $NS(X) \otimes \mathbb{Q}$ by the cup product). Then it defines a symmetric bilinear pairing on $E(K)$, which induces the structure of a positive definite lattice on $E(K)/E(K)_{tor}$.*
In particular if $P \in E(K)$, then P is a torsion section if and only if $\langle P, P\rangle = 0$.
For any $P, Q \in E(K)$ the pairing $\langle -, -\rangle$ is

$$\begin{array}{rcl} \langle P,Q\rangle &=& \chi + P\cdot s + Q \cdot s - P\cdot Q - \sum_{v\in Red} contr_v(P,Q) \\ \langle P,P\rangle &=& \chi + 2(P\cdot s) - \sum_{v\in Red} contr_v(P) \end{array}$$

where χ is the Euler characteristic of the surface and the rational number $contr_v(P,Q)$ are given in the table below

$$(2)\qquad \begin{array}{llllll} & I_2 & I_n & I_n^* & IV^* & III^* \\ contr_v(P) & 2/3 & i(n-i)/n & \begin{cases} 1 \text{ if } i=1 \\ 1+n/4 \text{ if } i=n-1 \text{ or } i=n \end{cases} & 4/3 & 3/2 \\ contr_v(P,Q) & 1/3 & i(n-j)/n & \begin{cases} 1/2 \text{ if } i=1 \\ 2+n/4 \text{ if } i=n-1 \text{ or } i=n \end{cases} & 2/3 & - \end{array}$$

where the numbering of the fibers is the one described before, P and Q meet the fiber in the component C_i and C_j and $i \leq j$.

The pairing defined in the theorem is called **height pairing**. This pairing will be used to determine the intersection of the torsion sections of the elliptic fibrations with the irreducible components of the reducible fibers.

2. Elliptic fibrations and symplectic automorphisms

Using elliptic fibrations one can describe the action of a symplectic automorphism induced by a torsion section on the Néron Severi group. Since a symplectic automorphism acts as the identity on the transcendental lattice we can describe the action of the symplectic automorphism over $NS(X) \oplus T_X$, which is a sublattice of a finite index of $H^2(X, \mathbb{Z})$. Moreover, knowing the discriminant form of $NS(X)$ and of T_X one can explicitly find a basis for the lattice $H^2(X, \mathbb{Z})$, and so one can describe the action of the symplectic automorphism on the lattice $H^2(X, \mathbb{Z})$.
Let X be a K3 surface admitting an elliptic fibration with section, then the Néron Severi group of X contains the classes F and s, which are respectively the class of the fiber and of the section. Let us suppose that t is an n-torsion section of an elliptic K3 surface and let σ_t be the automorphism, induced by t, which fixes the base of the fibration (so fixes each fiber) and acts on each fiber as translation by t. This automorphism is symplectic and if the section t is an n-torsion section (with respect to the group law of the elliptic fibration) the automorphism σ_t has order n. More in general let us consider a K3 surface X admitting an elliptic fibration $\mathcal{E}_X$ with torsion part of the Mordell Weil group equal to a certain abelian group G. Then the sections of $tors(MW(\mathcal{E}_X))$ induce symplectic automorphisms which commute. So we obtain the following:

Lemma 2.1. *If X is a K3 surface with an elliptic fibration $\mathcal{E}_X$ and $tors(MW(\mathcal{E}_X)) = G$, then X admits G as abelian group of symplectic automorphisms and these automorphisms are induced by the torsion sections of $\mathcal{E}_X$.*

Now we want to analyze the action of the symplectic automorphisms induced by torsion sections on the classes generating the Néron Severi group of the elliptic fibration. By definition, σ_t fixes the class F. Since it acts as a translation on each fiber, it sends on each fiber the intersection of the fiber with the zero section, in the intersection of the fiber with the section t. Globally σ_t sends the section s in the section t. More in general the automorphism σ_t sends sections in sections and the section r will be send in $r+t-s$, where the $+$ and $-$ are the operations in the Mordell Weil group. To complete the description of the action of σ_t on the Néron Severi group we have to describe its action on the reducible fibers. The automorphism σ_t restricted to a reducible fiber has to be an automorphism of it. This fact imposes restrictions on the existence of a torsion section and of certain reducible fibers in the same elliptic fibration.

Lemma 2.2. *Let t be a torsion section and F_v a reducible fiber of the fibration.*
1) If t meets the fiber F_v in a component C_i, then $\sigma_t(C_0)=C_i$.
2) If t meets the fiber F_v in the component C_0, then $\sigma_t(C_j)=C_j$ for each j.
3) If there is a fiber of type I_d with component C_i and there is an n-torsion section with $n\not| d$, then $t\cdot C_0=1$ and by 2) $\sigma_t(C_i)=C_i$ for each $i=0,\ldots,d$.
4) If there is a fiber I_d with component C_i and an n-torsion section t with $n|d$, such that $t\cdot C_i=1$, $i\neq 0$, then $i|d$ and $d|(n\cdot i)$. Moreover σ_t restricted to this fiber has order d/i and $\sigma_t(C_j)=C_{j+i}$.

Proof. 1) The section s meets the reducible fiber in the component C_0, so $1=s\cdot C_0=\sigma_t(s)\cdot\sigma_t(C_0)=t\cdot\sigma_t(C_0)$. But $\sigma_t(C_0)$ has to be a component of the same fiber, because σ_t fixes the fibers, and has to be the component with a non-trivial intersection with t, so $\sigma_t(C_0)=C_i$.
2) By 1) applied in the case $i=0$ we obtain $\sigma_t(C_0)=C_0$. The group law on the fibers of type I_n is $\mathbb{C}^*\times\mathbb{Z}/n\mathbb{Z}$ (cf. [Mi, Section VIII.3]). Since the automorphism fixes the component C_0 it acts as $(\omega_n,0)$ on I_n, where ω_n is a primitive n-th root of unity. It acts trivial on $\mathbb{Z}/n\mathbb{Z}$, so it fixes all the components C_i.
3-4) The automorphism group of the fiber of type I_d is $\mathbb{C}^*\times\mathbb{Z}/d\mathbb{Z}$. The automorphism σ_t either acts on I_d as in 2) or has order which divides d. □

In the following we always use this notation: if t_1 is an n-torsion section, then t_h corresponds to the sum, in the Mordell Weil group law, of h times t_1. Moreover $C_i^{(j)}$ is the i-th component of the j-th reducible fibers.

3. The cases $G=(\mathbb{Z}/2\mathbb{Z})^3$ and $G=(\mathbb{Z}/2\mathbb{Z})^4$

We have seen (cf. Lemma 2.1) that an example of a K3 surface with G as group of symplectic automorphisms is given by an elliptic fibration with G as torsion part of the Mordell Weil group. The groups which appear as torsion part of the Mordell Weil group of an elliptic fibration on a K3 surface are twelve. In particular the groups $G=(\mathbb{Z}/2\mathbb{Z})^3$ and $G=(\mathbb{Z}/2\mathbb{Z})^4$ are groups acting symplectically on a K3 surface, but they can not be realized as the torsion part of the Mordell Weil group of an elliptic fibration on a K3 surface (cf. [Shim]). Hence to find examples of K3 surfaces with one of these groups as group of symplectic automorphisms, we have to use a different construction. One possibility is to consider a K3 surface with elliptic fibration $\mathcal{E}_X$ with $MW(\mathcal{E}_X)=(\mathbb{Z}/2\mathbb{Z})^2$ and to find one (or two) other symplectic involutions which commute with the ones induced by torsion sections.

3.1. **The group $G=(\mathbb{Z}/2\mathbb{Z})^3$ acting symplectically on an elliptic fibration.** We start considering an elliptic K3 surface with two 2-torsion sections. An equation of such an elliptic fibration is given by

$$y^2=x(x-p(\tau))(x-q(\tau))\quad deg(p(\tau))=deg(q(\tau))=4,\ \tau\in\mathbb{C}. \tag{3}$$

Then we consider an involution on the base of the fibration $\mathbb{P}^1$, which preserves the fibration. This involution fixes two points of the basis. Up to the choice of the coordinates on $\mathbb{P}^1$, we can suppose that the involution on the basis is $\sigma_{\mathbb{P}^1,a}:\tau\mapsto-\tau$. So we consider on the K3 surface the involution $(\tau,x,y)\mapsto(-\tau,x,-y)$. Since this map has to be an involution of the surface, it has to fix the equation of the elliptic fibration. Moreover the involution $\sigma_{\mathbb{P}^1,a}$ has to commute with the involutions induced by the torsion sections. This implies that it has to fix the curves corresponding to the torsion sections $t:\tau\mapsto(p(\tau),0)$ and $u:\tau\mapsto(q(\tau),0)$. The equation of such an elliptic fibration is

$$\begin{array}{c} y^2=x(x-p(\tau))(x-q(\tau))\quad\text{with}\\ deg(p(\tau))=deg(q(\tau))=4\text{ and }p(\tau)=p(-\tau)\ \ q(\tau)=q(-\tau).\end{array} \tag{4}$$

i.e. $y^2=x(x-(p_4\tau^4+p_2\tau^2+p_0))(x-(q_4\tau^4+q_2\tau^2+q_0)), \quad p_4,p_2,p_0,q_4,q_2,q_0\in\mathbb{C}$.
The involution $\sigma_{\mathbb{P}^1,a}$ fixes the four curves corresponding to the sections in the torsion part of the Mordell Weil group and the two fibers over the points 0 and ∞ of $\mathbb{P}^1$. On these two fibers the automorphism is not the identity (because it sends y in $-y$). So it fixes only the eight intersection points between these four sections and the two fibers. It fixes eight isolated points, so it is a symplectic involution.
To compute the moduli of this family of surfaces we have to consider that the choice of the automorphism on $\mathbb{P}^1$ corresponds to a particular choice of the coordinates, so we can not act on the equation with all the automorphisms of $\mathbb{P}^1$. We choose coordinates such that the involution $\sigma_{\mathbb{P}^1,a}$ fixes the points $(1:0)$ and $(0:1)$ on $\mathbb{P}^1$.
The space of the automorphisms of $\mathbb{P}^1$ commuting with $\sigma_{\mathbb{P}^1,a}$ has dimension one. Moreover we can act on the equation (4) with the transformation $(x,y)\mapsto(\lambda^2x,\lambda^3y)$ and divide by λ^6. Since the parameters of the equation (4) are 6, we find that the number of moduli of this family is $6-2=4$.
Collecting these results, the properties of the family satisfying the equation (4) are the following:

discriminant	*singular fibers*	*moduli*
$p(\tau)^2q(\tau)^2(p(\tau)-q(\tau))^2$	$12I_2$	4

Since the number of moduli of this family is four, the Picard number of the generic surface in this family is 16. The trivial lattice of this fibration has rank 14, so there are two linearly independent sections of infinite order which generate the free part of the Mordell Weil group.

3.2. **The group $G=(\mathbb{Z}/2\mathbb{Z})^4$ acting symplectically on an elliptic fibration.** As in the previous section we construct an involution which commutes with the three symplectic involutions σ_t, σ_u (the involutions induced by the torsion section t and u) and $\sigma_{\mathbb{P}^1,a}$ of the surfaces described by the equation (4). So we consider two commuting involutions on $\mathbb{P}^1$ which commute with the involutions induced by the 2-torsion sections. Up to the choice of the coordinates of $\mathbb{P}^1$ we can suppose that the involutions on $\mathbb{P}^1$ are $\tau\mapsto-\tau$ and $\tau\mapsto 1/\tau$. We call the corresponding involution on the surface $\sigma_{\mathbb{P}^1,a}:(\tau,x,y)\mapsto(-\tau,x,-y)$ and $\sigma_{\mathbb{P}^1,b}:(\tau,x,y)\mapsto(1/\tau,x,-y)$. As before requiring that these involutions commute with the involutions induced by the torsion sections means that each of these involutions fixes the torsion sections. Elliptic K3 surfaces with the properties described have the following equation:

$$\begin{array}{c} y^2=x(x-p(\tau))(x-q(\tau)) \quad \text{with} \\ deg(p(\tau))=deg(q(\tau))=4 \ \ p(\tau)=p(-\tau)=p(\tfrac{1}{\tau}) \ \ q(\tau)=q(-\tau)=q(\tfrac{1}{\tau}). \end{array} \tag{5}$$

i.e. $y^2=x(x-(p_4\tau^4+p_2\tau^2+p_4))(x-(q_4\tau^4+q_2\tau^2+q_4)), \quad p_4,p_2,q_4,q_2\in\mathbb{C}$.
As before the choice of the involutions of $\mathbb{P}^1$ implies that the admissible transformations on the equation have to commute with $\sigma_{\mathbb{P}^1,a}$ and $\sigma_{\mathbb{P}^1,b}$. The only possible transformation is the identity so no transformations of $\mathbb{P}^1$ can be applied to the equation (5). The only admissible transformation on that equation is $(x,y)\mapsto(\lambda^2x,\lambda^3y)$.
Collecting these results, the properties of the family satisfying the equation (5) are the following:

discriminant	*singular fibers*	*moduli*
$p(\tau)^2q(\tau)^2(p(\tau)-q(\tau))^2$	$12I_2$	3

As in the previous case the comparison between the number of moduli of the family and the rank of the trivial lattice implies that the free part of the Mordell Weil group is $\mathbb{Z}^3$.

4. An elliptic fibration with six fibers of type I_4

An equation of an elliptic K3 surface with six fibers of type I_4 is the following

$$y^2=x(x-\tau^2\sigma^2)\left(x-\frac{(\tau^2+\sigma^2)^2}{4}\right). \tag{6}$$

This equation is well known, for example it is described in [TY, Section 2.3.1], where it is constructed considering the K3 surface as double cover of a rational elliptic surface with two fibers of type I_2 and two fibers of type I_4. One can find this equation also considering the equation of an elliptic K3 surface

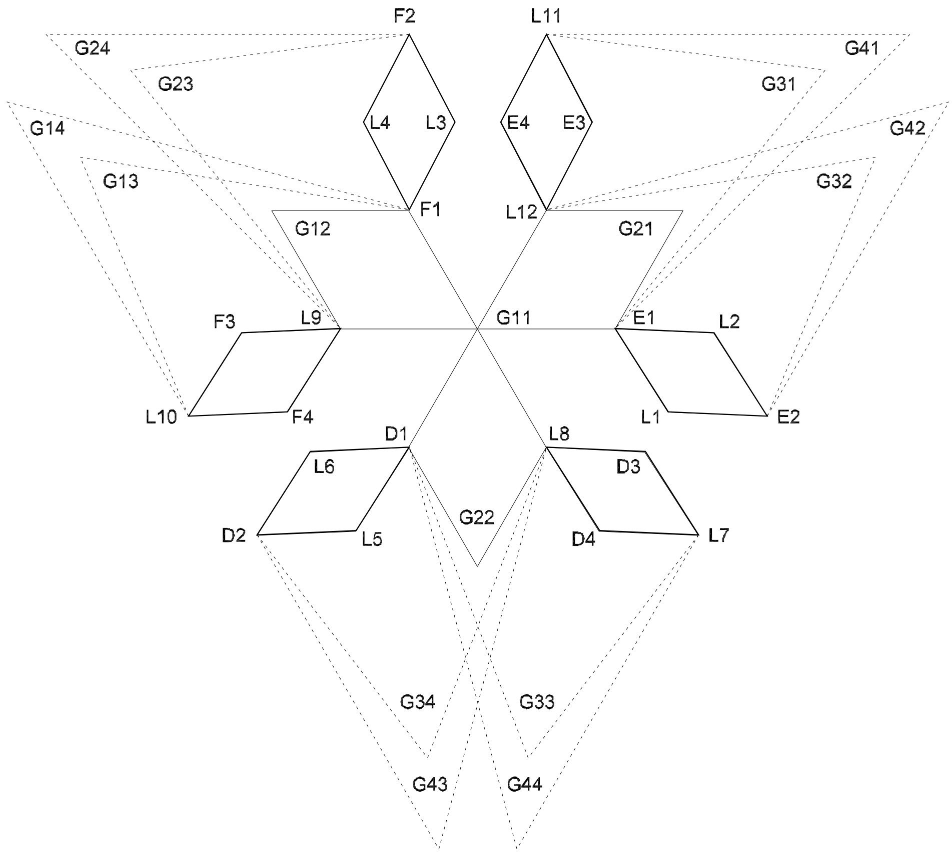

Figure 1

with two 2-torsion sections (i.e the equation (3)) and requiring that the tangent lines to the elliptic curve defined by the equation (3) in the two rational points of the elliptic surface pass respectively through the two points of order two $(p(\tau), 0)$ and $(q(\tau), 0)$.
We will call $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ the elliptic K3 surface described by the equation (6). It has two 4-torsion sections t_1 and u_1, which induce two commuting symplectic automorphisms σ_{t_1} and σ_{u_1}. We will analyze this surface to study not only the group of symplectic automorphisms $G = (\mathbb{Z}/4\mathbb{Z})^2$, but also its subgroups $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z} = \langle\sigma_{t_1}^2, \sigma_{u_1}\rangle$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} = \langle\sigma_{t_1}^2, \sigma_{s_1}^2\rangle$, $\mathbb{Z}/4\mathbb{Z} = \langle\sigma_{t_1}\rangle$, $\mathbb{Z}/2\mathbb{Z} = \langle\sigma_{t_1}^2\rangle$ which act symplectically too.
It is more surprising that the equation (6) appears as a specialization also of the surfaces described in (4) and (5). So it admits also the group $G = (\mathbb{Z}/2\mathbb{Z})^4$ as group of symplectic automorphisms. The automorphisms $\sigma_{\mathbb{P}^1,a} : (\tau, x, y) \mapsto (-\tau, x, -y)$ and $\sigma_{\mathbb{P}^1,b} : (\tau, x, y) \mapsto (1/\tau, x, -y)$ commute with the automorphism induced by the two 2-torsion sections t_2 and u_2.
The automorphisms group $Aut(X_{(\mathbb{Z}/4\mathbb{Z})^2})$ of the surface $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ is described in [KK]. Keum and Kondo prove that $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ is the Kummer surface $Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$. They consider forty rational curves on the surface: sixteen of them, (the ones called $G_{i,j}$), form a Kummer lattice, and the twentyfour curves $G_{i,j}$, E_i, $i = 1, 2, 3, 4$, F_j, $j = 1, 2, 3, 4$ form the so called *double Kummer*, a lattice which is a sublattice of the Néron Severi group of $Km(E_1 \times E_2)$ for each couple of elliptic curves E_1, E_2. These forty curves generate $NS(Km(E_{\sqrt{-1}} \times E_{\sqrt{-1}}))$ and describe five different elliptic fibrations on $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ which have six fibers of type I_4 each. A complete list of the 63 elliptic fibrations present on the surface $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ can be found in [Nis].
In the Figure 1 some of the intersections between the forty curves introduced in [KK] are shown. The

five elliptic fibrations are associated to the classes ([KK, Proof of Lemma 3.5])

$$E_1+E_2+L_1+L_2,\quad G_{11}+G_{12}+F_1+L_9,\quad G_{11}+G_{21}+E_1+L_{12},$$
$$G_{11}+G_{22}+D_1+L_8,\quad G_{33}+G_{44}+D_1+L_7.$$

We identify the fibration that we will consider with one of these, say the one with fiber $E_1+E_2+L_1+L_2$. Then we can define $s=G_{11}$, $t=G_{13}$, $u=G_{32}$, and so $C_0^{(1)}=E_1$, $C_0^{(2)}=L_{12}$, $C_0^{(3)}=L_8$, $C_0^{(4)}=D_1$, $C_0^{(5)}=E_1$, $C_0^{(6)}=L_9$.
We will consider the group $G=(\mathbb{Z}/2\mathbb{Z})^4$ generated by two involutions induced by torsion sections and two involutions induced by involutions on $\mathbb{P}^1$ (with respect to one of these elliptic fibrations). However one can choose the last two involutions as induced by the two torsion sections of different elliptic fibrations (cf. Remark 4.2).

Proposition 4.1. *Let $X_{(\mathbb{Z}/4\mathbb{Z})^2}$ be the elliptic K3 surface with equation* (6). *Let t_1 and u_1 be two 4-torsion sections of the fibration. Then $t_1\cdot C_1^{(j)}=1$ if $j=1,2,3,4$, $t_1\cdot C_2^{(5)}=t\cdot C_0^{(6)}=1$, $u_1\cdot C_1^{(h)}=1$ if $h=4,5,6$, $u_1\cdot C_3^{(3)}=1$, $u_1\cdot C_2^{(1)}=u_1\cdot C_0^{(2)}=1$.*
A $\mathbb{Z}$-basis for the lattice $NS(X_{(\mathbb{Z}/4\mathbb{Z})^2})$ is given by $F,\ s,\ t_1,\ u_1,\ C_i^{(j)},\ j=1,\ldots,6,\ i=1,2$ and $C_3^{(j)}$, $j=2,\ldots,5$.
The trivial lattice of the fibration is $U\oplus A_3^{\oplus 6}$. It has index 4^2 in the Néron Severi group of $X_{(\mathbb{Z}/4\mathbb{Z})^2}$. The lattice $NS(X_{(\mathbb{Z}/4\mathbb{Z})^2})$ has discriminant -4^2 and its discriminant form is $\mathbb{Z}_4(-\frac{1}{4})\oplus\mathbb{Z}_4(-\frac{1}{4})$.
The transcendental lattice is $T_{X_{(\mathbb{Z}/4\mathbb{Z})^2}}=\begin{bmatrix}4&0\\0&4\end{bmatrix}$, and has a unique primitive embedding in the lattice Λ_{K3}.

Proof. The singular fibers of this fibration are six fibers of type I_4 (the classification of the type of the singular fibers is determined by the zero-locus of the discriminant of the equation of the surface and can be found in [Mi, IV.3.1]). Hence the trivial lattice of this elliptic fibration is $U\oplus A_3^{\oplus 6}$. Since it has rank 20 which is the maximal Picard number of a K3 surface, there are no sections of infinite order on this elliptic fibration. The torsion part of the Mordell Weil group is $(\mathbb{Z}/4\mathbb{Z})^2$, generated by two 4-torsion sections. We will call these sections t_1 and u_1. By the height formula (cf. Theorem 1.2) the intersection between a four torsion section and the six fibers of type I_4 has to be of the following type: the section has to meet four fibers in the component C_i with i odd, a fiber in the component C_2 and a fiber in the component C_0. After a suitable numbering of the fibers we can suppose that the section t_1 has the intersection described in the statement. The section u_1 and the section v_1 which corresponds to t_1+u_1 in the Mordell Weil group must intersect the fibers in a similar way (four in the component C_i with an odd i, one in C_2 and one in C_0). If $t_1\cdot C_i^{(j)}=1$ and $u_1\cdot C_h^{(j)}=1$, then v_1 meets the fiber in the component $C_{i+h}^{(j)}$ (it is a consequence of the group law on the fibers of type I_n). The conditions on the intersection properties of u_1 and v_1 imply that $u_1\cdot C_1^{(h)}=1$ if $h=3,4,5,6$, $u_1\cdot C_2^{(1)}=u_1\cdot C_0^{(2)}=0$, and hence $v_1\cdot C_1^{(j)}=1$, $i=2,6$, $v_1\cdot C_2^{(4)}=1$, $v_1\cdot C_3^{(h)}=1$, $h=1,5$, $v_1\cdot C_0^{(3)}=1$.
The torsion sections t_1 and u_1 can be written as a linear combination of the classes in the trivial lattice with coefficient in $\frac{1}{4}\mathbb{Z}$ (because they are 4-torsion sections). Hence the trivial lattice has index 4^2 in the Néron Severi group and so $d(NS(X_{(\mathbb{Z}/4\mathbb{Z})^2}))=4^6/4^4=4^2$. In particular

$$\begin{array}{l} t_1=2F+s-\frac{1}{4}\left(\sum_{i=1}^4(3C_1^{(i)}+2C_2^{(i)}+C_3^{(i)})+2C_1^{(5)}+4C_2^{(5)}+2C_3^{(5)}\right)\\ u_1=2F+s-\frac{1}{4}\left(\sum_{i=4}^6(3C_1^{(i)}+2C_2^{(i)}+C_3^{(i)})+C_1^{(3)}+2C_2^{(3)}+3C_3^{(3)}+2C_1^{(1)}+4C_2^{(1)}+2C_3^{(1)}\right).\end{array}$$

It is now clear that $F,\ s,\ t_1,\ u_1,\ C_i^{(j)},\ j=1,\ldots,6,\ i=1,2$ and $C_3^{(j)}$, $j=2,\ldots,5$ is a $\mathbb{Q}$-basis for the Néron Severi group and since the determinant of the intersection matrix of this basis is 4^2, it is in fact a $\mathbb{Z}$-basis. The discriminant form of the Néron Severi lattice is generated by

$$\begin{array}{l} d_1=\frac{1}{4}(C_1^{(1)}+2C_2^{(1)}+3C_3^{(1)}-C_1^{(3)}-2C_2^{(3)}-3C_3^{(3)}+C_1^{(6)}+2C_2^{(6)}+3C_3^{(6)})\\ d_2=\frac{1}{4}(C_1^{(2)}+2C_2^{(2)}+3C_3^{(2)}+C_1^{(4)}+2C_2^{(4)}+3C_3^{(4)}-C_1^{(5)}-2C_2^{(5)}-3C_3^{(5)}),\end{array} \tag{7}$$

hence the discriminant form of $NS(X_{(\mathbb{Z}/4\mathbb{Z})^2})$ is $\mathbb{Z}_4(-\frac{1}{4}) \oplus \mathbb{Z}_4(-\frac{1}{4})$. The transcendental lattice has to be a rank 2 positive definite lattice with discriminant form $\mathbb{Z}_4(\frac{1}{4}) \oplus \mathbb{Z}_4(\frac{1}{4})$ (the opposite of the one of the Néron Severi group). This implies that $T_{X_{(\mathbb{Z}/4\mathbb{Z})^2}} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$. □

Up to now for simplicity we put $X_{(\mathbb{Z}/4\mathbb{Z})^2} = X$.

Proposition 4.2. *Let $G_{4,4}$ be the group generated by σ_{t_1} and σ_{u_1}. The invariant sublattice of the Néron Severi group with respect to $G_{4,4}$ is isometric to* $\begin{bmatrix} -8 & 8 \\ 8 & 0 \end{bmatrix}$.
Its orthogonal complement $(NS(X)^{G_{4,4}})^{\perp}$ is $\Omega_{(\mathbb{Z}/4\mathbb{Z})^2} := (H^2(X,\mathbb{Z})^{G_{4,4}})^{\perp}$. It is the negative definite eighteen dimensional lattice $\{\mathbb{Z}^{18}, Q\}$ where Q is the bilinear form obtained as the intersection form of the classes

$$b_1 = s - t_1,\ b_2 = s - u_1,\ b_{i+2} = C_1^{(i)} - C_1^{(i+1)},\ i = 1, \dots, 5,\ b_{j+7} = C_2^{(j)} - C_2^{(j+1)}, j = 1, \dots, 5,$$
$$b_{h+11} = C_1^{(h)} - C_3^{(h)},\ h = 2, \dots, 5,\ b_{17} = C_1^{(1)} - C_2^{(2)},\ b_{18} = f + s - t_1 - C_1^{(1)} - C_1^{(2)} - C_2^{(2)} - C_1^{(3)}.$$

The lattice $\Omega_{(\mathbb{Z}/4\mathbb{Z})^2}$ admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of $\Omega_{(\mathbb{Z}/4\mathbb{Z})^2}$ is 2^8 and its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^2$.
The group of isometries $G_{4,4}$ acts on the discriminant group $\Omega^{\vee}_{(\mathbb{Z}/4\mathbb{Z})^2}/\Omega_{(\mathbb{Z}/4\mathbb{Z})^2}$ as the identity.
The lattice $H^2(X,\mathbb{Z})^{G_{4,4},}$ is

$$\begin{bmatrix} 4 & 6 & 0 & 0 \\ 6 & 4 & 6 & 4 \\ 0 & 6 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

and it is an overlattice of $NS(X)^{G_{4,4}} \oplus T_X \simeq \begin{bmatrix} -8 & 8 \\ 8 & 0 \end{bmatrix} \oplus \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ of index four.

Proof. Let $v_1 = t_1 + u_1$, $w_1 = t_2 + u_1$, $z_1 = t_3 + u_1$. By the definition, the automorphisms induced by the torsion sections fix the class of the fiber, so $F \in NS(X)^{G_{4,4}}$. Moreover the group $G_{4,4}$ acts on the section fixing the class $s + \sum_{i=1}^{3} t_i + u_i + v_i + w_i + z_i$. The action of the group $G_{4,4}$ is not trivial on the components of the reducible fibers, so the lattice $\langle F, s + \sum_{i=1}^{3} t_i + u_i + v_i + w_i + z_i\rangle$ is a sublattice of $NS(X)^{G_{4,4}}$ of finite index. Since the orthogonal of a sublattice is always a primitive sublattice we have $\langle F, s + \sum_{i=1}^{3} t_i + u_i + v_i + w_i + z_i\rangle^{\perp} = (NS(X)^{G_{4,4}})^{\perp}$. In this way one can compute the classes generating the lattice $(NS(X)^{G_{4,4}})^{\perp}$. A basis for this lattice is given by the classes b_i, moreover the lattice $NS(X)^{G_{4,4}}$ is isometric to $((NS(X)^{G_{4,4}})^{\perp})^{\perp}$, and a computation shows that it is isometric to $\begin{bmatrix} -8 & 8 \\ 8 & 0 \end{bmatrix}$.
To find the lattice $H^2(X,\mathbb{Z})^{G_{4,4}}$ we consider the orthogonal complement of $(H^2(X),\mathbb{Z})^{G_{4,4}})^{\perp} \simeq (NS(X)^{G_{4,4}})^{\perp}$ inside the lattice $H^2(X,\mathbb{Z})$. Since we know the generators of the discriminant form of $NS(X)$ and of T_X we can construct a basis of $H^2(X,\mathbb{Z})$. Indeed let a_1 and a_2 be the generators of T_X, then the classes F, s, t_1, u_1, $C_i^{(j)}$ $j = 1, \dots, 6$, $i = 1, 2$, $C_3^{(j)}$, $j = 2, \dots, 5$, $a_1/4 + d_1$ and $a_2/4 + d_2$ (d_i as in (7)) form a $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})$. The classes b_i, $i = 1, \dots, 18$ generate $(H^2(X,\mathbb{Z})^{G_{4,4}})^{\perp}$ and are expressed as a linear combination of the $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})$ described above. The orthogonal of these classes in $H^2(X,\mathbb{Z})$ is the lattice $H^2(X,\mathbb{Z})^{G_{4,4}}$. A computation with the computer shows that the action of $G_{4,4}$ is trivial on the discriminant group $\Omega^{\vee}_{(\mathbb{Z}/4\mathbb{Z})^2}/\Omega_{(\mathbb{Z}/4\mathbb{Z})^2} = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^2$. Moreover the lattice $\Omega_{(\mathbb{Z}/4\mathbb{Z})^2}$ satisfies the hypothesis of [Nik2, Theorem 1.14.4] so it admits a unique primitive embedding in Λ_{K3}. □

Proposition 4.3. **1)** *Let $G_{2,4}$ be the group generated by σ_{t_2} and σ_{u_1}. The invariant sublattice of the Néron Severi group with respect to $G_{2,4}$ is isometric to $U(4) \oplus \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$. Its orthogonal complement $(NS(X)^{G_{2,4}})^{\perp}$ is $\Omega_{\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}} := (H^2(X,\mathbb{Z})^{G_{2,4}})^{\perp}$. It is the negative definite sixteen dimensional lattice*

$\{\mathbb{Z}^{16}, Q\}$ *where* Q *is the bilinear form obtained as the intersection form of the classes*

$$\begin{array}{l} b_1 = s - u_1, \quad b_i = C_3^{(i)} - C_1^{(i)},\ i = 2, \ldots, 5, \qquad b_{j+3} = C_1^{(j)} - C_1^{(j+1)},\ j = 3, 4, 5, \\ b_{h+6} = C_2^{(h)} - C_2^{(h)},\ h = 3, 4, 5,\ b_{12} = C_2^4 - C_1^3,\ b_{13} = C_0^{(3)} - C_1^{(4)},\ b_{14} = C_1^{(1)} + C_2^{(1)} - C_2^{(2)} - C_3^{(2)}, \\ b_{15} = C_2^{(2)} + C_3^{(2)} - C_2^{(3)} - C_3^{(3)},\ b_{16} = C_2^{(1)} - C_1^{(2)} - C_1^{(3)} + C_3^{(3)} - C_2^{(5)} + C_1^{(6)} - 2t_1 + 2u_1. \end{array}$$

The lattice $\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}$ *admits a unique primitive embedding in the lattice* Λ_{K3}.
The discriminant of $\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}$ *is* 2^{10} *and its discriminant group is* $(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^4$.
The group of isometries $G_{2,4}$ *acts on the discriminant group* $\Omega^\vee_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}/\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}}$ *as the identity.*
The lattice $H^2(X,\mathbb{Z})^{G_{2,4}}$ *is*

$$\left[\begin{array}{rrrrrr} 4 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & -64 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 80 & 4 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

and it is an overlattice of $NS(X)^{G_{2,4}} \oplus T_X \simeq U(4) \oplus \left[\begin{array}{rr} -4 & 0 \\ 0 & -4 \end{array}\right] \oplus \left[\begin{array}{rr} 4 & 0 \\ 0 & 4 \end{array}\right]$ *of index two.*

2) *Let* $G_{2,2}$ *be the group generated by* $\sigma_{t_2} (= \sigma_{t_1}^2)$ *and* $\sigma_{u_2} (= \sigma_{u_1}^2)$.
The invariant sublattice of the Néron Severi group with respect to $G_{2,2}$ *is isometric to*

$$\left[\begin{array}{rrrrrrrr} -4 & -4 & -2 & -2 & -4 & 0 & 0 & 0 \\ -4 & -4 & 0 & 2 & -2 & 0 & 8 & 4 \\ -2 & 0 & -4 & 0 & 0 & 0 & 4 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -2 & 0 & 0 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 2 & 0 \\ 0 & 8 & 4 & 0 & 6 & 2 & -16 & 2 \\ 0 & 4 & 0 & 0 & 0 & 0 & 2 & -4 \end{array}\right].$$

Its orthogonal complement $(NS(X)^{G_{2,2}})^\perp$ *is* $\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} := (H^2(X,\mathbb{Z})^{G_{2,2}})^\perp$. *It is the negative definite twelve dimensional lattice* $\{\mathbb{Z}^{12}, Q\}$ *where* Q *is the bilinear form obtained as the intersection form of the classes*

$$\begin{array}{l} b_1 = -C_1^{(1)} - C_1^{(2)} - C_1^{(3)} + C_3^{(3)} + C_1^{(5)} + C_1^{(6)} - 2t_1 + 2u_1,\ b_2 = -C_1^{(1)} - 2C_2^{(1)} - C_3^{(1)} + F, \\ b_3 = C_1^{(2)} - C_3^{(2)},\ b_4 = -C_1^{(1)} - C_2^{(1)} + C_1^{(2)} + C_2^{(2)}, \\ b_5 = C_1^{(1)} + 2C_2^{(1)} + C_3^{(1)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} + C_1^{(4)} + C_2^{(4)} + C_3^{(4)} + C_1^{(5)} + C_1^{(6)} - 3F - 2s + 2u_1, \\ b_6 = -C_1^{(1)} - C_2^{(1)} - C_1^{(3)} - C_2^{(3)} + F,\ b_7 = C_1^{(4)} - C_3^{(4)},\ b_8 = -C_1^{(1)} - C_2^{(1)} + C_2^{(4)} + C_3^{(4)}, \\ b_9 = C_1^{(5)} - C_3^{(5)},\ b_{10} = -C_1^{(1)} - C_2^{(1)} + C_1^{(5)} + C_2^{(5)}, \\ b_{11} = C_1^{(6)} - C_3^{(6)},\ b_{12} = -C_1^{(1)} - C_2^{(1)} + C_1^{(6)} + C_2^{(6)}. \end{array}$$

The lattice $\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ *admits a unique primitive embedding in the lattice* Λ_{K3}.
The discriminant of $\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ *is* 2^{10} *and its discriminant group is* $(\mathbb{Z}/2\mathbb{Z})^6 \oplus (\mathbb{Z}/4\mathbb{Z})^2$.
The group of isometries $G_{2,2}$ *acts on the discriminant group* $\Omega^\vee_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}/\Omega_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ *as the identity.*

The lattice $H^2(X,\mathbb{Z})^{G_{2,2}}$ *is*

$$\left[\begin{array}{cccccccccc} 0&4&2&0&0&2&0&-2&0&0\\ 4&0&6&-8&8&4&6&-20&8&2\\ 2&6&0&-1&2&1&2&-6&0&2\\ 0&-8&-1&-2&0&-1&2&1&2&0\\ 0&8&2&0&-4&4&0&2&0&0\\ 2&4&1&-1&4&-4&0&-1&0&-2\\ 0&6&2&2&0&0&-4&2&0&0\\ -2&-20&-6&1&2&-1&2&-4&0&4\\ 0&8&0&2&0&0&0&0&-4&0\\ 0&2&2&0&0&-2&0&4&0&-4 \end{array}\right]$$

and it is an overlattice of $NS(X)^{G_{2,2}}\oplus T_X$ *of index four.*

3) *Let* G_4 *be the group generated by* σ_{t_1}*. The invariant sublattice of the Néron Severi group with respect to* G_4 *is isometric to* $\langle -4\rangle\oplus\left[\begin{array}{ccccc} -2&1&0&0&0\\ 1&-2&4&0&0\\ 0&4&4&8&4\\ 0&0&8&0&4\\ 0&0&4&4&0\end{array}\right]$*. Its orthogonal complement* $(NS(X)^{G_4})^{\perp}$ *is* $\Omega_{\mathbb{Z}/4\mathbb{Z}}:=(H^2(X,\mathbb{Z})^{G_4})^{\perp}$*. It is the negative definite fourteen dimensional lattice* $\{\mathbb{Z}^{14},Q\}$ *where* Q *is the bilinear form obtained as the intersection form of the classes*

$$\begin{array}{ll} b_1=s-t_1, & b_{i+1}=C_1^{(i)}-C_1^{(i+1)},\ i=1,2,3,\\ b_{j+3}=C_1^{(j)}-C_1^{(j)},\ j=2,\ldots,5, & b_{h+7}=C_2^{(h)}-C_2^{(h+1)},\ h=2,3\\ b_{11}=C_2^{(2)}-C_1^{(1)} & b_{12}=C_2^{(1)}-C_1^{(2)}\\ b_{13}=F-C_1^{(2)}-C_2^{(2)}-C_1^{(5)}-C_2^{(5)} & b_{14}=C_1^{(2)}+C_2^{(2)}-C_1^{(5)}-C_2^{(5)}. \end{array}$$

The lattice $\Omega_{\mathbb{Z}/4\mathbb{Z}}$ *admits a unique primitive embedding in the lattice* Λ_{K3}*.*
The discriminant of $\Omega_{\mathbb{Z}/4\mathbb{Z}}$ *is* 2^{10} *and its discriminant group is* $(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/4\mathbb{Z})^4$*.*
The group of isometries G_4 *acts on the discriminant group* $\Omega_{\mathbb{Z}/4\mathbb{Z}}^{\vee}/\Omega_{\mathbb{Z}/4\mathbb{Z}}$ *as the identity.*
The lattice $H^2(X,\mathbb{Z})^{G_4}$ *is*

$$\left[\begin{array}{cccccccc} 0&4&0&2&0&-1&0&0\\ 4&0&4&4&-4&0&0&-4\\ 0&4&0&0&0&0&0&0\\ 2&4&0&0&0&-1&0&0\\ 0&-4&0&0&-2&-1&0&-2\\ -1&0&0&-1&-1&-2&1&1\\ 0&0&0&0&0&1&-2&0\\ 0&-4&0&0&-2&1&0&-2 \end{array}\right]$$

and it is an overlattice of $NS(X)^{G_4}\oplus T_X$ *of index two.*

Proof. The proof is similar to the proof of Proposition 4.3. □

Remark 4.1. The automorphisms σ_{t_1} and σ_{u_1} do not fix the other four elliptic fibrations described in [KK] (different from $E_1+E_2+L_1+L_2$). The involutions $(\sigma_{t_1})^2$ and $(\sigma_{u_1})^2$ fix the class of the fiber of those elliptic fibrations, however they are not induced by torsion sections on those fibrations, in fact they do not fix each fiber of the fibration. On the fibrations different from $E_1+E_2+L_1+L_2$, the actions of $(\sigma_{t_1})^2$ and $(\sigma_{u_1})^2$ are analogues to the ones of $\sigma_{\mathbb{P}^1,a}$ and $\sigma_{\mathbb{P}^1,b}$ on the fibration $E_1+E_2+L_1+L_2$.□

Proposition 4.4. 1) *Let* $G_{2,2,2}$ *be the group generated by* σ_{t_2}, σ_{u_2} *and* $\sigma_{\mathbb{P}^1,a}$*. The automorphism* $\sigma_{\mathbb{P}^1,a}$ *acts in the following way:*

$$t_1\leftrightarrow v,\quad u_1\leftrightarrow w,\quad C_i^{(1)}\leftrightarrow C_i^{(2)},\quad C_i^{(5)}\leftrightarrow C_i^{(6)},\ i=0,1,2,3,\ C_1^{(j)}\leftrightarrow C_3^{(j)},\ j=3,4$$

where w and v are respectively the section obtained as $t_1+u_1+u_1$ and $t_1+t_1+u_1$ with respect to the group law of the Mordell Weil group, and fixes the classes F, s and $C_i^{(j)}$, $i=0,2$, $j=3,4$. The invariant sublattice of the Néron Severi group with respect to $G_{2,2,2}$ is isometric to

$$\left[\begin{array}{rrrrrr} -4 & 2 & 0 & 0 & 0 & 0\\ 2 & -20 & 6 & 0 & 0 & 0\\ 0 & 6 & -4 & -2 & 0 & 0\\ 0 & 0 & -2 & 0 & -2 & 0\\ 0 & 0 & 0 & -2 & 0 & -4\\ 0 & 0 & 0 & 0 & -4 & -8 \end{array}\right].$$

Its orthogonal complement $(NS(X)^{G_{2,2,2}})^{\perp}$ is $\Omega_{(\mathbb{Z}/2\mathbb{Z})^3}:=(H^2(X,\mathbb{Z})^{G_{2,2,2}})^{\perp}$. It is the negative definite fourteen dimensional lattice $\{\mathbb{Z}^{14},Q\}$ where Q is the bilinear form obtained as the intersection form of the classes

$$\begin{array}{l} b_1=C_3^{(3)}-C_1^{(3)},\ b_2=C_3^{(4)}-C_1^{(4)},\ b_3=C_3^{(2)}-C_1^{(1)},\ b_4=C_2^{(2)}-C_2^{(1)},\ b_5=C_1^{(2)}-C_1^{(1)},\\ b_6=C_0^{(3)}+C_3^{(3)}-C_1^{(1)}-C_2^{(1)},\ b_7=C_1^{(3)}+C_2^{(3)}-C_1^{(1)}-C_2^{(1)},\quad b_8=C_1^{(4)}+C_2^{(4)}-C_1^{(3)}-C_2^{(3)},\\ b_9=C_2^{(5)}+C_3^{(5)}-C_1^{(4)}-C_2^{(4)},\ b_{10}=C_1^{(6)}+C_2^{(6)}-C_2^{(5)}-C_3^{(5)},\ b_{11}=-C_1^{(1)}+C_1^{(5)}-t_1+u_1,\\ b_{12}=C_1^{(6)}-C_1^{(1)}-t_1+u_1,\qquad b_{13}=C_2^{(5)}-C_2^{(1)}+t_1-u_1,\\ b_{14}=C_1^{(1)}+C_1^{(2)}+C_1^{(3)}+C_1^{(4)}+C_1^{(5)}+2C_2^{(5)}+C_3^{(5)}-2F-2s+2t_1. \end{array}$$

The lattice $\Omega_{(\mathbb{Z}/2\mathbb{Z})^3}$ admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of $\Omega_{(\mathbb{Z}/2\mathbb{Z})^3}$ is 2^{10} and its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^6\oplus(\mathbb{Z}/4\mathbb{Z})^2$.
The group of isometries $G_{2,2,2}$ acts on the discriminant group $\Omega^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^3}/\Omega_{(\mathbb{Z}/2\mathbb{Z})^3}$ as the identity.
The lattice $H^2(X,\mathbb{Z})^{G_{2,2,2}}$ is an overlattice of $NS(X)^{G_{2,2,2}}\oplus T_X$ of index two.

2) *Let $G_{2,2,2,2}$ be the group generated by σ_{t_2}, σ_{u_2}, $\sigma_{\mathbb{P}^1,a}$ and $\sigma_{\mathbb{P}^1,b}$. The automorphism $\sigma_{\mathbb{P}^1,b}$ acts in the following way:*

$$t_1\leftrightarrow z,\quad u_1\leftrightarrow w,\quad C_i^{(3)}\leftrightarrow C_i^{(4)},\quad C_i^{(5)}\leftrightarrow C_{4-i}^{(6)},\ i=0,1,2,3,\quad C_1^{(j)}\leftrightarrow C_3^{(j)},\ j=1,2$$

where w is as in 1) and z is the section obtained as $t_1+t_1+t_1+u_1+u_1$ with respect to the group law of the Mordell Weil group, and fixes the classes F, s and $C_i^{(j)}$, $i=0,2$, $j=1,2$. The invariant sublattice of the Néron Severi group with respect to $G_{2,2,2,2}$ is isometric to

$$\left[\begin{array}{rrrrr} -20 & -8 & -12 & -2 & 4\\ -8 & 8 & 2 & 2 & 4\\ -12 & 2 & -4 & 0 & 4\\ -2 & 2 & 0 & 0 & 0\\ 4 & 4 & 4 & 0 & -8 \end{array}\right].$$

Its orthogonal complement $(NS(X)^{G_{2,2,2,2}})^{\perp}$ is $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}:=(H^2(X,\mathbb{Z})^{G_{2,2,2,2}})^{\perp}$. It is the negative definite fifteen dimensional lattice $\{\mathbb{Z}^{15},Q\}$ where Q is the bilinear form obtained as the intersection form of the classes

$$\begin{array}{l} b_1=C_3^{(3)}-C_1^{(3)},\ b_2=C_3^{(2)}-C_1^{(1)},\ b_3=C_2^{(2)}-C_2^{(1)},\ b_4=C_1^{(2)}-C_1^{(1)},\ b_5=C_3^{(4)}-C_1^{(3)},\\ b_6=C_1^{(4)}-C_1^{(3)},\ b_7=C_0^{(3)}+C_3^{(3)}-C_1^{(1)}-C_2^{(1)},\qquad b_8=C_1^{(3)}+C_2^{(3)}-C_1^{(1)}-C_2^{(1)},\\ b_9=C_1^{(4)}+C_2^{(4)}-C_1^{(3)}-C_2^{(3)},\ b_{10}=C_2^{(5)}+C_3^{(5)}-C_1^{(4)}-C_2^{(4)},\ b_{11}=C_1^{(6)}+C_2^{(6)}-C_2^{(5)}-C_3^{(5)},\\ b_{12}=C_1^{(6)}-C_1^{(1)}-t_1+u_1,\qquad b_{13}=C_2^{(5)}-C_2^{(1)}+t_1-u_1,\qquad b_{14}=C_1^{(5)}-C_1^{(1)}-t_1+u_1,\\ b_{15}=-C_1^{(1)}-C_1^{(3)}-C_2^{(3)}-C_1^{(4)}+F+s-t_1. \end{array}$$

The lattice $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ admits a unique primitive embedding in the lattice Λ_{K3}.
The discriminant of $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ is -2^9 and its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^6\oplus(\mathbb{Z}/8\mathbb{Z})$.
The group of isometries $G_{2,2,2,2}$ acts on the discriminant group $\Omega^{\vee}_{(\mathbb{Z}/2\mathbb{Z})^4}/\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ as the identity.
The lattice $H^2(X,\mathbb{Z})^{G_{2,2,2,2}}$ is an overlattice of $NS(X)^{G_{2,2,2,2}}\oplus T_X$ of index two.

Proof. The proof is similar to the proof of Proposition 4.2, we describe here the action of $\sigma_{\mathbb{P}^1,a}$ on the fibration (the action of $\sigma_{\mathbb{P}^1,b}$ can be deduced in a similar way). The automorphism $\sigma_{\mathbb{P}^1,a}$ fixes the sections t_2 and u_2, the class of the fiber (because sends fibers in fibers) and fixes the third and the fourth reducible fiber. It acts on the basis of the fibration $\mathbb{P}^1$ switching the two points corresponding to the first and the second reducible fiber and the two points corresponding to the fifth and the sixth reducible fiber. Since the sections t_2 and u_2 are fixed, the component $C_2^{(1)}$ (which meet the section t_2) has to be sent in the component $C_2^{(2)}$ (which is the component of the second fiber which meets the same section). Similary we obtain $\sigma_{\mathbb{P}^1,a}(C_0^{(1)}) = C_0^{(2)}$ and $\sigma_{\mathbb{P}^1,a}(C_i^{(5)}) = C_i^{(6)}$ for $i = 0, 2$. The component $C_2^{(3)}$ is sent in a component of the same fiber and since $1 = C_2^{(3)} \cdot u_2 = \sigma_{\mathbb{P}^1,a}(C_2^{(3)}) \cdot \sigma_{\mathbb{P}^1,a}(u_2) = \sigma_{\mathbb{P}^1,a}(C_2^{(3)}) \cdot u_2$, we have $\sigma_{\mathbb{P}^1,a}(C_2^{(3)}) = C_2^{(3)}$. Analogously we obtain $\sigma_{\mathbb{P}^1,a}(C_i^{(j)}) = C_i^{(j)}$, $i = 0, 2$, $j = 3, 4$. The image of $C_1^{(3)}$ has to be a component of the first fiber, so it can only be $C_1^{(3)}$ or $C_3^{(3)}$. The curve $C_0^{(3)}$ is a rational curve fixed by an involution, so the involution has two fixed points on it (it cannot be fixed pointwise because $\sigma_{\mathbb{P}^1,a}$ is a symplectic involution and the fixed locus of a symplectic involution is made up of isolated points). These points are the intersection between the curve $C_0^{(3)}$ and the section $u_2 + t_2$ of order two and the 0-section s (which are fixed by the involution). Then the point of intersection between $C_0^{(3)}$ and $C_1^{(3)}$ is not fixed, and then $C_1^{(3)}$ is not fixed by the involution, so we conclude that $\sigma_{\mathbb{P}^1,a}(C_1^{(i)}) = C_3^{(i)}$, $i = 3, 4$. To determine the image of the curves C_i^j, $i = 1, 3$, $j = 3, 4, 5, 6$ and of u_1 and t_1 one sees that after an analysis of all possible combinations the only possibility is the action given in the statement. □

Remark 4.2. The automorphisms $\sigma_{\mathbb{P}^1,a}$, $\sigma_{\mathbb{P}^1,b}$ are induced by a 2-torsion section of the fibrations $G_{22} + G_{11} + D_1 + L_8$ and $G_{11} + G_{21} + E_1 + L_{12}$ respectively (with the notation of [KK]). □

5. Other cases

For each group G in the list (1) except for the group $G = (\mathbb{Z}/2\mathbb{Z})^i$ for $i = 3, 4$ there exists a K3 surface X with an elliptic fibration $\mathcal{E}_X$ such that $tors(MW(\mathcal{E}_X)) = G$. For each group G we will consider the K3 surface X_G with the minimal possible Picard number among those admitting such an elliptic fibration. One can find these K3 surfaces in the Shimada's list (cf. [Shim, Table 2]). In the tables 1 and 2 in the Appendix we describe the trivial lattice of these elliptic fibration, the intersection properties of the torsion sections (which can be deduced by the height formula) and we give the transcendental lattices in all the cases but $\mathbb{Z}/4\mathbb{Z}$. In this case seems to be more difficult to identify the transcendental lattice, however our first aim is to compute the lattice $H^2(X, \mathbb{Z})^{\mathbb{Z}/4\mathbb{Z}}$ and we did it in Proposition 4.3. In Table 3 we describe $(NS(X_G)^G)^\perp \simeq \Omega_G$, giving a basis for this lattice. The proof of the results is very similar to the proof of Proposition 4.1 and 4.2 except for the equation of the fibrations. Observe that in all the cases the Mordell Weil group has only a torsion part. In fact the number of moduli in the equation is exactly $20 - rank(Tr)$, so there are no sections of infinite order. The equation in the case of $G = \mathbb{Z}/2\mathbb{Z}$ is standard. We computed already in Section 3 the equations for $G = (\mathbb{Z}/2\mathbb{Z})^i$, $i = 2, 3, 4$. For $G = \mathbb{Z}/p\mathbb{Z}$, $p = 3, 5, 7$ equations are given e.g. in [GS]. We now explain briefly how to find equations in the other cases.

$G = \mathbb{Z}/4\mathbb{Z}$. This fibration has in particular a section of order two so has an equation of the kind $y^2 = x(x^2 + a(\tau)x + b(\tau))$, $deg(a(\tau)) = 4$, $deg(b(\tau)) = 8$. To determine $a(\tau)$ and $b(\tau)$ we observe that a smooth elliptic curve of the fibration has a point Q of order four if $Q + Q = P = (0, 0)$ which is the point of order two on each smooth fiber. Geometrically this means that the tangent to the elliptic curve through Q must intersect the curve exactly in the point P. With a similar condition one computes the equations for the groups $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

$G = \mathbb{Z}/6\mathbb{Z}$. An elliptic fibration admits a 6-torsion section if and only if it admits a 2-torsion section and a 3-torsion section, so it is enough to impose to the fibration $y^2 = x(x^2 + a(\tau)x + b(\tau))$ to have an inflectional point on the generic fiber (different from $(0 : 0 : 1)$) (cf. [C, Ex 6, p.38]). By using this condition and the equation of a fibration with two 2-torsion sections one computes the equation of a fibration with $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$.

$G=\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. We consider the rational surface:

$$y^2=x^3+12(u_0^3u_1-u_1^4)x+2(u_0^6-20u_0^3u_1^3-8u_1^6),\ (u_0:u_1)\in\mathbb{P}^1 \tag{8}$$

which has two 3-torsion sections and four singular fibers I_3 (cf. [AD],[B] for a description). We can consider a double cover of $\mathbb{P}^1$ branched over two generic points $(\alpha:1)$, $(\beta:1)$, which is realized by the rational map $u\mapsto(\alpha\tau^2+\beta)/(\tau^2+1)$. The image is again a copy of $\mathbb{P}^1$. This change of coordinate in the equation (8) induces an elliptic fibration over the second copy of $\mathbb{P}^1$. This elliptic surface is a K3 surface, in fact the degree of the discriminant of this Weierstrass equation is 24 and admits two 3-torsion sections, which are the lift of the torsion sections of (8). The Weierstrass equation is for $\alpha,\beta\in\mathbb{C}$

$$y^2=x^3+12x(\tau^2+1)[(\alpha\tau^2+\beta)^3-(\tau^2+1)^3]+2[(\alpha\tau^2+\beta)^6-20(\alpha\tau^2+\beta)^3(\tau^2+1)^3-8(\tau^2+1)^6]. \tag{9}$$

In all the cases the lattices $(NS(X_G)^G)^{\perp}$ are isometric to the lattices Ω_G. Moreover since we know the transcendental lattice of the K3 surfaces considered (except for $G=\mathbb{Z}/4\mathbb{Z}$) we can compute also $H^2(X_G,\mathbb{Z})^G$ as in the proof of Proposition 4.2. Here we summarize the main properties of the lattices Ω_G (the basis are given in Table 3) and of their orthogonal complement:

Proposition 5.1. *For any K3 surface X with a group G of symplectic automorphisms, the action on $H^2(X,\mathbb{Z})$ decomposes as $(H^2(X,\mathbb{Z})^G)^{\perp_{H^2(X,\mathbb{Z})}}\simeq\Omega_G$ and $H^2(X,\mathbb{Z})^G=(\Omega_G)^{\perp_{H^2(X,\mathbb{Z})}}$. The following table lists the main properties of Ω_G and $(\Omega_G)^{\perp_{\Lambda_{K3}}}$.*

G	$rk(\Omega_G)$	$d(\Omega_G)$	$\Omega_G^\vee/\Omega_G$	$rk(\Omega_G^{\perp_{\Lambda_{K3}}})$	$\Omega_G^{\perp_{\Lambda_{K3}}}$
$\mathbb{Z}/2\mathbb{Z}$	8	2^8	$(\mathbb{Z}/2\mathbb{Z})^8$	14	$E_8(-2)\oplus U^{\oplus 3}$
$\mathbb{Z}/3\mathbb{Z}$	12	3^6	$(\mathbb{Z}/3\mathbb{Z})^6$	10	$U\oplus U(3)^{\oplus 2}\oplus A_2^{\oplus 2}$
$\mathbb{Z}/4\mathbb{Z}$	14	2^{10}	$(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/4\mathbb{Z})^4$	8	Q_4
$\mathbb{Z}/5\mathbb{Z}$	16	5^4	$(\mathbb{Z}/5\mathbb{Z})^4$	6	$U\oplus U(5)^{\oplus 2}$
$\mathbb{Z}/6\mathbb{Z}$	16	6^4	$(\mathbb{Z}/6\mathbb{Z})^4$	6	$U\oplus U(6)^{\oplus 2}$
$\mathbb{Z}/7\mathbb{Z}$	18	7^3	$(\mathbb{Z}/7\mathbb{Z})^3$	4	$U(7)\oplus\begin{bmatrix}4&1\\1&2\end{bmatrix}$
$\mathbb{Z}/8\mathbb{Z}$	18	8^3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}\oplus(\mathbb{Z}/8\mathbb{Z})^2$	4	$U(8)\oplus\begin{bmatrix}2&0\\0&4\end{bmatrix}$
$(\mathbb{Z}/2\mathbb{Z})^2$	12	2^{10}	$(\mathbb{Z}/2\mathbb{Z})^6\oplus(\mathbb{Z}/4\mathbb{Z})^2$	10	$U(2)^{\oplus 2}\oplus Q_{2,2}$
$(\mathbb{Z}/2\mathbb{Z})^3$	14	2^{10}	$(\mathbb{Z}/2\mathbb{Z})^6\oplus(\mathbb{Z}/4\mathbb{Z})^2$	8	$U(2)^{\oplus 3}\oplus\langle -4\rangle^{\oplus 2}$
$(\mathbb{Z}/2\mathbb{Z})^4$	15	-2^9	$(\mathbb{Z}/2\mathbb{Z})^6\oplus\mathbb{Z}/8\mathbb{Z}$	7	$\langle -8\rangle\oplus U(2)^{\oplus 3}$
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	16	2^{10}	$(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/4\mathbb{Z})^4$	6	$Q_{2,4}$
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	18	$2^4 3^3$	$(\mathbb{Z}/3\mathbb{Z})\oplus(\mathbb{Z}/12\mathbb{Z})^2$	4	$\begin{bmatrix}0&6&0&0\\6&0&-3&0\\0&-3&6&6\\0&0&6&8\end{bmatrix}$
$(\mathbb{Z}/3\mathbb{Z})^2$	16	3^6	$(\mathbb{Z}/3\mathbb{Z})^4\oplus\mathbb{Z}/9\mathbb{Z}$	6	$U(3)^{\oplus 2}\oplus\begin{bmatrix}2&3\\3&0\end{bmatrix}$
$(\mathbb{Z}/4\mathbb{Z})^2$	18	2^8	$(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/8\mathbb{Z})^2$	4	$\begin{bmatrix}4&6&0&0\\6&4&6&4\\0&6&4&0\\0&4&0&0\end{bmatrix}$

where

$$Q_4=\begin{bmatrix}0&4&0&2&0&-1&0&0\\4&0&4&4&-4&0&0&-4\\0&4&0&0&0&0&0&0\\2&4&0&0&0&-1&0&0\\0&-4&0&0&-2&-1&0&-2\\-1&0&0&-1&-1&-2&1&1\\0&0&0&0&0&1&-2&0\\0&-4&0&0&-2&1&0&-2\end{bmatrix},$$

$$Q_{2,2}=\begin{bmatrix}0&1&0&0&0&0\\1&-2&2&0&0&0\\0&2&-4&2&0&0\\0&0&2&-4&2&0\\0&0&0&2&-4&4\\0&0&0&0&4&-8\end{bmatrix},\quad Q_{2,4}=\begin{bmatrix}4&-2&0&0&0&0\\-2&0&-2&0&0&0\\0&-2&-64&-4&0&0\\0&0&-4&0&-4&0\\0&0&0&-4&80&4\\0&0&0&0&4&0\end{bmatrix}.$$

The lattices Ω_G are even negative definite lattices, do not contain vectors of length -2 and are generated by vectors of maximal length, i.e. by vectors of length -4.

Proof. The proof is for each case similar to the one of Proposition 4.2. The fact that the lattices Ω_G do not contain vector of length -2 was proved by Nikulin (cf. [Nik1]). The fact that these lattices are generated by vectors of length -4 is a direct consequence of the choice of the basis in Table 3, Appendix, and in Propositions 4.2, 4.3, 4.4. In fact all these bases are made up of classes with self intersection -4. □

Remark 5.1. The lattices $\Omega_{(\mathbb{Z}/2\mathbb{Z})^2}$, $\Omega_{\mathbb{Z}/4\mathbb{Z}}$, $\Omega_{(\mathbb{Z}/2\mathbb{Z})^4}$ and $\Omega_{(\mathbb{Z}/3\mathbb{Z})^2}$ are isometric respectively to the laminated lattices (with the bilinear form multiplied by -1) $\Lambda_{12}(-1)$, $\Lambda_{14.3}(-1)$, $\Lambda_{15}(-1)$ and to the lattice $K_{16.3}(-1)$ (a special sublattice of the Leech-lattice). All these lattices are described in [PP], [CS1] and [NS] and the isometries can be proved applying an algorithm described in [PP] about isometries of laminated lattices. We did the computations by using the package *Automorphism group and isometry testing* of MAGMA and the computation can be done at the web page MAGMA-Calculator (cf. [M]).
In particular the lattices $\Lambda_{12}(-1)$, $\Lambda_{14.3}(-1)$, $\Lambda_{15}(-1)$ and $K_{16.3}(-1)$ are primitive sublattices of the lattice Λ_{K3}.
The lattice $\Omega_{\mathbb{Z}/3\mathbb{Z}}$ is isometric to the lattice $K_{12}(-2)$ described in [CS2] and [CT]. The proof of this isometry can be found in [GS] where a description of the lattices $\Omega_{\mathbb{Z}/5\mathbb{Z}}$ and $\Omega_{\mathbb{Z}/7\mathbb{Z}}$ is also given.

Remark 5.2. We have computed here for a group of symplectic automorphisms in the list of Nikulin the invariant sublattice and its orthogonal in the K3 lattice. It is in general difficult to find explicitely the action of a symplectic automorphism on Λ_{K3}. This is done by Morrison [Mo] in the case of involutions, otherwise this is not known. By using an elliptic fibration with a fiber I_{16}, a fiber I_4, four fibers I_1 and a 4-torsion section s_1 (cf. [Shim, No.3171]) one can identify an operation of a symplectic automorphisms of order four on the K3 lattice whose square is the operation described by Morrison. In fact it is easy to find two copies of the lattice $E_8(-1)$ in the fiber I_{16} together with the sections s_0, s_2. Then the operation of s_2 interchanges these two lattices and after the identification with Λ_{K3}, it fixes a copy of U^3. Since the computations are quite involved we omit them.

6. Families of K3 surfaces admitting symplectic automorphisms

Now let us consider an algebraic K3 surface X with a group of symplectic automorphisms G. Since the action of G is trivial on the transcendental lattice, we have $T_X \subset H^2(X,\mathbb{Z})^G$ and $NS(X) \supset (H^2(X,\mathbb{Z})^G)^\perp \simeq \Omega_G$. The lattice Ω_G is negative definite, and the signature of $NS(X)$ is $(1, \rho-1)$, so $\Omega_G^{\perp_{NS(X)}}$ has to contain a class with a positive self-intersection. In particular the minimal possible Picard number of X is $\mathrm{rank}(\Omega_G)+1$. Here we want to consider the possible Néron Severi groups of K3 surfaces admitting a finite abelian group of symplectic automorphisms. The first step is to find all lattices $\mathcal{L}$ such that: $\mathrm{rank}(\mathcal{L})=\mathrm{rank}(\Omega_G)+1$, Ω_G is primitively embedded in $\mathcal{L}$ and $\Omega_G^{\perp_\mathcal{L}}$ is positive definite. We prove that if a K3 surface admits one of these lattices as Néron Severi group, then it admits the group G as symplectic group of automorphisms and viceversa if a K3 surface with Picard number $\mathrm{rank}(\Omega_G)+1$ admits G as symplectic group of automorphisms, then its Néron Severi group is isometric to one of those lattices.
To find the lattice $\mathcal{L}$ we need the following remark.

Remark 6.1. We recall here the correspondence between even overlattices of a lattice and totally isotropic subgroups of its discriminant group (cf. [Nik2, Proposition 1.4.1]). Let L_H be an overlattice of L such that $\mathrm{rank}L=\mathrm{rank}L_H$. If $L \hookrightarrow L_H$ with a finite index, then $[L_H : L]^2 = d(L)/d(L_H)$ and $L \subsetneqq L_H \subsetneqq L^\vee$. The lattice L_H corresponds to a subgroup of the discriminant group H of $L^\vee$. Viceversa a subgroup H of $L^\vee/L$ corresponds to a $\mathbb{Z}$-module L_H such that $L \subsetneqq L_H$. On $L^\vee$ is defined a bilinear form which is the $\mathbb{Q}$-linear extension of the bilinear form on L and so on every subgroup of $L^\vee$ is defined a biliner form induced by the one on L.
Let L be an even lattice with bilinear form b and H be a subgroup of $L^\vee/L$. We call b also the form induced on H and L_H by the one on L. If H is such that $b(h,h) \in 2\mathbb{Z}$ for each $h \in H$, then L_H is an even overlattice of L.
In fact each element $x \in L_H$ is $x = h+l$, $h \in H$, $l \in L$. The value of the quadratic form $q(x) = b(x,x)$ on x is in $2\mathbb{Z}$, in fact $b(x,x) = b(h+l,h+l) = b(h,h)+2b(h,l)+b(l,l)$ and $b(h,h), b(l,l) \in 2\mathbb{Z}$ by hypothesis, $b(h,l) \in \mathbb{Z}$ because $h \in L^\vee$. For each $h,h' \in H$ we have $b(h+h',h+h') = b(h,h)+$

$2b(h,h')+b(h',h')\in 2\mathbb{Z}$ by hypothesis and so $b(h,h')\in\mathbb{Z}$. Moreover $b(x,x')=b(h+l,h'+l')=b(h,h')+b(h,l')+b(l,h')+b(l,l')$ and $b(h,l'),\,b(l,h')\in\mathbb{Z}$ because $h\in L^\vee$, $b(l,l')\in\mathbb{Z}$ because $l,l'\in L$ and we have already proved that $b(h,h')\in\mathbb{Z}$. So the bilinear form on L_H takes values in $\mathbb{Z}$ and the quadratic form induced by b takes values in $2\mathbb{Z}$, i.e. L_H is an even lattice. □

Proposition 6.1. *Let X be an algebraic K3 surface and let Υ be a negative definite primitive sublattice of $NS(X)$ such that $\Upsilon^{\perp_{NS(X)}}=\mathbb{Z}L$ where L is a class in $NS(X)$, $L^2=2d$ with $d>0$. Let the discriminant group of Υ be $\oplus_{j=1}^m(\mathbb{Z}/h_j\mathbb{Z})^{n_j}$, with $h_j|h_{j+1}$ for each $j=1,\ldots,m-1$ and put $\mathbb{Z}L\oplus\Upsilon=\mathcal{L}$. Then $NS(X)$ is one of the following lattices:*
i) if $\gcd(2d,h_j)=1$ for all j, then $NS(X)=\mathcal{L}$;
ii) if $L^2\equiv 0,\mod r$ with $r|h_m$, then either $NS(X)=\mathcal{L}$, or $NS(X)=\mathcal{L}'_r$ is an overlattice of $\mathcal{L}$ of index r with Υ primitively embedded in $\mathcal{L}'_r$. If $NS(X)=\mathcal{L}'_r$ then there is an element of type $(L/r,v/r)\in\mathcal{L}'_r$ which is not in $\mathcal{L}$ where $v/r\in\Upsilon^\vee$, $v^2\equiv -2d\mod 2r^2$.

Proof. Let us suppose that $L^2\equiv 0\mod r$ where $r|h_i$ for a certain i. If there exists an element v in Υ such that $v/r\in\Upsilon^\vee$ and $v^2\equiv -2d\mod 2r^2$, then the element $v=(L/r,v/r)$ has an integer intersection with all the classes of $\mathcal{L}$ and has an even self-intersection. Then (cf. Remark 6.1) adding v to the lattice $\mathcal{L}$ we find an even overlattice of $\mathcal{L}$ of index r.
We prove that if there exists an even overlattice $\mathcal{L}'_r$ of $\mathcal{L}$ with Υ primitively embedded in $\mathcal{L}'_r$, then there is a j with $\gcd(2d,h_j)>1$ (i.e. there exists an integer r such that $L^2\equiv 0\mod r$ and $r|h_m$) and the overlattice $\mathcal{L}'_r$ is constructed by adding a class $(L/r,v/r)$ to $\mathcal{L}$. Let $(\alpha L/r,\beta v'/s)$, $v'\in\Upsilon$, α, β, r, $s\in\mathbb{Z}$ be an element in $NS(X)$, then its intersection with L and with the classes of Υ is an integer. This implies that $\alpha L/r$ and $\beta v'/s$ are classes in $(\mathbb{Z}L\oplus\Upsilon)^\vee$, w.l.o.g we assume that $\gcd(\alpha,r)=1$, $\gcd(\beta,s)=1$. In particular $\alpha L/r=kL/(2d)\in(\mathbb{Z}L)^\vee$ for a certain $k\in\mathbb{Z}$ and $\beta v'/s\in\Upsilon^\vee$ so $r|2d$ and $s|h_m$.
By the relations

$$r(\alpha L/r,\beta v'/s)-\alpha L=r\beta v'/s\in NS(X)$$
$$s(\alpha L/r,\beta v'/s)-\beta v'=s\alpha L/r\in NS(X)$$

and the fact that Υ is a primitive sublattice of $NS(X)$ we obtain that $r=s$. The class is now $(\alpha L/r,\beta v'/r)$ with $r|2d$ (so $L^2=2d\equiv 0\mod r$) and $r|h_m$. Since $\gcd(\alpha,r)=1$ there exist $a,b\in\mathbb{Z}$ such that $a\alpha+br=1$, so the class $a(\alpha L/r,\beta v'/r)-bL=(L/r,a\beta v'/r)$ is in the Néron Severi group. Hence we can assume $\alpha=1$ and we take the class $(L/r,v/r)$, where $v=a\beta v'\in\Upsilon$. Observe that the self-intersection of a class in $NS(X)$ is an even integer, so $(L/r,v/r)^2=(2d+v^2)/r^2\in 2\mathbb{Z}$, hence $v^2\equiv -2d\mod 2r^2$.
Now we prove that all the possible overlattices of $\mathbb{Z}L\oplus\Upsilon$ containing Υ as a primitive sublattice, are obtained in this way. Essentially we need to prove that if there exists an overlattice $\mathcal{L}'$ of a lattice $\mathcal{L}'_r$ (containing Υ primitively) then it is a lattice $\mathcal{L}'_{r'}$ with $r|r'$, so all the possible overlattices $\mathcal{L}'$ of $\mathbb{Z}L\oplus\Upsilon$ with Υ primitive in $\mathcal{L}'$ are of type $\mathcal{L}'_k$ for a certain k. Assume that $(L/t,v/t)$, $(L/s,w/s)$ and $\mathbb{Z}L\oplus\Upsilon$ generate the lattice $\mathcal{L}'$. If $t=s$ then we have $(v-w)/t\in NS(X)$ and so $\mathcal{L}'$ is generated by $(L/t,v/t)$ and $\mathcal{L}$. If $s\neq t$ consider the element $(L/t,v/t)-(L/s,w/s)=((s-t)L/ts,(sv-wt)/(ts))$, if $\gcd(s-t,ts)=1$ then we can replace $((s-t)L/ts,(sv-wt)/(ts))$ by an element $(L/ts,v'/ts)$. Then it is an easy computation to see that the lattice $\mathcal{L}'$ is generated by this element and $\mathcal{L}$. If $\gcd((s-t),ts)\neq 1$ then we reduce the fraction $(t-s)/ts$ and we apply the same arguments. □

Proposition 6.2. *Let X_G be an algebraic K3 surface with a finite abelian group G as group of symplectic automorphisms and with $\rho(X)=rank(\Omega_G)+1$. Then the Néron Severi group of X is one of the following (we write $a\equiv_d b$ for $a\equiv b\mod d$)*

- $G=\mathbb{Z}/2\mathbb{Z}$: $\begin{cases} L^2\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \text{ or} \\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\end{cases} \\ L^2\not\equiv_4 0, & NS(X)=L\oplus\Omega_G;\end{cases}$

- $G=\mathbb{Z}/p\mathbb{Z}$, $p=3,5,7$: $\begin{cases} L^2\equiv_{2p} 0, & \begin{cases} NS(X)=L\oplus\Omega_G \text{ or} \\ NS(X)/(L\oplus\Omega_G)=\langle(L/p,v/p)\rangle,\ v/p\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{2p^2} 0,\end{cases} \\ L^2\not\equiv_{2p} 0, & NS(X)=L\oplus\Omega_G;\end{cases}$

$$\bullet\ G = \begin{array}{l}\mathbb{Z}/4\mathbb{Z}\\ (\mathbb{Z}/2\mathbb{Z})^2\\ (\mathbb{Z}/2\mathbb{Z})^3\\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}\end{array} : \begin{cases} L^2\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/4,v/4)\rangle,\ v/4\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{32} 0,\end{cases}\\ L^2\not\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0;\end{cases}\end{cases}$$

$$\bullet\ G = \mathbb{Z}/6\mathbb{Z} : \begin{cases} L^2\equiv_6 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/3,v/3)\rangle,\ v/3\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{18} 0,\\ NS(X)/(L\oplus\Omega_G)=(L/6,v/6),\ v/6\in\Omega_G^\vee/\Omega_G,\ L^2+v^2\equiv_{72} 0,\end{cases}\\ L^2\not\equiv_6 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=(L/2,v/2),\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0;\end{cases}\end{cases}$$

$$\bullet\ G = \begin{array}{l}\mathbb{Z}/8\mathbb{Z}\\ (\mathbb{Z}/2\mathbb{Z})^4\\ (\mathbb{Z}/4\mathbb{Z})^2\end{array} : \begin{cases} L^2\equiv_8 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/4,v/4)\rangle,\ v/4\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{32} 0,\\ NS(X)/(L\oplus\Omega_G)=(L/8,v/8),\ v/8\in\Omega_G^\vee/\Omega_G,\ L^2+v^2\equiv_{128} 0,\end{cases}\\ L^2\equiv_4 0,\ L^2\not\equiv_8 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/4,v/4)\rangle,\ v/4\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{32} 0,\end{cases}\\ L^2\not\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=(L/2,v/2),\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0;\end{cases}\end{cases}$$

$$\bullet\ G = (\mathbb{Z}/3\mathbb{Z})^2 : \begin{cases} L^2\equiv_{18} 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/3,v/3)\rangle,\ v/3\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{18} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/9,v/9)\rangle,\ v/9\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{162} 0,\end{cases}\\ L^2\not\equiv_{18} 0,\ L^2\equiv_6 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=(L/3,v/3),\ v/3\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{72} 0,\end{cases}\\ L^2\not\equiv_6 0,\ NS(X)=L\oplus\Omega_G; & \end{cases}$$

$$\bullet\ G = \mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} : \begin{cases} L^2\equiv_{12} 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/12,v/12)\rangle,\ v/12\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{288} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/6,v/6)\rangle,\ v/6\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{72} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/4,v/4)\rangle,\ v/4\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{32} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/3,v/3)\rangle,\ v/3\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{18} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\end{cases}\\ L^2\not\equiv_{12} 0, L^2\equiv_6 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/6,v/6)\rangle,\ v/6\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{72} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/3,v/3)\rangle,\ v/3\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{18} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\end{cases}\\ L^2\not\equiv_{12} 0, L^2\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=\langle(L/4,v/4)\rangle,\ v/4\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_{32} 0,\\ NS(X)/(L\oplus\Omega_G)=\langle(L/2,v/2)\rangle,\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0,\end{cases}\\ L^2\not\equiv_6 0,\ L^2\not\equiv_4 0, & \begin{cases} NS(X)=L\oplus\Omega_G \ \mathit{or}\\ NS(X)/(L\oplus\Omega_G)=(L/2,v/2),\ v/2\in\Omega_G^\vee/\Omega_G, L^2+v^2\equiv_8 0.\end{cases}\end{cases}$$

Moreover the class L can be choosen as an ample class. We will denote by $\mathcal{L}_G^{2d}$ *the lattice* $\mathbb{Z}L\oplus\Omega_G$ *where* $L^2=2d$ *and by* $\mathcal{L}'^{2d}_{G,r}$ *the overlattices of* $\mathcal{L}_G^{2d}$ *of index* r.

Proof. The statement follows from the application of Proposition 6.1 to the lattices $\mathbb{Z}L\oplus\Omega_G$, where Ω_G has the discriminant group given in Proposition 5.1. The cases $G=\mathbb{Z}/p\mathbb{Z}$, $p=3,5,7$ are described in [GS]. The case $G=\mathbb{Z}/2\mathbb{Z}$ is described in [vGS]. Applying Proposition 6.1 one obtains that there are no conditions on L^2 to obtain an overlattice of index two of $\mathbb{Z}L\oplus\Omega_G$. However if $G=\mathbb{Z}/2\mathbb{Z}$ and $L^2\not\equiv_4 0$ there are no possible overlattices of $\mathbb{Z}L\oplus\Omega_{\mathbb{Z}/2\mathbb{Z}}$. This depends by the properties of $\Omega_{\mathbb{Z}/2\mathbb{Z}}$. In fact if there were an overlattice of $\mathbb{Z}L\oplus\Omega_{\mathbb{Z}/2\mathbb{Z}}$, $L^2\not\equiv_4 0$, then there would be an element of the form $(L/2,v/2)$ such that $L^2+v^2\equiv 0 \mod 8$. Since $L^2\not\equiv_4 0$ then $L^2\equiv_8 2$ or $L^2\equiv_8 6$. This implies that

$v^2 \equiv_8 \pm 2$. But all the elements in $\Omega_{\mathbb{Z}/2\mathbb{Z}} = E_8(-2)$ have self-intersection which is a multiple of four. Since $L^2 > 0$ by the Riemann Roch theorem we can assume that L or $-L$ effective. Hence we assume L to be effective. Let D be an effective (-2)-curve, then $D = \alpha L + v'$, with $v' \in \Omega_G$ and $\alpha > 0$ since Ω_G does not contain (-2)-curves. We have $L \cdot D = \alpha L^2 > 0$, and so L is ample. □

Proposition 6.3. *Let $\mathcal{L}_G$ be either $\mathcal{L}^{2d}_G$, or $\mathcal{L}'^{2d}_{G,r}$. Let X_G be an algebraic K3 surface such that $NS(X_G) = \mathcal{L}_G$. Then X_G admits G as group of symplectic automorphisms (cf. also* [Nik1, Theorem 4.15]*,* [vGS, Proposition 2.3]*,* [GS, Proposition 5.2]*).*

Proof. Denote by $\widetilde{G}$ a group of isometries of $E_8(-1)^2 \oplus U^3$ acting as in the case of the K3-surfaces with an elliptic fibration and a group of symplectic automorphisms G.
Step 1: the isometries of $\widetilde{G}$ fix the sublattice $\mathcal{L}_G$. Since $\widetilde{G}(\Omega_G) = \Omega_G$ and $\widetilde{G}(L) = L$ (because $L \in \Omega_G^{\perp}$ which is the invariant sublattice of Λ_{K3}), if $\mathcal{L}_G = \mathcal{L}^{2d}_G = \mathbb{Z}L \oplus \Omega_G$ it is clear that $\widetilde{G}(\mathcal{L}_G) = \mathcal{L}_G$. Now we consider the case $\mathcal{L}_G = \mathcal{L}'^{2d}_G$. The isometry $\widetilde{G}$ acts trivially on $\Omega_G^{\vee}/\Omega_G$ (it can be proved by a computer-computation on the generators of the discriminant form of the lattices Ω_G and on $(\mathbb{Z}L)^{\vee}/\mathbb{Z}L)$.
Let $\frac{1}{r}(L, v') \in \mathcal{L}_G$, with $v' \in \Omega_G$. This is also an element in $(\Omega_G \oplus L\mathbb{Z})^{\vee}/(\Omega_G \oplus L\mathbb{Z})$. So we have $\widetilde{G}(\frac{1}{r}(L, v')) \equiv \frac{1}{r}(L, v') \mod (\Omega_G \oplus \mathbb{Z}L)$, which means $\widetilde{G}(\frac{1}{r}(L, v')) = \frac{1}{r}(L, v') + (\beta L, v'')$, $\beta \in \mathbb{Z}$, $v'' \in \Omega_G$. Hence in any case we have $\widetilde{G}(\mathcal{L}_G) = \mathcal{L}_G$.
Step 2: The isometries of $\widetilde{G}$ preserve the Kähler cone $\mathcal{C}^+_X$. We recall that the Kähler cone of a K3 surface X can be described as the set

$$\mathcal{C}^+_X = \{x \in V(X)^+ \mid (x, d) > 0 \text{ for each } d \in NS(X) \text{ such that } (d, d) = -2,\ d \text{ effective}\}$$

where $V(X)^+$ is the connected component of $\{x \in H^{1,1,}(X, \mathbb{R}) \,|\, (x, x) > 0\}$ containing a Kähler class. First we prove that $\widetilde{G}$ fixes the set of the effective (-2)-classes. Since there are no (-2)-classes in Ω_G, if $N \in \mathcal{L}_G$ has $N^2 = -2$ then $N = \frac{1}{r}(aL, v')$, $v' \in \Omega_G$, for an integer $a \neq 0$ (recall that $r \in \mathbb{Z}_{>0}$). Since $\frac{1}{r} aL^2 = L \cdot N > 0$, because L and N are effective divisor, we obtain $a > 0$. The curve $N' = \widetilde{G}(N)$ is a (-2)-class because $\widetilde{G}$ is an isometry, hence N' or $-N'$ is effective. Since $N' = \widetilde{G}(N) = (aL/r, \widetilde{G}(v')/r)$ we have $-N' \cdot L = -a/rL^2 < 0$ and so $-N'$ is not effective. Hence $N' = \widetilde{G}(N)$ is an effective (-2)-class. Using the fact that $\widetilde{G}$ has finite order it is clear that $\widetilde{G}$ fixes the set of the effective (-2)-classes. Now let $x \in \mathcal{C}^+_X$ then $\widetilde{G}(x) \in \mathcal{C}^+_X$, in fact $(\widetilde{G}(x), \widetilde{G}(x)) = (x, x) > 0$ and for each effective (-2)-class d there exists an effective (-2)-class d' with $d = \widetilde{G}(d')$, so we have $(\widetilde{G}(x), d) = (\widetilde{G}(x), \widetilde{G}(d')) = (x, d') > 0$. Hence $\widetilde{G}$ preserves $\mathcal{C}^+_X$ as claimed.
Step 3: The isometries of $\widetilde{G}$ are induced by automorphisms of the surface X. The isometries of $\widetilde{G}$ are the identity on the sublattice $\mathcal{L}_G^{\perp}$ of Λ_{K3}, so they are Hodge isometries. By the Torelli theorem, an effective Hodge isometry of the lattice Λ_{K3} is induced by an automorphism of the K3 surface (cf. [BPV, Theorem 1.11]). By [BPV, Corollary 3.11], $\widetilde{G}$ preserves the set of the effective divisors if and only if it preserves the Kähler cone. The isometries of $\widetilde{G}$ preserve the Kähler cone, from Step 2, and so they are effective, hence induced by automorphisms of the surface. The latter are symplectic by construction (they are the identity on the transcendental lattice $T_X \subset \Omega_G^{\perp}$). □

The previous proposition implies the following:

Proposition 6.4. *Let $\mathcal{L}_G$ be either $\mathcal{L}^{2d}_G$, or $\mathcal{L}'^{2d}_{G,r}$. Let X be an algebraic K3 surface with Picard number equal to $1 + rank(\Omega_G)$ for a certain G in the list* (1)*. The surface X admits G as group of symplectic automorphisms if and only if $NS(X) = \mathcal{L}_G$.*

In general is an open problem to understand if for a particular lattice $\mathcal{L}_G$ there exists a K3 surface X such that $NS(X) = \mathcal{L}_G$.
For the lattices $\mathcal{L}^{2d}_{\mathbb{Z}/pZ}$ with $p = 2, 3, 5$ the existence is proved in [vGS] and in [GS] as for the lattices $\mathcal{L}'^{2d}_{\mathbb{Z}/pZ,r}$ for $(p, r) = (2, 2), (3, 3), (5, 5), (7, 7)$. Observe that an even lattice of signature $(1, n)$, $(n < 20)$ is the Néron Severi group of a K3 surface if and only if it admits a primitive embedding in Λ_{K3} (cf.

[Mo, Corollary 1.9]). The lattices $\mathcal{L}'^{2d}_{G,r}$ for $G=\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^2$ and any r satisfy the conditions of [Nik2, Theorem 1.14.4] so they admits a primitive embedding in the K3 lattice. For the lattices $\mathcal{L}^{2d}_{\mathbb{Z}/7Z}$, $d\equiv 0 \mod 7$, $\mathcal{L}^{2d}_{G}$, $G=\mathbb{Z}/8\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^m$ $m=3,4$, $(\mathbb{Z}/4\mathbb{Z})^2$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $\mathcal{L}^{2d}_{G}$ for $G=(\mathbb{Z}/3\mathbb{Z})^2$,$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ for $d\equiv 0 \mod 3$ a carefully analysis shows that the discriminant group admits a number of generators which would be bigger than the rank of the transcendental lattice (if a K3 surface would exists) so this is not possible. The other cases needs a detailed analysis. We discuss the case $G=\mathbb{Z}/7\mathbb{Z}$ and show the following proposition (some other cases are described in [G1]). The following lemma is completely trivial, but it is useful for the next proposition.

Lemma 6.1. *Let S be a lattice primitively embedded in Λ_{K3} and $s\in S$. Then either $T=\mathbb{Z}s\oplus S^{\perp_{\Lambda_{K3}}}$ is primitively embedded in Λ_{K3}, or there exists an overlattice T' of T such that T' is primitively embedded in Λ_{K3} and $T\hookrightarrow T'$ with finite index. Viceversa if $\Gamma=\mathbb{Z}L\oplus S^{\perp_{\Lambda_{K3}}}$ or an overlattice with finite index of Γ is primitively embedded in Λ_{K3}, then $L\in S$ and so there exists an element of square L^2.*

Proposition 6.5. *There exists a K3 surface with Néron Severi group isomorphic to $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ if and only if $d\equiv 1,2,4 \mod 7$ and a K3 surface with Néron Severi group isomorphic to $\mathcal{L}'^{2d}_{\mathbb{Z}/7\mathbb{Z},7}$ if and only if $d\equiv 0 \mod 7$.*

Proof. We apply Lemma 6.1 in our case. First we consider the lattice $\Omega^{\perp}_{\mathbb{Z}/7\mathbb{Z}}\simeq\begin{pmatrix} 4 & 1\\ 1 & 2\end{pmatrix}\oplus U(7)$. Let d be a positive integer, then if

$$d\equiv 0,1,2,4 \mod 7 \tag{10}$$

there exists an element in $\Omega^{\perp}_{\mathbb{Z}/7\mathbb{Z}}$ with square $2d$. These elements are:

$$\begin{array}{ll}(0,0,1,k), \text{ if } d=7k, & (1,1,1,k) \text{ if } d=7k+4,\\ (0,1,1,k) \text{ if } d=7k+1, & (1,0,1,k) \text{ if } d=7k+2.\end{array}$$

If d does not satisfy (10) there is no element L of square $2d$. Indeed the bilinear form on the vector (p,q,r,s) is $4p^2+2pq+2q^2+14rs$. If $L=(p,q,r,s)$, then $d=2p^2+pq+q^2+7rs$. Considering this equation modulo 7 we obtain $d=(4p+q)^2$, but 3, 5 and 6 are not square in $\mathbb{Z}/7\mathbb{Z}$. Hence the lattices $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ with $d\equiv 3,5,6 \mod 7$ do not admit a primitive embedding in Λ_{K3} by Lemma 6.1. Since the lattice $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ with $d\equiv 1,2,4 \mod 7$ do not admits overlattices (Proposition 6.2), they admits a primitive embedding in Λ_{K3}, again by Lemma 6.1. We observed before that $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ for $d\equiv 0 \mod 7$ is not primitively embedded in Λ_{K3}. On the other hand, by Lemma 6.1, $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ or an overlattice of $\mathcal{L}^{2d}_{\mathbb{Z}/7\mathbb{Z}}$ has to be primitively embedded in Λ_{K3}. Hence the lattice $\mathcal{L}'^{2d}_{\mathbb{Z}/7\mathbb{Z},7}$ is primitively embedded in Λ_{K3} for each $d\equiv 0 \mod 7$. □

In the cases where the existence is proved one can give a description of the coarse moduli space similar to [vGS, Proposition 2.3], [GS, Corollary 5.1]. In fact under this condition, Propositions 6.2, 6.3 imply that the coarse moduli space of algebraic K3 surfaces admitting a certain group of symplectic automorphisms is the coarse moduli space of the $\mathcal{L}$-polarized K3 surfaces, for a certain lattice $\mathcal{L}$ (for a precise definition of $\mathcal{L}$-polarized K3 surfaces see [D]).

Remark 6.2. The dimension of the coarse moduli space of the algebraic K3 surfaces admitting G as group of symplectic automorphisms is $19-rank(\Omega_G)$.
In fact for each group G there exist K3 surfaces such that their Néron Severi group is either $\mathcal{L}^{2d}_{G}$ or $\mathcal{L}'^{2d}_{G,r}$ (cf. [G1] and in fact is a similar computation as in Proposition 6.5), so the generic K3 surface admitting G as group of symplectic automorphisms has Picard number $1+rk(\Omega_G)$. The dimension of the moduli space of a family of K3 surfaces such that the generic one has Picard number ρ is $20-\rho$.

□

7. Appendix

In the Tables 1, 2 we number the fibers according to the column "singular fibers" (the fiber 1 is the first fiber mentioned in the column "singular fibers") and we write the intersections of the section and the components of the fibers which are not zero.

Table 1

G	equation	trivial lattice	singular fibers	intersection torsion section	T_X
$\mathbb{Z}/2\mathbb{Z}$	$y^2=x(x^2+a(\tau)x+b(\tau))$ $deg(a)=4\ deg(b)=8$	$U\oplus A_1^{\oplus 8}$	$8I_2+8I_1$	t_1 has order 2 $t_1\cdot C_1^{(i)}=1,\ i=1,\dots,8$	$N\oplus U(2)^{\oplus 2}$
$\mathbb{Z}/3\mathbb{Z}$	$y^2=x^3+A(\tau)x+B(\tau))$ $A=\frac{6dc+d^4}{3},\ B=\frac{27c^2-d^6}{3^3}$	$U\oplus A_2^{\oplus 6}$	$6I_3+6I_1$	t_1 has order 3 $t_1\cdot C_1^{(i)}=1,\ i=1,\dots,6$	$U\oplus U(3)\oplus A_2(-1)^{\oplus 2}$
$\mathbb{Z}/4\mathbb{Z}$	$y^2=x(x^2+(e^2(\tau)-2f(\tau))x+f^2(\tau))$ $deg(f)=4\ \ deg(e)=2$	$U\oplus A_3^{\oplus 4}\oplus A_1^{\oplus 2}$	$4I_4+2I_2+4I_1$	t_1 has order 4 $t_1\cdot C_1^{(i)}=1,\ i=1,\dots,6$	
$\mathbb{Z}/5\mathbb{Z}$	$y^2=x^3+A(\tau)x+B(\tau),$ $A=\frac{(-g^4+g^2h^2-h^4-3hg^3+3h^3g)}{3},$ $B=\frac{(g^2+h^2)(19g^4-34g^2h^2+19h^4)}{108}$ $\frac{+18hg^3-18h^3g}{108}$	$U\oplus A_4^{\oplus 4}$	$4I_5+4I_1$	t_1 has order 5 $t_1\cdot C_1^{(i)}=1,\ i=1,2,$ $t_1\cdot C_2^{(j)}=1,\ j=3,4$	$U\oplus U(5)$
$\mathbb{Z}/6\mathbb{Z}$	$y^2=x(x^2+(3k^2(\tau)-l^2(\tau))x+k^3(\tau)(3k(\tau)-2l(\tau)))$ $deg(k)=deg(l)=2$	$U\oplus A_5^{\oplus 2}\oplus A_2^{\oplus 2}\oplus A_1^{\oplus 2}$	$2I_6+2I_3+2I_2+2I_1$	t_1 has order 6 $t_1\cdot C_1^{(i)}=1,\ i=1,\dots,6$	$U\oplus U(6)$
$\mathbb{Z}/7\mathbb{Z}$	$y^2+(1+\tau-\tau^2)xy+(\tau^2-\tau^3)y=x^3+(\tau^2-\tau^3)x^2$	$U\oplus A_6^{\oplus 3}$	$3I_7+3I_1$	t_1 has order 7 $t_1\cdot C_1^{(1)}=t_1\cdot C_2^{(2)}=1$ $t_1\cdot C_3^{(3)}=1$	$\begin{bmatrix}4&1\\1&2\end{bmatrix}$
$\mathbb{Z}/8\mathbb{Z}$	$y^2=x\left(x^2+\left(-2m(\tau)^2n(\tau)^2+\frac{(m(\tau)-n(\tau))^4}{4}\right)x\right)$ $+x\left(m^4(\tau)n^4(\tau)\right)$ $deg(m)=1\ \ deg(n)=1$	$U\oplus A_7^{\oplus 2}\oplus A_3\oplus A_1$	$2I_8+I_4+I_2+2I_1$	t_1 has order 8 $t_1\cdot C_1^{(i)}=1,\ i=1,3,4$ $t_1\cdot C_3^{(2)}=1$	$\begin{bmatrix}2&0\\0&4\end{bmatrix}$

TABLE 2

G	equation	trivial lattice	singular fibers	intersection torsion section	T_X
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$y^2 = x(x - p(\tau))(x - q(\tau))$ $deg(p(\tau)) = 4$ $deg(q(\tau)) = 4$	$U \oplus A_1^{\oplus 12}$	$12I_2$	t_1, u_1 have order 2 , $t_1 \cdot C_1^{(i)} = 1,\ i = 1, \ldots, 8$ $u_1 \cdot C_1^{(j)} = 1$ $j = 5, \ldots, 12$	$U(2)^{\oplus 2} \oplus A_1^{\oplus 4}$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$y^2 = x(x - r^2(\tau))(x - s^2(\tau))$ $deg(r) = 2$ $deg(s) = 2$	$U \oplus A_1^{\oplus 4} \oplus A_3^{\oplus 4}$	$4I_4 + 4I_2$	t_1 has order 4 , u_1 has order 2 $t_1 \cdot C_1^{(i)} = 1,\ i = 1, \ldots, 6$ $t_1 \cdot C_0^{(i)} = 1,\ i = 7, 8$ $u_1 \cdot C_2^{(i)} = 1,\ i = 1, 2$ $u_1 \cdot C_0^{(i)} = 1,\ i = 3, 4$ $u_1 \cdot C_1^{(i)} = 1$ $i = 5, 6, 7, 8$	$U(2) \oplus U(4)$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	$y^2 = x[x - (3w(\tau) - z(\tau))(w(\tau) + z(\tau)^3][x - (3w(\tau)$ $+z(\tau))(w(\tau) - z(\tau))^3]$ $deg(w) = 1$ $deg(z) = 1$	$U \oplus A_5^{\oplus 3} \oplus A_1^{\oplus 3}$	$3I_2 + 3I_6$	t_1 has order 6 , u_1 has order 2 $t_1 \cdot C_1^{(i)} = 1,\ i = 1, 2, 4, 5$ $t_1 \cdot C_2^{(3)} = t_1 \cdot C_0^{(6)} = 1,$ $u_1 \cdot C_3^{(i)} = 1,\ i = 1, 3$ $u_1 \cdot C_0^{(i)} = 1$ $i = 2, 4$ $u_1 \cdot C_1^{(i)} = 1$ $i = 5, 6$	$\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$y^2 = x^3 + 12x[(\tau^2 + 1)(\alpha\tau^2 + \beta)^3 - (\tau^2 + 1)^4]+$ $2[(\alpha\tau^2 + \beta)^6 - 20(\alpha\tau^2 + \beta)^3(\tau^2 + 1)^3 - 8(\tau^2 + 1)^6]$ $\alpha, \beta \in \mathbb{C}$	$U \oplus A_2^{\oplus 8}$	$8I_3$	t_1, u_1 have order 3 $t_1 \cdot C_1^{(i)} = 1,\ i = 1, \ldots, 6$ $t_1 \cdot C_0^{(i)} = 1$ $i = 7, 8$ $u_1 \cdot C_0^{(i)} = 1,\ i = 1, 2$ $u_1 \cdot C_1^{(i)} = 1$ $i = 3, 4, 7, 8$ $u_1 \cdot C_2^{(i)} = 1$ $i = 5, 6$	$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$y^2 = x(x - u^2(\tau)v^2(\tau))(x - (1/4)(u^2(\tau) + v^2(\tau))^2)$ $deg(u) = 1 \deg(w) = 1$	$U \oplus A_3^{\oplus 6}$	$6I_4$	t_1, u_1 have order 4 $t_1 \cdot C_1^{(i)} = 1,\ i = 1, 2, 3, 4$ $t_1 \cdot C_2^{(5)} = t_1 \cdot C_0^{(6)} = 1$ $u_1 \cdot C_1^{(i)} = 1,\ i = 3, 4, 5, 6$ $u_1 \cdot C_2^{(1)} = t_1 \cdot C_0^{(2)} = 1$	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

In the Table 3 we give a set of generators for the lattice Ω_G for all the group G in the list (1) except $G=(\mathbb{Z}/4\mathbb{Z})$, $i=1,2$, $G=(\mathbb{Z}/2\mathbb{Z})^i$, $i=2,3,4$, $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ which are presented explicitly in Propositions 4.2, 4.3, 4.4. We always refer to the elliptic fibration given in Table 2. In each case the sublattice of the Néron Severi group which is fixed by G is generated, at least over $\mathbb{Q}$, by the class of the fiber and by the sum of the classes of the sections.

TABLE 3

G	generators of the lattice $\Omega_G=(NS(X_G)^G)^\perp$
$\mathbb{Z}/2\mathbb{Z}$	$b_i=C_1^{(i)}-C_1^{(i+1)},\ i=1,\ldots,6,\ b_7=F-C_1^{(1)}-C_1^{(2)},\ b_8=s-t$
$\mathbb{Z}/3\mathbb{Z}$	$g_1=s-t_1,\quad g_2=s-t_2,\quad g_{i+2}=C_1^{(i)}-C_1^{(i+1)},\quad i=1,2,3,4,$ $g_{j+6}=C_1^{(j)}-C_2^{(j+1)},\quad j=1,2,3,4,\quad g_{11}=C_2^{(1)}-C_2^{(2)},\quad g_{12}=C_0^{(1)}-C_2^{(2)}.$
$\mathbb{Z}/5\mathbb{Z}$	$b_1=s-t_1,\ b_2=t_1-t_2,\ b_3=t_2-t_3,\ b_4=t_3-t_4,\ b_5=C_0^{(3)}-C_1^{(2)},\ b_{i+5}=C_1^{(i)}-C_3^{(i)},\ i=1,2,3,$ $b_{j+8}=C_2^{(j)}-C_4^{(j)},\ j=1,2,3,\ b_{h+11}=C_1^{(h)}-C_4^{(h)},\ h=1,2,3,\ b_{k+14}=C_1^{(k)}C_1^{(k+1)},\ k=1,2.$
$\mathbb{Z}/6\mathbb{Z}$	$b_i=C_i^{(2)}-C_{i+2}^{(2)},\ i=1,\ldots,4,\ b_5=C_2^{(2)}-C_1^{(1)},\ b_6=C_1^{(2)}-C_1^{(1)},\ b_7=C_1^{(1)}-C_3^{(1)},\ b_8=C_1^{(1)}-C_4^{(1)},\ b_9=C_2^{(1)}-C_4^{(1)},\ b_{10}=C_1^{(3)}-C_1^{(4)},$ $b_{11}=C_1^{(3)}-C_2^{(4)},\ b_{12}=C_2^{(3)}-C_1^{(4)},\ b_{13}=C_1^{(5)}-C_1^{(6)},\ b_{14}=C_1^{(6)}-C_1^{(1)}-C_2^{(1)}-C_3^{(1)},\ b_{15}=C_2^{(4)}-C_1^{(1)}-C_2^{(1)},\ b_{16}=s-t$
$\mathbb{Z}/7\mathbb{Z}$	$b_1=s-t_1,\ b_2=C_0^{(2)}-C_1^{(1)},\ b_{i+2}=t_i-t_{i+1},\ i=1,\ldots,5,\ b_{j+7}=C_j^{(1)}-C_{j+2}^{(1)},\ j=1,\ldots,4,$ $b_{12}=C_3^{(1)}-C_6^{(1)},\ b_{13}=C_3^{(2)}-C_6^{(2)},\ b_{14}=C_1^{(1)}-C_1^{(2)},\ b_{h+14}=C_h^{(2)}-C_{h+2}^{(2)},\ h=1,\ldots,4$
$\mathbb{Z}/8\mathbb{Z}$	$b_1=t_1-s,\quad b_2=t_3-t_1,\quad b_3=t_4-t_2,\quad b_4=C_3^{(3)}-C_1^{(1)}-C_2^{(1)},$ $b_i=C_{i-2}^{(1)}-C_{i-4}^{(1)}$ with i=5,6,7,8, $\quad b_j=C_{j-6}^{(1)}-C_{j-4}^{(1)}$ with j=9,…,13, $b_{14}=C_4^{(1)}-C_1^{(1)},\quad b_{15}=C_4^{(2)}-C_1^{(2)},\quad b_{16}=C_0^{(2)}-C_1^{(1)},\quad b_{17}=C_1^{(2)}-C_1^{(1)},\ b_{18}=C_1^{(3)}-C_3^{(3)}$
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	$b_1=s-t_1,\quad b_2=s-u_1,\quad b_{i+2}=C_1^{(i)}-C_3^{(i)},\ i=1,2,3,\quad b_{j+5}=C_2^{(j)}-C_4^{(j)},\ j=1,2,3,$ $b_9=C_3^{(1)}-C_5^{(1)}\quad b_{10}=C_3^{(3)}-C_5^{(3)}\quad b_{11}=C_1^{(1)}-C_1^{(2)}\quad b_{12}=C_1^{(4)}-C_1^{(5)}$ $b_{13}=C_1^{(2)}-C_2^{(3)}\quad b_{14}=C_0^{(1)}-C_1^{(2)}\quad b_{15}=C_2^{(1)}-C_2^{(2)}\quad b_{16}=C_1^{(1)}-C_2^{(2)}$ $b_{17}=C_1^{(1)}-C_3^{(3)}\quad b_{18}=C_1^{(5)}-C_1^{(2)}-C_2^{(2)}-C_3^{(2)}$
$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$	$b_1=s-t_1\quad b_2=s-u_1\quad b_{i+2}=C_1^{(i)}-C_1^{(i+1)}\quad i=1,\ldots 7$ $b_{j+8}=C_2^{(j)}-C_2^{(j+1)}\quad j=2\ldots 6\quad b_{15}=C_1^{(1)}-C_2^{(2)}\quad b_{16}=C_0^{(1)}-C_2^{(2)}.$

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