Triple lines on cubic threefolds

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Overview

Introduction

Cubic threefolds are smooth hypersurfaces of degree three in the projective-four space $\mathbb{P}^4$.

Examples:

1. The Fermat cubic:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \tag{1}$$

2. The Klein cubic:

$$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0 = 0. \tag{2}$$

Introduction

In what follows we denote by X a smooth cubic threefold defined over $\mathbb{C}$. Let $\mathbb{G}(1,4)$ be the grassmannian of lines in $\mathbb{P}^4$.

Definition

The Fano surface $\mathrm{F}(X)$ of X is the set parametrizing the lines of $\mathbb{G}(1,4)$ which lie entirely in X.

We write

$$\mathrm{F}(X) = \{\ell \in \mathbb{G}(1,4) \mid \ell \subset X\}.$$

Remark: The Fano surface $\mathrm{F}(X)$ of a smooth cubic threefold X is smooth.

Lines on cubic threefolds

- Clemens and Griffiths in 1972.

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Definition

Let ℓ be a line on X. Lines with $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell \bigoplus \mathcal{O}_\ell$ are called lines of the first type and those with $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell(1) \bigoplus \mathcal{O}_\ell(-1)$ are called lines of the second type.

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An alternative description is given by the following lemma:

Lemma

The line $\ell \subset X$ is of the second type if and only if there exists a unique 2-plane $P \supset \ell$ in $\mathbb{P}^4$ tangent to X at every point of ℓ. If $\ell \subset X$ is a line of the first type, then there is no 2-plane tangent to X in all points of ℓ.

Remark: The generic line $\ell \subset X$ is of the first type.

Lines on cubic threefolds

Some notations:

- $[x_0 : x_1 : x_2 : x_3 : x_4]$ the homogeneous coordinates on $\mathbb{P}^4$.
- $p_{i,j}, 0 \leq i < j \leq 4$ the Plücker coordinates of $\mathbb{G}(1,4) \subset \mathbb{P}^9$.
- $F(x_0, x_1, x_2, x_3, x_4) = 0$ the equation of $X \subset \mathbb{P}^4$.
- On the affine chart $p_{0,1} = 1$ of $\mathbb{P}^9$, a point $\ell \in \mathbb{G}(1,4)$ corresponds to a line spanned by two points $v_0 = [1 : 0 : -p_{1,2} : -p_{1,3} : -p_{1,4}]$ and $v_1 = [0 : 1 : p_{0,2} : p_{0,3} : p_{0,4}]$ in $\mathbb{P}^4$.
- $(p_{0,2}, p_{0,3}, p_{0,4}, p_{1,2}, p_{1,3}, p_{1,4})$ the local coordinates on $\mathbb{G}(1,4)$.

Lines on cubic threefolds

An arbitrary point $p \in \ell$ has coordinates

$$\begin{aligned} x_0 &= t_0 \\ x_1 &= t_1 \\ x_2 &= -p_{1,2}t_0 + p_{0,2}t_1 \\ x_3 &= -p_{1,3}t_0 + p_{0,3}t_1 \\ x_4 &= -p_{1,4}t_0 + p_{0,4}t_1 \end{aligned}$$

with $[t_0 : t_1] \in \mathbb{P}^1$. The line ℓ is on X if and only if

$$\begin{aligned} 0 &= F(t_0, t_1, -p_{1,2}t_0 + p_{0,2}t_1, -p_{1,3}t_0 + p_{0,3}t_1, -p_{1,4}t_0 + p_{0,4}t_1) \quad (3) \\ &= t_0^3\phi^{3,0}(\ell) + t_0^2t_1\phi^{2,1}(\ell) + t_0t_1^2\phi^{1,2}(\ell) + t_1^3\phi^{0,3}(\ell) \quad (4) \end{aligned}$$

for all $[t_0 : t_1] \in \mathbb{P}^1$.

Lines on cubic threefolds

This implies

$$\phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0$$

which are the local equations of the Fano surface $\mathrm{F}(X)$ on the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4)$.

Eckardt points

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A point $p \in X$ is called an Eckardt point if it is a point of multiplicity 3 for the intersection $X \cap \mathrm{T}_p X$, where $\mathrm{T}_p X$ denotes the projective tangent space of X at p.

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Proposition

A point $p \in X$ is an Eckardt point if it is contained in infinitely many lines.

Eckardt points

Some features on Eckardt points on cubic threefolds:

- Clemens and Griffiths proved in 1972 that a smooth cubic threefold can contain at most finitely many Eckardt points, and in fact at most 30 according to Canonero, Catalisano and Serpico in 1997.
- The Fermat cubic is the unique cubic threefold that contains 30 Eckardt points.
- A generic cubic threefold does not contain Eckardt points.

Eckardt points

- Each Eckardt point $p \in X$ determines an elliptic curve E_p on its Fano surface, which is the base of the cone $X \cap \mathrm{T}_p X$ (Tjurin in 1971).
- Lines going through Eckardt points are of the second type.
- In 2009 Roulleau proved that the number of elliptic curves on a Fano surface $\mathrm{F}(X)$ is at most 30. He also proved that the Fano surface of the Fermat cubic is the unique one that contains 30 elliptic curves.

Multiple lines on cubic threefolds

Let ℓ be a line on X and P a 2-plane containing it. We look at the intersection $P \cap X$. We write $P \cap X = \ell \cup C$, where C is a conic.

Figure 1: Conic and a line

Multiple lines on cubic threefolds

If the conic degenerates, then we have $P \cap X = \ell \cup \ell' \cup \ell''$ where ℓ, ℓ' and ℓ'' are three distinct lines.

Figure 2: Three distinct lines

Multiple lines on cubic threefolds

If $P \cap X = 2\ell \cup \ell'$ with ℓ distinct from ℓ', then ℓ is called a double line on X and ℓ' the residual line of ℓ. We say that the 2-plane P is tangent to X at every point of ℓ.

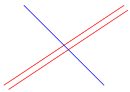

Figure 3: Double line and a line

- In 1972 Murre showed that the double lines on cubic threefolds are exactly the lines of the second type.

Multiple lines on cubic threefolds

If $P \cap X = 3\ell$, then we say that ℓ is a triple line on X.

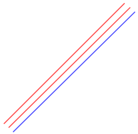

Figure 4: Triple line

- Until now almost nothing is known about triple lines on cubic threefolds except that the set of triple lines in a cubic threefold is finite, which was proved by Clemens and Griffiths in 1972.

Curves on the Fano surface

- In 1972 Murre showed that the set

$$\mathrm{F}_0(X) = \{\ell \in \mathrm{F}(X), \exists P \simeq \mathbb{P}^2 | P \cap X = 2\ell \cup \ell^{'}\}$$

of lines of the second type is an algebraic curve on the Fano surface $\mathrm{F}(X)$.

- Another curve on the the Fano surface $\mathrm{F}(X)$ is the set

$$\mathrm{R}(X) = \{\ell^{'} \in \mathrm{F}(X), \exists P \simeq \mathbb{P}^2 | P \cap X = 2\ell \cup \ell^{'}\}$$

of residual lines.

Curves on the Fano surface

Let us consider the incidence variety

$$\mathrm{I} = \{(\ell, \ell') \in \mathrm{F}_0(X) \times \mathrm{R}(X), \exists P \simeq \mathbb{P}^2 \mid P \cap X = 2\ell \cup \ell'\}.$$

- In 2017 Naranjo, Ortega and Verra proved that:

1) $\mathrm{I} \to \mathrm{R}(X)$ is finite-to-one.
2) $\mathrm{I} \to \mathrm{F}_0(X)$ is bijective.
3) For a generic cubic threefold X the curves $\mathrm{F}_0(X)$ and $\mathrm{R}(X)$ are irreducible, smooth and isomorphic.

Curves on the Fano surface

Remark

The curve $\mathrm{F}_0(X)$ is defined by the equations

$$m(\ell) = 0, \phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0$$

where $m(\ell) = 0$ is the local equation of $\mathrm{F}_0(X)$ in $\mathrm{F}(X)$ on the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

Triple lines and the curve of lines of the second type

Question

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The curve $\mathrm{F}_0(X)$ is nonsingular.

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Later on, people who cited his work precised that:

- The curve $\mathrm{F}_0(X)$ is nonsingular for a generic cubic threefold X.
- What are the singular points of $\mathrm{F}_0(X)$ when it is not smooth?

Triple lines and the curve of lines of the second type

Theorem

The triple lines on cubic threefolds are exactly the singular points of the curve $\mathrm{F}_0(X)$.

Proposition

The Fermat cubic in $\mathbb{P}^4$ contains 135 triple lines.

Triple lines and the curve of lines of the second type

- The curve of lines of the second type of the Fermat cubic in $\mathbb{P}^4$ is reducible.
- Its irreducible components correspond to the elliptic curves of the Fano surface of the Fermat cubic.
- The intersection points of the irreducible components of the curve of lines of the second type of the Fermat cubic in $\mathbb{P}^4$ are exactly the triple lines.
- Each triple line of the Fermat cubic in $\mathbb{P}^4$ goes through two Eckardt points.

Triple lines and the curve of lines of the second type

- The elliptic curves on the Fano surface $\mathrm{F}(X)$ parametrize the lines on the cones and the triple lines correspond to the inflection points of the elliptic curves.
- The curve $\mathrm{F}_0(X)$ of lines of the second type of a cubic threefold may contain a main component, which is not parametrised by an Eckardt point.

Thank you for your attention!