On special cubic 4-folds and EPW sextics associated to Enriques surfaces 20/07/2021

(jointly Igor Dolgachev)

Def. E.S. $S = Y/\iota$, Y K3, $\iota: Y \to Y$ b.p.f. involution

$K_S \equiv 0$ (here), $K_S \neq$ Pic(S), $2K_S = 0$, Tors Pic$(S) = \langle K_S \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

$S \xrightarrow[\sim]{|\Delta|} S_\Delta \subset \mathbb{P}^5$, Δ Fano polarization
generic ES
$\Delta^2 = 10$ (one orbit)

$S_\Delta \supset F_1, \dots, F_{10}, F_{-1}, \dots, F_{-10}$, 20 plane cubics

$F_i \cdot F_j = \begin{cases} 1 & \text{if } |i| \neq |j| \\ 0 & \text{if } i = \pm j \end{cases}$ $\quad F_{-i} \sim F_i + K_S$;

$\sum_{i=1}^{10} F_i \sim 3\Delta$; $|2F_i| = |2F_{-i}|$ is an ell pencil with 2 double fibers

$\wr$ $2^9 \cdot 10!$ choices

$\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_{10})$ Lagrangian tree of planes

$\Lambda_i = \langle F_i \rangle$, $\#(\Lambda_i \cap \Lambda_j) = 1$ if $i \neq j$

Enriques-Fano cubic

$\mathcal{C}_{\underline{\Lambda}} \supset \underline{\Lambda}$

$\wr$

$F(\mathcal{C}_{\underline{\Lambda}})$ Fano scheme of lines on $\mathcal{C}_{\underline{\Lambda}}$, an ISV of dim 4

EPW sextic $X_{\underline{\Lambda}} \subset \mathbb{P}^5$

$\Lambda_i \rightsquigarrow w_i \in \Lambda^3\mathbb{C}^6$, $A = \langle w_1, \dots, w_{10} \rangle$
10 dim'l Lagrangian subspace of $\Lambda^3\mathbb{C}^6$ w.r.t.

$\underbrace{\Lambda^3\mathbb{C}^6}_{\text{of dim } 20} \times \Lambda^3\mathbb{C}^6 \to \Lambda^6\mathbb{C}^6 = \mathbb{C}$

$$X_{\underline{\Lambda}} = X_A = \{[v] \in \mathbb{P}(\mathbb{C}^6) \mid \underbrace{(v \wedge \Lambda^2\mathbb{C}^6)}_{\dim 10} \cap \underbrace{A}_{\dim 10} \neq 0\}$$

$$\begin{array}{ccccc}
\tilde{X}_A & \xrightarrow{\text{res. sing.}} & \hat{X}_A & \xrightarrow{2:1} & X_A = X_{\underline{\Lambda}} \\
\uparrow & & & & \cup \\
\text{IHS } V & & S_i & \longrightarrow & \Lambda_i \\
\text{of } K3^{[2]} & & \text{sing } K3 \text{ surfaces} & &
\end{array}$$

Def. S an E.S.

a) A marking on S is a choice of a basis $(e_1, \dots, e_{10})$ of $H^2(S,\mathbb{Z}) = \mathrm{Num}(S) = \mathrm{Pic}(S)/\langle K_S \rangle$

b) A supermarking: one of the 2^{10} liftings of $(e_1, \dots, e_{10})$ to $(E_1, \dots, E_{10}) \in (\mathrm{Pic}\, S)^{10}$

Result: The following 10-dim'l var. are irred. & birat. isom.:

$$\mathcal{M}^{sm}_{En} \underset{bir}{\simeq} \mathcal{M}^{EF}_{L10} \underset{bir}{\simeq} \mathcal{M}^{EF}_{cub} = \mathcal{M}^{M\text{-pol}}_{cub}$$

moduli of s.m. Enr. surfaces

$$\cap$$

$$\mathbb{L}(G(3,6)^{10}) // PGL(6)$$

(several components of dim ≥ 10)

$$M = \left(\mathbb{Z}^{11}, \begin{pmatrix} 3 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}\right)$$

$$\simeq \mathcal{M}^{M'-pol}_{K3^{[2]}} \underset{\text{BEI}}{\simeq} \mathcal{M}^{EF}_{EPW} \subset \mathcal{M}^{M''-pol}_{K3^{[2]}}$$

$$M' = \left(\mathbb{Z}^{11}, \begin{pmatrix} 6 & 2 & \cdots & 2 \\ 2 & -2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 2 & 0 \cdots & 0 & -2 \end{pmatrix}\right) \qquad M'' = \left(\mathbb{Z}^{11}, \begin{pmatrix} 2 & & 0 \\ & -2 & \\ 0 & & \ddots \\ & & & -2 \end{pmatrix}\right)$$

<u>Rk</u>. $(\Lambda_1, \ldots, \Lambda_{10}) \rightsquigarrow (\Lambda_{-1}, \Lambda_2, \ldots, \Lambda_{10})$ 2^{nd} 10-dim.
irred. comp. $\mathcal{M}^{ps-EF}_{L10}$, $\mathcal{M}^{ps-EF}_{EPW}$

Markings and isotropic tens

$$\mathrm{Num}(S) = \mathrm{Pic}(S)/\langle K_S \rangle \simeq \mathbb{E} = E_8 \oplus U$$

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

E_8 negative def.

<u>Def</u>. $(f_1, \ldots, f_k)$ is an isotropic k-sequence if the $f_i \in \mathrm{Num}(S)$ are primitive, lin. independent, $f_i^2 = 0$, $f_i \cdot f_j = 1$ $(i \neq j)$

$1 \leq k \leq 10$

For each k only one orbit

A model of $\mathbb{E}$, $(f_1, \ldots, f_{10})$:

$$I^{1,10} = \left(\mathbb{Z}^{11}, \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}\right)$$

$e_0 \; e_1 \ldots e_{10}$

$k_{10} = -3e_0 + e_1 + \cdots + e_{10}$, $k_{10}^2 = -1$

$E = k_{10}^{\perp} \subset I^{1,10}$, $\underline{f_i = e_i - k_{10}}$ $(i = 1, \ldots, 10)$ stand. isotropic tenn

$f_1 + \cdots + f_{10} = 3\Delta$, $\Delta \in E$, $\Delta^2 = 10$

<u>Prop.</u> $\{\text{Markings}\} \xleftrightarrow{1:1} \{\text{Isotropic tenns}\}$

<u>Prop</u>. There are 3 orbits of $v \in E$ with $v^2 = 10$; the one containing Δ is distinguished by the condition

$$\forall f \in E,\ f \neq 0,\ f^2 = 0 \Rightarrow |v \cdot f| \geq 3$$

<u>Def</u>. A Fano polarisation on S is a (semi)-ample class H from the orbit of $\varphi(\Delta)$ for some marking $\varphi : E \xrightarrow{\sim} \mathrm{Num}(S)$

- A marking is (semi)-ample $\Leftrightarrow$ $\dfrac{f_1 + \cdots + f_{10}}{3}$ is (semi)-ample

The moduli spaces of ES

K3s with (-2)-curve

$$\mathcal{M}^{sm}_{E_n} \xrightarrow{2^{10}:1} \mathcal{M}^{m}_{E_n} = \underbrace{\mathcal{M}^{E(2)\text{-pol}}_{K3}} \setminus \underline{\begin{pmatrix} \text{Coble} \\ \text{divisor} \end{pmatrix}}$$

$\uparrow$

Irreducible

Cubic 4-folds

$X = \mathcal{C}_{\underline{a}}$ EF cubic, H hyperplane section

$(H^2, n_1, \dots, n_{10})$ have int. matrix $\underbrace{\begin{pmatrix} 3 & 1 & \cdots & 1 \\ 1 & 3 & & 1 \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}}_{M}$

$H^4(X, \mathbb{Z}) \simeq I^{21,2}$

Prop. Two primitive embeddings $\phi : M \hookrightarrow I^{21,2}$

$\phi' : M \hookrightarrow I^{21,2}$ s.t. $\phi(H^2) = \phi'(H^2) = h$ are conjugate by an isometry of $I^{21,2}$ fixing h.

(Use Nikulin's result)

Cor. $H^4(X, \mathbb{Z}) \supset M^{\perp} = T = \underline{E(-2) \oplus \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}}$

Pf. : Find a model

$$I^{21,2} = I^{1,10}_{(-1)} \oplus I^{1,10}_{(-1)} \oplus \langle 1 \rangle$$

$$h = (k_{10}, k_{10}', e)$$

$$M = \langle h, P_1, \dots, P_{10} \rangle, \quad P_i = (e_i, e_i', -e)$$

$M^\perp$ is as stated. $\square$

$$\boxed{\mathcal{M}^{M\text{-pol}}_{\mathrm{cub}} \simeq \Gamma_T \backslash \mathcal{D}^{\circ}_T}$$

$$\Gamma_T = \{ u \in O(I^{21,2}) \mid u|_M = \mathrm{id}_M \}.$$

K3

$$\begin{array}{ccc} Y & \xrightarrow[/\iota]{2:1} & X \leftarrow \text{rational Coble surface} \\ \cup & & \cup \\ C_Y & \xrightarrow{\sim} & C_X \\ (C_Y)^2 = -2 & & (C_X)^2 = -4 \end{array}$$