

Symmetric Surfaces with many Singularities

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Abstract

Let $G \subset SO(4)$ denote a finite subgroup containing the Heisenberg group. In these notes we classify all these groups, we find the dimension of the spaces of G -invariant polynomials and we give equations for the generators whenever the space has dimension two. Then we complete the study of the corresponding G -invariant pencils of surfaces in \mathbb{P}_3 which we started in [S]. It turns out that we have five more pencils, two of them containing surfaces with nodes.

0 Introduction

Consider the Klein four group $V \subseteq SO(3)$. Let \tilde{V} denote its inverse image in $SU(2)$ under the universal covering $SU(2) \rightarrow SO(3)$. The image of the direct product $\tilde{V} \times \tilde{V}$ in $SO(4)$ under the double covering $SU(2) \times SU(2) \rightarrow SO(4)$ is the Heisenberg group H . In this note we classify all the subgroups G of $SO(4)$ which contain H . First we classify all the subgroups of $SU(2) \times SU(2)$ which contain $\tilde{V} \times \tilde{V}$, then their images in $SO(4)$ are the groups G (cf. proposition 1.1 and section 1.4). They operate in a natural way on $\mathbb{C}[x_0, x_1, x_2, x_3]$, the ring of polynomials in four variables with complex coefficients. In section 3 we give generators for the spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ of homogeneous G -invariant polynomials of degree j whenever this dimension is two. Since the groups G contain H , we have invariant polynomials only in even degree. When the dimension is two the generators are the multiple quadric $q^{j/2} = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{j/2}$ (trivial invariant) and another polynomial of degree j which we denote by f . The pencils

$$f + \lambda q^{j/2} = 0, \quad \lambda \in \mathbb{P}_1,$$

of surfaces in the three dimensional complex projective space \mathbb{P}_3 have then a large symmetry group (this is the reason why we consider just subgroups of $SO(4)$ containing H). We describe them in section 4. In particular we find the

singular surfaces contained in it. In [S] we considered the case of $G = TT, OO, II$ which are the images in $SO(4)$ of the direct products $\tilde{T} \times \tilde{T}, \tilde{O} \times \tilde{O}, \tilde{I} \times \tilde{I}$ where \tilde{T} denotes the binary tetrahedral group, \tilde{O} the binary octahedral group, \tilde{I} the binary icosahedral group in $SU(2)$. We denoted the groups there by G_6, G_8 and G_{12} and we called them bi-polyhedral groups. We found pencils containing surfaces with many nodes (=ordinary double points). In particular the degree twelve II -invariant pencil contains a surface with 600 nodes which improves the previous lower bound for the maximal number of nodes of a surface of degree twelve in \mathbb{P}_3 (cf. [C]). Here we describe the other G -invariant pencils and show that we have two more pencils which contain surfaces with nodes (the others do not contain surfaces with isolated singularities at all). We list the groups G and the degrees j below, as well as the number of nodes on the singular surfaces. In each pencil we have four of these singular surfaces and the nodes there form just one G -orbit. For the convenience of the reader we recall the results about the TT -, OO -, and II -invariant pencils too.

G	order	j	nodes			
$(OO)'$	192	4	4	12	16	8
TT	288	6	12	48	48	12
OO	1152	8	24	72	144	96
IO	2880	12	240	360	240	120
II	7200	12	300	600	360	60

The group IO is the image in $SO(4)$ of the direct product of the binary icosahedral group with the binary octahedral group in $SU(2) \times SU(2)$ and $(OO)'$ is a subgroup of OO which we describe in section 1.2 and 1.4. This is an index two subgroup of the reflection group $[3,3,4]$ (cf. [Co2], p. 226) and has the same invariant polynomials. In all the cases but one ($G = IO$), the singular surfaces contain just isolated singularities (i.e. the nodes). The surfaces in the IO -invariant pencil contain two double lines in the base locus. The $[3,3,4]$ -invariant polynomials of degree two and four were already known by Coxeter in [Co1]. Here we show that in the pencil of degree four we have a surface with the maximal number possible of nodes (=16) which is a so called Kummer surface. Finally in section 6 we give a computer picture of the IO -invariant surface of degree 12 with 360 nodes.

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1 Symmetry groups

Denote by $H \subseteq SO(4)$ the Heisenberg group (with 32 elements). We want to collect in a systematic way all the finite subgroups of $SO(4)$ containing H , and their polynomial invariants of low degree. These are invariants of the Heisenberg group with extra symmetries.

1.1 Ternary groups

We specify the following matrices in $SO(3)$

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_n := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix},$$

$$S := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, U := \frac{1}{2} \begin{pmatrix} \tau - 1 & -\tau & 1 \\ \tau & 1 & \tau - 1 \\ -1 & \tau - 1 & \tau \end{pmatrix},$$

where $\tau := \frac{1}{2}(1 + \sqrt{5}) = 2 \cdot \cos(\frac{\pi}{5})$, $a := \cos\frac{2\pi}{n}$, $b := \sin\frac{2\pi}{n}$. These matrices generate the following subgroups of $SO(3)$

	generators	order	group	name
V	A_1, A_2	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Klein four
D_n	A_2, R_n	$2n$	D_n	dihedral
T	A_1, S	12	$Alt(4)$	tetrahedral
O	A_1, R_4, S	24	$Sym(4)$	octahedral
I	A_1, S, U	60	$Alt(5)$	icosahedral

Where $Alt(4)$ and $Alt(5)$ denote the group of even permutations of four and five elements and $Sym(4)$ denotes the permutation group of four elements. Whenever $n \in \mathbb{N}$, $n \neq 0$, is *even*, V is contained in each of the above groups. By the classification of the finite subgroups of $SO(3)$, these are all such subgroups.

The groups T , O and I are the rotation groups of tetrahedron, octahedron and icosahedron. An identification with the permutation groups is given in [Co2], p. 46-50 as well as an identification of D_n with the symmetry group

of a regular polygon with n vertices (ibid. p. 46). Sometimes it is useful for the computations to identify the matrices above with the cycles of the permutation groups. Indeed the identification of T and O with subgroups of the permutation group $Sym(4)$ is obtained by letting them act on the four space diagonals of the unit cube. Let these lines and vectors generating them be

$$d_1 : (1, 1, 1), \quad d_2 : (-1, 1, 1), \quad d_3 : (1, -1, 1), \quad d_4 : (1, 1, -1)$$

The matrices in O permute these lines by

$$A_1 : (12)(34), \quad A_2 : (13)(24), \quad R_4 : (1423), \quad S : (123).$$

Using this correspondence with permutation groups, it is easy to write down their conjugacy classes. We write the conjugacy classes of the dihedral group D_n , $n = 2l$, $l \geq 2$ too. In the next table we characterize a conjugacy class by one of its elements. Under each representative we write the number of elements in the conjugacy class.

group	repr. of a conj. class and number of elements				
V	1	A_1	A_2	A_3	
	1	1	1	1	
D_n	1	R_n^k	R_n^l	A_2	$A_2 R_n$
	1	2	1	1	1
T	1	A_1	S	S^2	
	1	3	4	4	
O	1	A_1	$R_4 A_2$	R_4	S
	1	3	6	6	8

where $k = 1, \dots, l - 1$.

The symmetries of T obviously leave invariant the icosahedron (cf. [Co2], p. 52) with vertices

$$(\pm 1, \pm \tau, 0), \quad (0, \pm 1, \pm \tau), \quad (\pm \tau, 0, \pm 1)$$

The matrix U permutes these vertices as

$$\begin{aligned} & \pm(0, 1, \tau) \mapsto \pm(0, 1, \tau), \\ & \pm(0, 1, -\tau) \mapsto \pm(-\tau, 0, 1) \mapsto \pm(-1, -\tau, 0) \mapsto \pm(1, -\tau, 0) \mapsto \pm(\tau, 0, -1) \mapsto \pm(0, 1, -\tau). \end{aligned}$$

So together with the group T the symmetry U generates a group of order at least $12 \cdot 5 = 60$, contained in the symmetry group of the icosahedron specified. Therefore it coincides with the icosahedral group $I \cong A_5$. The action of I on the five cosets of I/T defines the map $I \mapsto A_5$. Using

$$\begin{aligned} A_1 \cdot U &= U^4 \cdot A_1, & A_1 \cdot U^2 &= U^3 \cdot A_1, \\ S \cdot U &= U^3 \cdot A_2, & S \cdot U^2 &= U \cdot A_2 S, & S \cdot U^3 &= U^2 \cdot S^2, & S \cdot U^4 &= U^4 \cdot S^2 A_2, \end{aligned}$$

one finds that under this map

$$A_1 \mapsto (14)(23), \quad S \mapsto (132), \quad U \mapsto (12345).$$

Using this correspondence, one enumerates the conjugacy classes in I

group	repr. of a conj. class and number of elements				
I	1	A_1	S	U	U^2
	1	15	20	12	12

1.2 Subgroups of products of ternary groups

Here we classify *subdirect products* $G \subseteq G_1 \times G_2$, where G_1 and G_2 are finite ternary groups V, D_n (n even), T, O or I . We assume that G contains the subgroup $V \times V \subseteq G_1 \times G_2$. We are interested in these subgroups only up to interchanging the factors G_1 and G_2 . So we assume that we are in one of the following cases

- $|G_1| \geq |G_2|$ and $G_1, G_2 \neq D_n$,
- $G_1 \neq D_n, G_2 = D_n$,
- $G_1 = D_n$ and $G_2 = D_m$.

Additionally, passing to smaller subgroups $G'_i \subseteq G_i$ if necessary, we may assume that both projections

$$p_1 : G \rightarrow G'_1, \quad p_2 : G \rightarrow G'_2$$

are surjective, so that $G \subseteq G'_1 \times G'_2$. Finally, we do not distinguish between groups conjugate in $SO(3) \times SO(3)$. In the table below we assume $n \neq m$, $n = 2l$, $m = 2l'$ and let $s := \text{lcm}(n, m)$.

Proposition 1.1 *The following list is a complete list of subgroups $G \subseteq G_1 \times G_2$ under the assumptions above:*

G_1	G_2	G	$ G /16$	$G/V \times V$	G_1	G_2	G	$ G /16$	$G/V \times V$
V	V	$V \times V$	1	1	V	D_n	$V \times D_n$	$n/2$	$\mathbb{Z}_1 \times 1$
T	V	$T \times V$	3	$\mathbb{Z}_3 \times 1$	T	D_n	$T \times D_n$	$3n/2$	$\mathbb{Z}_3 \times \mathbb{Z}_1$
	T	$T \times T$	9	$\mathbb{Z}_3 \times \mathbb{Z}_3$		$3 n$	$(T \times D_n)'$	$n/2$	\mathbb{Z}_1
		$(T \times T)'$	3	\mathbb{Z}_3	O	D_n	$O \times D_n$	$6n/2$	$D_3 \times \mathbb{Z}_1$
O	V	$O \times V$	6	$D_3 \times 1$		$4 n$	$(O \times D_n)'$	$3n/2$	$D_{3l/2}$
	T	$O \times T$	18	$D_3 \times \mathbb{Z}_3$	I	D_n	$I \times D_n$	$15n/2$	
	O	$O \times O$	36	$D_3 \times D_3$	D_n	D_n	$D_n \times D_n$	$n^2/4$	$\mathbb{Z}_1 \times \mathbb{Z}_1$
	O	$O \times O$	36	$D_3 \times D_3$			$(D_n \times D_n)'$	$n/2$	\mathbb{Z}_1
		$(O \times O)'$	6	D_3	D_n	D_m	$D_n \times D_m$	$nm/4$	$\mathbb{Z}_1 \times \mathbb{Z}_1'$
		$(O \times O)''$	18	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$			$(D_n \times D_m)'$	$s/2$	\mathbb{Z}_s
I	V	$I \times V$	15						
	T	$I \times T$	45						
	O	$I \times O$	90						
	I	$I \times I$	225						

Proof. We discuss the cases one-by-one. But before that, we observe that the kernels $K_1 \subseteq G_1 \times 1$ and $K_2 \subseteq 1 \times G_2$ of both projections $p_i : G \rightarrow G_i$ are normal subgroups. This follows by conjugating component-wise from the surjectivity of both projections.

The groups $G_1 \times V$ do not have proper subgroups containing $V \times V$ and mapping surjectively onto G_1 , (for $G_1 = D_n$ too). So we do not need to consider the cases $G_2 = V$.

First consider the case of $G_1 = I$. Since I is simple, the kernel $K_1 \subseteq I \times 1$ either coincides with $I \times 1$, or is trivial. The latter cannot be the case, because this kernel contains $V \times 1$. The only possibilities are the product cases $I \times V, I \times T, I \times O, I \times I$ and $I \times D_n$.

Let now $Q := G/V \times V$ and $Q_1 := G_1/V, Q_2 := G_2/V$. We consider Q a proper subdirect product of $Q_1 \times Q_2$.

T, T : $Q \subseteq \mathbb{Z}_3 \times \mathbb{Z}_3$ mapping surjectively onto both factors is either the diagonal or the anti-diagonal. The two corresponding subgroups are not conjugate in $T \times T$, but in $T \times O$ they are. The inverse image of the diagonal $\mathbb{Z}_3 \subseteq \mathbb{Z}_3 \times \mathbb{Z}_3$ in $T \times T$ is $(T \times T)'$.

O, T : $Q \subseteq D_3 \times \mathbb{Z}_3$ mapping surjectively onto both factors would have order six and be isomorphic with D_3 under p_1 . But there is no epimorphism of D_3 onto \mathbb{Z}_3 . Such a group does not exist.

O, O : $Q \subseteq D_3 \times D_3$ must have order six, twelve or 18. If it has order six, it

is a graph of an isomorphism between both factors D_3 . Then it is conjugate to the diagonal, and this leads to the subgroup $(O \times O)' \subseteq O \times O$. The case $|Q|=12$ cannot occur, because then the kernel $K_1 \subseteq D_3 \times 1$ would have order two, and could not be normal. If Q has order 18 both the kernels K_1 and K_2 have order three, and coincide with the unique proper normal subgroup of D_3 . This implies that G contains $T \times T \subseteq O \times O$ and is the inverse image of a subgroup $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 = O \times O / T \times T$. By surjectivity of projections this can only be the diagonal. Its inverse image is $(O \times O)''$.

T, D_n : if $Q \subseteq \mathbb{Z}_3 \times \mathbb{Z}_l$ maps surjectively onto both the components then three divides l . Let its inverse image be $(T \times D_n)'$.

O, D_n : if $Q \subseteq D_3 \times \mathbb{Z}_l$ maps surjectively onto both factors then two divides l . We denote the inverse image by $(O \times D_n)'$.

D_n, D_n : $Q \subseteq \mathbb{Z}_l \times \mathbb{Z}_l$ mapping surjectively onto \mathbb{Z}_l is conjugate to the diagonal. We call its inverse image $(D_n \times D_n)'$.

D_n, D_m : $Q \subseteq \mathbb{Z}_l \times \mathbb{Z}_{l'}$ mapping surjectively onto \mathbb{Z}_l and $\mathbb{Z}_{l'}$ is generated by an element of order $s/2$. We call its inverse image $(D_n \times D_m)'$.

□

1.3 Binary groups

We consider the standard double cover $SU(2) \rightarrow SO(3)$ (cf. [DV] p. 39–42). Let \tilde{G} denotes the pre-image in $SU(2)$ of $G \subseteq SO(3)$. We specify the following matrices, \tilde{M} , which are in the pre-image of $M \in SO(3)$:

$$\begin{aligned} \tilde{A}_1 &:= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \tilde{A}_2 &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \tilde{A}_3 &:= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ \tilde{S} &:= \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, & \tilde{U} &:= \frac{1}{2} \begin{pmatrix} \tau & \tau-1+i \\ 1-\tau+i & \tau \end{pmatrix}, & \tilde{R}_n &:= \begin{pmatrix} e^{\frac{i\pi}{n}} & 0 \\ 0 & e^{-\frac{i\pi}{n}} \end{pmatrix}. \end{aligned}$$

since $\tilde{M}^{\text{ord}(M)} = -1$, they have order $2 \cdot \text{ord}(M)$. By an argumentation as in [S] section 2, we can write the conjugacy classes in the binary groups:

group	repr. of a conj. class and number of elements								
\tilde{V}	1	-1	\tilde{A}_1	\tilde{A}_2	\tilde{A}_3				
	1	1	2	2	2				
\tilde{D}_n	1	-1	\tilde{A}_2	$\tilde{A}_2\tilde{R}_n$	\tilde{R}_n^l	\tilde{R}_n^k	$-\tilde{R}_n^k$		
	1	1	2l	2l	2	2	2		
\tilde{T}	1	-1	\tilde{A}_1	\tilde{S}	$-\tilde{S}$	\tilde{S}^2	$-\tilde{S}^2$		
	1	1	6	4	4	4	4		
\tilde{O}	1	-1	\tilde{A}_1	$\tilde{R}_4\tilde{A}_2$	\tilde{R}_4	$-\tilde{R}_4$	\tilde{S}	$-\tilde{S}$	
	1	1	6	12	6	6	8	8	
\tilde{I}	1	-1	\tilde{A}_1	\tilde{S}	$-\tilde{S}$	\tilde{U}	$-\tilde{U}$	\tilde{U}^2	$-\tilde{U}^2$
	1	1	30	20	20	12	12	12	12

where $k = 1, \dots, l - 1$.

1.4 Quaternary groups

Here we consider the images of the finite groups $\tilde{G}_1 \times \tilde{G}_2 \subseteq SU(2) \times SU(2)$ under the double covering map

$$SU(2) \times SU(2) \rightarrow SO(4), \quad (q, q') : p \mapsto q \cdot p \cdot q'^{-1}$$

(cf. [DV] p. 42–45), we abbreviate there G_1G_2 . Since the corresponding subgroups of $SO(3) \times SO(3)$ contain the group $V \times V$, these contain the Heisenberg group $VV \subseteq SO(4)$, and by proposition 1.1 these are all such subgroups. We specify now the matrices:

$$(A_1, 1) := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (1, A_1) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$(A_2, 1) := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (1, A_2) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
(A_3, 1) &:= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & (1, A_3) &:= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
(R_n, 1) &:= \begin{pmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}, & (1, R_n) &:= \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}, \\
(S, 1) &:= \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, & (1, S) &:= \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \\
(U, 1) &:= \frac{1}{2} \begin{pmatrix} \tau & 0 & 1-\tau & -1 \\ 0 & \tau & -1 & \tau-1 \\ \tau-1 & 1 & \tau & 0 \\ 1 & 1-\tau & 0 & \tau \end{pmatrix}, & (1, U) &:= \frac{1}{2} \begin{pmatrix} \tau & 0 & \tau-1 & 1 \\ 0 & \tau & -1 & \tau-1 \\ 1-\tau & 1 & \tau & 0 \\ -1 & 1-\tau & 0 & \tau \end{pmatrix},
\end{aligned}$$

where $\alpha := \cos \frac{\pi}{n}$, $\beta := \sin \frac{\pi}{n}$.

We can write the conjugacy classes of the groups $G_1 G_2 \subseteq SO(4)$ in $SO(4)$ and their number of elements. These are the images of the conjugacy classes of $\tilde{G}_1 \times \tilde{G}_2$ in $G_1 G_2 \subseteq SO(4)$ under the double covering map. Observe that the matrices $(g_1, g_2) \in SO(4)$ with the same eigenvalues are conjugate (cf. (1.1) of [S]), this fact simplifies the computations considerably. In this section and in the next one we omit the groups $D_n D_m$ and $(G_1 G_2)'$ with $G_2 = D_n$. We return to those groups later. In the tables we use the following conventions:

- we omit the conjugacy classes of $+1, -1$ (these contain one element each)
- whenever the conjugacy classes (g_1, g_2) and its s -th power (g_1^s, g_2^s) are distinct we write them just one time since they have the same number of elements.

G_1G_2	order	$A_2, 1$	A_2, A_2	$R_4, 1$	R_4, A_2	R_4, R_4	$S, 1$	S, A_2	S, R_4	S, S
VV	32	12	18	0	0	0	0	0	0	0
TV	96	12	18	0	0	0	8	48	0	0
TT	288	12	18	0	0	0	16	96	0	64
OV	192	24	54	6	36	0	8	48	0	0
OT	576	24	54	6	36	0	16	192	48	64
OO	1152	36	162	12	216	36	16	288	96	64
IV	480	36	90	0	0	0	20	120	0	0
IT	1440	36	90	0	0	0	28	360	0	160
IO	2880	48	270	6	180	0	28	600	120	160
II	7200	60	450	0	0	0	40	1200	0	400

G_1G_2	order	$A_2, 1$	A_2, A_2	$R_4, 1$	R_4, A_2	$S, 1$	S, A_2
VD_n	$16n$	$8 + 4l$	$6(2l + 1)$	0	0	0	0
TD_n	$48n$	$8 + 4l$	$6(2l + 1)$	0	0	8	$16(2l + 1)$
OD_n	$96n$	$20 + 4l$	$18(2l + 1)$	6	$12(2l + 1)$	8	$16(2l + 1)$
ID_n	$240n$	$32 + 4l$	$30(2l + 1)$	0	0	20	$40(2l + 1)$

G_1G_2	$U, 1$	U, A_2	U, R_4	U, S	U, U	U, U^2	U, R_n^k
IV	12	72	0	0	0	0	0
IT	12	72	0	96	0	0	0
IO	12	216	72	96	0	0	0
II	24	720	0	480	144	144	0
ID_n	12	$24(2l + 1)$	0	0	0	0	24

G_1G_2	A_2, R_n^k	$1, R_n^k$	S, R_n^k	R_4, R_n^k
VD_n	12	2	0	0
TD_n	12	2	16	0
OD_n	36	2	16	12
ID_n	60	2	40	0

where $k = 1, \dots, l - 1$. For the groups which are not products it is a little more complicated to write down the sizes of their conjugacy classes. But using the description from proposition 1.1 one finds

group	order	$A_2, 1$	A_2, A_2	R_4, A_2	R_4, R_4	$S, 1$	S, A_2	S, S
$(TT)'$	96	12	18	0	0	0	0	32
$(OO)'$	192	12	42	48	12	0	0	32
$(OO)''$	576	12	90	144	36	16	96	64

2 Poincaré series

In this section we want to find the dimension of the spaces of homogeneous invariant polynomials of a given degree. We consider the Poincaré series

$$p(t) := \sum_{j=0}^{\infty} \dim \mathbb{C}[x_0, x_1, x_2, x_3]_j^G \cdot t^j$$

where G is a group as in section 1.4. By a theorem of Molien ([B] p. 21) and an easy computation as in [S], (2.1), it can be written as

$$p(t) = \frac{1}{|G|} \sum \frac{n_g}{\det(g - 1 \cdot t)}$$

where the sum runs over all the conjugacy classes of G and n_g denote their number of elements. At the denominator we have the characteristic polynomials. Using the numbers of conjugacy classes (under $SO(4)$) given in section 1.4 and computing their characteristic polynomials, the power series package of MAPLE produces the following table of dimensions m_d of invariant polynomials in degree d of the groups $G \subseteq G_1 G_2$. Observe that since G contains the Heisenberg group we do not have invariant polynomials of odd degree.

First we consider the case of $G_i \neq D_n$:

group	m_2	m_4	m_6	m_8	m_{10}	m_{12}
VV	1	5	6	15	19	35
TV	1	1	2	5	5	13
TT	1	1	2	3	3	7
$(TT)'$	1	3	4	7	9	15
OV	1	1	1	4	4	8
OT	1	1	1	2	2	4
OO	1	1	1	2	2	3
$(OO)'$	1	2	3	5	6	9
$(OO)''$	1	1	2	3	3	5
IV	1	1	1	1	1	5
IT	1	1	1	1	1	3
IO	1	1	1	1	1	2
II	1	1	1	1	1	2

Whenever $G_1 \neq D_n$ and $G_2 = D_n$, we can write the finite sums $p(t)$ for each n . Here we compute the first coefficients of the Poincaré series for $n = 4, 6, 8$.

group	m_2	m_4	m_6	m_8	m_{10}	m_{12}
VD_4	1	3	3	9	11	19
VD_6	1	3	3	6	6	14
VD_8	1	3	3	6	6	10
TD_4	1	1	1	3	3	7
TD_6	1	1	1	2	2	6
TD_8	1	1	1	2	2	4
OD_4	1	1	1	3	3	5
OD_6	1	1	1	2	2	4
OD_8	1	1	1	2	2	3
ID_4	1	1	1	1	1	3
ID_6	1	1	1	1	1	3
ID_8	1	1	1	1	1	2

Of course, whenever $m_d = 1$, the space of invariant polynomials is spanned by the $d/2$ -th power of the invariant quadric (*trivial invariant*)

$$q := x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

3 Invariants

In this section we want to compute a system of generators for the spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ whenever this space has dimension two and $G \subseteq \text{SO}(4)$ is a finite subgroup containing the Heisenberg group. We distinguish two cases.

(3.1) First case. Assume that $G \subseteq G_1 G_2$ with $G_i \neq D_n$, $i = 1, 2$. We do some remarks on the groups G which simplify the computations of the invariant polynomials.

- the group $TV \subseteq TT$.
- The group TT is contained in $(OO)''$. In fact the generators modulo VV are

group	TT	$(OO)''$
generators	$(1, S)$	$(1, S), (S, 1)$
	$(S, 1)$	$(R_4 A_2, R_4 A_2)$

with

$$(R_4 A_2, R_4 A_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Similarly to [S] section 3, $(OO)''$ is a subgroup of index two in the reflection group of the $\{3, 4, 3\}$ -cell (cf. [Co2] p. 149 for the definition of this polytope), if we add the generator

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we get the whole reflection group.

- Finally since $OT \subseteq OO$, they have the same two invariant polynomials in degree eight. Those of degree ten are obtained just by multiplication with the quadric q .

By these remarks and considering G as in the assumption, it follows that we have to compute the generators of six two-dimensional spaces $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ for the following pairs (G, j) :

$$((OO)', 4) \quad (TT, 6) \quad (OO, 8) \quad (IO, 12) \quad (II, 12).$$

The generators of the TT -, OO - and II -invariant spaces are given in [S] as well as a description of the corresponding pencils of surfaces in \mathbb{P}_3 (base locus and singular surfaces). Here we examine the remaining cases and in the next section we describe the corresponding pencils of surfaces in \mathbb{P}_3 . The basic idea to find generators of the invariant spaces is the same as in [S]. For the computations here we use the matrix representation of the groups given in section 1.4 (see also [ST1] and [ST2]).

$(OO)'$ -invariants. We start with the space of Heisenberg invariant quartics. It has dimension five, being spanned by

$$\begin{aligned} f_0 &:= x_0^4 + x_1^4 + x_2^4 + x_3^4, \\ f_1 &:= 2(x_0^2x_1^2 + x_2^2x_3^2), \quad f_2 := 2(x_0^2x_2^2 + x_1^2x_3^2), \quad f_3 := 2(x_0^2x_3^2 + x_1^2x_2^2), \\ f_4 &:= 4x_0x_1x_2x_3. \end{aligned}$$

In terms of these invariants

$$q^2 = f_0 + f_1 + f_2 + f_3.$$

Modulo VV , the group $(OO)'$ is generated by (R_4A_2, R_4A_2) and (S, S) . Tracing the action of these generators on f_0, \dots, f_4 one finds the invariants q^2 and f_0 .

IO -invariants. The generators $(U, 1)$, $(1, S)$ and $(1, R_4)$ of IO operate on the space $\mathbb{C}[x_0, x_1, x_2, x_3]_{12}^{VV}$ which is 35-dimensional. A computation with MAPLE shows that it contains the non trivial IO -invariant polynomial

$$\begin{aligned} S_{IO} &:= -\sum_i f_i^3 + 11 \sum_{i,j} f_i^2 f_j + (f_0^2 - 14f_4^2) \sum_i f_i + 30f_0f_4^2 - 2f_0 \sum_{i,j} f_i f_j \\ &\quad - 30f_1f_2f_3 + 3\sqrt{5}f_4(2f_0 \sum_i f_i - \sum_{i,j} f_i f_j - f_0^2 - \sum_i f_i^2 + 4f_4^2) \\ &\quad + 6\sqrt{5}F_a \end{aligned}$$

where the sums run over all the indices $i, j = 1, 2, 3$, $i \neq j$, and

$$F_a := f_1^2f_2 + f_3^2f_1 + f_2^2f_3 - f_1f_2^2 - f_3f_1^2 - f_2f_3^2$$

is the anti-symmetric part of S_{IO} .

(3.2) Second case. Assume that $G \subseteq G_1 G_2$ with $G_1 = D_n$ or $G_2 = D_m$, $n, m \geq 4$, even. With the help of the table in section 1.4 we discuss the following cases:

- $D_n D_m$: observe that $V D_m$ is contained in $D_n D_m$. A direct computation shows that the generator $(1, R_n)$ leaves invariant a three-dimensional family of degree four VV -invariant polynomials. Generators in this case are $f_2 + f_3$, q^2 , $f_2 - f_4$. Now the matrix $(R_n, 1)$ of $D_n D_m$ operates on these three-dimensional space leaving invariant q^2 and $f_2 + f_3$.
- since $(D_n D_m)' \subseteq D_n D_m$ these groups have already an at least two-dimensional family of invariant polynomials of degree four, which is generated by q^2 and $f_2 + f_3$.
- $T D_n$, $n \geq 6$: we have a five-dimensional family of TV -invariant in degree eight. By the action of the generator $(1, R_n)$, $n \geq 6$ we have a two-dimensional family of invariants. Put

$$K_i := x_0^2 + x_1^2 - x_2^2 - x_3^2 + (-1)^i 2x_0(x_2 + x_3) + 2x_1(x_2 - x_3), \quad i = 0, 3$$

$$K_i := x_0^2 + x_1^2 - x_2^2 - x_3^2 + (-1)^i 2x_0(x_2 + x_3) - 2x_1(x_2 - x_3), \quad i = 1, 2$$

then generators are

$$q^4 \text{ and } P_8 := \prod_{i=0}^3 K_i.$$

- Suppose three divides n . Since the groups $(T D_n)'$ are contained in $T D_n$, they have at least a two-dimensional family of invariant polynomials in degree eight. We consider now the action of the extra generators (S, R_n) of $(T D_n)'$ on the space of degree four, resp. six VV -invariant polynomials. A direct computation with MAPLE shows that we have no invariants other then the quadric q^2 , resp. q^3 .
- $O D_n$, $n \geq 6$: we have $T D_n \subseteq O D_n$ and the extra generator $(R_4, 1)$ leaves the previous polynomials invariant.
- Suppose now four divides n . Since the groups $(O D_n)'$ are contained in $O D_n$, they have at least a two-dimensional family of invariant polynomials in degree eight. Modulo VV the groups $(O D_n)'$ have generators $(S, 1)$ and (R_4, R_n) , hence they contain the group TV . This has no non-trivial invariant polynomials of degree four and has a two-dimensional

family of invariant polynomials in degree six, generated by q^3 and $S_6(x)$ (cf. [S] p. 437). For $n \geq 6$, $S_6(x)$ is not $(OD_n)'$ -invariant, for $n = 4$ it is. In any case we do not get new invariants.

- ID_n , $n \geq 8$: we have a five-dimensional family of IV -invariant polynomials in degree twelve. The action of $(1, R_n)$, $n \geq 8$ on this space produces a two-dimensional family of invariant polynomials. Put

$$\begin{aligned} K'_0 &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2(1 - \tau)(x_0x_2 - x_1x_3) \\ K'_1 &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2\tau(x_0x_3 + x_1x_2) \\ K'_2 &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 - 2(1 - \tau)(x_0x_2 - x_1x_3) \\ K'_3 &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 - 2\tau(x_0x_3 + x_1x_2) \\ K'_4 &:= 2(1 - \tau)(x_0x_3 + x_1x_2) + 2(x_0x_2 - x_1x_3) \\ K'_5 &:= -2\tau(x_0x_3 + x_1x_2) + 2(x_0x_2 - x_1x_3) \end{aligned}$$

then generators are

$$q^6 \text{ and } P_{12} := \prod_{i=0}^5 K'_i.$$

In conclusion, the dimension of $\mathbb{C}[x_0, x_1, x_2, x_3]_j^G$ is two, for (G, j) equal to

$$(D_n D_m, 4) \quad n, m \geq 4, \quad (OD_n, 8) \quad n \geq 6, \quad (ID_n, 12) \quad n \geq 8.$$

Remark 3.3 Observe that the polynomials q^2 and $f_2 + f_3$ are $D_n D_n$ -invariant even if n is an odd integer. As in the even case, generators of these groups are $(A_2, 1)$, $(1, A_2)$ and $(R_n, 1)$, $(1, R_n)$.

4 Invariant pencils

4.1 The pencil of $(OO)'$ -invariant quartics

We take the generators q^2 and $f_0 := x_0^4 + x_1^4 + x_2^4 + x_3^4$. The base locus is a double curve of degree eight. Let C_8 denote the curve $\{q = 0, f_0 = 0\}$. The pencil is invariant under the matrix C too, hence by symmetry reasons the curve C_8 has bi-degree $(4, 4)$ on q (cf. [S], section 5). Moreover we have the following

Lemma 4.1 *The curve C_8 is smooth and irreducible.*

Proof. The Jacobian matrix of C_8 has rank two in each point, hence the curve is smooth. If $C_8 = C' \cup C''$, the curves C' , C'' would meet in some point on q (observe that they cannot be lines of the same ruling), but this is impossible because C_8 is smooth. \square

Now a singular point on a surface of the pencil ($\neq q^2$) is not contained on q (cf. (6.1) of [S]). Hence by this fact, lemma 4.1 and Bertini's theorem the general surface in the pencil is smooth. All the other surfaces but q^2 are irreducible and reduced and the singular ones have only isolated singularities. *The symmetry group of $\{3, 3, 4\}$.* Consider the "cross polytope" $\beta_4 = \{3, 3, 4\}$ in \mathbb{R}^4 with vertices the permutations of $(\pm 1, 0, 0, 0)$ as in [Co2] p. 156 and edge $\sqrt{2}$. The generators of $(OO)'$ permutes these points, hence $(OO)'$ is contained in the symmetry group $[3, 3, 4]$ of $\{3, 3, 4\}$, more precisely it is an index two subgroup. In fact the symmetry group of β_4 has order $2^4 \cdot 4! = 384 = 2 \cdot 192$ and by adding the generator C to $(OO)'$ we get the whole symmetry group $[3, 3, 4]$ (cf. [Co2] p. 226). In particular, observe that they have the same invariant polynomials. The polytope β_4 has $N_0 = 8$, $N_1 = 24$, $N_2 = 32$, $N_3 = 16$ the reciprocal "measure polytope", $\gamma_4 = \{4, 3, 3\}$ has $N_0 = 16$, $N_1 = 32$, $N_2 = 24$, $N_3 = 8$. Hence we get four $[3, 4, 3]$ -orbits of points: the vertices and the middle points of the edges of the β_4 and the vertices and the middle points of the edges of the reciprocal γ_4 . These have coordinates the permutation of $(\pm 1, 0, 0, 0)$, $(\pm 1, \pm 1, 0, 0)$, resp. $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 1, 0)$. As points of \mathbb{P}_3 these are singular on the surfaces $f_0 + \lambda q^2$ for $\lambda = -1$, $-\frac{1}{2}$ resp. $-\frac{1}{4}$, $-\frac{1}{3}$ and a direct computation shows that they are all ordinary double points. We do it for $\lambda = -1$ and $(1 : 0 : 0 : 0)$. In the affine chart $\{x_0 \neq 0\}$ the equation becomes

$$\begin{aligned} 0 &= 1 + x^4 + y^4 + z^4 - (1 + x^2 + y^2 + z^2)^2 \\ &= -2x^2 - 2y^2 - 2z^2 + \text{terms of degree } \geq 4, \end{aligned}$$

hence the rank of the Hessian matrix at $(0, 0, 0)$ is three. This shows that $(1 : 0 : 0 : 0)$ is an ordinary double point and so are all the points in its orbit. We collect the results on the singular surfaces in the following table. In the middle column we write just one point, but we mean all its permutations.

λ	nodes	number\description
-1	$(1 : 0 : 0 : 0)$	4
$-\frac{1}{2}$	$(\pm 1 : \pm 1 : 0 : 0)$	12
$-\frac{1}{3}$	$(\pm 1 : \pm 1 : \pm 1 : 0)$	16, Kummer surface
$-\frac{1}{4}$	$(\pm 1 : \pm 1 : \pm 1 : \pm 1)$	8
∞	—	double quadric

In the case of $\lambda = -\frac{1}{3}$, we get a surface with 16 nodes which is the maximal number possible for a surface of degree four. This is a Kummer surface. Observe that in this case too as in [S] the nodes are fix points under the action of some matrices in [3, 3, 4], resp. in $(OO)'$ and they are contained on lines of fix points (see (5.2) and (6.3) of [S]). Moreover an estimation as in section 8 of [S] shows that whenever the lines of fix points do not meet the base locus, they contain exactly four nodes. It is natural to expect that these are all the singular surfaces in the $(OO)'$ -invariant pencil (as in the case of the TT -, OO -, and II -invariant pencils). This is a direct consequence of the following

Proposition 4.1 1. *The conjugacy classes in $(OO)'$ (under $(OO)'$) with eigenvalues ± 1 are the following*

<i>conj. class</i>	(A_2, A_2)	(A_2, A_1)	$(R_4 A_2, R_4 A_2)$	(S, S)	(R_4, R_4)
<i>number of elements</i>	6	12	24	32	12
<i>number of fix lines</i>	6	12	24	16	6

2. *The fix lines of the matrices in these conjugacy classes contain the maximal number possible of node.*

Proof. Choosing a fix line for each of the representative above and intersecting with the singular surfaces we find

matrix and fix line	value of λ		int. points	
$(A_2, A_2) :$ $x_1 = x_3 = 0$	-1	$-\frac{1}{2}$	$(1 : 0 : 0 : 0)$ $(0 : 0 : 1 : 0)$	$(\pm 1 : 0 : 1 : 0)$
$(A_2, A_1) :$ $x_0 = x_3, x_1 = x_2$	$-\frac{1}{2}$	$-\frac{1}{4}$	$(0 : 1 : 1 : 0)$ $(1 : 0 : 0 : 1)$	$(1 : 1 : 1 : 1)$ $(-1 : 1 : 1 : -1)$
$(R_4 A_2, R_4 A_2) :$ $x_1 = 0, x_2 = x_3$	-1	$-\frac{1}{2}$	$(1 : 0 : 0 : 0)$ $(\pm 1 : 0 : 1 : 1)$	$(0 : 0 : 1 : 1)$
$(S, S) :$ $x_1 = x_2 = x_3$	-1	$-\frac{1}{3}$	$(1 : 0 : 0 : 0)$ $(\pm 1 : 1 : 1 : 1)$	$(0 : 1 : 1 : 1)$

where we do not write the matrix (R_4, R_4) , since the fix lines of the matrices in its conjugacy class are the same as those of the matrices in the conjugacy class of (A_2, A_2) . From this table follows that:

- the fix lines above meet different surfaces, hence the conjugacy classes of these matrices are in fact, all distinct (cf. also [S] (7.3)).
- The fix lines contain the maximal number possible (four), of nodes (this shows 2.)

About the number of fix lines: observe that the conjugacy classes of (S, S) has order 32 and contains the elements (S^2, S^2) hence we have $\frac{32}{2} = 16$ distinct fix lines. Finally the conjugacy class of (A_2, A_2) contains six elements, that of (A_2, A_1) contains twelve elements and that of (R_4A_2, R_4A_2) contains 24 elements. Since each element in these conjugacy classes has two fix lines and the elements with minus sign are in the same conjugacy class, the number of fix lines is the same as the number of matrices. \square

We know that the singular points form $[3, 4, 3]$ -orbits, but we can now show something more.

Lemma 4.2 *The nodes on each singular surface form one $(OO)'$ -orbit:*

λ	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
<i>orbit</i>	4	12	16	8
<i>fix group mod. ± 1</i>	S_4	D_4	D_3	A_4
<i>order</i>	24	8	6	12

Proof. The situation is easy for $\lambda = -1$. In fact the matrix $C \in [3, 4, 3]$ leaves each singular point fix, hence the group $(OO)'$ must permute them. Consider $\lambda = -\frac{1}{2}, -\frac{1}{3}$ resp. $-\frac{1}{4}$ and assume that the orbit's length of singular points is less or equal then 12, 16, resp. 8. Then the fix group in $(OO)'$ mod. ± 1 has order bigger or equal then 8, 6 resp. 12. Checking in the table given in the previous page we see that in fact the only possibility is to have equality. \square

Put now N_0 =number of nodes on a surface in the pencil, N_1 =number of fix lines of matrices in the same conjugacy class, n_0 =number of nodes on a line, n_1 =number of line through a point. Knowing the fix groups of the singular points and using the formula

$$N_0 \cdot n_1 = N_1 \cdot n_0 \tag{1}$$

for a configuration of lines and points (cf. [S] section 11.), we can write the table:

Repr. of the conj. class	value of λ			Configuration		
(A_2, A_2)	-1	$-\frac{1}{2}$		$(4_3, 6_2)$	$(12_1, 6_2)$	
(A_2, A_1)	$-\frac{1}{2}$	$-\frac{1}{4}$		$(12_2, 12_2)$	$(8_3, 12_2)$	
(R_4A_2, R_4A_2)	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$(4_6, 24_1)$	$(12_2, 24_1)$	$(16_3, 24_2)$
(S, S)	-1	$-\frac{1}{3}$	$-\frac{1}{4}$	$(4_4, 16_1)$	$(16_1, 16_1)$	$(8_4, 16_2)$

where in the second column we mean the nodes of the surfaces with the given λ .

4.2 The pencil of IO -invariant 12-ics

We take the generators q^6 and S_{IO} . We compute first the base locus. Consider the groups $(1, O)$, resp. $(I, 1)$, which operate on the two rulings of the quadric q , each element there has two lines of fix points (cf. (5.4) of [S]). For the convenience of the reader we recall the table on the length of the orbits under the action of the octahedral group O , and of the icosahedral group I :

octahedron	icosahedron
24, 12, 8, 6	60, 30, 20, 12

Now we show:

Lemma 4.3 *The variety $q \cap S_{IO}$ consists of an $(I, 1)$ -orbit of twelve lines and of an $(1, O)$ -orbit of six singular lines of S_{IO} .*

Proof. We have $\deg(q \cap S_{IO})=24$. By the table above it can only have bi-degree $(12, 12)$ or $(12, 6)$. An argumentation as in [S] (5.5), (b), shows that $q \cap S_{IO}$ splits into the union of lines of the two rulings of q . More precisely it contains the $(I, 1)$ -orbit of twelve lines and the $(1, O)$ -orbit of twelve or of six lines. In the last case the lines have multiplicity two in the intersection. So take the fix line $(\lambda : \mu : i\lambda : i\mu)$, $(\lambda : \mu) \in \mathbb{P}_1$, of the matrix $(1, A_2)$ in the orbit of length six. A direct computation (with MAPLE) shows that it is singular on S_{IO} , so we are done. □

Lemma 4.4 *Let p be a singular point on a surface of the pencil $S_{IO} + \lambda q^6$ and assume that p is on the quadric q . Then $p \in L_i$, $i = 1, \dots, 6$, where the L_i 's denote the singular lines of the base locus.*

Proof. As in the proof of (6.1) in [S], if p is singular on a surface in the pencil and $p \in q$, then p is a singular point of $q \cap S_{IO}$. Hence $p \in L_i$. \square

By this fact and Bertini's theorem follows that:

Lemma 4.5 *1. The general surface in the pencil is smooth away from the lines L_1, \dots, L_6 .*

Moreover we have:

2. The singular surfaces have only isolated singularities away from the singular lines L_1, \dots, L_6 . In particular they are irreducible and reduced.

Proof of 2. The surface q^6 is the only not reduced surface in the pencil. Indeed, assume that there is another not reduced surface $\{f_\alpha^m = 0\}$, $\alpha \cdot m = 12$, and $\deg f_\alpha = \alpha$. Then the twelve lines of the base locus in the intersection of f_α^m and q would be singular too, which is not the case. Assume now that there is a surface S in the pencil which contains a singular curve C . The latter meets the quadric in at least one point, which by lemma 4.4 is on some L_i . This point is a "pinch point" of S (for the definition of "pinch point" cf. [SR] p. 423). We compute explicitly the pinch points along a line L_i . Because of symmetry reason it is enough to do it just for one line. We take $L_1: \{x_2 = ix_0, x_3 = ix_1\}$. First we consider the transformation $x_0 \mapsto y_0$, $x_1 \mapsto y_1$, $x_2 \mapsto y_2 + i \cdot y_0$, $x_3 \mapsto y_3 + i \cdot y_1$, which maps L_1 to $\{y_2 = y_3 = 0\}$. We call the transform L_1 again.

Let $F := F(y_0, y_1, y_2, y_3, \lambda)$ be the equation of the pencil after the transformation. We can write the Taylor expansion of F along L_1

$$F = A(y_0, y_1, \lambda) \cdot y_2^2 + B(y_0, y_1, \lambda) \cdot y_2 \cdot y_3 + C(y_0, y_1, \lambda) \cdot y_3^2 \\ + \text{terms of order } \geq 3,$$

the pinch points are solution of

$$\det \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix} = 0$$

which splits into the product

$$(x_1^4 + 2x_1^3x_0 + 2x_0^2x_1^2 - 2x_0^3x_1 + x_0^4)(x_1^4 - 2x_1^3x_0 + 2x_0^2x_1^2 + 2x_0^3x_1 + x_0^4) \\ (x_0^2 + 3x_0x_1 + \sqrt{5}x_0x_1 - x_1^2)(x_0^2 - 3x_0x_1 - \sqrt{5}x_0x_1 - x_1^2) \\ (3x_0^4 - 2\sqrt{5}x_0^2x_1^2 + 3x_1^4)(x_0^4 + 16x_0^2x_1^2 - 6\sqrt{5}x_0^2x_1^2 + x_1^4) = 0.$$

this has twenty distinct solutions (independent from λ), therefore we have “simple” pinch points on L_i , hence no singular curves. \square

In this case as in the case of the pencils of [S] the singular points are contained on lines of fix points (cf. (6.3) of [S]) and we get an estimate for the number of isolated singularities as follows: Denote by S a surface of the IO -invariant pencil then $\deg(S \cap \partial_i S) = 12 \cdot 11$ and $S \cdot \partial_i S = C + 2L_1 + \dots + 2L_6$ where $\deg C = 12 \cdot 11 - 12$. Since C cannot be singular, there is a $j = 0, 1, 2, 3$ s. t. $C \not\subseteq \partial_j S$. Hence $\deg(C \cap \partial_j S) = (12 \cdot 11 - 12)(12 - 1) = 120 \cdot 11$. Since the singular points are computed two times in the intersection their number is $\leq \frac{120 \cdot 11}{2} = 660$.

Denote by L a line of fix points and assume that it meets the base locus in two points z_1, z_2 . Then for each surface $S \neq q^6 \text{mult}_{z_j}(L \cdot S) = 2$. In fact a line like L there can be only a fix line of a matrix in the conjugacy class of (A_2, A_2) and it meets the base locus at some line L_i . Moreover an argumentation as in (7.1) of [S] shows that L cannot be tangent to S at z_i . By this fact and by a computation as in section 8 of [S], we find that a fix line contains ≤ 8 singular points if it meets the base locus, and ≤ 12 singular points otherwise. Finally, before giving an exact description of the singular points on the fix lines (cf. table below) we remark that the fix lines of elements in the same conjugacy class form one orbit under the action of IO and the points where they meet are real (cf. (7.3) and (7.5) of [S]).

The singular surfaces. We proceed by a direct computation using the lines of fix points. The matrices of IO with fix lines containing singular points are in the conjugacy class of (A_2, A_2) , (S, S) or $(A_2, R_4 A_2)$. The total number of distinct fix lines is 90, 80, resp. 180. In the table below we give the singular surfaces and the singular points on the fix lines. We choose a representative in each conjugacy class and a fix line of it, moreover we put $a' := \sqrt{2} - 1$, $a := \sqrt{2} + 1$, $\gamma_1 = 2\sqrt{2} - \sqrt{5}$, $\gamma'_1 = 2\sqrt{2} + \sqrt{5}$, $\gamma_2 := 2 + \sqrt{2} + \sqrt{5} + \sqrt{10}$, $\gamma'_2 := -2 + \sqrt{2} - \sqrt{5} + \sqrt{10}$, $\gamma_3 := -2 + \sqrt{2} + \sqrt{5} - \sqrt{10}$, $\gamma'_3 := 2 + \sqrt{2} - \sqrt{5} - \sqrt{10}$, $\alpha_1 := \frac{1}{2}(-1 + \sqrt{2} - \sqrt{10})$, $\alpha'_1 := \frac{1}{2}(1 + \sqrt{2} - \sqrt{10})$, $\beta_1 := \frac{1}{2}(4 - 3\sqrt{2} - 2\sqrt{5} + \sqrt{10})$, $\alpha_2 := \frac{1}{2}(1 + \sqrt{2} + \sqrt{10})$, $\alpha'_2 := \frac{1}{2}(-1 + \sqrt{2} + \sqrt{10})$, $\alpha_3 := -2 + \frac{3}{2}\sqrt{2} - \sqrt{5} + \frac{1}{2}\sqrt{10}$, $\alpha'_3 := 2 + \frac{3}{2}\sqrt{2} + \sqrt{5} + \frac{1}{2}\sqrt{10}$, $\beta_2 := \frac{1}{2}(-10 + 7\sqrt{2} - 4\sqrt{5} + 3\sqrt{10})$, $\beta_3 := \frac{1}{2}(\sqrt{2} - 2\sqrt{5} + \sqrt{10})$, $\beta_4 := \frac{1}{2}(2 + \sqrt{2} + \sqrt{10})$, $c_1 := -\frac{74}{972} + \frac{4}{243}\sqrt{10}$,

$$c_2 := -\frac{74}{972} - \frac{4}{243}\sqrt{10}:$$

matrix and fix line	value of λ		int. points	
$(A_2, A_2) :$ $x_1 = x_3 = 0$	0	$-\frac{1}{8}$	$(1 : 0 : 0 : 0)$ $(0 : 0 : 1 : 0)$ $(\pm 1 : 0 : 1 : 0)$	$(\pm a' : 0 : 1 : 0)$ $(1 : 0 : \pm a' : 0)$
$(A_2, R_4 A_2) :$ $x_0 = -ax_1, x_3 = a'x_2$	$-\frac{1}{8}$		$(-a : 1 : 0 : 0)$ $(0 : 0 : 1 : a')$ $(1 : \pm a' : 1 : a')$	
	c_1	c_2	$(\alpha_1 : -\beta_1 : 1 : a')$ $(-\alpha_2 : \beta_1 : 1 : a')$ $(\alpha_3 : \beta_2 : 1 : a')$ $(-\alpha_3 : -\beta_2 : 1 : a')$	$(\alpha'_1 : -\beta_3 : 1 : a')$ $(-\alpha'_2 : \beta_3 : 1 : a')$ $(\alpha'_3 : \beta_4 : 1 : a')$ $(-\alpha'_3 : -\beta_4 : 1 : a')$
$(S, S) :$ $x_1 = x_2 = x_3$	0		$(1 : 0 : 0 : 0)$ $(0 : 1 : 1 : 1)$ $(-1 : 1 : 1 : 1)$ $(-\sqrt{5} : 1 : 1 : 1)$ $(\sqrt{5} \pm 2 : 1 : 1 : 1)$	
	c_1	c_2	$(\gamma_1 : 1 : 1 : 1)$ $(\gamma_2 : 1 : 1 : 1)$ $(\gamma_3 : 1 : 1 : 1)$	$(\gamma'_1 : 1 : 1 : 1)$ $(\gamma'_2 : 1 : 1 : 1)$ $(\gamma'_3 : 1 : 1 : 1)$

Here the fix line of (A_2, A_2) contains two points of the base locus. Observe that the number of singular points is maximal on the lines, hence the four given λ 's are the only values corresponding to singular surfaces.

Proposition 4.2 *In the pencil $S_{IO} + \lambda q^6$ we have the following IO-orbits of nodes*

λ	0	c_1	$-\frac{1}{8}$	c_2
orbit	120	240	360	240
fix group mod. ± 1	A_4	D_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_3
order	12	6	4	6

Proof. We analyze case by case the four different surface. Then a direct computation as in section 4.2, shows that the singular points are nodes.

I. $\lambda = 0$. Observe that $(1 : 0 : 0 : 0) \in S_{IO}$ is contained in three fix lines of elements in the conjugacy class of (A_2, A_2) . Moreover

(*) there is a matrix of order four $(1, \gamma) \in IO$, with $\gamma^2 = A_2$,

Hence $(1, \gamma)$ and (A_2, A_2) commute and the four points on the fix line of (A_2, A_2) form one orbit under the action of $(1, \gamma)$. By this fact follows that we have a configuration of lines and points. By the formula (1), we get $N_0 \cdot 3 = 90 \cdot 4$, hence $N_0 = 120$. The point $(1 : 0 : 0 : 0)$ is on the fix line of (S, S) too, more precisely it is contained on four such lines. The formula $120 \cdot 4 = 80 \cdot n_0$ shows $n_0 = 6$ so the six points on the fix line are in the orbit of length 120 too. In conclusion we have just one IO -orbit of singular double points.

II. $\lambda = -\frac{1}{16}$. The four points on the fix line of (A_2, A_2) form one orbit by (*). The formula (1) gives $N_0 \cdot n_1 = 360$. If $n_1 \geq 2$ then $N_0 \leq 180$ and the fix group in IO mod. ± 1 has order ≥ 8 . Checking in the table on page 24 this is not possible. Hence $N_0 = 360$ and $n_1 = 1$. The fix group has order four and it is isomorphic with $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each singular point is contained in two fix lines of matrices in the conjugacy class of $(A_2, R_4 A_2)$. Now the formula (1) with $N_0 = 360$, $n_1 = 2$ and $N_1 = 180$ gives $n_0 = 4$. So we conclude that the singular points on these fix lines are in the orbit of length 360 too.

III. $\lambda = c_i$, $i = 1, 2$. The three points on the fix line of (S, S) form one orbit under the action of $(S, 1)$. The formula (1) gives $N_0 \cdot n_1 = 240$. If $n_1 \geq 2$ then $N_0 \leq 120$ and the fix group mod. ± 1 has order ≥ 12 , which is not possible (check in the table on page 24 again). Hence $N_0 = 240$ and $n_1 = 1$. The fix group has order six and it is isomorphic with D_3 , hence three fix lines of matrices in $(A_2, R_4 A_2)$ contain a singular points. The formula (1) in this case gives $n_0 = 4$, which shows that the singular points on these fix lines are in the previous orbit too. \square

We give now the table of the configurations of lines and points:

Repr. of the conj. class	value of λ		Configuration	
(A_2, A_2)	0	$-\frac{1}{8}$	$(120_3, 90_4)$	$(360_1, 90_4)$
$(A_2, R_4 A_2)$	$-\frac{1}{8}$	c_i	$(360_2, 180_4)$	$(240_3, 180_4)$
(S, S)	0	c_i	$(120_4, 80_6)$	$(240_1, 80_3)$

4.3 The pencil of $D_n D_m$ -invariant quartics

We take the generators q^2 and $f_2 + f_3 = (x_0 - ix_1)(x_0 + ix_1)(x_2 - ix_3)(x_2 + ix_3)$. The latter is the union of four complex planes meeting each other at the four complex lines $\{x_0 \pm ix_1 = 0, x_2 \pm ix_3 = 0\}$ on q . The pencil contains the

multiple quadric $q'^2 := q^2 - 2(f_2 + f_3) = (x_0^2 + x_1^2 - x_2^2 - x_3^2)^2$ too. Hence with this new generator the equation of the pencil becomes

$$q'^2 + \lambda q^2 = (q' + i\sqrt{\lambda}q)(q' - i\sqrt{\lambda}q).$$

A surface in the pencil is the union of two quadrics which meet each other at the four lines given above. An easy computation shows that none of the surface in the pencil contains isolated singularities.

4.4 The pencil of OD_n -invariant octics ($n \geq 6$) and of ID_n -invariant 12-ics ($n \geq 8$)

We take generators q^4 , q^6 and $P_8 = \prod_{i=0}^3 K_i$, $P_{12} = \prod_{i=0}^5 K'_i$ (cf. (3.1.2)) and we denote the pencils by $\Pi_m(\lambda) := P_m + \lambda q^{\frac{m}{2}}$, $m = 8, 12$. We have

Lemma 4.6 *The base locus of the pencil is $\frac{m}{2}$ -times the intersection $q \cap P_m$, $m = 8, 12$. Where (as sets)*

$$\begin{aligned} q \cap P_8 &= \text{fix lines of } (1, R_n), \\ &\quad \text{orbit of eight lines under } (O, 1), \\ q \cap P_{12} &= \text{fix lines of } (1, R_n), \\ &\quad \text{orbit of twelve lines under } (I, 1). \end{aligned}$$

Proof. The element $(1, R_n)$ has fix lines $L_1 : \{x_0 = -ix_1, x_2 = ix_3\}$, $L_2 : \{x_0 = ix_1, x_2 = -ix_3\}$ for each n . These are contained in each quadric K_s , $s = 0, \dots, 3$, resp. K'_t , $t = 0, \dots, 5$, hence they have at least multiplicity four, resp. six in the intersection $q \cap P_m$, $m = 8, 12$. By this fact and an argumentation as in [S], (5.5), it follows that the lines of the length $m = 8, 12$ -orbit under $(O, 1)$, resp. $(I, 1)$ are in the base locus too. Moreover the previous multiplicity of intersection are exactly four, resp. six. \square

Lemma 4.7 *The pencil does not contain surfaces with isolated singular points.*

Proof. The groups have order $96n$, resp. $240n$, since an isolated singular point has orbit of finite length, it is fixed by $(1, R_n)$. This shows that it is on the lines of the base locus, hence not isolated. A contradiction to the assumption. \square

Proposition 4.3 *In the pencil we have the following singular surfaces, with one orbit of double lines which are fix lines for the elements of some conjugacy class:*

value of λ	$OD_n :$	-1	$\frac{1}{3}$	0
	$ID_n :$	0		
orbit	$OD_n :$	6	8	12
	$ID_n :$	30		
fix lines in the conj. class of		$(A_3, R_n^{\frac{n}{2}})$	(S, R_6^2)	$(R, R_n^{\frac{n}{2}})$

Proof. It is a direct computation using the equations $x_0 = x_2, x_1 = -x_3; x_0 = -(1 + \sqrt{3})x_3 + x_1, x_2 = x_3 + (1 - \sqrt{3})x_1$ and $x_2 = (1 - \sqrt{2})x_0, x_3 = (\sqrt{2} - 1)x_1$ of a fix line of $(A_3, R_n^{\frac{n}{2}}), (S, R_6^2),$ resp. $(R, R_n^{\frac{n}{2}}).$ \square

5 Final Remarks

1) By [Co2] p. 292 there are sixteen regular polytopes $\{p, q, r\}$ in four dimensions. These polytopes correspond to four distinct symmetry groups $[p, q, r]$ listed in the table below. Three of these symmetry groups can be obtained from groups which we described above by adding extra generators C and C' , where C denotes the matrix given in section 3, C' is the matrix of [S] p. 433.

symmetry groups	$[3, 3, 3]$	$[3, 3, 4]$	$[3, 4, 3]$	$[3, 3, 5]$
our groups		$(OO)'$	TT	II
extra generators		C	C, C'	C

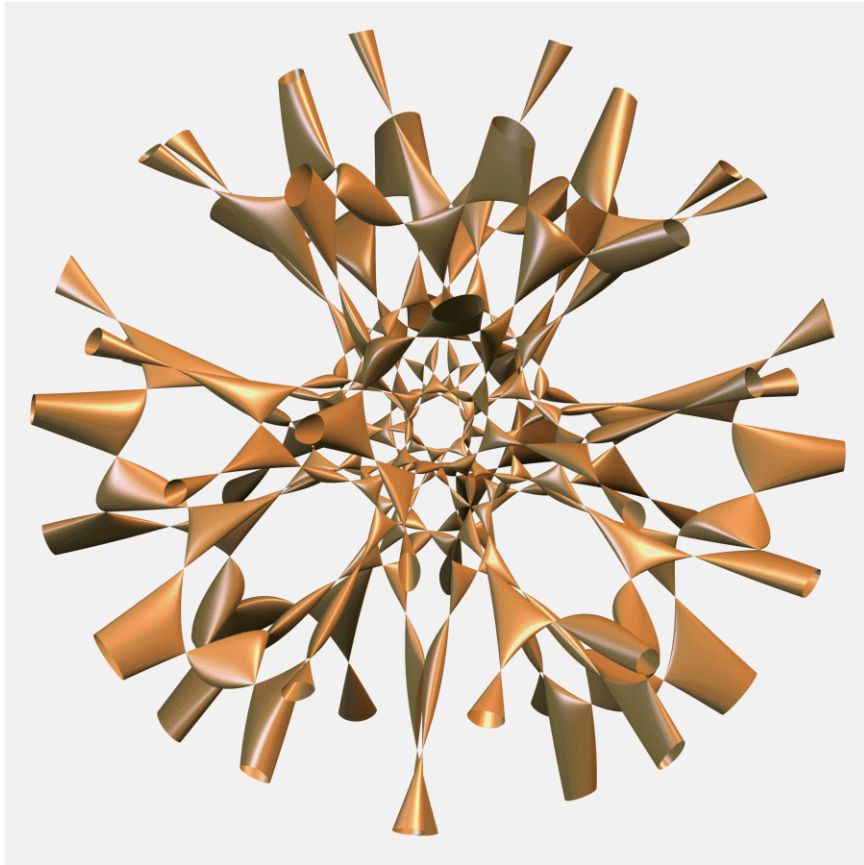
The group $[3, 3, 3] \cong Sym(5)$ does not contain the Heisenberg group H , in fact it has some invariant polynomials of odd degree (cf. [Co1] p. 780). In these notes we complete the description of the G -invariant pencils of surfaces whenever G is the symmetry group of a regular four dimensional polytope and G contains H .

2) In remark 3.3, with $n = 3$, we give the degree four invariant polynomials of the *bi-polyhedral dihedral group* $D_3D_3 \subseteq SO(4)$. By Mukai [M] the quotient \mathbb{P}_3/D_3D_3 is isomorphic with the Satake compactification of the moduli space of abelian surfaces with $(1, 2)$ -polarization, hence this invariant polynomials should be related to modular forms.

3) In [BS] the quotients $X/G, G = TT, OO, II$ are described. It would be interesting to examine the quotients in the remaining cases.

6 Computer Picture

We exhibit a computer picture of the $I \times O$ -symmetric surface of degree 12 with 360 nodes. This has been realized with the program SURF written by S. Endraß.



$I \times O$ -symmetric surface of degree 12
with 360 nodes

References

- [BS] Barth, W., Sarti, A., Polyhedral Groups and Pencils of K3-Surfaces with Maximal Picard Number, math. AG/0301329.
- [B] Benson, D. J., Polynomial Invariants of Finite Groups, London Math. Society LNS 190, Cambridge University Press (1993).

- [C] Chmutov, S. V., Examples of Projective Surfaces with many Singularities, *Journal of Alg. Geom.* 1, (1992) 191–196.
- [Co1] Coxeter, H. M. S., The product of the generators of a finite group generated by reflections, *Duke Math. J.* Vol. 18 (1951) 765–782.
- [Co2] Coxeter, H. M. S., *Regular polytopes* (second edition), The Macmillan company, New York (1963).
- [DV] Du Val, P., *Homographies Quaternions and Rotations*, Oxford Mathematical Monographs (1964).
- [M] Mukai, S., Moduli of Abelian surfaces and regular polyhedral groups, in: *Proceedings of Moduli of algebraic varieties Symposium*, Sapporo January 1999.
- [S] Sarti, A., Pencils of symmetric surfaces in $\mathbb{P}^3(\mathbb{C})$, *J. of Alg.* 246, 429–452 (2001).
- [ST1] Seifert, H., Threfall, W., Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes, *Math. Ann.* 104, (1931) 1–70.
- [ST2] Seifert, H., Threfall, W., Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß), *Math. Ann.* 107, (1932) 543–586.
- [SR] Semple, J. G., Roth, L., *Introduction to Algebraic Geometry*, Oxford Univeristy Press (1949).