A GEOMETRICAL CONSTRUCTION FOR THE POLYNOMIAL INVARIANTS OF SOME REFLECTION GROUPS

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ABSTRACT. We construct invariant polynomials for the reflection groups $[3,4,3]$ and $[3,3,5]$ by using some special sets of lines on the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$. Then we give a simple proof of the well known fact that the ring of invariants are rationally generated in degree 2,6,8,12 and 2,12,20,30.

Key Words: Polynomial invariants, Reflection and Coxeter groups, Group actions on varieties.

Mathematics Subject Classification: Primary 20F55, 13F20; Secondary 14L30.

0. INTRODUCTION

There are four groups generated by reflections which operate on the four-dimensional Euclidean space. These are the symmetry groups of some regular four dimensional polytopes and are described in [C2, p. 145 and Table I p. 292-295]. With the notation there the polytopes, the groups and their orders are

Polytope	$5-$cell	$16-$cell	$24-$cell	$600-$cell
Group	$[3,3,3]$	$[3,3,4]$	$[3,4,3]$	$[3,3,5]$
Order	120	384	1152	14400

They operate in a natural way on the ring of polynomials $R = \mathbb{R}[x_0, x_1, x_2, x_3]$ and it is well known that the ring of invariants R^G (G one of the groups above) is algebraically generated by a set of four independent polynomials (cf. [B, p. 357]). Coxeter shows in [C1] that the rings R^G, $G = [3,3,3]$ or $[3,3,4]$ are generated in degree $2,3,4,5$ resp. $2,4,6,8$ and since the product of the degrees is equal to the order of the group, any other invariant polynomial is a combination with real coefficients of products of these invariants (i.e., in the terminology of [C1], the ring R^G is *rationally generated* by the polynomials). Coxeter also gives equations for the generators. In the case of the groups $[3,4,3]$ and $[3,3,5]$ he recalls a result of Racah,(cf. [R]), who shows with the help of the theory of Lie groups that the rings R^G are rationally generated in degree $2,6,8,12$ resp. $2,12,20,30$.
Equations for these generating polynomials can be found e.g. in [M], [Sm, p. 218], [CS, p. 203] and most recently in [IKM] (the groups are often denoted in the literature by F_4 and H_4). The method used by Metha in [M] is simple: He considers the equations of the reflecting hyperplanes and he finds a set of linear forms which are invariant under the action of the groups $[3,4,3]$ resp. $[3,3,5]$, then he uses these to give equations for the polynomial invariants (a similar method is used by Coxeter in the case of the groups $[3,3,3]$ and $[3,3,4]$). In [Sm, p. 218] Smith explains how to obtain equations for the invariant polynomials of the rings R^G,

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but he refers to [CS] for the explicit equations, however only in the case of the group $[3,4,3]$. In fact Conway and Sloane use coding theory to construct the invariants of this group, but they do not consider the case of $[3,3,5]$. In [IKM] the authors find the invariants by solving a special system of partial differential equations. But as they say the method is quite elaborated and they need the support of computer-algebra.

In this paper we give a different construction, which should be interesting in particular from the point of view of algebraic geometry: We consider some special $[3,4,3]$-, resp. $[3,3,5]$-orbit of lines on the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$ and construct the invariant polynomials by using the action of the group and geometric considerations. We remark that in our construction of the polynomials we use very little computer-algebra, in fact only MAPLE for some computation in Proposition 2.1 and 3.2 (cf. Section 4). Otherwise everything is proved by hand and by geometric considerations. This construction seems to be interesting for the following reasons:
1. We can give a simple proof of Racah's result,
2. We establish relations between the invariants of the groups $[3,4,3]$ and $[3,3,5]$ and the invariants of some binary subgroups of $SU(2)$,
3. The construction may be helpful in the study of the geometry of the algebraic surfaces defined by the zero sets of the invariant polynomials. We have in fact families of surfaces with many symmetries and by the construction, for example it is possible to determine immediately the base locus of the families, which consists of sets of lines on $\mathbb{P}_1 \times \mathbb{P}_1$.

We explain now briefly our method and also the structure of the paper: Denote by T, O and I the rotations subgroups in $SO(3,\mathbb{R})$ of the platonic solids: tetrahedron, octahedron and icosahedron, it is well known that $SO(4,\mathbb{R})$ contains central extensions G_6 of $T \times T$, G_8 of $O \times O$ and G_{12} of $I \times I$ by ± 1. Then G_6 is an index four subgroup of $[3,4,3]$ and G_{12} is an index two subgroup of $[3,3,5]$ (cf. e.g. [Sa], Section 3). These two groups, and G_8 too, act on the three dimensional projective space $\mathbb{P}_3$, and in particular on the two ruling of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ (this action is studied in [Sa]). The quadric can be described as the zero set of the quadratic form:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2$$

which is $[3,4,3]$- and $[3,3,5]$-invariant. By considering some special orbits of lines of $\mathbb{P}_1 \times \mathbb{P}_1$ under G_6, G_{12} and G_8, it is possible to construct explicitly $[3,4,3]$- and $[3,3,5]$-invariant polynomials, this is done and explained in details in Section 2. In Section 3 we show that our polynomials generate the rings of invariants R^G by showing some relations between them and the invariant forms of the binary tetrahedral group and of the binary icosahedral group in $SU(2)$. More precisely we define a surjective map between polynomials of degree d on $\mathbb{P}_3$ and polynomials of be-degree (d,d) on $\mathbb{P}_1 \times \mathbb{P}_1$. Then we show that the image of a G_n-invariant polynomial $n = 6, 12$ splits into the product of two invariant polynomials of the same degree under the action of the binary subgroup in $SU(2)$ corresponding to G_n (there are classical $2:1$ maps $SU(2) \longrightarrow SO(3)$, $SU(2) \times SU(2) \longrightarrow SO(4)$ which we recall in Section 1). This corresponds in some sense to the fact that G_n contains the product $G \times G$ (for $n = 6$ is $G = T$ and for $n = 12$ is $G = I$) and each copy $G \times 1$, $1 \times G$ operates on one ruling of $\mathbb{P}_1 \times \mathbb{P}_1$ and leaves the other ruling invariant. This relation is the main ingredient in our proof of the result of Racah (Corollary 2.1). It seems to be however interesting by itself. Finally Section 4 contains explicit computations and in Section 5 we present open problems and possible applications of the results of the paper.

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1. NOTATIONS AND PRELIMINARIES

Denote by R the ring of polynomials in four variables with real coefficients $\mathbb{R}[x_0, x_1, x_2, x_3]$, by G a finite group of homogeneous linear substitutions, and by R^G the ring of invariant polynomials.

1. A set of polynomials $F_1, \ldots, F_n$ in R is called *algebraically dependent* if there is a non trivial relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_n^{i_n}) = 0,$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $\alpha_I \in \mathbb{R}$.

2. The polynomials are called *algebraically independent* if they are not dependent. For the ring R^G, there always exists a set of four algebraically independent polynomials (cf. [B], thm. I, p. 357).

3. We say that R^G is *algebraically generated* by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have an algebraic relation

$$\sum \alpha_I (P^{i_0} \cdot F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = 0.$$

4. We say that the ring R^G is *rationally generated* by a set of polynomials $F_1, \ldots, F_4$, if for any other polynomial $P \in R^G$ we have a relation

$$\sum \alpha_I (F_1^{i_1} \cdot \ldots \cdot F_4^{i_4}) = P, \ \alpha_I \in \mathbb{R}$$

5. The four polynomials of 3 are called a *basic set* if they have the smallest possible degree (cf. [C1]).

6. There are two classical 2 : 1 coverings

$$\rho : SU(2) \to SO(3) \text{ and } \sigma : SU(2) \times SU(2) \to SO(4),$$

we denote by T, O, I the tetrahedral group, the octahedral group and the icosahedral group in $SO(3)$ and by $\tilde{T}$, $\tilde{O}$, $\tilde{I}$ the corresponding binary groups in $SU(2)$ via the map ρ. The σ-images of $\tilde{T} \times \tilde{T}$, $\tilde{O} \times \tilde{O}$ and $\tilde{I} \times \tilde{I}$ in $SO(4)$ are denoted by G_6, G_8 and G_{12}. By abuse of notation we write (p, q) for the image in $SO(4)$ of an element $(p, q) \in SU(2) \times SU(2)$. As showed in [Sa] (3.1) p. 436, the groups G_6 and G_{12} are subgroups of index four respectively two in the reflections groups $[3, 4, 3]$ and $[3, 3, 5]$.

2. GEOMETRICAL CONSTRUCTION

Denote by $\tilde{G}$ one of the groups $\tilde{T}$, $\tilde{O}$ or $\tilde{I}$. Clearly, the subgroups $\tilde{G} \times 1$ and $1 \times \tilde{G}$ of $SO(4)$ are isomorphic to $\tilde{G}$. Moreover, each of them operates on one of the two rulings of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ and leaves invariant the other ruling (as shown in [Sa]). We recall the lengths of the orbits of points under the action of the groups T, O and I

group	T	O	I
lengths of the orbits	12, 6, 4	24, 12, 8, 6	60, 30, 20, 12

These lines are fixed by elements $(p, 1) \in \tilde{G} \times 1$ on one ruling, resp. $(1, p') \in 1 \times \tilde{G}$ on the other ruling of the quadric. Recall that these elements have two lines of fix points with eigenvalues α, $\bar{\alpha}$ which are in fact the eigenvalues of p and p'. We call two lines L, L' of $\mathbb{P}_1 \times \mathbb{P}_1$ a *couple* if L is fixed by $(p, 1)$ with eigenvalue α and L' is fixed by $(1, p)$ with the same eigenvalue.

2.1. **The invariant polynomials of G_6 and of G_{12}.** Consider the six couples of lines $L_1, L_1', \dots, L_6, L_6'$ in $\mathbb{P}_1 \times \mathbb{P}_1$ which form one orbit under the action of $\tilde{T} \times 1$, resp. $1 \times \tilde{T}$, and denote by $f_{11}^{(6)}, \dots, f_{66}^{(6)}$ the six planes generated by such a couple of lines (and by abuse of notation their equation, too). Now set

$$F_6 = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(6)} \cdot f_{22}^{(6)} \cdot \dots \cdot f_{66}^{(6)}) = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(6)}) \cdot g(f_{22}^{(6)}) \cdot \dots \cdot g(f_{66}^{(6)}).$$

Observe that an element $g \in \tilde{T} \times 1$ leaves each line of one ruling invariant and operates on the six lines of the other ruling. A similar action is given by an element of $1 \times \tilde{T}$. Since we sum over all the elements of $\tilde{T} \times 1$, the action of $1 \times \tilde{T}$ does not give anything new, hence F_6 is G_6-invariant. Furthermore observe that F_6 has real coefficients. In fact, in the above product, for each plane generated by the lines L_i, L_i' we also take the plane generated by the lines which consist of the conjugate points. The latter has equation $\bar{f_{ii}}^{(6)}$, i.e., we have an index $j \neq i$ with $f_{jj}^{(6)} = \bar{f_{ii}}^{(6)}$ and the products $f_{ii}^{(6)} \cdot \bar{f_{ii}}^{(6)}$ have real coefficients.
Consider now the orbits of lengths eight and twelve under the action of $\tilde{O} \times 1$ and $1 \times \tilde{O}$ and the planes $f_{ii}^{(8)}$, $f_{jj}^{(12)}$ generated by the eight, respectively by the twelve couples of lines. As before the polynomials

$$F_8 = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(8)} \cdot \dots \cdot f_{88}^{(8)}),$$

$$F_{12} = \sum_{g \in \tilde{T} \times 1} g(f_{11}^{(12)} \cdot \dots \cdot f_{1212}^{(12)})$$

are G_6-invariant and have real coefficients.
Finally we consider the lines of $\mathbb{P}_1 \times \mathbb{P}_1$ which form orbits of length $12, 20$ and 30 under the action of $\tilde{I} \times 1$ resp. $1 \times \tilde{I}$. The planes generated by the couples of lines produce the G_{12}-invariant real polynomials

$$\Gamma_{12} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(12)} \cdot \dots \cdot h_{1212}^{(12)}),$$

$$\Gamma_{20} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(20)} \cdot \dots \cdot h_{2020}^{(20)}),$$

$$\Gamma_{30} = \sum_{g \in \tilde{I} \times 1} g(h_{11}^{(30)} \cdot \dots \cdot h_{3030}^{(30)}).$$

2.2. **The invariant polynomials of the reflection groups.** We consider the matrices

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

as in [Sa] (3.1) p. 436, the groups generated by G_6, C, C' and G_{12}, C are the reflections groups $[3,4,3]$ respectively $[3,3,5]$.

Proposition 2.1. *1. The polynomials F_6, F_8, F_{12}, Γ_{12}, Γ_{20}, Γ_{30} are C invariant.*
2. The polynomials F_6, F_8, F_{12} are C' invariant.

Proof. 1. The matrix C interchanges the two rulings of the quadric, hence the polynomials F_i and Γ_j are invariant by construction. We prove 2 by a direct computation in the last Section. □

From this fact we obtain

Corollary 2.1. *The polynomials q, F_6, F_8, F_{12} are $[3,4,3]$-invariant and the polynomials $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ are $[3,3,5]$-invariant.*

Here we denote by q the quadric $\mathbb{P}_1 \times \mathbb{P}_1$.

3. The rings of invariant forms

Identify $\mathbb{P}_3$ with $\mathbb{P}M(2\times 2, \mathbb{C})$ by the map

$$(x_0 : x_1 : x_2 : x_3) \quad \mapsto \quad \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}. \tag{1}$$

Furthermore consider the map

$$\begin{array}{rcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longrightarrow & M(2\times 2, \mathbb{C}) \\ ((z_0, z_1), (z_2, z_3)) & \longmapsto & \begin{pmatrix} z_0 z_2 & z_0 z_3 \\ z_1 z_2 & z_1 z_3 \end{pmatrix} = \mathcal{Z}. \end{array} \tag{2}$$

Then $\mathcal{Z}$ is a matrix of determinant $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ which is the equation of q. Now denote by $\mathcal{O}_{\mathbb{P}_3}(n)$ the sheaf of regular functions of degree n on $\mathbb{P}_3$ and by $\mathcal{O}_q(n,n)$ the sheaf of regular function of be-degree (n,n) on the quadric q. We obtain a surjective map between the global sections

$$\phi : \quad H^0(\mathcal{O}_{\mathbb{P}_3}(n)) \quad \longrightarrow \quad H^0(\mathcal{O}_q(n,n)) \tag{3}$$

by doing the substitution

$$x_0 = \frac{z_0 z_2 + z_1 z_3}{2}, \quad x_1 = \frac{z_0 z_2 - z_1 z_3}{2i},$$

$$x_2 = \frac{z_0 z_3 - z_1 z_2}{2}, \quad x_3 = \frac{z_0 z_3 + z_1 z_2}{2i}$$

in a polynomial $p(x_0, x_1, x_2, x_3) \in H^0(\mathcal{O}_{\mathbb{P}_3}(n))$. Observe that $\phi(q) = 0$. Now let

$$\begin{array}{rcl} t & = & z_0 z_1 (z_0^4 - z_1^4), \\ W & = & z_0^8 + 14 z_0^4 z_1^4 + z_1^8, \\ \chi & = & z_0^{12} - 33(z_0^8 z_1^4 + z_0^4 z_1^8) + z_1^{12} \end{array}$$

denote the $\tilde{T}$-invariant polynomials of degree $6, 8$ and 12 and let

$$\begin{array}{rcl} f & = & z_0 z_1 (z_0^{10} + 11 z_0^5 z_1^5 - z_1^{10}), \\ H & = & -(z_0^{20} + z_1^{20}) + 228(z_0^{15} z_1^5 - z_0^5 z_1^{15}) - 494 z_0^{10} z_1^{10}, \\ \mathcal{T} & = & (z_0^{30} + z_1^{30}) + 522(z_0^{25} z_1^5 - z_0^5 z_1^{25}) - 10005(z_0^{20} z_1^{10} + z_0^{10} z_1^{20}) \end{array}$$

be the $\tilde{I}$-invariant polynomials of degree $12, 20, 30$ given by Klein in [K] p. 51-58. Put $t_1 = t(z_0, z_1)$, $t_2 = t(z_2, z_3)$, $W_1 = W(z_0, z_1)$, $W_2 = W(z_2, z_3)$ and analogously for the other invariants.

Proposition 3.1. *If $p \in H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ is G_6-invariant, then:*

$$\phi(p) = \sum_I \alpha_I t_1^{\alpha_1} t_2^{\alpha_1'} W_1^{\alpha_2} W_2^{\alpha_2'} \chi_1^{\alpha_3} \chi_2^{\alpha_3'}$$

If p is G_{12}-invariant, then:

$$\phi(p) = \sum_J \beta_J f_1^{\beta_1} f_2^{\beta_1'} H_1^{\beta_2} H_2^{\beta_2'} \mathcal{T}_1^{\beta_3} \mathcal{T}_2^{\beta_3'}$$

where

$$I = \{(\alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3') | \alpha_i, \alpha_i' \in \mathbb{N}, 6\alpha_1 + 8\alpha_2 + 12\alpha_3 = n,\ 6\alpha_1' + 8\alpha_2' + 12\alpha_3' = n\},$$
$$J = \{(\beta_1, \beta_1', \beta_2, \beta_2', \beta_3, \beta_3') | \beta_i, \beta_i' \in \mathbb{N}, 12\beta_1 + 20\beta_2 + 30\beta_3 = n,\ 12\beta_1' + 20\beta_2' + 30\beta_3' = n\}.$$

Proof. Put

$$\phi(p) = p'(z_0, z_1, z_2, z_3).$$

An element $g = (g_1, g_2)$ in G_6 or G_{12} operates on $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_3$ by the matrix multiplication

$$g_1 \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} g_2^{-1}$$

and on the matrix $\mathcal{Z}$ of (2) by

$$g_1 \begin{pmatrix} z_0 z_2 & z_0 z_3 \\ z_1 z_2 & z_1 z_3 \end{pmatrix} g_2^{-1} \quad = \quad g_1 \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} z_2 & z_3 \end{pmatrix} g_2^{-1}.$$

Clearly if p is G_6- or G_{12}-invariant then also the projection $\phi(p)$ with the previous operation is. In particular for $g = (g_1, 1)$ in $\tilde{T} \times 1$, resp. in $\tilde{I} \times 1$ the polynomial p' is $\tilde{T} \times 1$-, respectively $\tilde{I} \times 1$-invariant as polynomial in the coordinates $(z_0 : z_1) \in \mathbb{P}_1$ and for any $(z_2 : z_3) \in \mathbb{P}_1$. On the other hand for $g = (1, g_2)$ in $1 \times \tilde{T}$, resp. in $1 \times \tilde{I}$ the polynomial p' is $1 \times \tilde{T}$-, respectively $1 \times \tilde{I}$-invariant as polynomial in the coordinate $(z_2 : z_3) \in \mathbb{P}_1$ and for any $(z_0 : z_1) \in \mathbb{P}_1$. Hence p' must be in the form of the statement.

□

By a direct computation in Section 4 we prove the following

Proposition 3.2. *The quadric q does not divide the polynomials F_i, Γ_j. Moreover, F_6 does not divide F_{12}.*

Corollary 3.1. *We have $\phi(q) = 0$, $\phi(F_6) = t_1 \cdot t_2$, $\phi(F_8) = W_1 \cdot W_2$, $\phi(F_{12}) = \chi_1 \cdot \chi_2$, $\phi(\Gamma_{12}) = f_1 \cdot f_2$, $\phi(\Gamma_{20}) = H_1 \cdot H_2$, $\phi(\Gamma_{30}) = T_1 \cdot T_2$ (up to some scalar factor).*

Proof. This follows from Proposition 3.1 and 3.2

Proposition 3.3. *The polynomials q, F_6, F_8, F_{12}, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ are algebraically independent.*

Proof. Let $\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4} = 0$ and $\sum_J \beta_J q^{j_1} \Gamma_{12}^{j_2} \Gamma_{20}^{j_3} \Gamma_{30}^{j_4} = 0$ be algebraic relations, $I = (i_1, i_2, i_3, i_4) \in \mathbb{N}^4$, $J = (j_1, j_2, j_3, j_4) \in \mathbb{N}^4$, $\alpha_I, \beta_J \in \mathbb{R}$, then

$$\begin{aligned} 0 &= \phi(\textstyle\sum_I \alpha_I q^{i_1} F_6^{i_2} F_8^{i_3} F_{12}^{i_4}) \\ &= \textstyle\sum_{I'} \alpha_{I'} \phi(F_6)^{i_2'} \phi(F_8)^{i_3'} \phi(F_{12})^{i_4'} \\ &= \textstyle\sum_{I'} \alpha_{I'} t_1^{i_2'} t_2^{i_2'} W_1^{i_3'} W_2^{i_3'} \chi_1^{i_4'} \chi_2^{i_4'} \end{aligned} \tag{4}$$

similarly

$$
\begin{aligned}
0 &= \phi(\textstyle\sum_J \beta_J q^{j_1}\Gamma_{12}^{j_2}\Gamma_{20}^{j_3}\Gamma_{30}^{j_4}) \\
&= \textstyle\sum_{J'} \beta_{J'}\phi(\Gamma_{12})^{j'_2}\phi(\Gamma_{20})^{j'_3}\phi(\Gamma_{30})^{j'_4} \\
&= \textstyle\sum_{J'} \beta_{J'} f_1^{j'_2} f_2^{j'_2} H_1^{j'_3} H_2^{j'_3} \mathcal{T}_1^{j'_4}\mathcal{T}_2^{j'_4}.
\end{aligned} \tag{5}
$$

If the polynomials t_1, W_1, χ_1 are fixed, we obtain a relation between t_2, W_2 and χ_2, which is the same relation as for t_1, W_1 and χ_1 if we fix t_2, W_2 and χ_2. The same holds for the polynomials $f_1, H_1, \mathcal{T}_1$ and $f_2, H_2, \mathcal{T}_2$. From [K] p. 55 and p. 57 there are only the relations

$$108\, t_1^4 - W_1^3 + \chi_1^2 = 0,\ 108\, t_2^4 - W_2^3 + \chi_2^2 = 0$$

and

$$\mathcal{T}_1^2 + H_1^3 - 1728 f_1^5 = 0,\ \mathcal{T}_2^2 + H_2^3 - 1728 f_2^5 = 0$$

between these polynomials. By multiplying these relations, however, it is not possible to obtain expressions like (4) and (5). □

Corollary 3.2. *The polynomials q, F_6, F_8, F_{12}, resp. $q, \Gamma_{12}, \Gamma_{20}, \Gamma_{30}$ generate rationally the ring of invariant polynomials of* $[3,4,3]$*, resp.* $[3,3,5]$.

Proof. (cf. [C1] p. 775) By Proposition 3.3 and Proposition 3.2 these are algebraically independent, moreover the products of their degrees are

$$2\cdot 6\cdot 8\cdot 12 = 1152 \text{ and } 2\cdot 12\cdot 20\cdot 30 = 14400,$$

which are equal to the order of the groups $[3,4,3]$ and $[3,3,5]$. By [C1] this implies the assertion. □

4. EXPLICIT COMPUTATIONS

We recall the following matrices of $SO(4)$ (cf. [Sa]):

$$(q_2,1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\quad (1,q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(p_3,1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},\quad (1,p_3) = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$(p_4,1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},\quad (1,p_4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$(p_5,1)=\frac{1}{2}\begin{pmatrix}\tau & 0 & 1-\tau & -1\\ 0 & \tau & -1 & \tau-1\\ \tau-1 & 1 & \tau & 0\\ 1 & 1-\tau & 0 & \tau\end{pmatrix},$$
$$(1,p_5)=\frac{1}{2}\begin{pmatrix}\tau & 0 & \tau-1 & 1\\ 0 & \tau & -1 & \tau-1\\ 1-\tau & 1 & \tau & 0\\ -1 & 1-\tau & 0 & \tau\end{pmatrix},$$

where $\tau=\frac{1}{2}(1+\sqrt{5})$. Then we have

Group	Generators
G_6	$(q_2,1),(1,q_2),(p_3,1),(1,p_3)$
G_8	$(q_2,1),(1,q_2),(p_3,1),(1,p_3),(p_4,1),(1,p_4)$
G_{12}	$(q_2,1),(1,q_2),(p_3,1),(1,p_3),(p_5,1),(1,p_5)$

Now we can write down the equations of the fix lines on $\mathbb{P}_1\times\mathbb{P}_1$ and those of the planes which are generated by a couple of lines. The products of planes of Section 2.1 in the case of the group G_6 are

$$\begin{aligned}
f_{11}^{(6)}\cdot f_{22}^{(6)}\cdot\ldots\cdot f_{66}^{(6)} &= (x_2-ix_3)(x_1+ix_3)(x_2+ix_3)(x_1-ix_2)(x_1-ix_3)(x_1+ix_2),\\
f_{11}^{(8)}\cdot f_{22}^{(8)}\cdot\ldots\cdot f_{88}^{(8)} &= (x_1+ax_2-bx_3)(x_1+bx_2-ax_3)(x_1-ax_2-bx_3)(x_1-ax_3-bx_2)\\
&\quad (x_2+bx_1-ax_3)(x_2+ax_1-bx_3)(x_2-bx_1+ax_3)(x_2+bx_3-ax_1),\\
f_{11}^{(12)}\cdot f_{22}^{(12)}\cdot\ldots\cdot f_{1212}^{(12)} &= (x_3-x_1+cx_2)(x_3-x_1-cx_2)(x_2+x_3-cx_1)(x_2+x_3+cx_1)\\
&\quad (x_3-x_2+cx_1)(x_3-x_2-cx_1)(x_1+x_2+cx_3)(x_1+x_2-cx_3)\\
&\quad (x_1+x_3-cx_2)(x_1+x_3+cx_2)(x_1-x_2+cx_3)(x_1-x_2-cx_3),
\end{aligned}$$

with $a=(1/2)(1+i\sqrt{3}), b=(1/2)(1-i\sqrt{3}), c=i\sqrt{2}$.
Then the G_6-invariant polynomials F_6, F_8 and F_{12} have the following expressions

$$\begin{aligned}
F_6 &= x_0^6+x_1^6+x_2^6+x_3^6+5x_0^2x_1^2(x_0^2+x_1^2)+5x_1^2x_3^2(x_1^2+x_3^2)+5x_1^2x_2^2(x_1^2+x_2^2)\\
&\quad +6x_0^2x_2^2(x_0^2+x_2^2)+6x_0^2x_3^2(x_0^2+x_3^2)+6x_3^2x_2^2(x_2^2+x_3^2)+2x_0^2x_2^2x_3^2,
\end{aligned}$$

$$F_8 = 3\sum x_i^8+12\sum x_i^6x_j^2+30\sum x_i^4x_j^4+24\sum x_i^4x_j^2x_k^2+144x_0^2x_1^2x_2^2x_3^2,$$

$$\begin{aligned}
F_{12} &= \frac{123}{8}\sum x_i^{12}+\frac{231}{4}\sum x_i^{10}x_j^2+\frac{21}{8}\sum x_i^8x_j^4-\sum\frac{255}{2}\sum x_i^6x_j^6+\frac{949}{2}\sum x_i^8x_j^2x_k^2\\
&\quad +\frac{1839}{2}\sum x_i^6x_j^4x_k^2+\frac{6111}{4}\sum x_i^4x_j^4x_k^4+1809\sum x_i^6x_j^2x_k^2x_h^2+\frac{7281}{2}\sum x_i^4x_j^4x_k^2x_h^2.
\end{aligned}$$

Here the sums run over all the indices $i, j, k, h = 0, 1, 2, 3$, always being different when appearing together. By applying the map ϕ, a computer computation with MAPLE shows that

$$\begin{aligned} \phi(F_6) &= -\frac{13}{16} t_1 \cdot t_2, \\ \phi(F_8) &= \frac{3}{64} W_1 \cdot W_2, \\ \phi(F_{12}) &= \frac{3}{256} \chi_1 \cdot \chi_2 \end{aligned}$$

as claimed in Corollary 3.1.
Proof of Proposition 2.1, 2. The polynomials $F_6,\ F_8,\ F_{12}$ remain invariant by interchanging x_2 with x_3, which is what the matrix C' does. □

Proof of Proposition 3.2. We write the computations just in the case of the $[3,4,3]$-invariant polynomials. Consider the points $p_1 = (i\sqrt{2} : 1 : 1 : 0)$ and $p_2 = (1 : i : 0 : 0)$, then $q(p_1) = q(p_2) = 0$ and by a computer computation with MAPLE we get $F_6(p_1) = 26$, $F_8(p_2) = 12$ and $F_{12}(p_2) = 32$. This shows that q does not divide the polynomials. Since $F_6(p_2) = 0$, F_6 does not divide F_{12}. □

Remark 4.1. Observe that an equation for a $[3,4,3]$-invariant polynomial of degree six and for a $[3,3,5]$-invariant polynomial of degree twelve was given by the author in [Sa] by a direct computer computation with MAPLE.

5. Final Remarks

1. The zero sets of the polynomials which are described in this paper define algebraic surfaces in $\mathbb{P}_3(\mathbb{C})$ with many symmetries. Such surfaces are expected to have many interesting geometrical properties: many lines, many singularities, etc. In [Sa] it is shown that the projective one-dimensional families of surfaces with equations $F_6 + \lambda q^3 = 0$ and $\Gamma_{12} + \lambda q^6 = 0$, $\lambda \in \mathbb{P}_1$ contain each four surfaces with many nodes. The article also describes a one-dimensional $[3,4,3]$-invariant family of surfaces of degree 8. The family contains four surfaces with A_1-singularities and it is also G_8-symmetric. In Figure 1 we show the picture of a surface with 144 nodes. But in fact the whole $[3,4,3]$-invariant family of surfaces of degree 8 is projectively two-dimensional with equation $F_8 + \lambda F_6 \cdot q + \mu q^4 = 0$, $(\lambda, \mu) \in \mathbb{P}_2$. It would be interesting to describe more surfaces in this family and in the families of $[3,4,3]$-symmetric surfaces of degree 12 and of $[3,3,5]$-symmetric surfaces of degree 20 and 30.

2. Another interesting problem is to study the quotients of the previous surfaces by the groups. In [BS] it is shown that the G_6-quotient, resp. the G_{12}-quotient of a surface in the family defined by $F_6 + \lambda q^3 = 0$, resp. defined by $\Gamma_{12} + \lambda q^6 = 0$ is a K3-surface. It would be interesting to identify the quotients by the groups $[3,4,3]$, resp. $[3,3,5]$ which contain the groups G_6, resp. G_{12}. And in general, to describe more quotients.

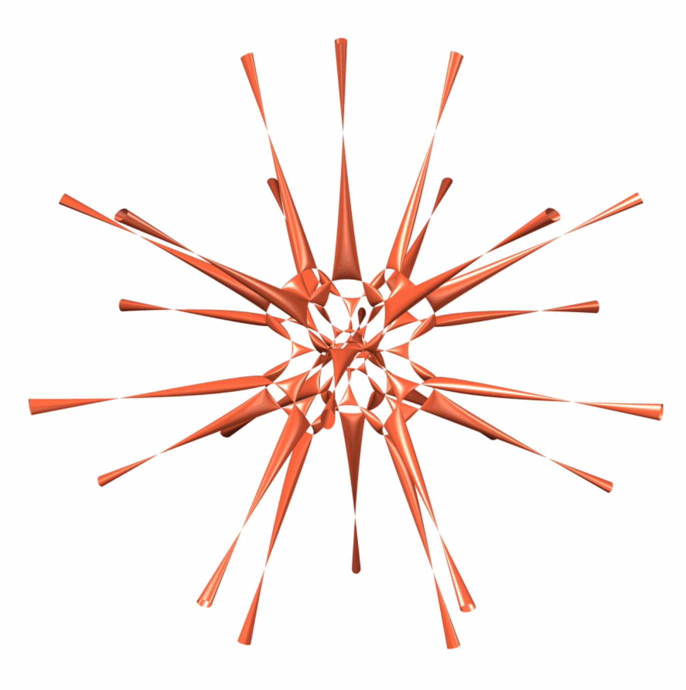

Fig. 1. [3, 4, 3]-symmetric octic with 144 nodes

$$x_0^8 + x_1^8 + x_2^8 + x_3^8 + 14\left(x_0^4x_1^4 + x_0^4x_2^4 + x_0^4x_3^4 + x_1^4x_2^4 + x_1^4x_3^4 + x_2^4x_3^4\right) +$$
$$+168x_0^2x_1^2x_2^2x_3^2 - \tfrac{9}{16}(x_0^2 + x_1^2 + x_2^2 + x_3^2)^4 = 0$$

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