Pencils of symmetric surfaces in $\mathbb{P}_3$

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0. Introduction. The rotations leaving invariant a platonic solid form a finite subgroup of SO(3). Since the cube is dual to the octahedron, the icosahedron dual to the dodecahedron, there are three such rotation groups $G\subseteq$SO(3), isomorphic with A_4, S_4, and A_5 respectively. By *binary polyhedral group* we denote the inverse image $\tilde{G}\subseteq$SU(2) of G under the universal covering SU(2)$\rightarrow$SO(3). The 2 : 1 image of $\tilde{G}\times\tilde{G}$ in SO(4) under SU(2)$\times$SU(2)$\rightarrow$SO(4) will be called *bi-polyhedral group* G_n:

group:	G_6	G_8	G_{12}
order:	288	1152	7200

The aim of this note is to study the simplest non-trivial G_n-invariant polynomials.
As subgroups of SO(4) the three groups G_n admit the trivial invariant $Q(x) = x_0^2+x_1^2+x_2^2+x_3^2$. In section **2.** we compute the Poincaré series of the three groups G_n acting on $\mathbb{C}[x_0,x_1,x_2,x_3]$ and find that the first non-trivial invariant is homogeneous of degree n. This is the reason for choosing the notation G_n. In section **4.** such a non-trivial G_n-invariant S_n is given explicitly. The equation

$$S_n(x)+\lambda Q(x)^{\frac{n}{2}}=0, \quad \lambda\in\mathbb{P}_1(\mathbb{C})$$

defines a pencil of G_n-invariant surfaces of degree n in $\mathbb{P}_3(\mathbb{C})$. We compute the singular surfaces in each pencil as well as their singularities.
It turns out that each pencil contains, except for the multiple quadric $Q(x)^{\frac{n}{2}}=0$, precisely four singular surfaces. Their singularities are ordinary double points and form one G_n-orbit. The parameters λ for these surfaces and their numbers of singularities are:

n	λ				singularities			
6	-1	$-\frac{2}{3}$	$-\frac{7}{12}$	$-\frac{1}{4}$	12	48	48	12
8	-1	$-\frac{3}{4}$	$-\frac{9}{16}$	$-\frac{5}{9}$	24	72	144	96
12	$-\frac{3}{32}$	$-\frac{22}{243}$	$-\frac{2}{25}$	0	300	600	360	60

Very often argumentations use explicit computations in the coordinates. These usually can be done by hand, but it is safer to use a computer programm like MAPLE. In some point, however they are so involved that I cannot avoid the use of the computer.
The results seem to be interesting for the following reasons:

1. S. Mukai [10] identified the quotient spaces $\mathbb{P}_3/G_n$ with the Satake compactification ($n = 6, 8$), resp. with a certain modification of it ($n = 12$), of the moduli spaces of abelian surfaces admitting a certain polarization. Our surfaces descend to these quotients and they should therefore be related to modular forms.

2. The reflection group of the regular four-dimensional 24-cell, resp. 600-cell (cf. [4] chapter VIII p. 145) contains the group G_6, resp. G_{12}. The invariants of G_6, resp. G_{12} in fact are invariant under this bigger group. The existence of these invariants for the reflection groups is known (cf. [3], [11]), explicit equations however seem not to have been computed before.

3. V. Goryunov at Europroj '96 observed that G_{12} admits an invariant surface of degree 12 with 600 nodes. Our explicit computations confirm his announcement. This gives the lower bound $\mu(12) \geq 600$, where $\mu(d)$ denotes the maximal number of nodes on a projective surface of degree d. Upper bounds for $\mu(d)$ are given in [13], [9]. In particular $\mu(12) \leq 645$ by Miyaoka's bound [9]. By a result of Kreiß [8] it was known that $\mu(12) \geq 576$. Our result explicitly improves this bound to show

$$600 \leq \mu(12) \leq 645.$$

We exhibit in section **12.** a computer picture of this surface.

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1. Rotation groups. We start by considering the rotation groups of the platonic solids. These are tetrahedron, octahedron, cube, icosahedron and dodecahedron, where octahedron and cube, icosahedron and dodecahedron are reciprocal (cf. [4] p. 127), hence they have the same rotation group.

We consider tetrahedron, octahedron and icosahedron in the same position as in [4] p. 52, where for the icosahedron we interchange the axes x and y. The rotation groups have representations as matrices of SO(3) and are called *polyhedral groups* (cf. [4] p. 46). We denote them by T, O and I. These are isomorphic to the permutation groups A_4, S_4 and A_5 respectively ([4] p. 46-50, [7] p. 14-19). We denote by V the Klein four group contained in each group T, O, I. Consider now the classical surjective $2:1$ maps of [12] p. 77-78

$$\rho : \mathrm{SU}(2) \to \mathrm{SO}(3) \quad \text{and} \quad \sigma : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4),$$

they transform the polyhedral groups into the *binary polyhedral groups* and into subgroups of SO(4) respectively. We call the latter *bi-polyhedral groups.* Denote by $\tilde{V} := \rho^{-1}(V)$, $\tilde{A}_4 := \rho^{-1}(T)$, $\tilde{S}_4 := \rho^{-1}(O)$ and $\tilde{A}_5 := \rho^{-1}(I)$ the binary groups and by $\mathcal{H} := \sigma(\tilde{V} \times \tilde{V})$, $G_6 := \sigma(\tilde{A}_4 \times \tilde{A}_4)$, $G_8 := \sigma(\tilde{S}_4 \times \tilde{S}_4)$, $G_{12} := \sigma(\tilde{A}_5 \times \tilde{A}_5)$ the bi-polyhedral groups (these last notations will be clear succesively). Before giving the generators of the groups, we specify some matrices that we will use several times in the sequel. Let $\tau := \frac{1}{2}(1+\sqrt{5})$.

- In SO(3)

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$R_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R_5 := \frac{1}{2}\begin{pmatrix} \tau-1 & -\tau & 1 \\ \tau & 1 & \tau-1 \\ -1 & \tau-1 & \tau \end{pmatrix},$$

$A_i^2 = R_j^j = id, \; i = 1,2,3; \; j = 3,4,5.$

- In SU(2) the quaternions

$$q_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad q_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$q_i^2 = -id, \; i = 1,2,3$

and the matrices

$$p_3 := \tfrac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, \qquad p_4 := \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix},$$

$$p_5 := \tfrac{1}{2}\begin{pmatrix} \tau & \tau-1+i \\ 1-\tau+i & \tau \end{pmatrix}, \quad p_j^j = -id, \ j = 3,4,5.$$

- In SO(4)

$$\sigma_1 := \sigma(q_1, id) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 := \sigma(q_2, id) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\sigma_3 := \sigma(id, q_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_4 := \sigma(id, q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\pi_3 := \sigma(p_3, id) = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\pi_3' := \sigma(id, p_3) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$\pi_4 := \sigma(p_4, id) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\pi_4' := \sigma(id, p_4) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\pi_5 := \sigma(p_5, id) = \tfrac{1}{2}\begin{pmatrix} \tau & 0 & 1-\tau & -1 \\ 0 & \tau & -1 & \tau-1 \\ \tau-1 & 1 & \tau & 0 \\ 1 & 1-\tau & 0 & \tau \end{pmatrix},$$

$$\pi'_5 := \sigma(id, p_5) = \tfrac{1}{2}\begin{pmatrix} \tau & 0 & \tau-1 & 1 \\ 0 & \tau & -1 & \tau-1 \\ 1-\tau & 1 & \tau & 0 \\ -1 & 1-\tau & 0 & \tau \end{pmatrix},$$

$\sigma_i^2 = \pi_j^j = \pi_j'^j = -id,\ i = 1,2,3,4;\ j = 3,4,5.$

- In O(4)

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$C^2 = C'^2 = id.$

With this notation the group V is generated by A_1, A_2; T is generated by A_1, A_2 and R_3; O is generated by A_2, R_3, R_4 and I by A_1, A_2 and R_5. The binary groups are generated by the pre-images in SU(2) of the previous generators, hence $\tilde{V}$ is generated by q_1, q_2; the group $\tilde{A}_4$ is generated by q_1, q_2, p_3; $\tilde{S}_4$ by q_2, p_3, p_4 and $\tilde{A}_5$ by q_1, q_2, p_5 . Taking the images via σ of these matrices we get the generators of the bi-polyhedral groups. We have σ_i, $i = 1,2,3,4$ for $\mathcal{H}$; σ_i, $i = 1,2,3,4$, π_3, π'_3 for G_6; σ_i, $i = 2,4$, π_j, π'_j, $j = 3,4$ for G_8 and σ_i, $i = 1,2,3,4$, π_5, π'_5 for G_{12}. These groups have order $\frac{1}{2}|\tilde{G}|^2$, where by $\tilde{G}$ we denote one of the binary groups in SU(2), hence $|G_6| = 288$, $|G_8| = 1152$, $|G_{12}| = 7200$.

(1.1) *Eigenvalues and eigenvectors of the matrices* $\sigma(p,q)$. The tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be identified with the space of complex 2×2-matrices. The simple tensors $v \otimes w \in \mathbb{C}^2 \otimes \mathbb{C}^2$, $v = (x_1, x_2)$, $w = (y_1, y_2)$ then are identified with the zero determinant matrices via:

$$\begin{array}{lcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^4 \\ ((x_1,x_2),(y_1,y_2)) & \longmapsto & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{pmatrix}. \end{array}$$

We have $\sigma(p,q)\ v \otimes w = pv \cdot (wq^{-1})^t$. Using $q^{-1} = \bar{q}^t$, we get $\sigma(p,q)\ v \otimes w = pv \otimes \bar{q}w$. By this formula follows:

The eigenvalues, resp. the eigenvectors, of $\sigma(p,q)$ are the products, resp. the tensor products, of the eigenvalues, resp. of the eigenvectors, of p and $\bar{q}$.

2. Poincaré series. The groups $G_n \subseteq \mathrm{SO}(4)$ act on $\mathbb{C}^4$ in a natural way, this induces an action on $\mathbb{C}[x_0,x_1,x_2,x_3]$. We want to calculate the dimension of $\mathbb{C}[x_0,x_1,x_2,x_3]_j^{G_n}$, the vector space of G_n-invariant homogeneous polynomials of degree j. We consider the *Poincaré series*

$$\begin{aligned} p(\mathbb{C}[x_0,x_1,x_2,x_3]^{G_n},t) &:= \sum_{j=0}^{\infty} t^j \dim \mathbb{C}[x_0,x_1,x_2,x_3]_j^{G_n} \\ &= \frac{1}{|G_n|} \sum_{g\in G_n} \frac{1}{\det(id - g^{-1}t)} \end{aligned}$$

by *Molien's theorem* [1], p. 21.
Since $\det g = 1$, we can replace the denominator of the previous expression by the characteristic polynomial of g. Moreover, since elements in the same conjugacy class have the same characteristic polynomial

$$(\mathbf{2.1}) \quad p(\mathbb{C}[x_0,x_1,x_2,x_3]^{G_n},t) = \frac{1}{|G_n|} \sum \frac{n_g}{\det(g - id\cdot t)}$$

where the sum runs over all the conjugacy classes of G_n and n_g denotes the number of elements in the conjugacy class of g in $\mathrm{GL}(4,\mathbb{C})$ (cf. also [2] p. 300, for a similar computation). We calculate now the conjugacy classes in G_n. We start with the conjugacy classes of T, O and I, which are those of the permutation groups A_4, S_4 and A_5. We specify a representative and the size of the corresponding conjugacy class below. For convenience we give the conjugacy classes of V as well. Moreover in the case of T we put the elements R_3, and R_3^2 in bracket since they have the same eigenvalues and so are conjugate in $\mathrm{GL}(3,\mathbb{C})$:

group	conj. classes
$V:$	$id,1; A_2,3;$
$T:$	$id,1; A_2,3; \{R_3,4; R_3^2,4\};$
$O:$	$id,1; A_2,3; R_3,8; R_4,6; R_3R_4,6;$
$I:$	$id,1; A_2,15; R_5,12; R_5^2,12; R_3,20.$

Let now G denote one of the groups $V,\ T,\ O,\ I{\subseteq}\mathrm{SO}(3)$, and $\tilde{G}$ one of the binary groups $\tilde{V},\ \tilde{A}_4,\ \tilde{S}_4,\ \tilde{A}_5$. For the conjugacy class of an element $g{\in}G$ we have two conjugacy classes in $\tilde{G}=\rho^{-1}(G)$, namely those of $+\tilde{g}$ and $-\tilde{g}$, the two elements in $\rho^{-1}(g)$. If they are equal then $\tilde{g}$ is a traceless matrix. The only traceless matrices in the binary groups are those in the conjugacy classes of q_2 or p_3p_4. We can write the product $p_3p_4=\frac{1}{\sqrt{2}}(q_1+q_3)$, so using the multiplication rules for the quaternions $q_1,\ q_2,\ q_3$ (cf. [12] p. 77) we find in fact $-q_2{\in}[q_2]$ and $-p_3p_4\in[p_3p_4]$.

We give now the conjugacy classes of the binary groups. In the case of $\tilde{A}_4$ and $\tilde{S}_4$ the elements p_3 and $-p_3^2$, $-p_3$ and p_3^2, resp. q_2 and p_3p_4 are conjugate in SU(2), in fact they have the same eigenvalues, so we put them in bracket. We use the notation $\pm q$ to indicate the representatives $+q,\ -q$. The number given after such a pair of representatives is the size of the corresponding conjugacy classes.

group	conj. classes
$\tilde{V}$:	$\pm id, 1; q_2, 6;$
$\tilde{A}_4$:	$\pm id, 1; q_2, 6; \{p_3, 4; -p_3^2, 4\}; \{-p_3, 4; p_3^2, 4\};$
$\tilde{S}_4$:	$\pm id, 1; \{q_2, 6; p_3p_4, 12\}; \pm p_3, 8; \pm p_4, 6;$
$\tilde{A}_5$:	$\pm id, 1; q_2, 30; \pm p_5, 12; \pm p_5^2, 12; \pm p_3, 20.$

The conjugacy classes of $\mathcal{H}$, resp. G_n, $n=6,8,12$, are the images via σ of the conjugacy classes of $\tilde{V}\times\tilde{V}$, $\tilde{A}_4\times\tilde{A}_4$, $\tilde{S}_4\times\tilde{S}_4$, $\tilde{A}_5\times\tilde{A}_5$. Their order is the product of the order of the conjugacy classes in the binary groups except for the conjugacy classes of q_2 and p_3p_4, where we have to divide by two. We will not to list them now. Our aim is to write the series **(2.1)** for the groups $\mathcal{H}$ and G_n, so we need just the conjugacy classes in $\mathrm{GL}(4,\mathbb{C})$. Here the elements with the same eigenvalues have the same conjugacy class so using **(1.1)** we can find them. We give a representative with the size of the respective conjugacy class below. By σ_{24} we denote the product of σ_2 and

σ_4, the other notations have been introduced in the first paragraph,

group	conj. classes
$\mathcal{H}:$	$\pm id, 1; \sigma_2, 12; \sigma_{24}, 18;$
$G_6:$	$\pm id, 1; \sigma_2, 12; \sigma_{24}, 18; \sigma_2\pi_3', 96; \pm\pi_3, 16; \pm\pi_3\pi_3', 64;$
$G_8:$	$\pm id, 1; \sigma_2, 36; \sigma_{24}, 162; \sigma_2\pi_3', 288; \sigma_2\pi_4', 216; \pm\pi_3, 16; \pm\pi_3\pi_3', 64;$
	$\pm\pi_3\pi_4', 96; \pm\pi_4, 12; \pm\pi_4\pi_4', 36;$
$G_{12}:$	$\pm id, 1; \sigma_2, 60; \sigma_{24}, 450; \sigma_2\pi_5', 720; \sigma_2\pi_5'^2, 720; \sigma_2\pi_5'^2\sigma_4, 720; \pm\pi_5, 24;$
	$\pm\pi_5\pi_5', 144 \pm \pi_5\pi_5'^2\sigma_4, 480; \pm\pi_5^2, 24; \pm\pi_5^2\pi_5', 288; \pm\pi_5^2\pi_5'^2, 144;$
	$\pm\pi_5^2\pi_5'^2\sigma_4, 480; \pm\pi_5^2\sigma_2, 40; \pm\pi_5^2\sigma_2\pi_5'^2\sigma_4, 400.$

Now we can calculate the characteristic polynomial of the previous representatives and we write the sums **(2.1)** for $\mathcal{H}$ and G_n. By expanding these sums in power series with MAPLE we get:

group	Poincaré series
$\mathcal{H}:$	$1 + t^2 + 5t^4 + 6t^6 + 15t^8 + 19t^{10} + 35t^{12} + 44t^{14} + O(t^{16}),$
$G_6:$	$1 + t^2 + t^4 + 2t^6 + 3t^8 + 3t^{10} + 7t^{12} + 8t^{14} + O(t^{16}),$
$G_8:$	$1 + t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + 3t^{12} + 3t^{14} + O(t^{16}),$
$G_{12}:$	$1 + t^2 + t^4 + t^6 + t^8 + t^{10} + 2t^{12} + 2t^{14} + O(t^{16}).$

These series show that:

- there are invariant polynomials just in even degree (this is due to the well known fact (cf. e.g. [5], [6]) that the $\mathcal{H}$-invariant polynomials are just in even degree and $\mathcal{H}\subseteq G_n$, $n = 6, 8, 12$).
- In each degree we have the trivial invariant quadric surface $Q_j(x) := (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{j}{2}}$. In degree $n = 6, 8, 12$ appears the first not trivial G_n-invariant polynomial (this explains the notation). We call it $S_n(x)$ and we calculate an expression of it later.

3. Reflection groups. The group generated by G_{12} and C together (notation of paragraph **1.**) is the reflection group of order 14400 (cf. [1] p.

80-81), which is the symmetry group of the regular 600-cell $\{3,3,5\}$ (cf. [4] p. 153 and [3]). The reflection group of order 1152 which appears on the table of [1] p. 81 is the reflection group of the 24-cell $\{3,4,3\}$ (cf. [4] p. 149 and [3]), which we take with vertices the permutations of $(\pm 1, \pm 1, 0, 0)$ as in [4] p. 156. The generators of G_6 permute these vertices, hence the group is contained in the reflection group of the 24-cell. We now show

(3.1) *the group* $< G_6, C, C' >$ *is the reflection group of* $\{3,4,3\}$, *where* C, C' *are the matrices given in the first paragraph.*

Proof. We have $C^2 = id$ and an easy calculation shows that $C\sigma(p,q)C = \sigma(q,p)$, hence $\mid < C, G_6 > \mid = 2 \cdot 288 = 576$. Clearly C' is a symmetry transformation of the 24-cell. Assume moreover that there are σ, σ' in G_6 with $C'\sigma = C\sigma'$ then $CC' = \sigma'\sigma^{-1}$. It follows that $\sigma'\sigma^{-1}$ has order two, hence it is in $\mathcal{H}$. A long calculation shows that we have no matrices in $\mathcal{H}$ which are equal to CC', so $C'G_6 \neq CG_6$, $\mid < G_6, C, C' > \mid = 1152$ and the assertion follows. □

The situation is quite different in the case of G_8. It cannot be contained in any symmetry group of a four-dimensional regular polyhedron since the latter, with notation of [4] are only α_4, β_4, γ_4, $\{3,3,5\}$, $\{5,3,3\}$, $\{3,4,3\}$ and ten star polytopes (cf. [4] p. 145, p. 276). The first three have too small symmetry groups (cf. [4] p. 133) and the ten star polytopes have the same symmetry group as $\{3,3,5\}$. The group G_8 interchanges, in fact, the $\{3,4,3\}$ and its reciprocal, which we call $\{3,4,3\}'$ (it is a 24-cell again, [4] p. 149). We take it in a similar position as [4] p. 156, with vertices the permutations of $(\pm\sqrt{2}, 0, 0, 0)$ and $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$, i.e. it is the reciprocal with respect to the sphere $x_0^2 + x_1^2 + x_2^2 + x_3^2 = \sqrt{2}$ (cf. [4] p. 126). The rotations of order four π_4, π_4' send in fact the vertices of $\{3,4,3\}$ to the vertices of $\{3,4,3\}'$.

(3.2) *Invariant polynomials of the reflection groups.* It is well known (cf. [3]) that the invariant polynomials under $< G_{12}, C >$ and $< G_6, C, C' >$ are just in even degree. A basic set for the invariants in sense of [3] p. 774 is given by polynomials in degrees $2, 12, 20, 30$, respectively $2, 6, 8, 12$ (cf. [11], [3]). Clearly the invariants under G_{12} are exactly those of $< G_{12}, C >$. In the case of G_6 we have to be more careful, because of the matrix C', which describes an odd permutation of the coordinates. In degree six, however it turns out that the groups G_6 and $< G_6, C, C' >$ have the same invariant polynomials.

4. Equation of the invariant polynomials. We describe briefly the method to calculate $S_n(x)$. We start with a basis of $\mathbb{C}[x_0, x_1, x_2, x_3]_n^{\mathcal{H}}$, $n = 6, 8, 12$, which is well known (cf. e.g. [5], [6]). Then we write a polynomial as a linear combination of the elements of this basis and we impose it to be

invariant under the remaining generators of $G_n\backslash\mathcal{H}$. In degree 6 and 8 the computations are not difficult if left to MAPLE. In degree 12 it is however difficult, because of the relatively high number of parameters involved.
We say that a polynomial is *totally symmetric* if it is invariant under each coordinate permutation. In degree 6 and 8 the invariant polynomials are

$$S_6(x)=\sum x_i^6+15\sum x_i^2x_j^2x_k^2,\quad S_8(x)=\sum x_i^8+14\sum x_i^4x_j^4+168x_0^2x_1^2x_2^2x_3^2,$$

where the sums run over all the indices $i,j,k=0,1,2,3$, and if some indices appear together they are different from each other. These polynomials are totally symmetric. Before giving the expression of $S_{12}(x)$ we introduce some abbreviations. Let $y_i:=x_i^2$ and

$$\begin{array}{lll} S_{51}:=\sum y_i^5y_j; & S_{42}:=\sum y_i^4y_j^2; & S_{33}:=\sum y_i^3y_j^3;\\ S_{411}:=\sum y_i^4y_jy_k; & S_{321}:=\sum y_i^3y_j^2y_k; & S_{222}:=\sum y_i^2y_j^2y_k^2;\\ S_{3111}:=\sum y_i^3y_jy_ky_h; & S_{2211}:=\sum y_i^2y_j^2y_hy_k, & \end{array}$$

where for the sums we use the same conventions as before. The totally symmetric part of $S_{12}(x)$ is

$$f_s:=2S_{51}-6S_{42}-12S_{411}+14S_{33}+9S_{321}+348S_{3111}+30S_{222}-270S_{2211}.$$

In a similar way we say that a polynomial P is *anti-symmetric* if it is invariant under each even coordinate permutation and $\gamma\cdot P=-P$ for each odd permutation γ. The anti-symmetric part of $S_{12}(x)$ is

$$33\sqrt{5}f_a$$

with

$$\begin{aligned} f_a \;:=\;& y_0^3(y_1^2y_2-y_1y_2^2+y_2^2y_3-y_2y_3^2+y_3^2y_1-y_3y_1^2)-y_1^3(y_2^2y_3-y_2y_3^2\\ &+y_3^2y_0-y_3y_0^2+y_0^2y_2-y_0y_2^2)+y_2^3(y_0^2y_1-y_0y_1^2+y_1^2y_3-y_1y_3^2\\ &+y_3^2y_0-y_3y_0^2)-y_3^3(y_0^2y_1-y_0y_1^2+y_1^2y_2-y_1y_2^2+y_2^2y_0-y_2y_0^2). \end{aligned}$$

In conclusion

$$S_{12}(x):=f_s+33\sqrt{5}f_a.$$

Observe that $Q_2(x)$ and $S_n(x)$ are algebraically independent. In fact for the point $p=(i\sqrt{2},1,1,0)$ in $\mathbb{C}^4$ holds $Q_2(p)=0$ but $S_n(p)\neq 0$, $n=6,8,12$. This has in particular an interesting consequence. The polynomial $S_8(x)$ is clearly invariant under $<G_6,C,C'>$ and is not the product $S_6(x)Q_2(x)$, hence it is an equation for the non-trivial invariant polynomial of degree

eight in the basic set of invariants under $< G_6, C, C' >$.
The homogeneous invariant polynomials define pencils of symmetric surfaces in $\mathbb{P}_3$:

$$F_n(\lambda) : S_n(x) + \lambda Q_n(x) = 0, \qquad \lambda \in \mathbb{P}_1$$

The aim of the next paragraphs is to investigate the base locus of the pencils and to find the singular surfaces.

5. Base locus. Let $\mathcal{G}$ be a group acting on $\mathbb{P}_3$.

(5.1) Definition. A point $z \in \mathbb{P}_3$ is called a *fix point* if there is a $\sigma \in \mathcal{G}$ $(\sigma \neq \pm\, id)$, s.t. $\sigma z = z$. We call

$$\mathrm{Fix}(z) = \mathrm{Fix}_{\mathcal{G}}(z) := \{g \in \mathcal{G} \mid gz = z\} \subseteq \mathcal{G}$$

fix group of z and

$$O(z) = O_{\mathcal{G}}(z) := \{gz \mid g \in \mathcal{G}\} \subseteq \mathbb{P}_3$$

orbit of z. We have the formula:

$$|\mathrm{Fix}(z)| \cdot |O(z)| = |\mathcal{G}|$$

(5.2) Definition. We call a line $L \subseteq \mathbb{P}_3$, a *line of fix points* (or *fix line*) if there is a $\sigma \in \mathcal{G}$ $(\sigma \neq \pm\, id)$ s.t. $\sigma x = x$ for all $x \in L$.

(5.3) As in [10] we identify $\mathbb{P}_3 \backslash Q_2$ with $\mathbb{P}\mathrm{GL}(2)$, the projective space of invertible complex 2×2-matrices. More explicitly a point $(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_3$ corresponds to a matrix

$$x = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in \mathbb{P}\mathrm{GL}(2)$$

The quadric $Q_2 = \mathbb{P}_1 \times \mathbb{P}_1$ corresponds to the rank one complex 2×2-matrices. We use this identification to show:
(5.4) *The matrices $\sigma(p, id), \sigma(id, q) \in G_n$ have in $\mathbb{P}_3$ two disjoint lines of fix points each. These are contained in the quadric Q_2 and belong to one ruling, respectively to the other ruling of Q_2.*
Proof. By **(1.1)** it follows that the matrices $\sigma(p, id), \sigma(id, q)$ have two eigenvalues with multiplicity two each, the eigenspaces are lines of $\mathbb{P}_3$. These are spanned by points which correspond to matrices of rank one in the identification **(5.3)**, hence they are contained in Q_2. Again using **(1.1)** we see that these are lines of one ruling resp. of the other ruling of Q_2. $\square$

The base locus of the pencil $F_n(\lambda)$ is the variety

$$\{x \in \mathbb{P}_3 | S_n(x) = Q_n(x) = 0\}$$

Clearly it is not reduced and is invariant under the group action. Put $\mathcal{B}_n := Q_2 \cap S_n$. We consider now the groups $\sigma(\tilde{G}, id)$ and $\sigma(id, \tilde{G})$, ($\tilde{G}= \tilde{A}_4$, $\tilde{S}_4$, $\tilde{A}_5$, as usual) which modulo $\{\pm id\}$ are isomorphic to the subgroups T, O and I $\subseteq$ SO(3). It is a well known fact that under the action of these groups there are orbits of the following lengths,

tetrahedron	octahedron	icosahedron
12, 6, 4	24, 12, 8, 6	60, 30, 20, 12

Moreover, observe that

1. by **(1.1)** it follows that the group $\sigma(\tilde{G}, id)$ acts on the lines of the first ruling $\{v\} \times \mathbb{P}_1$ and lets invariant each line of the second ruling $\mathbb{P}_1 \times \{w\}$. Vice versa $\sigma(id, \tilde{G})$ acts on $\mathbb{P}_1 \times \{w\}$ and lets invariant each line of $\{v\} \times \mathbb{P}_1$.

2. Denote by $\mathcal{L}_n$, $\mathcal{L}'_n$ the sets of lines in $\{v\} \times \mathbb{P}_1$, resp. $\mathbb{P}_1 \times \{w\}$ of the orbit of length n. The matrix C maps lines of $\mathcal{L}_n$ to lines of $\mathcal{L}'_n$ (recall that $C\sigma(p, q)C= \sigma(p, q)$).

Using these facts we show
(5.5) (a) *the variety $\mathcal{B}_n$ is reduced, i.e. does not contain multiple components.*
(b) *The base locus splits in $2n$ lines, n of each ruling of Q_2. In particular these are fix lines for elements in G_n.*

Proof of (a). By Bezout's theorem $\deg(Q_2 \cap S_n) = 2n$. If $Q_2 \cap S_n$ is not reduced then there is a component $V \subseteq Q_2 \cap S_n$ s.t. Q_2 and S_n meet with multiplicity at least two. This is the case when V is singular on S_n, or S_n and Q_2 are tangent at V. Consider a line L of one of the two rulings, not in $\mathcal{B}_n$, and which meets V in at least one point. W.l.o.g. assume L in the ruling $\{v\}\times\mathbb{P}_1$. Let $x \in L \cap V$. We have $\mathrm{mult}_x(L \cdot S_n) \geq 2$. The group $\sigma(id, \tilde{G})$ acts on L, so we consider the orbit of x under this group. By the table above, we see that L and S_n meet at more than n points computed with multiplicity, so $L \subseteq S_n$. This contradicts the assumption. We have shown that $Q_2 \cap S_n$ is reduced. $\square$

Proof of (b). Take a line L of the first or of the second ruling, $L \not\subseteq \mathcal{B}_n$. The curve $\mathcal{B}_n$ has bi-degree (n,n) on Q_2, so $|L \cap \mathcal{B}_n| = n$. W.l.o.g. assume that L is in the ruling $\{v\} \times \mathbb{P}_1$. The group $\sigma(id, \tilde{G})$ acts on the points of L. Let $x \in L \cap \mathcal{B}_n$, then by the table on the previous page, the orbit of x under $\sigma(id, \tilde{G})$ must have length n. Hence x belongs to a line of $\mathcal{L}'_n$. As we have infinitely many lines like L, the lines $\mathcal{L}'_n$ are contained in $\mathcal{B}_n$. By 2. the lines in $\mathcal{L}_n$ are contained in $\mathcal{B}_n$ too.

6. Singular points. We give now some general results on the singular points of the pencils.

(6.1) *Let p be a singular point on a surface of the pencil $F_n(\lambda)$ (not on Q_n), then p is not contained in the complex quadric.*

Proof. Let p be a singular point on the surface $F_n(\lambda_0){:=}S_n(x) + \lambda_0 Q_n(x) = 0$ and assume that $p{\in}Q_2$. Then $p \in S_n$, moreover since $\partial_i Q_n(p) = 0$, for $i = 0,1,2,3$, we have $\partial_i S_n(p) = 0$, $i = 0,1,2,3$, too, so p is a singular point on $Q_2 \cap S_n$. This consists of $2n$ lines which meet each other at n^2 points. Hence p must be an intersection point of two lines. If S_n is singular at p it follows that S_n is singular at all the n^2 points of intersection of the lines in the base locus, in fact they form one orbit under the action of $\sigma(\tilde{G}, id)$ and $\sigma(id, \tilde{G})$ (notation of paragraph **5.**). In particular S_n has n singular points on a line L in $Q_2 \cap S_n$. A hypersurface $\{\partial_i S_n = 0\}$ $(i = 0,1,2,3)$ has degree $n-1$, therefore it intersects L in $n-1$ points. So S_n has at most $n-1$ singular points on L. It follows that L is singular on S_n. Hence S_n and Q_2 meet at L and so at all the $2n$ lines of $Q_2 \cap S_n$ with multiplicity at least two. This is not possible, in fact $\deg(Q_2 \cap S_n){=}\ 2n$. This shows that $p{\notin}Q_2$. □

This lemma has many consequences:

(6.2) (a) *the general surface in the pencil is smooth,*

(b) *the singular surfaces, not Q_n, have only isolated singularities,*

(c) *the surfaces different from Q_n are irreducible and reduced.*

Proof. (a) It is a consequence of Bertini's theorem.

(b) Assume that $S{:=}\ \{Q_n(x) + \lambda_0 S_n(x) = 0\}$ contains a singular curve. This meets Q_2 in at least one point p, which is singular on S. By **(6.1)**, this is not possible.

(c) Is like (b). □

(6.3) *A singular point on a surface in the pencil $F_n(\lambda)$, $n = 6,8,12$, is a fix point under G_n in sense of definition* **(5.1)**. *Moreover, as vector of $\mathbb{C}^4$, it is eigenvector of a matrix with eigenvalue $+1$ or -1.*

Proof. It is possible to obtain a first rough estimate of the maximal number of singular points on a surface S of degree n in $\mathbb{P}_3$ in the following way: if S has equation $\{F=0\}$ then a point on S is singular if and only if $\partial_i F(p)=0$ for all $i=0,1,2,3$. These surfaces meet in at most $n(n-1)^2$ points with multiplicity. Since these are singular on S, they are counted at least two times in the intersection, so the effective bound is $\frac{n}{2}(n-1)^2$. We give these numbers for $n=6,8,12$ in the table below. In the second row we give the length of the orbit of a point under G_n, which is not a fix point,

n	6	8	12
$\frac{n}{2}(n-1)^2$	75	196	726
orbit	144	576	3600

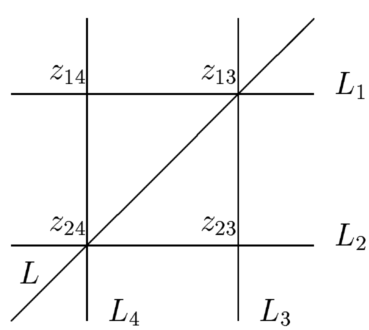
Clearly such a point cannot be singular. Let now x denote a singular point and consider it as vector in $\mathbb{C}^4$. Let $\sigma := \sigma(p,q) \in G_n$ s.t. $\sigma x = \lambda x$ equivalently

$$pxq^{-1} = \lambda x.$$

Consider x as matrix of $\mathbb{P}\mathrm{GL}(2)$ as in the identification **(5.3)** and take the determinant on both sides of the previous equation. We get $\det(x) = \lambda^2 \det(x)$. In fact $\det(p) = \det(q) = 1$ since they are matrices in SU(2). The equality holds only when $\det(x)=0$ or $\lambda^2=1$. If $\det(x)=0$ then $x \in Q_n$ and this is not possible by **(6.1)**. $\square$

By this fact follows
(6.4) *the singular points are contained in fix lines of G_n, $n=6,8,12$.*

7. Lines of fix points. Let L_1, L_2, L_3, L_4 be the fix lines of the elements $\sigma(p,id)$, resp. $\sigma(id,q) \in G_n$. Let $z_{ij} := L_i \cap L_j$, $i=1,2$, $j=3,4$, denote the intersection points. If the matrix $\sigma(p,q)$ has the eigenvalue 1 or -1 then it has at least one line, L, of fix points, which form the following configuration with the fix lines of $\sigma(p,id)$ and of $\sigma(id,q)$



With this notation we give the following result that we will use later,

(7.1) *assume that L meets the base locus of the pencil $F_n(\lambda)$. W.l.o.g. let these intersection points be $z_{13},\ z_{24}$ as in the picture. Then for each surface $S \neq Q_n$ in the pencil we have $mult_{z_{ij}}(L \cdot S) = 1,\ (i,j) = (1,3),\ (2,4)$.*
Proof. By **(6.1)** the points z_{13}, z_{24} are smooth points on each surface (not Q_n) in the pencil $F_n(\lambda)$. The lines of the two rulings of Q_2 which meet at these points, are lines of the base locus, hence are contained in S. The tangent space of S at the z_i is the plane spanned by these two lines. Clearly this plane does not contain L, hence L cannot be tangent to S at z_{ij}. □

(7.2) We give below a representative of each of the conjugacy classes in G_n (under the action of G_n itself), which have eigenspaces of dimension two in $\mathbb{C}^4$ (and so fix lines in $\mathbb{P}_3$) with eigenvalue 1 or -1. As usual we write $\pm\sigma$ to indicate the representatives $+\sigma$ and $-\sigma$. Clearly these elements have the same fix lines. Observe that the elements in the conjugacy classes of $\pi_3\pi'_3$, $\pi_3^2\pi_3'^2$; $\pi_5\pi'_5$, $\pi_5^2\pi_5'^2$; $\pi_3\pi_3'^2$, $-\pi_3^2\pi'_3$ have two by two the same fix lines, so we consider them together. The elements in the conjugacy classes of σ_{24} and $\pi_4\pi'_4$ have the same fix lines too. This follows from the fact that we can write each element in $[\sigma_{24}]$ as the square of an element in $[\pi_4\pi'_4]$. Moreover since $C\sigma(p,q)C{=}\ \sigma(q,p)$, the elements in the conjugacy classes of $\sigma_2\pi'_3\pi'_4$ and of $\pi_3\pi_4\sigma_4$ have the same fix lines. Next to each representative we write the number of distinct fix lines in the conjugacy class.

group	fix lines
G_6 :	$\sigma_{24}, 18; \pm\pi_3\pi'_3, \pm\pi_3^2\pi_3'^2, 16; \pm\pi_3^2\pi'_3, \pm\pi_3\pi_3'^2, 16;$
G_8 :	$\sigma_{24}, \pm\pi_4\pi'_4, 18; \pm\pi_3\pi'_3, 32; \pi_3\pi_4\pi'_3\pi'_4, 72; \pi_3\pi_4\sigma_4, \sigma_2\pi'_3\pi'_4, 36;$
G_{12} :	$\sigma_{24}, 450; \pm\pi_3\pi'_3, 200; \pm\pi_5\pi'_5, \pm\pi_5^2\pi_5'^2, 72.$

(7.3) *If $[\sigma]$ denotes a conjugacy class above, then the fix lines of the elements in $[\sigma]$ form one orbit under G_n.*
Proof. The statement is clear when σ has just one line of fix points. We prove that the two fix lines of the elements in $[\sigma_{24}]$, $[\pi_3\pi_4\pi'_3\pi'_4]$ or $[\pi_3\pi_4\sigma_4]$ are equivalent under G_n. The latter are eigenspaces of $\mathbb{C}^4$ with eigenvalues 1 and -1. Remember that if π is an element in one of the previous classes then $-\pi$ is in the same conjugacy class too. It has the same eigenspaces but with eigenvalues interchanged. So we can find a matrix in G_n which maps one line to the other and vice versa. □

This shows in particular that every G_n-invariant property, which holds for a special fix line of an element in a conjugacy class above, holds for each other fix line of the elements in the same conjugacy class.
(7.4) We give now the generators of the fix line(s) of the representatives above, which we will need later, $\tau = \frac{1}{2}(1+\sqrt{5})$:

$$\begin{array}{l}
\sigma_{24} :< (0:0:1:0), (1:0:0:0) >, < (0:0:0:1), (0:1:0:0) >; \\
\pi_3\pi_3' :< (1:0:0:0), (0:1:-1:1) >; \pi_3\pi_3'^2 :< (0:1:1:0), (0:-1:0:1) >; \\
\pi_3\pi_4\pi_3'\pi_4 :< (1:0:0:0), (0:1:0:1) >, < (0:1:0:-1), (0:0:1:0) >; \\
\pi_3\pi_4\sigma_4 :< (1:\sqrt{2}:1:0), (0:1:\sqrt{2}:1) >, < (\sqrt{2}:-1:0:1), (1:-\sqrt{2}:1:0) >; \\
\pi_5\pi_5' :< (1:0:0:0), (0:0:\tau-1:1) > .
\end{array}$$

We can formulate a result about the intersection points of the fix lines: **(7.5)** *the intersection points of the previous lines are real. In particular they are not on the quadric.*
Proof. Let $L \neq L'$ be some fix lines and let $x \in L \cap L'$. Since the matrices of G_n which fix the lines are real, it follows that $\bar{x} \in L \cap L'$ too. Hence $x = \bar{x}$. $\square$

8. Coverings of $\mathbb{P}_1$. A pencil in $\mathbb{P}_3$ defines a morphism (away from the base locus)

$$\begin{array}{ccc}
\mathbb{P}_3 & \longrightarrow & \mathbb{P}_1 \\
x & \mapsto & (S_n(x) : Q_n(x)).
\end{array}$$

Let L be a fix line which does not meet the base locus. This morphism restricts on L to a $n:1$ cover of $\mathbb{P}_1$, we call it f. If L meets the base locus of $F_n(\lambda)$, then f is not defined at the points of intersection z_{13}, z_{24} (notation of paragraph **7.**). By **(7.1)** the line L meets each surface in $F_n(\lambda)$ with multiplicity one at these points. So f extends to a cover

$$\bar{f} : L \longrightarrow \mathbb{P}_1$$

of degree $n-2$, having branch points of order $\frac{n}{2}-1$ at z_{13}, z_{24}. Using Hurwitz's formula for the degree of the ramification locus of a curve morphism, we get **(8.1)** *the degree of the ramification locus of the morphism f is $2n-2$ and of $\bar{f}$ it is $2n-6$ $(n = 6, 8, 12)$.*
Of course the singular points of the surfaces in the pencil are ramification points of the previous morphisms. So the degree of the ramification locus gives an estimate for the number of singular points on the fix lines. In fact if we do not consider the points on the multiple quadric, we find that these points are at most n if L does not intersect the base locus, these are $n-2$ otherwise.

9. Singular surfaces. The aim of this paragraph is to find the singular surfaces in the pencil and their number of singular points.

(9.1) *The pencils* $F_6(\lambda)$ *and* $F_{12}(\lambda)$. We have seen that the surfaces are invariant under the reflection groups of the 24-cell and of the 600-cell. We consider these in position as in [4] p. 157, where by the 600-cell and its reciprocal we interchange the x_0 and x_1 coordinate. The vertices of $\{3,4,3\}$ are the permutations of $(\pm 1, \pm 1, 0, 0)$, those of $\{3,3,5\}$ are the permutations of $(\pm 2, 0, 0, 0)$, $(\pm 1, \pm 1, \pm 1, \pm 1)$ and the even permutations of $(\pm 1, \pm\tau, \pm\tau^{-1}, 0)$. Of course the N_0-vertices, N_1-edges, N_2-faces and N_3-cells form one orbit each under the reflection group action. We consider besides the vertices and the middle points of the edges, the vertices and the middle points of the edges of the reciprocal of $\{3,4,3\}$, which we denote by $\{3,4,3\}'$ and of the reciprocal of $\{5,3,3\}$ (their number is equal respectively to N_3 and N_2). We recall the coordinates of the vertices, which are given in [4] p. 156-157. For $\{3,4,3\}'$ these are the permutations of $(\pm 2, 0, 0, 0)$ and $(\pm 1, \pm 1, \pm 1, \pm 1)$, for $\{5,3,3\}$ we have the permutations of $(\pm 2, \pm 2, 0, 0)$, $(\pm\sqrt{5}, \pm 1, \pm 1, \pm 1)$, $(\pm\tau, \pm\tau, \pm\tau, \pm\tau^{-2})$, $(\pm\tau^2, \pm\tau^{-1}, \pm\tau^{-1}, \pm\tau^{-1})$ and the even permutations of $(\pm\tau^{-2}, \pm\tau^2, \pm 1, 0)$, $(\pm\tau^{-1}, \sqrt{5}, \tau, 0)$, $(\pm 1, \pm 2, \pm\tau, \pm\tau^{-1})$. In these way we get four distinct orbits of points. If we consider these in $\mathbb{P}_3$, the orbits have half length, and we can multiply by some non zero scalar factor without changing the coordinates of the points. Substituting one point of each orbit in the equations of the pencil we get the values of λ, s.t. the corresponding surface contains the whole orbit. Since the partial derivatives of the equation vanish, these points are in fact singular (one needs to consider just one point in each orbit and then the calculations with MAPLE are easy; in degree six, however, one can do them by hand). We give below the length of the orbits in $\mathbb{P}_3$ and the values of λ of the surfaces $F_6(\lambda)$, resp. $F_{12}(\lambda)$, which contain the orbits.

24-cell					600-cell				
orbit:	12	48	48	12	orbit:	300	600	360	60
λ:	-1	$-\frac{2}{3}$	$-\frac{7}{12}$	$-\frac{1}{4}$	λ:	$-\frac{3}{32}$	$-\frac{22}{243}$	$-\frac{2}{25}$	0

We show now that we have no more singular points and singular surfaces in the pencils. To do this we consider the lines of fix points for elements of G_6, resp. G_{12}. In fact, for our aim (cf. 1., 2., and 3. below), it is enough to consider just one line in each G_n-orbit and its intersection points with the above given singular surfaces. By **(7.3)** the other lines meet the same surfaces in the same number of singular points and these form one G_n-orbit. We take the lines of **(7.4)** and the fix line $< (0:0:1:0), (1:0:0:0) >$ of

σ_{24}:

fix line of	surface		int. points	
σ_{24}	$F_6(-1)$	$F_6(-\frac{1}{4})$	$(0:0:1:0)$, $(1:0:0:0)$	$(1:0:\pm 1:0)$
$\pi_3\pi_3'$	$F_6(-1)$	$F_6(-\frac{2}{3})$	$(\pm 1:1:-1:1)$, $(1:0:0:0)$	$(\pm 3:1:-1:1)$, $(0:1:-1:1)$
$\pi_3\pi_3'^2$	$F_6(-\frac{7}{12})$	$F_6(-\frac{1}{4})$	$(0:2:1:-1)$, $(0:-1:1:2)$, $(0:1:2:1)$	$(0:1:1:0)$, $(0:-1:0:1)$, $(0:0:1:1)$

In the case of the pencil $F_{12}(\lambda)$, we take the representative $\pi_5^2\sigma_2\pi_5'^2\sigma_4$ instead of $\pi_3\pi_3'$ (it is more convenient for the computations that we are going to do). Its fix line is $<(1:0:0:0),(0:\tau^2:1:0)>$:

fix line of	surface		int. points	
σ_{24}	$F_{12}(-\frac{3}{32})$	$F_{12}(-\frac{22}{243})$	$(1:0:\pm 1:0)$	$(1:0:\pm\tau^2:0)$, $(\pm\tau^2:0:1:0)$
	$F_{12}(-\frac{2}{25})$	$F_{12}(0)$	$(1:0:\pm\tau:0)$, $(\pm\tau:0:1:0)$	$(1:0:0:0)$, $(0:0:1:0)$
$\pi_5^2\sigma_2\pi_5'^2\sigma_4$	$F_{12}(-\frac{3}{32})$	$F_{12}(-\frac{22}{243})$	$(\pm\tau^2:1:\tau^{-2}:0)$, $(\pm\tau^{-2}:\tau^2:1:0)$, $(\pm\sqrt{5}:\tau:\tau^{-1}:0)$	$(0:\tau^2:1:0)$, $(\pm 3:\tau:\tau^{-1}:0)$
	$F_{12}(0)$		$(1:0:0:0)$, $(\pm 1:\tau:\tau^{-1}:0)$	
$\pi_5\pi_5'$	$F_{12}(-\frac{2}{25})$	$F_{12}(0)$	$(0:0:1:\tau)$, $(\pm\tau^2:0:\tau^{-1}:1)$, $(\pm\sqrt{5}:0:\tau:\tau^2)$	$(1:0:0:0)$, $(\pm\tau^{-1}:0:1:\tau)$, $(\pm\tau:0:\tau^{-1}:1)$

By the observation above and the tables follows that:

1. the points of intersection of the fix lines with the surfaces $F_6(-\frac{1}{4})$ and $F_6(-1)$, resp. $F_{12}(0)$ and $F_{12}(-\frac{3}{32})$ are in fact vertices of the 24-cell $\{3,4,3\}$, resp. of the reciprocal $\{3,4,3\}'$, and of the $\{3,3,5\}$ and its reciprocal $\{5,3,3\}$.

2. The points of intersection with the surfaces $F_6(-\frac{7}{12})$ and $F_6(-\frac{2}{3})$ are mid points of the edges of the $\{3,4,3\}$, resp. of the $\{3,4,3\}'$.

3. In the case of the two remaining surfaces $F_{12}(-\frac{2}{25})$ and $F_{12}(-\frac{22}{243})$ we have to do some more consideration. The middle point of the edge connecting the two neighboring vertices $(1, \tau, \pm\tau^{-1}, 0)$ is $(1, \tau, 0, 0)$. So all the permutations (with + or - sign) of these points are middle points of the edges of the $\{3, 3, 5\}$. Similarly the middle points of the edge connecting the two points $(\pm\tau^{-2}, \tau^2, 1, 0)$ is $(0, \tau^2, 1, 0)$, hence the permutations of these points (with + and - sign) are middle points of the edges of the $\{5, 3, 3\}$. So we find the intersection points of the fix line of σ_{24} and the surfaces $F_{12}(-\frac{2}{25})$ and $F_{12}(-\frac{22}{243})$. The fix lines of the remaining two matrices $\pi_5^2\sigma_2\pi_5'^2\sigma_4$ and $\pi_5\pi_5'$ contain the points $(0 : \tau^2 : 1 : 0) \in F_{12}(-\frac{22}{243})$ resp. $(0 : 0 : 1 : \tau) \in F_{12}(-\frac{2}{25})$. The other points of intersection form one orbit under $\pi_5^2\sigma_2$, resp. π_5 which let the line invariant. Hence all the points of intersection are middle points of the edges of one polyhedron, resp. of the other.

Now we are almost done. The fix line of σ_{24}, resp. $\pi_5\pi_5'$ contains the points $(\pm i : 0 : 1 : 0)$, resp. $(\pm i\sqrt{\tau^2 - 2\tau + 2} : 0 : \tau - 1 : 1)$ of the base locus of $F_6(\lambda)$, resp. of $F_{12}(\lambda)$. Using the estimations **(8.1)** for the number of singular points on the fix lines, we see that in fact we have found the maximal number possible. We conclude that

- we have no more singular surfaces and singular points in the pencils,
- the points which we have found are all double points,
- the singular points form one orbit already under the action of the groups G_6 and G_{12}.
  Proof. In the case of the surfaces $F_{12}(-\frac{2}{25})$ and $F_{12}(-\frac{22}{243})$, the assertion follows by 3. above and **(7.3)**. In the case of the surfaces $F_{12}(-\frac{3}{32})$ and $F_{12}(0)$ we proceed in a similar way. We consider the two points of intersection of each surface with the fix line of σ_{24} and we observe that the matrix σ_2 maps one point to the other. Hence we conclude again by **(7.3)**. $\square$

(9.2) *The pencil $F_8(\lambda)$.* As we remarked in paragraph **3.**, the group G_8 maps the vertices of a $\{3, 4, 3\}$ in those of a $\{3, 4, 3\}'$, (the vertices are taken now with coordinates as in **3.**). The same holds for the middle points of the edges. These two orbits belong respectively to the surfaces $F_8(-1)$, $F_8(-\frac{5}{9})$ and as in **(9.1)** the points are singular. Obviously these are the points of the surfaces $F_6(-1)$ and $F_6(-\frac{1}{4})$ together, resp. of $F_6(-\frac{2}{3})$ and $F_6(-\frac{7}{12})$ together. We look now for some more orbits of points. For instance consider the middle points of the segments connecting the vertices of $\{3,4,3\}$ with its reciprocal $\{3,4,3\}'$. These are the permutations of:

(1) $(\pm\frac{\sqrt{2}+1}{2}, \pm\frac{1}{2}, 0, 0), (\pm\frac{\sqrt{2}-1}{2}, \pm\frac{1}{2}, 0, 0)$;
(2) $\frac{1}{2}(\pm(1+\frac{\sqrt{2}}{2}), \pm(1+\frac{\sqrt{2}}{2}), \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}), \frac{1}{2}(\pm(1-\frac{\sqrt{2}}{2}), \pm(1-\frac{\sqrt{2}}{2}), \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$;
(3) $\frac{1}{2}(\pm 1, \pm 1, \pm\sqrt{2}, 0)$;
(4) $\frac{1}{2}(\pm(1+\frac{\sqrt{2}}{2}), \pm(1-\frac{\sqrt{2}}{2}), \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$.

A computation as in **(9.1)** shows that the points (1), (2), which now we consider in $\mathbb{P}_3$, are singular on the surface $F_8(-\frac{3}{4})$ and the points (3), (4), in $\mathbb{P}_3$, are singular on $F_8(-\frac{9}{16})$. We take, in fact just one point in each group. Then because of symmetry reasons the points in the same group are singular on the same surface. We want to understand in how many G_8-orbits do these points split. We know already that we have at least two orbits since the points belong to two different surfaces. To proceed we use the lines of fix points. As in **(9.1)** we give the intersections of the fix lines of **(7.4)** with the found singular surfaces. For σ_{24} we take the same fix line as in **(9.1)**, for $\pi_3\pi_4\sigma_4$ we take $<(1:\sqrt{2}:1:0),(0:1:\sqrt{2}:1)>$ and for $\pi_3\pi_4\pi_3'\pi_4'$ we take $<(1:0:0:0),(0:1:0:1)>$. Put $a := 1+\sqrt{2}$.

fix line of	surface		int. points	
σ_{24}	$F_8(-1)$	$F_8(-\frac{3}{4})$	$(0:0:1:0)$, $(1:0:0:0)$, $(1:0:\pm 1:0)$	$(1:0:\pm a:0)$, $(\pm a:0:1:0)$
$\pi_3\pi_3'$	$F_8(-1)$	$F_8(-\frac{5}{9})$	$(\pm 1:1:-1:1)$, $(1:0:0:0)$	$(\pm 3:1:-1:1)$, $(0:1:-1:1)$
$\pi_3\pi_4\sigma_4$	$F_8(-\frac{3}{4})$	$F_8(-\frac{9}{16})$	$(a:1:-1:-a)$, $(1:a:a:1)$, $(-1:1:a:a)$, $(a:a:1:-1)$	$(1:\sqrt{2}:1:0)$, $(0:1:\sqrt{2}:1)$, $(1:0:-1:-\sqrt{2})$, $(\sqrt{2}:1:0:-1)$
$\pi_3'\pi_4'$	$F_8(-1)$	$F_8(-\frac{5}{9})$	$(1:0:0:0)$, $(0:1:0:1)$	$(\pm 1:1:0:1)$, $(\pm 2:1:0:1)$
	$F_8(-\frac{9}{16})$		$(\pm\sqrt{2}:1:0:1)$	

We show that the points (1), (2) on $F_8(-\frac{3}{4})$, resp. the points (3), (4) on $F_8(-\frac{9}{16})$ form one G_8-orbit.
Proof. By the tables above the fix line of σ_{24} and of $\pi_3\pi_4\sigma_4$ meets the surface $F_8(-\frac{3}{4})$ at points which are in (1), resp. (2) (we have just to multiply the points by some scalar factor). Hence we cannot have other singular points

besides the points (1), (2) together. We have to show:
(a) the points on the fix lines form one orbit under the fix group of the line,
(b) the fix lines of the elements in $[\sigma_{24}]$ meet some fix line of the elements in $[\pi_3\pi_4\sigma_4]$ at the singular points of $F_8(-\frac{3}{4})$.
We remarked in paragraph **7.**, **(7.2)**, that a matrix of order two in $[\sigma_2]$ is the square of a matrix of order four in $[\pi_4]$. This holds clearly for σ_4 too. Put $\sigma_4 = \gamma^2$, with $\gamma \in [\pi_4]$, then γ and σ_4 commute, hence the four points on the fix lines of σ_{24}, resp. $\pi_3\pi_4\sigma_4$ form one orbit under γ. This shows (a). Consider now the fix line $<(1:0:0:1),(0:1:1:0)>$ of the element $\sigma_2\sigma_3 \in [\sigma_{24}]$, it contains the point $(1:a:a:1)$ of the fix line of $\pi_3\pi_4\sigma_4$. This shows (b) and completes the case of $F_8(-\frac{3}{4})$. We do the same for the points (3), (4) on $F_8(-\frac{9}{16})$. We consider the fix line of $\pi_3\pi_4\sigma_4$ and of $\pi_3\pi_4\pi_3'\pi_4'$. These meet $F_8(-\frac{9}{16})$ in four resp. two points, an argumentation as before shows that they form one orbit under the fix group of the line, so we get (a). To show (b), we consider the fix line $<(0:0:0:1),(-1:0:1:0)>$ of $\sigma_2\pi_3\pi_4\pi_3'\pi_4' \in [\pi_3\pi_4\pi_3'\pi_4']$ it contains the point $(1:0:-1:-\sqrt{2})$ of the fix line of $\pi_3\pi_4\sigma_4$ and we have done. □

The fix line of $\pi_3\pi_3'$ meets the base locus of $F_8(\lambda)$ at the points $(\pm i\sqrt{3}:1:-1:1)$, hence by using the estimations for the number of singular points on the fix lines and the table on the previous page, we see that:

- we have no more singular surfaces and singular points in the pencil $F_8(\lambda)$,
- the singular points are all double points,
- the singular surfaces have just one orbit of singular points each. We resume below their length in $\mathbb{P}_3$ and the values of λ s.t. $F_8(\lambda)$ is singular.

orbit :	24	72	144	96
λ :	-1	$-\frac{3}{4}$	$-\frac{9}{16}$	$-\frac{5}{9}$

10. The singular points are nodes. We take now an arbitrary singular point in each orbit and we compute the Hessian matrix at this point. In each case the matrix has rank three, hence the point is a node and so are all the points in its orbit.

11. Configurations of lines and points. We recall the following

(11.1) Definition. A *space configuration* of lines and points is a system of l lines and p points s.t. each line contains π of the given points and each point belongs to λ lines. We say that we have a (p_λ, l_π) configuration. Moreover we have the formula

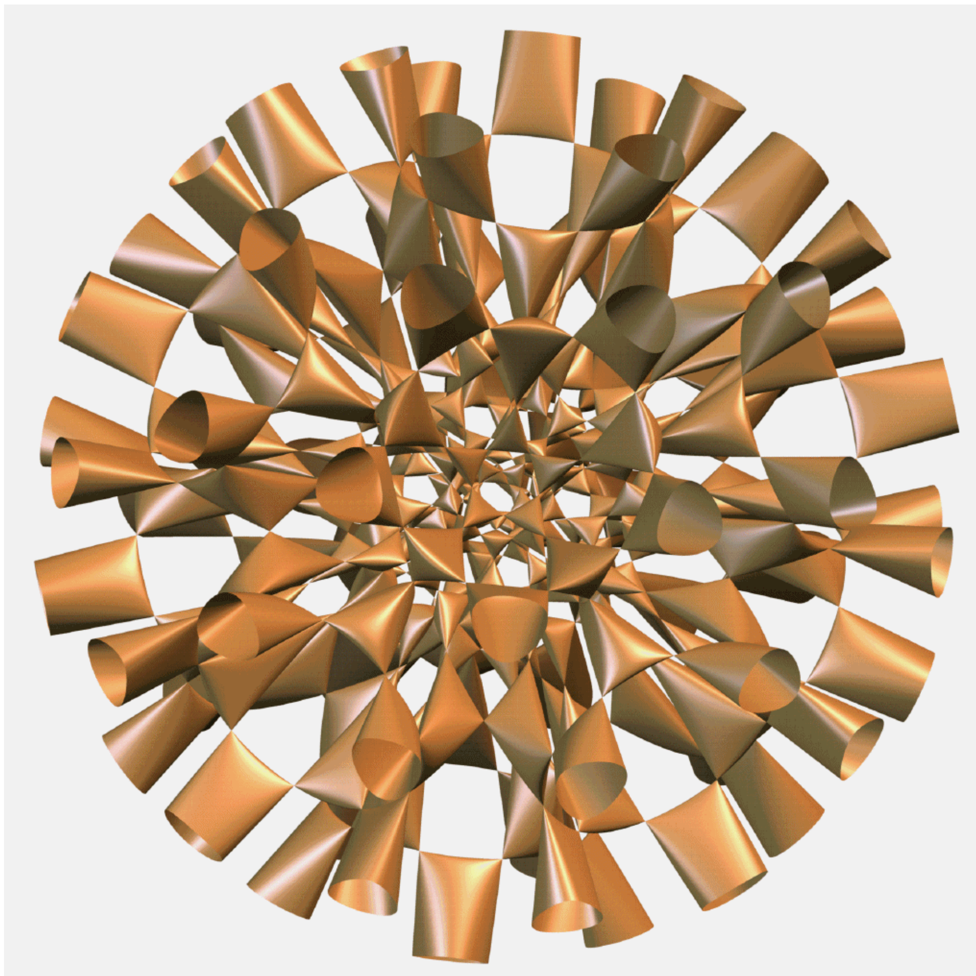
$$p \cdot \lambda = l \cdot \pi.$$

The fix lines of elements in the same conjugacy class and the singular points on a singular surface form a space configuration of lines and points. By the formula above and the results of paragraph **10.**, we can compute how many fix lines of the same orbit pass through each singular point. We list the configurations below. In the first column we give a representative of the conjugacy class which we consider.

repr. of the conj. class	sing. points of		configuration	
σ_{24}	$F_6(-1)$	$F_6(-\frac{1}{4})$	$(18_2, 12_3)$	$(18_2, 12_3)$
	$F_8(-1)$	$F_8(-\frac{3}{4})$	$(18_4, 24_3)$	$(18_4, 72_1)$
	$F_{12}(-\frac{3}{32})$	$F_{12}(-\frac{22}{243})$	$(450_2, 300_3)$	$(450_4, 600_3)$
	$F_{12}(-\frac{2}{25})$	$F_{12}(0)$	$(450_4, 360_5)$	$(450_2, 60_{15})$
$\pi_3\pi_3'$	$F_6(-1)$	$F_6(-\frac{2}{3})$	$(16_3, 12_4)$	$(16_3, 48_1)$
	$F_8(-1)$	$F_8(-\frac{5}{9})$	$(32_3, 24_4)$	$(32_3, 96_1)$
	$F_{12}(-\frac{3}{32})$	$F_{12}(-\frac{22}{243})$	$(200_6, 300_4)$	$(200_3, 600_1)$
	$F_{12}(0)$		$(200_3, 60_{10})$	
$\pi_3\pi_3'^2$	$F_6(-\frac{7}{12})$	$F_6(-\frac{1}{4})$	$(16_3, 48_1)$	$(16_3, 12_4)$
$\pi_3\pi_4\sigma_4$	$F_8(-\frac{3}{4})$	$F_8(-\frac{9}{16})$	$(36_4, 72_2)$	$(36_4, 144_1)$
$\pi_3\pi_4\pi_3'\pi_4'$	$F_8(-1)$	$F_8(-\frac{9}{16})$	$(72_2, 24_6)$	$(72_2, 144_1)$
	$F_8(-\frac{5}{9})$		$(72_4, 96_3)$	
$\pi_5\pi_5'$	$F_{12}(-\frac{2}{25})$	$F_{12}(0)$	$(72_5, 360_1)$	$(72_5, 60_6)$

It is interesting to observe that the singular points are intersection points of the fix lines, hence **(7.5)** confirms that they are all real points.

12. Computer picture. We exhibit a computer picture of the surface of degree 12 with 600 nodes. This has been realized by the program SURF written by S. Endraß.



$A_5 \times A_5$–symmetric surface of degree 12

with 600 nodes

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