Stability of Halphen pencils of index two

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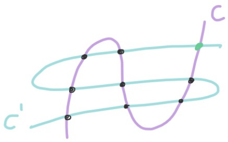
Plan for the talk:

- Motivation & overview of the problem
- The mathematical setup
- Main results
- Worked-out example

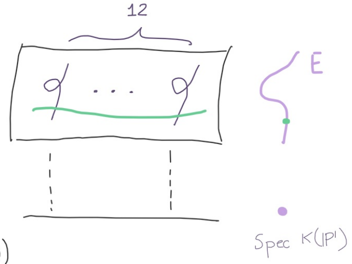
Motivation & overview of the problem



$\lambda C + \mu C' = 0$

$$\begin{tikzcd}
 & Y \arrow[dl, dashed, "Bl_9 \mathbb{P}^2"'] \arrow[d] \\
\mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1
\end{tikzcd}$$

$p \longmapsto (C(p) : C'(p))$



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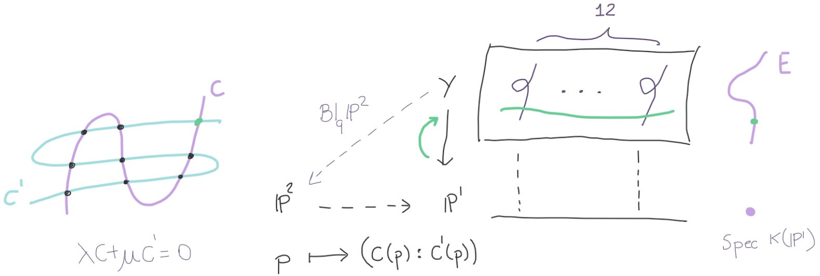

Pencils of plane cubics	$\leftrightarrow$	RES with section
???	$\leftrightarrow$	RES with a multiple fiber

Miranda's work (1980)

(GIT) stability of pencils of plane cubics in terms of the types of singular fibers in the associated RES.

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(Z.)

(GIT) Stability of Halphen pencils of index two in terms of ...

Some Background

Definition

A smooth rational surface Y is called a **rational elliptic surface** (RES) if it admits a genus one fibration $f : Y \to \mathbb{P}^1$ which is relatively minimal.

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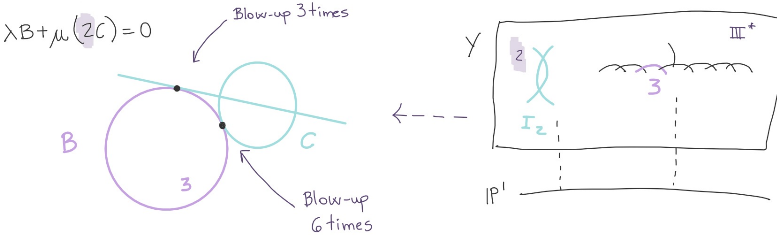
Proposition (e.g. Dolgachev & Cossec)

Given any RES $f : Y \to \mathbb{P}^1$, there exists $m \geq 1$ and a birational map $\pi : Y \to \mathbb{P}^2$ so that $f \circ \pi^{-1}$ is a **Halphen pencil** (of index m) which can be written as $\lambda B + \mu(mC) = 0$, where C is a cubic.

Pencil of curves of deg $3m$
with 9 base pts of mult. m

Example (2.)



$\lambda B + \mu (2C) = 0$
Blow-up 3 times
B
3
C
Blow-up 6 times
Y
2
I_2
3
III^*
$\mathbb{P}^1$

The GIT setup

Definition

Let G be a reductive group acting on a projective variety X. Choose an ample line bundle $\mathcal{L}$ (on X) together with a G-linearization. Then the associated GIT quotient is the projective variety:

$$X//G \doteq Proj \oplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})^G$$

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- The natural quotient map $\pi : X \to X//G$ is only rational. It is not defined at the points where all G-invariant functions vanish. These "bad points" are called unstable, and we remove them.
- Points where π is defined are called semistable.
- Points $x \in X^{ss}$ whose orbits $G \cdot x$ are closed in X^{ss} and of maximal dimension are called stable.

Tools for the GIT analysis

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    i) x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$
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- When $G = \mathbb{C}^\times$, then the action amounts to a finite dimensional representation $\lambda : \mathbb{C}^\times \to GL(V)$, where $V = \oplus V_i$ and on each one-dimensional space V_i we have $\lambda(t) \cdot v = t^i v$

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- It turns out this is the typical situation (Hilbert-Munford criterion)

GIT stability of pencils of plane sextics

- $G = SL(3)$
- $\mathscr{P}_6$: the space of pencils of plane curves of degree 6
- $G \curvearrowright \mathscr{P}_6 \simeq \underbrace{Gr(2,n)}_{\doteq X} \hookrightarrow \mathbb{P}^N$ (Plücker embedding)

- Here $n = \binom{6+2}{2}$ and $N = \binom{n}{2} - 1$
- A pencil $\mathcal{P} \in \mathscr{P}_6$, with generators $C_f : \sum f_{ij} x^i y^j z^{6-i-j} = 0$ and $C_g : \sum g_{ij} x^i y^j z^{6-i-j} = 0$, has Plücker coordinates given by all the 2×2 minors:

$$m_{ijkl} \doteq \begin{vmatrix} f_{ij} & f_{kl} \\ g_{ij} & g_{kl} \end{vmatrix}$$

A normalized one-parameter subgroup $t \mapsto \begin{pmatrix} t^{a_x} & 0 & 0 \\ 0 & t^{a_y} & 0 \\ 0 & 0 & t^{a_z} \end{pmatrix}$ acts on the Plücker coordinates m_{ijkl} of a pencil $\mathcal{P}$ as follows:

$$m_{ijkl} \mapsto t^{r_{ijkl}} \cdot m_{ijkl}$$

where $r_{ijkl} \doteq a_x(i+k) + a_y(j+l) + a_z(12-i-k-j-l)$.

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Definition

$$\omega(\mathcal{P}, \lambda) \doteq \min\{(a_x - a_z)(i+k) + (a_y - a_z)(j+l) \, : \, m_{ijkl} \neq 0\}$$

Hilbert-Mumford Criterion

A pencil $\mathcal{P} \in \mathscr{P}_6$ is unstable (resp. nonstable) if and only if there exists λ such that

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} > 4 \quad \text{(resp. } \geq \text{)}$$

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Similarly, we can define

$$\omega(f,\lambda) \doteq \min\{(a_x - a_z)i + (a_y - a_z)j \, : \, f_{ij} \neq 0\}$$

and we can compare $\omega(\mathcal{P},\lambda)$ and $\omega(f,\lambda)$ for $C_f \in \mathcal{P}$

As a consequence we can prove:

Theorem 1 (Z.)

Assume $\mathcal{P}$ contains a curve C_f such that $lct(\mathbb{P}^2, C_f) = \alpha$. If $\mathcal{P}$ is unstable (resp. not stable), then $\mathcal{P}$ contains a curve C_g such that $lct(\mathbb{P}^2, C_g) < \frac{\alpha}{4\alpha - 1}$ (resp. $\leq$).

$lct(X, \Delta) \doteq \sup\{t \in \mathbb{Q}_{>0} \ ; \ (X, t\Delta) \text{ is l.c.}\}$

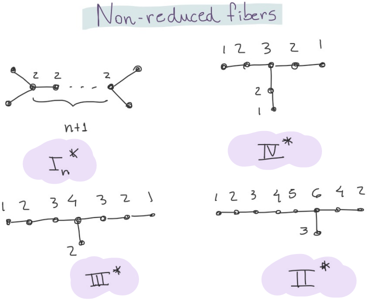
$\in \mathbb{Q} \cap (0, 1]$

Stability criteria

If $\mathcal{P}$ is a Halphen pencil of index two and Y denotes the corresponding RES, we can prove the following:

Theorem 2 (Z.)

If $\mathcal{P}$ is nonstable, then Y contains a non-reduced fiber. Further, if $\mathcal{P}$ is unstable, then Y contains a fiber of type II^*, III^* or IV^*.



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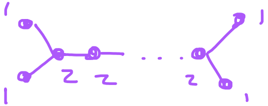
F of type $I_n^* \Rightarrow lct(Y, F) = 1/2$



What about the converse?

A sample result:

Theorem 3 (Z.)

A Halphen pencil $\mathcal{P}$ is unstable if and only if Y contains a fiber F of type II^* and $B \doteq \pi(F)$ is unstable.

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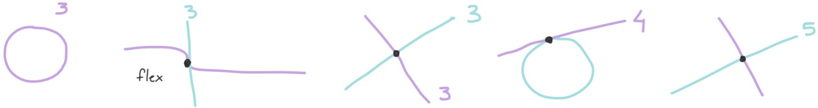
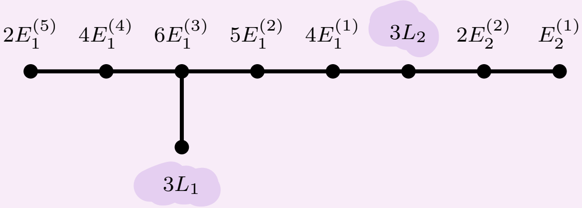
The curve B when F is of type II^* (Z.)



Example (two triple lines)

Let C be a smooth cubic. Let L_1 be an inflection line of C at a point P_1 and let L_2 be a line through P_1 which is tangent to C at another point P_2. Then the pencil $\mathcal{P}$ generated by $B = 3L_1 + 3L_2$ and $2C$ is a Halphen pencil of index two which yields a fiber of type II^*:



Example (Continued)

We can find coordinates in $\mathbb{P}^2$ so that B is given $x^3y^3 = 0$ and C is given by $z^2x - y(y-x)(y-\alpha x) = 0$, where $\alpha \in \mathbb{C}\backslash\{0,1\}$. Thus, the only (possibly) non-zero Plücker coordinates of $\mathcal{P}$ are

$$m_{0633}, m_{1333}, m_{1533}, m_{2033}, m_{2233}, m_{2433}, m_{3133}, m_{3342}$$

which implies $\mathcal{P}$ is unstable:

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which implies $\mathcal{P}$ is unstable: Let $a_x = 1, a_y = a, a_z = -1-a$, for some $a \in \mathbb{Q} \cap (-1/3, 1]$. Then

$$\begin{aligned}
\omega(\mathcal{P},\lambda) &= \min\{(a_x - a_z)(i+k) + (a_y - a_z)(j+l)\ ;\ m_{ijkl} \neq 0\} \\
&= \min\{(2+a)(i+k) + (1+2a)(j+l)\ ;\ m_{ijkl} \neq 0\} \\
&= 3(5+7a) \\
&\Rightarrow \frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{3(5+7a)}{3+3a} = \frac{5+7a}{1+a} > 4
\end{aligned}$$

Halphen pencils of higher index

Let $\mathcal{P}$ be a Halphen pencil of index $m > 1$

Theorem 4 (Z.)

If $\mathcal{P}$ contains a curve C_f such that $m_p(C_f) = 3m$ at some base point p, then $\mathcal{P}$ is not stable.

Theorem 5 (Z.)

If $\mathcal{P}$ is not stable, then Y contains a non-reduced fiber (type I_n^*, II^*, III^* or IV^*).

References

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Thank you!