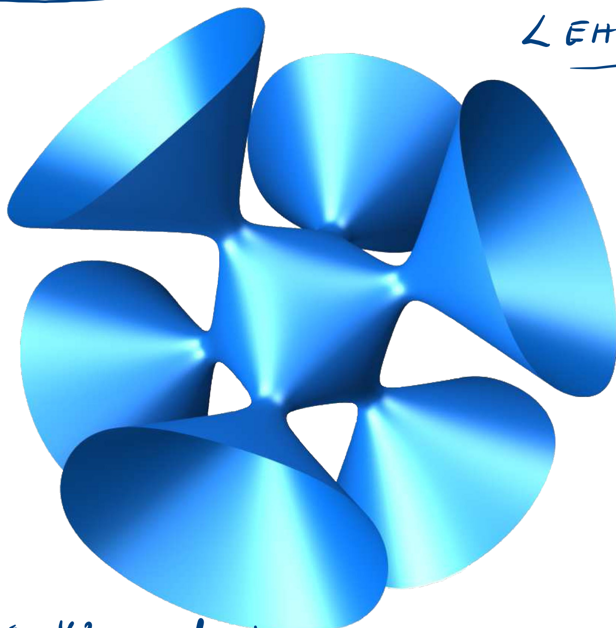
COMPLEX REFLECTION GROUPS, K3 SURFACES AND LEHRER-SPRINGER THEORY

(joint work with Cédric Bonnafé)



„a K3 surface"

A long project in 4 parts:

0) Cplx reflection groups and autom. of K3 surfaces

1) Construction of K3 surfaces

2) Geometric description of the K3 surfaces: elliptic fibrations (G_{29}, G_{30}, G_{31})

3) Do part 2) for G_{28} (Work in progress...)

§ Introduction :

<u>Motivation</u> : Work with W. Barth in 2003

$G \subset SO(4)$ s.t. $\frac{G}{\{\pm id\}} = T\times T,\ O\times O,\ I\times I$ G is a bipolyhedral group.

T, O, I = rotations groups of tetrahedron, octahedron and Icosahedron in $SO(3)$

$G \curvearrowright \mathbb{P}_3(\mathbb{C})$ one can construct families of dimension 1 of G-invariant spaces.

$$\{X_\lambda^G\}_{\lambda\in\mathbb{P}_1} \subset \mathbb{P}_3(\mathbb{C})$$

Theorem * (Barth - S. 2003)

The minimal resolution:

$$Y_\lambda^G \longrightarrow X_\lambda^G/G$$ is a K3 surface

with picard number $\rho(Y_\lambda^G) \in \{19, 20\}$ (19 for λ gen.)

Interesting case

$G = \mathcal{G}_{12}$, $\mathcal{G}_{12} = I \times I / \{\pm 1\}$

Bonnafé 2018: $\mathcal{G}_{12}$ is a subgroup of a cplx reflection group.

Def $W \subset GL_{\mathbb{C}}(V)$ finite,

V is a $\mathbb{C}$-vector space, $\dim V = n$

$Ref(W) = \{ s \in W \mid \dim V^s = n-1 \}$, $V^s = \{ x \in V \mid s(x) = x \}$

If $W = \langle Ref(W) \rangle$ we call W a complex reflection group.

Fact Assume that $W \subseteq GL_{\mathbb{C}}(V)$ is irreducible $\Rightarrow$ $\exists$ classification by Shephard–Todd (1954), there is an infinite family and 34 exceptional groups

$G_4, \dots, G_{37}$

Bourgé 2018 : $\mathcal{G}_{/2} = G_{30}^{SL} = G_{30} \cap SL_{\mathbb{C}}$

G_{30} = Coxeter group of dimension 4, it is the isometry group of a regular polytope in dim. 4 (DODECAPLEX)

$|G_{30}| = 14 \cdot 400.$

Another important result: Shephard-Todd-Chevalley-Serre thm.:

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$$

f_i are homog. $\deg f_i = d_i$

$\dim V = n$

$$|W| = d_1 \cdots d_n$$

[Michel-Morin 2010: one can take
$f_i \in \mathbb{Q}[x_1, \dots, x_n]$]

§ Main results

Part 0 of the work: G_{23} appears in the study of finite group acting on K3 surfaces.

Q: If $G \curvearrowright X$ K3, how big can be G?
(G finite)

Recall If G acts symplectically
$g^* \omega_X = \omega_X$ $\forall g \in G$, ω_X = holo 2-form on X.
$\Rightarrow$ $|G| \leq 960$, if $|G| = 960$ (Mukai 1988, Xiao 1996)
$\Rightarrow$ $G = M_{20}$ Mathieu group.

And there are 11 maximal groups acting symplectically.

Theorem 0 (Kondo 1999, Bonnafé-S. 2020, Brandhorst-Hashimoto 2020)

X K3, $G \curvearrowright X$ finite then.

(1) $|G| \leq 3840$ (Kondo)

(2) If $|G| = 3840 \Rightarrow$

$X = \mathrm{Km}(E_i \times E_i)$, E_i elliptic curve with cplx mult by i

and $G = G_{Ko}$ is unique (Kondo)

(3) the next biggest groups have $|G| = 1920$ and there are exactly two pairs (X_i, G_i) $i = 1, 2$ with this property, the X_i are Kummer surfaces.

BS + BH

One of the G_i is G_{2g} !!

Remarks (1) for all these groups

$M_{20} \leq G$, acts symplectically.

(2) proof uses some lattice theory, Transcendental lattice

(3) In the case of G_{29} in fact is $\mathbb{P}G_{29}$ that acts on the K3 surface:

$$X_{M_4} = \{ x_0^4 + \dots + x_3^4 - 6(x_0^2 x_1^2 + \dots + x_2^2 x_3^2) = 0 \}.$$

↑ already studied by Mukai

Part 1 of the work: Construction of K3 surfaces, generalization of

Thm ★

$W \subset GL_{\mathbb{C}}(V)$ (irreducible), $\dim V = 4$, W cplex refl. group.

$W = \langle Ref(W) \rangle$, $\mathbb{C}[V]^W = \mathbb{C}[f_1, f_2, f_3, f_4]$

Theorem 1 (Bonnafé–S. 2021)

Assume that

(a) $S \in \mathrm{Ref}(W) \Rightarrow \sigma(S) = 2$

(b) d_i is "well chosen"

(c) $\Gamma = W^{SL} = W \cap SL_{\mathbb{C}}$ or $\Gamma = W' = D(W)$

(d) $\mathcal{Z}(f_i) = \{x \in \mathbb{P}(V) \mid f_i(x) = 0\}$ has only ADE-sing.

$d_i = \deg f_i$

∃ examples with $\sigma(S) > 2$

we love surfaces in w.p.s. and we want the canonical class to be trivial.

we can work with that!

to get only ADE in the quotient

$\Rightarrow \dfrac{\mathcal{Z}(f_i)}{\Gamma}$ is a K3 surface with at most ADE singularities (i.e. min res is K3)

- We could find 15-families of K3 surfaces.
- The proof is <u>NOT</u> a case-by-case proof.

↳ <u>main</u> <u>ingredient</u> is **Lehrer–Springer theory**.

<u>Part 2 of the work</u>: Study elliptic fibrations and compute Picard number.

Take G_{29}, $X_{Mu} \circlearrowleft G_{29} \Rightarrow X_{Mu}/G_{29} = \mathbb{P}(d_1, d_2, d_3)$ w.p.p.

But if one takes $G'_{29} = G_{29}^{SL} = G_{29} \cap SL_{\mathbb{C}}$

We have $\widetilde{X}_{2\rho} \xrightarrow{\text{min. res.}} \dfrac{X_{nu}}{G_{2\rho}^{SL}}$ is K3

and.

Theorem 2 (BS 2022, Xiao 1996)

① $\widetilde{X}_{2\rho}$ has picard number 20 (Xiao)

② $\widetilde{X}_{2\rho}$ admits an elliptic fibration with fibers $\widetilde{E}_6 + \widetilde{D}_6 + 2\widetilde{A}_2 + \widetilde{A}_1$ and

$$T_{\widetilde{X}_{2\rho}} = \begin{pmatrix} 6 & 0 \\ 0 & 60 \end{pmatrix}$$

(BS)

← already stated by M. Schütt 2007

Remarks (1) for the proof we use the model of $X_{2\rho} = X_{nu}/G_{2\rho}$ in w.p.s.

$$X_{2\rho} = \{ (x_2 : x_3 : x_4 : j) \in \mathbb{P}(2,3,5,10) \mid j^2 = f(x_2, x_3, x_4) \}$$

double cover of a w.p.p. $\mathbb{P}(2,3,5)$

total degree 20

$G_i^{SL} \cong G_i'$ $\quad i = 30, 31$

(2) Similar results for G_{30} and G_{31}

(3) Left out G_{28} : $G_{28}^{SL} \neq G_{28}'$ (work in progress)

§ Some Lehrer - Springer theory

$W = \langle Ref(W) \rangle < GL_{\mathbb{C}}(V)$, $\dim V = n$

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n].$$

f_k homogeneous, $\deg f_k = d_k$

Let $e \in \mathbb{Z}_{>0}$

$$\Delta(e) = \{1 \leq k \leq n \mid e \text{ divides } d_k\}$$

$$\delta(e) = \# \Delta(e)$$

$V(w, \zeta_e)$ = eigenspace of $w \in W$ for the eigenvalue ζ_e (if ζ_e not e.v. put $V(w, \zeta_e) = 0$)

Theorem (Springer 1974, Lehrer–Springer 1999)

(0) $\delta(e) = \max_{w \in W} (\dim V(w, \zeta_e))$

$\Delta(e) = \{ 1 \leq k \leq m \mid e \mid d_k \}$

$\delta(e) = \# \Delta(e)$

in part ζ_e is e.v. of $w \in W \iff \delta(e) \neq 0$
$\iff$ e divides some d_k

(1) Let $w_e \in W$ s.t. $\dim V(w_e, \zeta_e) = \delta(e)$ and
let $w \in W \Rightarrow \exists x \in W$ s.t. $x(V(w, \zeta_e)) \subset$
$\subset V(w_e, \zeta_e)$

(2) $\bigcup_{w \in W} V(w, \zeta_e) = \bigcup_{x \in W} x(V(w_e, \zeta_e)) = \{ v \in V \mid$
$\forall k \in \{1, \dots, m\} \setminus \Delta(e)$
$f_k(v) = 0 \}$

Remarks (i) the v.s $V(w, J_e)$ give projective spaces $\mathbb{P}(V(w,J_e)) \subset \mathbb{P}(V)$ stable for the action of $w \in W$.

(ii) The part (2) of theorem allows to find W-orbit of points and lines in $\mathbb{P}_n(\mathbb{C}) = \mathbb{P}(V)$

Example $W = G_{30}$, $V = \mathbb{C}^4$ $\quad \mathbb{C}[W]^W = \mathbb{C}[f_1, f_2, f_3, f_4]$

$e = 10 \qquad \delta(e) = \delta(10) = \#\{1 \leq i \leq 4 \,/\, 10 \,/\, d_i\} = 2 \qquad 2 \quad 12 \quad \underline{20 \quad 30}$

$2 = \dim V(w_e, J_e)$ no lines in $\mathbb{P}_3(\mathbb{C})$

$$\mathcal{X}_{10} := \bigcup_{w \in G_{30}} V(w, \mathcal{T}_{10}) = \bigcup_{x \in G_{30}} x\left(V(v_{10}, \mathcal{T}_{10})\right)$$

↑ stein 2

$$= \{ v \in V \mid f_1(v) = f_2(v) = 0 \}$$

f_1: deg 2, f_2: deg 12

⇒ we have found a G_{30}-orbit of lines in the base locus of the pencil $\{ f_1^6 + \lambda f_2 = 0 \} = \{ X_\lambda^{G_{30}} \}$, $\lambda \in \mathbb{P}^1$ (f_2: deg 12)

with a bit more effort one finds 24 lines

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