Complex ball quotients and moduli spaces of some irreducible holomorphic symplectic fourfolds

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The aim of the talk was to show a relation between the moduli space of some IHS fourfolds carrying a non-symplectic automorphism of order three and the moduli space of smooth cubic 3-folds, that was described by Allcock, Carlson and Toledo in a famous paper of 2011, [1].

1. Non–symplectic automorphisms on IHS manifolds

We start by recalling the following:

Definition 1.1. *An irreducible holomorphic symplectic (IHS) manifold X is a compact, complex, Kähler manifold which is simply connected and admits a unique (up to scalar multiplication) everywhere non-degenerate holomorphic 2-form.*

Assume that X is equivalent by deformation to the Hilbert scheme of n points $\mathrm{Hilb}^{[n]}(S)$, where S is a K3 surface (we will say for simplicity that X is of type $K3^{[n]}$), let $\sigma \in \mathrm{Aut}(X)$ be an automorphism, and assume that σ has prime order p. This induces an action on $H^{2,0}(X) = \mathbb{C}\omega_X$. If $\sigma^*\omega_X = \zeta\omega_X$, ζ a primitive p–root of unity, we say that σ acts *non–symplectically* on X.

Recall that by using the Beauville-Bogomolov-Fujiki (BBF) quadratic form on the second cohomology with integer coefficients we have an isometry $H^2(X,\mathbb{Z}) = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle$, where U is the hyperbolic plane and $E_8(-1)$ is the even negative definite lattice associated to the root system E_8. So $H^2(X,\mathbb{Z})$ is a lattice of signature $(3,20)$ and an automorphism $\sigma \in \mathrm{Aut}(X)$ induces an isometry of $H^2(X,\mathbb{Z})$. We can consider two important sublattices

$$T = H^2(X,\mathbb{Z}) = \{x \in H^2(X,\mathbb{Z}) \mid \sigma^*(x) = x\}, \qquad S = T^\perp \cap H^2(X,\mathbb{Z}),$$

we call T the *invariant sublattice* and one can easily show that

$$T \subset \mathrm{NS}(X), \qquad \mathrm{Trans}_X \subset S$$

where $\mathrm{NS}(X)$ is the *Néron-Severi group* of X and Trans_X the *transcendental lattice* of X. We recall that since X is projective (see [2]) then $\mathrm{sgn}(T) = (1, \rho-1)$ and $\mathrm{sgn}(S) = (2, 21-\rho)$, where $\rho = \mathrm{rank}\, T$. These two lattices play an important role if one wants to classify automorphisms, in fact one starts by classifying S and T. This was done in the case of $n = 2$ by Boissière, Camere, Sarti and Tari (see [4], [10]) and for $n > 2$ it is a work in progress by Camere and Cattaneo (see [6]).

Properties of S. Let X be of type $K3^{[2]}$ (i.e. X is an IHS fourfold) carrying a non–symplectic automorphism σ of prime order p, then

$$\mathrm{rank}\, S = m(p-1), \ m \in \mathbb{Z}_{>0} \tag{1}$$

in fact the action of σ^* on $S \otimes \mathbb{C}$ is by primitive roots of unity, and the discriminant group satisfies

$$S^\vee / S \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus a}, \qquad a \in \mathbb{Z}_{\geq 0} \tag{2}$$

where $S^\vee = \{v \in S \otimes \mathbb{Q} \mid (v,z) \in \mathbb{Z}, \forall z \in S\}$.

2. A key example

Let $V \subset \mathbb{P}^5$ be a smooth cubic 4-fold, and assume that V has equation

$$x_5^3 + f_3(x_0, \ldots, x_4) = 0$$

so that V is the triple cover of $\mathbb{P}^4$ ramified on a smooth cubic threefold $C: \ \{f_3(x_0, \ldots, x_4) = 0\}$. The covering automorphism is

$$\sigma : \mathbb{P}^5 \longrightarrow \mathbb{P}^5, (x_0 : \ldots : x_4 : x_5) \mapsto (x_0 : \ldots : x_4 : \zeta x_5)$$

with $\zeta = e^{\frac{2\pi i}{3}}$ so that σ has order 3. We consider now the Fano variety of lines

$$F(V) = \{l \in Gr(1,5) \,|\, l \subset V\}$$

where $Gr(1,5)$ is the Grassmannian variety of lines of $\mathbb{P}^5$. It was shown by Beauville and Donagi (see [3]) that $F(V)$ is of $K3^{[2]}$ type and it is $\langle 6 \rangle$-polarized ample (i.e. there is a primitive embedding of the rank one lattice $\langle 6 \rangle$ in $\mathrm{NS}(F(V))$ so that the image contains an ample class). The automorphism σ on V induces an automorphism $\bar{\sigma}$ on $F(V)$ of the same order and one can show that $\bar{\sigma}$ acts non–symplectically. Moreover the fixed locus $F(V)^{\bar{\sigma}}$ is the Fano surface of lines $F(C)$ of the smooth cubic threefold C, this is a surface of general type with Hodge numbers: $h^{1,0} = h^{0,1} = 5$, $h^{2,0} = h^{0,2} = 10$, $h^{1,1} = 25$. By using the topology of the fixed locus one computes in the formulas (1), (2) that $m = 11$ and $a = 1$, so that combining with results on lattices by Nikulin (see [8]) and by Rudakov-Shafarevich (see [9]) one computes that

$$S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1), \qquad T = \langle 6 \rangle \tag{3}$$

where $A_2(-1)$ is the negative definite lattice associated to the root system A_2.

3. A relation between V, S and $C = F(V)^{\bar{\sigma}}$

For $V \subset \mathbb{P}^5$ a smooth cubic 4-fold, recall that $H^4(V, \mathbb{Z})$ is a lattice of signature $(21, 2)$ which is odd, unimodular (see [7]), i.e. with the usual intersection pairing it is isometric to $\langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$. Let $h \in H^2(\mathbb{P}^5, \mathbb{Z})$ be the class of a hyperplane and define $\theta(V) := h^2_{|V} \in H^4(V, \mathbb{Z})$ then one computes $\theta(V)^2 = 3$. Take now the *primitive cohomology*

$$H^4_0(V, \mathbb{Z}) := \theta(V)^{\perp} \cap H^4(V, \mathbb{Z}) \cong S(-1)$$

where the last isometry is shown by Hassett in [7] and S is the lattice we defined in (3). Recall that $H^{3,1}(V)$ is one-dimensional generated by a $(3,1)$-form, that we denote by v. In the case of a cubic 4-fold V as in the example above, one checks that $\sigma(v) = \zeta v$, with $\zeta = e^{\frac{2\pi i}{3}}$, so that

$$H^{3,1}(V) \subset H^4_0(V, \mathbb{Z})_{\zeta} \cong S(-1)_{\zeta}$$

where the last two spaces denote the eigenspaces for the eigenvalue ζ of the action of σ on $H^4_0(V, \mathbb{Z}) \otimes \mathbb{C}$ respectively $S(-1) \otimes \mathbb{C}$. Recall that by [1] the one–dimensional space $H^{3,1}(V)$ is called the *period point* of the smooth cubic threefold C.

Let us now go back to IHS fourfolds with non–symplectic automorphism. Consider $H^{2,0}(F(V)) = \mathbb{C}\omega_{F(V)}$, since $\bar{\sigma}^*\omega_{F(V)} = \zeta\omega_{F(V)}$ then $\omega_{F(V)} \in H^2(F(V),\mathbb{Z})_\zeta = S_\zeta \subset S\otimes\mathbb{C}$ where again $H^2(F(V),\mathbb{Z})_\zeta$ and S_ζ denotes the eigenspaces with respect to ζ for the action of $\bar{\sigma}$. More precisely the period point belongs to

$$\omega_{F(V)} \in \{[\omega] \in \mathbb{P}(S_\zeta) \,|\, (\omega,\bar{\omega}) > 0\}$$

which is an open analytic subset in a 10-dimensional complex projective space. The BBF–quadratic form restricts to an hermitian form on S_ζ of signature $(1,10)$ so that by an easy computation one can see that in fact $\omega_{F(V)}$ belongs to a 10-dimensional complex ball, that we denote by $\mathbb{B}_{10}$. On the other hand not any point in $\mathbb{B}_{10}$ gives an IHS manifolds X of type $K3^{[2]}$ with a non–symplectic automorphism of order three. So

$$\omega_{F(V)} \in \frac{\mathbb{B}_{10}\backslash\mathcal{H}}{\Gamma} =: \Omega_{10}$$

where $\mathcal{H}$ is a hyperplanes arrangment and Γ is an arithmetic subgroup of the group of the isometries of the lattice S. In fact Ω_{10} can be identified with the moduli space $\mathcal{M}^{\rho,\zeta}_{\langle 6\rangle}$ of IHS of $K3^{[2]}$ type where:

- we fix the representation $\rho : \mathbb{Z}/3\mathbb{Z} \longrightarrow O(\Lambda)$, that fixes the action of the automorphism on the $K3^{[2]}$-lattice $\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$,
- we fix the embedding of the lattice $\langle 6\rangle$ in Λ so that the orthogonal complement is S,
- we fix the action on the holomorphic two form of the IHS manifold of $K3^{[2]}$ type as the multiplication by ζ.

Observe that a priori not all X in this moduli space are of the type $F(V)$ with the non–symplectic automorphism $\bar{\sigma}$.

We denote now by $\mathcal{C}_3^{sm}$ the moduli space of smooth cubic threefolds as described in [1] and we have the following result obtained in a work in progress in collaboration with S. Boissière and C. Camere, [5]:

Theorem 3.1. (1) *We have an isomorphism of moduli spaces:* $\mathcal{C}_3^{sm} \cong \mathcal{M}^{\rho,\zeta}_{\langle 6\rangle}$.

(2) *Let X be of type $K3^{[2]}$ and let $\langle 6\rangle \hookrightarrow \mathrm{NS}(X)$ be an ample polarization. Then X admits a non–symplectic automorphism of order three with invariant lattice $\langle 6\rangle$ if and only if $X \cong F(V)$, and V is the triple cover of $\mathbb{P}^4$ ramified on a smooth cubic threefold and the automorphism is induced by the covering automorphism.*

The proof uses the results of [1] and in particular the description of the period point of a smooth cubic threefold that we recalled above, the study of the hyperplane arrangment $\mathcal{H}$ and the study of the group Γ.

References

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