ON THE CONE CONJECTURE FOR ENRIQUES MANIFOLDS

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À Claire, avec gratitude et admiration

Abstract. Enriques manifolds are non simply connected manifolds whose universal cover is irreducible holomorphic symplectic, and as such they are natural generalizations of Enriques surfaces. The goal of this note is to prove the Morrison-Kawamata cone conjecture under some assumptions on the universal cover and then deduce it for very general Enriques manifolds when the degree of the cover is prime. The proof uses the analogous result (established by Amerik-Verbitsky) for their universal cover. We also verify the conjecture for the known examples.

1. Introduction

An Enriques manifold is a connected complex manifold X which is not simply connected and whose universal covering $\widetilde{X}$ is an irreducible holomorphic symplectic (IHS) manifold. Enriques manifolds were simultaneously and independently introduced in [BNWS11] and [OS11a] as a natural generalization of Enriques surfaces. The fundamental group of an Enriques manifold is a cyclic and finite group $G = \langle g \rangle$ and its order is called the *index* of X, which is also the order of the torsion canonical class $K_X \in \operatorname{Pic}(X)$. From the definition it follows that an Enriques manifold is even dimensional. Moreover, as $X = \widetilde{X}/G$, it is compact and, since $h^{2,0}(X) = h^{0,2}(X) = 0$, it turns out that an Enriques manifold is always projective (cf [BNWS11, Proposition 2.1, (4)] and [OS11a, Corollary 2.7]) and so is its universal cover. In those papers examples of Enriques manifolds of index 2, 3 and 4 are constructed, while their periods are studied in [OS11b].

The Morrison-Kawamata cone conjecture (see [Mor93, Ka97]) concerns the action of the automorphism group of manifolds (and more generally pairs) with numerically trivial canonical class on the cone of rational nef classes and predicts the existence of a rational, polyhedral fundamental domain for such action. More precisely, and more generally, we have the following.

Conjecture 1.1 (The Morrison-Kawamata Cone conjecture)**.** *Let $X \to S$ be a K-trivial fiber space, that is a proper surjective morphism $f : X \to S$ with connected fibers between normal varieties such that X has $\mathbb{Q}$-factorial and terminal singularities and K_X is zero in $N^1(X/S)$.*

(1) *There exists a rational polyhedral convex cone Π which is a fundamental domain for the action of $\operatorname{Aut}(X/S)$ on the convex hull $\operatorname{Nef}^+(X)$ of $\operatorname{Nef}(X) \cap \operatorname{Pic}(\widetilde{X})_{\mathbb{Q}}$ inside $\operatorname{Pic}(X)_{\mathbb{R}}$ in the sense that*
   (a) $\operatorname{Nef}^+(X/S) = \cup_{g \in \operatorname{Aut}(X/S)} g^*\Pi$.
   (b) *$int(\Pi) \cap int(g^*\Pi) = \emptyset$, unless $g^* = id$ in $\operatorname{GL}(N^1(X/S))$.*

(2) *There exists a rational polyhedral convex cone Π' which is a fundamental domain in the sense above for the action of $\operatorname{Bir}(X/S)$ on the convex hull $\operatorname{Mov}^+(X)$ of $\operatorname{Mov}(X) \cap \operatorname{Pic}(X)_{\mathbb{Q}}$ inside $\operatorname{Pic}(X)_{\mathbb{R}}$.*

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Item (2) above is also known as the birational Cone conjecture. Notice that in the literature there is also a version of the conjecture where $\mathrm{Nef}^+(\widetilde{X})$ is replaced by $\mathrm{Nef}^e(X/S) := \mathrm{Nef}(X/S) \cap \mathrm{Eff}(X/S)$ (same for $\mathrm{Mov}^+(X)$ which is replaced by $\overline{\mathrm{Mov}}^e(X/S) := \overline{\mathrm{Mov}}(X/S) \cap \mathrm{Eff}(X/S)$). One can see that $\mathrm{Nef}^e(\widetilde{X}) \subset \mathrm{Nef}^+(\widetilde{X})$ (cf. [MY15, Remark 1.4]). This statement is analogous to the classical one for the 4 known deformations type of IHS manifolds (and equivalent in general modulo the SYZ-conjecture).

The conjecture (we will not specify which version of it, and refer the interested readers to the papers we quote) has been proved in dimension 2 by Sterk-Looijenga, Namikawa, Kawamata, and Totaro (see [Ste85, Na85, Ka97, Tot10]), by Prendergast-Smith [PS12] for abelian varieties, by Amerik-Verbitsky [AV17, AV20] for IHS manifolds, building upon the birational cone conjecture established by Markman in [Mar11], see also Markman-Yoshioka [MY15] and by Lehn-Mongardi-Pacienza [LMP22] for singular IHS varieties. For a recent extension to a not necessarily closed field of characteristic 0 see [Ta21, Theorem 1.0.5]. For Calabi-Yau varietes the conjecture is open in general. For a recent result in this direction, see [GLW22] and the reference therein. For recent results in the relative case, in particular on the birational cone conjecture for families of $K3$ surfaces, see [LZ22] and [L23]. We refer the reader to [Tot12, LOP18] for nice introductions to this topic. The conjecture is deeply related with birational geometry. Item (1) of the conjecture yields the finiteness, up to automorphisms, of birational contractions and fiber space structures of the initial variety, while item (2) implies, modulo standard conjectures of the MMP, the finiteness of minimal models, up to isomorphisms, of any $\mathbb{Q}$-factorial and terminal variety with non-negative Kodaira dimension (cf. [CL14, Theorem 2.4]).

From now on we will restrict ourselves to the absolute and smooth case. By the Beauville-Bogomolov decomposition theorem (see [Be83a]) we know that any K-trivial variety V admits a finite étale cover $\widetilde{V} \to V$, where $\widetilde{V}$ is a product of Calabi-Yau manifolds, IHS manifolds and an abelian variety. A general question is: suppose we know Conjecture 1.1 for $\widetilde{V}$, can we deduce it for V? Our main result provides a positive answer when we have only one factor of IHS type and the IHS has the smallest possible Picard group. More precisely we show the following:

Theorem 1.2. *Let $X = \widetilde{X}/G$ be an Enriques manifold, where $G = \langle g \rangle$ is a finite cyclic group acting freely on an IHS manifold $\widetilde{X}$. Assume that the action of G on $\mathrm{Pic}(\widetilde{X})$ is the identity, i.e. $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$. Then there exists a rational and polyhedral cone which is a fundamental domain for the action of $\mathrm{Aut}(X)$ (respectively of $\mathrm{Bir}(X)$) on $\mathrm{Nef}^+(X)$ (resp. on $\mathrm{Mov}^+(X)$).*

Notice that we show that the fundamental domain is also convex whenever the map $\mathrm{Aut}(\widetilde{X}) \to O(H^2(\widetilde{X}, \mathbb{Z}))$ is injective.

The hypothesis $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$ can be verified in several cases.

Theorem 1.3. *Let $X = \widetilde{X}/G$ be an Enriques manifold, where $G = \langle g \rangle$ is a cyclic group of order d acting freely on an IHS manifolds $\widetilde{X}$. Then $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$ in the following cases:*

1. *The index $|G| = p$ is prime and the Enriques manifold is very general in the moduli space.*
2. *X is one of the examples provided in* [BNWS11] *and* [OS11a].
3. *$\widetilde{X}$ is of $K3^{[n]}$-type (resp. of $\mathrm{Kum_n}$-type) and the index $d = 13, 17, 19, 23, 46$ (resp. $d = 5, 7, 9, 14, 18$).*

In particular, by Theorem 1.2, for these Enriques manifolds there exists a rational and polyhedral cone which is a fundamental domain for the action of $\mathrm{Aut}(X)$ (respectively of $\mathrm{Bir}(X)$) on $\mathrm{Nef}^+(X)$ (resp. on $\mathrm{Mov}^+(X)$).

As for the proof of Theorem 1.2, if $\pi : \widetilde{X} \to X = \widetilde{X}/G$ is the covering map, by Amerik-Verbitsky [AV17, AV20] there exists a rational polyhedral convex cone $\widetilde{D}$ which is a fundamental domain for the action of $\mathrm{Aut}(\widetilde{X})$ on $\mathrm{Nef}^+(\widetilde{X})$. We set

$$D := \widetilde{D} \cap \pi^* N^1(X).$$

Where recall that $N^1(X)$ denotes the Néron-Severi group of X which coincides with the Picard group of X. The proof of Theorem 1.3 then consists in showing that $\pi_*(D)$ is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$: the rationality and polyhedrality of D are implied by those of $\widetilde{D}$. The main point is to show that if $\xi \in \pi^* N^1(X)$ and φ is an automorphism of $\widetilde{X}$ such that $\varphi^*(\xi) \in \widetilde{D}$, then φ "descends" to an automorphism of X, i.e. φ commutes with G. To show this commutativity we check it on cohomology and this is where we use the assumtion $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$. Some care has to be taken, as it is well known that there are non-trivial automorphisms acting trivially on cohomology for the 2 deformation classes coming from abelian surfaces (see Remark 2.1 for details). As for the proof of Theorem 1.3, item (1) is proved in Proposition 2.3, after having recalled the construction of moduli spaces of marked Enriques manifolds, while items (2) and (3) are the content of several propositions in Sections 4.1 and 4.2.

We conclude the introduction by mentioning that, motivated by the approach followed in this paper, Monti and Quedo have recently proved the cone conjecture for all étale quotients of abelian varieties, cf. [MQ24].

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2. Preliminaries

2.1. **Basic facts on IHS manifolds.** An irreducible holomorphic symplectic manifold is a compact Kähler manifold $\widetilde{X}$ which is simply connected and carries a holomorphic symplectic 2–form σ, such that $H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) = \mathbb{C} \cdot \sigma$. For a general introduction of the subject, we refer to [Huy99].

Let $\widetilde{X}$ be an irreducible holomorphic symplectic manifold of dimension $2n \geq 2$. Let $\sigma \in H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ such that $\int_{\widetilde{X}} \sigma^n \bar{\sigma}^n = 1$. Then, following [Be83a], the second cohomology group $H^2(\widetilde{X}, \mathbb{C})$ is endowed with a quadratic form $q = q_{\widetilde{X}}$ defined as follows

$$q(a) := \frac{n}{2} \int_{\widetilde{X}} (\sigma\bar{\sigma})^{n-1} a^2 + (1-n) \left(\int_{\widetilde{X}} \sigma^n \bar{\sigma}^{n-1} a \right) \cdot \left(\int_{\widetilde{X}} \sigma^{n-1} \bar{\sigma}^n a \right), \ a \in H^2(\widetilde{X}, \mathbb{C}),$$

which is non-degenerate and, up to a positive multiple, is induced by an integral nondivisible quadratic form on $H^2(\widetilde{X}, \mathbb{Z})$ of signature $(3, b_2(\widetilde{X}) - 3)$. The form q is called the Beauville-Bogomolov-Fujiki quadratic form of $\widetilde{X}$. By Fujiki [Fuj87], there exists a positive rational number $c = c_{\widetilde{X}}$ (the Fujiki constant of $\widetilde{X}$) such that

$$c \cdot q^n(\alpha) = \int_{\widetilde{X}} \alpha^{2n}, \ \forall \alpha \in H^2(\widetilde{X}, \mathbb{Z}).$$

Recall that for an IHS manifold $\widetilde{X}$, we have $\mathrm{Pic}(\widetilde{X}) = \mathrm{NS}(\widetilde{X})$ and we will indifferently use both the notations.

Remark 2.1. As usual $\widetilde{X}$ denotes an IHS manifold. Recall that $\nu : \mathrm{Aut}(\widetilde{X}) \longrightarrow \mathrm{Aut}(H^2(\widetilde{X}, \mathbb{Z}))$ is finite, by [Huy99, Proposition 9.1]. Notice moreover that by a result of Hassett and Tschinkel, [HT13, Theorem 2.1] the kernel of the homomorphism ν is invariant under smooth deformations of the manifold $\widetilde{X}$. This allows to compute it for all 4 known deformation types of IHS manifolds, thanks to [Be83a, Proposition 10], [BNWS11, Corollary 3.3] and [MW17, Theorems 3.1 and 5.2]. We have that ν is injective for deformations of punctual Hilbert schemes of $K3$ surfaces and OG10 manifolds, while the kernel is generated by the group of translations by points of order n on an abelian surface and by $-\mathrm{id}$ (respectively equal to $(\mathbb{Z}/2\mathbb{Z})^{\times 8}$) if $\widetilde{X}$ is a deformation of a generalized Kummer of dimension $2n$ (respectively of an OG6 manifold).

2.2. **Basic facts on Enriques manifolds.** Throughout this section, X denotes an arbitrary Enriques manifold, $\widetilde{X}$ its IHS universal cover and G its fundamental group. Consider now the quotient $\pi : \widetilde{X} \to X = \widetilde{X}/G$, where $|G| = d$. Notice that the action of $g \in G$ on $\widetilde{X}$ cannot be symplectic, i.e. we cannot have $g^*\sigma = \sigma$, otherwise X would have $h^{2,0}(X) \neq 0$ (where $H^{2,0}(\widetilde{X}) = \mathbb{C}\sigma$). Therefore there exists a d-th root of unity $\lambda \neq 1$ such that $g^*\sigma = \lambda\sigma$. Moreover the action must be purely non–symplectic, i.e. λ is a primitive root of unity of the same order of g. Otherwise, if G contained symplectic automorphisms, there would be points with a non trivial stabilizer and the covering would not be étale (see [BNWS11, Section 2.2]). Observe that if G is of prime order then a non–symplectic action is the same as a purely non–symplectic one. Recall that by [Be83b, Section 4] a group acting purely non–symplectically on an IHS manifold is cyclic, we denote by g its generator.

Recall that the automorphisms (resp. the birational transformations) of X identify to the quotient by G of the normalizer group of G in $\mathrm{Aut}(\widetilde{X})$ (resp. in $\mathrm{Bir}(\widetilde{X})$), i.e.

$$\mathrm{Aut}(X) = \{\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X}) : \widetilde{\tau} \circ G \circ \widetilde{\tau}^{-1} = G\}/G \tag{1}$$

and

$$\mathrm{Bir}(X) = \{\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X}) : \widetilde{\tau} \circ G \circ \widetilde{\tau}^{-1} = G\}/G. \tag{2}$$

Indeed, if $\widetilde{\tau} \in \{\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X}) : \widetilde{\tau} \circ G \circ \widetilde{\tau}^{-1} = G\}/G$ then one obviously recovers an automorphism on X. For the other inclusion, if $\tau \in \mathrm{Aut}(X)$ then, by the universal property of the universal cover, the morphism $\tau \circ \pi$ factorizes through $\widetilde{X}$, namely there exists a morphism $\widetilde{\tau} : \widetilde{X} \to \widetilde{X}$ sitting in the following commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\tau}"] \arrow[d, "\pi"] & \widetilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "\tau"] & X. \end{tikzcd}$$

By construction $\widetilde{\tau}$ is bijective and yields the lifting of the automorphism τ. For birational transformations again one inclusion is obvious. For the other inclusion if $\tau \in \mathrm{Bir}(X)$, then the largest open subset $U \subset X$ over which τ is defined has complement of codimension ≥ 2. The same is then true for $\pi^{-1}(U) =: \widetilde{U}$. Since $\pi_1(\widetilde{U}) = \pi_1(\widetilde{X})$ we have that $\widetilde{U}$ is simply connected and the argument given before works the same and yields and automorphism $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{U})$ lifting τ, which we see as $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$. We will show that in some cases this is even equal to the quotient of the centralizer of G in $\mathrm{Aut}(X)$.

Lemma 2.2. *For any Enriques manifold X we have $H^2(X, \mathbb{Z}) = \mathrm{Pic}(X)$. Moreover we have $\pi^*H^2(X, \mathbb{Z}) = H^2(\widetilde{X}, \mathbb{Z})^G \subset \mathrm{Pic}(\widetilde{X})$. In particular*

$$\dim_{\mathbb{R}} \pi^* N^1(X) = \dim_{\mathbb{R}} \pi^* \mathrm{Pic}(X) = \mathrm{rk}\, H^2(\widetilde{X}, \mathbb{Z})^G.$$

Proof. Consider $\xi \in H^2(X,\mathbb{Z})$, we have

$$q(\pi^*\xi,\sigma) = q(g^*\pi^*\xi, g^*\sigma) = q(\pi^*\xi,\lambda\sigma) = \lambda q(\pi^*\xi,\sigma)$$

with $\lambda \neq 1$ as g is not a symplectic automorphism. Then $q(\pi^*\xi,\sigma) = 0$, which implies that $\pi^*\xi \in \mathrm{Pic}(\widetilde{X})$, as $\mathrm{Pic}(\widetilde{X}) = \sigma^{\perp_q} \cap H^2(\widetilde{X},\mathbb{Z})$. In particular we have that $\xi \in \mathrm{Pic}(X)$. We have shown that $H^2(X,\mathbb{Z}) \subset \mathrm{Pic}(X) \subset H^2(X,\mathbb{Z})$ so we get the first equality (notice that this equality also follows immediately from the fact that $H^{2,0}(X) = 0$, but we will need the argument below). By construction $\pi^*H^2(X,\mathbb{Z}) \subset H^2(\widetilde{X},\mathbb{Z})^G$ with finite index and moreover by [OS11a, proof of Proposition 2.8, and proof of Proposition 5.1] we get in fact equality. Finally since $\pi^*\mathrm{Pic}(X) \subset \mathrm{Pic}(\widetilde{X})$ the previous equality implies that $H^2(\widetilde{X},\mathbb{Z})^G \subset \mathrm{Pic}(\widetilde{X})$ (which is in fact a more general fact for non–symplectic automorphisms acting on IHS manifold, but this gives an easy proof). □

The examples of Enriques manifolds from [BNWS11] and [OS11a], which to our knowledge are all the known examples so far, will be recalled in detail in Section 4.

2.3. **Examples.** We recall the following examples from [BNWS11] and [OS11a] which to our knowledge are all the known examples so far.

2.3.1. *From* $K3^{[n]}$ *manifolds.* Consider a K3 surface S with an Enriques (i.e. a fixed point free) involution ι. On the Hilbert scheme $S^{[n]}$ consider the natural involution $\iota^{[n]}$. Notice that $\iota^{[n]}$ has no fixed points if n is odd (notice that if n is even there are always fixed points) and $S^{[n]}/\langle\iota^{[n]}\rangle$ is an Enriques manifold of dimension $2n$ and index 2. Moreover in this case we know the invariant sublattice for the action of $\iota^{[n]}$. In fact $H^2(S,\mathbb{Z})^\iota = U(2) \oplus E_8(2)$ where U is the hyperbolic plane with bilinear form multiplied by 2 and $E_8(2)$ is the negative definite lattice associated to the corresponding Dynkin diagram and bilinear form multiplied by 2. Since $\iota^{[n]}$ is a natural automorphism, we have

$$H^2(S^{[n]},\mathbb{Z})^{\iota^{[n]}} = H^2(S,\mathbb{Z})^\iota \oplus \langle -2(n-1)\rangle = U(2) \oplus E_8(2) \oplus \langle -2(n-1)\rangle.$$

Notice that the punctual Hilbert scheme $(S/\iota)^{[n]}$ of the Enriques surface S/ι *is not* an Enriques manifold, in the sense of our definition, as its universal cover is a Calabi-Yau and not an IHS manifold, see [OS11a, Theorem 3.1]

2.3.2. *From* $\mathrm{Kum_n}$ *manifolds.* These examples arise from a bielliptic surface, i.e. a surface S with torsion canonical class of order $d \in \{2,3,4\}$ admitting a finite étale covering by an abelian surface $A \to S$, and specifically, under certain conditions on d and n, from a free action of a finite group on $A^{[n+1]}$ which preserves $\mathrm{Kum_n(A)} \subset \mathrm{A^{[n+1]}}$. We refer the reader to [OS11a, Section 6] and [BNWS11, Section 4.2] for further details, in particular about the fact that the same construction can not produce Enriques manifolds of index 6. We recall in detail the example for $d = 2,3,4$ in Section 4.1.

2.4. **Moduli spaces of marked Enriques manifolds.** We follow the presentation in [BCS16]. See also [OS11b] for another equivalent approach and for the study of the period map.

Let L be a lattice. Let $\widetilde{X}$ be an IHS manifold. Recall that a marking for $\widetilde{X}$ is an isometry $L \to H^2(\widetilde{X},\mathbb{Z})$. If d is an integer we will denote by λ a primitive d-th root of unity.

Let M be an even non-degenerate lattice of rank $\rho \geq 1$ and signature $(1,\rho-1)$. An *M-polarized* IHS manifold is a pair $(\widetilde{X},j)$ where $\widetilde{X}$ is a projective IHS manifold and j is a primitive embedding of lattices $j : M \hookrightarrow \mathrm{NS}(\widetilde{X})$. Two M-polarized IHS manifolds $(\widetilde{X}_1,j_1)$ and $(\widetilde{X}_2,j_2)$ are called *equivalent* if there exists an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $j_1 = f^* \circ j_2$. As in [DK07, Section 10] and [D96] one can construct a moduli space of *marked* M-polarized IHS manifolds as follows. We fix a primitive embedding of M in L and we identify M with its image in L. A *marking* of $(\widetilde{X}, j)$ is an isomorphism of lattices $\eta : L \to H^2(\widetilde{X}, \mathbb{Z})$ such that $\eta_{|M} = j$. As observed in [D96, p.11], if the embedding of M in L is unique up to an isometry of L then every M-polarization admits a compatible marking. Two M-polarized marked IHS manifolds $(\widetilde{X}_1, j_1, \eta_1)$ and $(\widetilde{X}_2, j_2, \eta_2)$ are called *equivalent* if there exists an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\eta_1 = f^* \circ \eta_2$ (this clearly implies that $j_1 = f^* \circ j_2$). Let $T := M^\perp \cap L$ be the orthogonal complement of M in L and set

$$\Omega_M := \{x \in \mathbb{P}(T \otimes \mathbb{C}) \,|\, q(x) = 0,\, q(x + \bar{x}) > 0\}.$$

Since T has signature $(2, \mathrm{rk}(T) - \rho)$ the period domain Ω_M is a disjoint union of two connected components of dimension $\mathrm{rk}(T) - \rho$. For each M-polarized marked IHS manifolds (X, j, η), since $\eta(M) \subset \mathrm{NS}(\widetilde{X})$, we have $\eta^{-1}(H^{2,0}(\widetilde{X})) \in \Omega_M$. On the other hand, by the surjectivity of the period map [Huy99, Theorem 8.1] restricted to any connected component $\mathfrak{M}_L^0$ of $\mathfrak{M}_L$ (the moduli space of IHS manifolds with second integral cohomology isometric to a given lattice L) we can associate to each point $\omega \in \Omega_M$ an M-polarized IHS manifold as follows: there exists a marked pair $(\widetilde{X}, \eta) \in \mathfrak{M}_L^0$ such that $\eta^{-1}(H^{2,0}(\widetilde{X})) = \omega \in \mathbb{P}(T \otimes \mathbb{C})$ so $M = T^\perp \subset \omega^\perp \cap L$, hence $\eta(M) \subset H^{2,0}(\widetilde{X})^\perp \cap H^{1,1}_{\mathbb{Z}}(\widetilde{X}) = \mathrm{NS}(\widetilde{X})$ and we take $(\widetilde{X}, \eta_{|M}, \eta)$.

By the Local Torelli Theorem for IHS manifolds, an M-polarized IHS manifold $(\widetilde{X}, j)$ has a local deformation space $\mathrm{Def}_M(\widetilde{X})$ that is contractible, smooth of dimension $\mathrm{rk}(T) - \rho$, such that the (local) period map $\mathfrak{p}\colon \mathrm{Def}_M(\widetilde{X}) \to \Omega_M$ is a local isomorphism (see [D96]). By gluing all these local deformation spaces one obtains a moduli space K_M of marked M-polarized IHS manifolds that is a non-separated analytic space, with a (global) period map $\mathfrak{p}\colon K_M \to \Omega_M$.

To construct a period domain for Enriques manifolds we have to take the non–symplectic action into account.

Let $(\widetilde{X}, j)$ be an M-polarized IHS manifold and $G = \langle g \rangle$ a cyclic group of prime order $p \geq 2$ acting non-symplectically on $\widetilde{X}$. Assume that the action of G on $j(M)$ is the identity and that there exists a group homomorphism $\rho : G \longrightarrow O(L)$ such that

$$M = L^\rho := \{x \in L \,|\, \rho(g)(x) = x, \forall g \in G\}.$$

We define a *(ρ, M)-polarization* of $(\widetilde{X}, j)$ as the data of a marking $\eta\colon L \to H^2(\widetilde{X}, \mathbb{Z})$ such that $\eta_{|M} = j$ and $g^* = \eta \circ \rho(g) \circ \eta^{-1}$.

Two (ρ, M)-polarized IHS manifolds $(\widetilde{X}_1, j_1)$ and $(\widetilde{X}_2, j_2)$ are *isomorphic* if there are markings, $\eta_1\colon L \to H^2(\widetilde{X}_1, \mathbb{Z})$ and $\eta_2\colon L \longrightarrow H^2(\widetilde{X}_2, \mathbb{Z})$ such that $\eta_{i|M} = j_i$, and an isomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\eta_1 = f^* \circ \eta_2$.

Recall that by construction $\mathbb{C}\sigma$ is identified to the line in $L \otimes \mathbb{C}$ defined by $\eta^{-1}(H^{2,0}(\widetilde{X}))$. Let $\lambda \in \mathbb{C}^*$ such that $\rho(g)(\sigma) = \lambda\sigma$. Observe that $\lambda \neq 1$ since the action is non-symplectic and it is a primitive p-th root of unity since p is prime. By construction σ belongs to the eigenspace of $T \otimes \mathbb{C}$ relative to the eigenvalue λ, where $T = M^\perp \cap L$. We denote it by $T(\lambda)$ (if $p = 2$, we have $\lambda = -1$ and we denote $T(\lambda) = T_{\mathbb{R}}(\lambda) \otimes \mathbb{C}$, where $T_{\mathbb{R}}(\lambda)$ is the real eigenspace relative to $\lambda = -1$).

Assume that $\lambda \neq -1$, then the period belongs to the space

$$\Omega_M^{\rho,\lambda} := \{x \in \mathbb{P}(T(\lambda)) \,|\, q(x + \bar{x}) > 0\}$$

of dimension $\dim T(\lambda) - 1$ which is a complex ball if $\dim T(\lambda) \geq 2$. By using the fact that $\lambda \neq -1$ it is easy to check that every point $x \in \Omega_M^{\rho,\lambda}$ satisfies automatically the condition $q(x) = 0$.

If $\lambda = -1$ we set $\Omega_M^{\rho,\lambda} := \{x \in \mathbb{P}(T(\lambda)) \mid q(x) = 0,\, q(x+\bar{x}) > 0\}$. It has dimension $\dim T(\lambda) - 2$. Clearly in both cases $\Omega_M^{\rho,\lambda} \subset \Omega_M$.

Let X be a *marked* Enriques manifold of index d, that is the data of a (ρ, M)*-polarization* of $(\widetilde{X}, j)$ with a marking $\eta\colon L \to H^2(\widetilde{X}, \mathbb{Z})$ such that $\eta_{|M} = j$ and $g^* = \eta \circ \rho(g) \circ \eta^{-1}$, so that $X = \widetilde{X}/G$. Recall that by the Bogomolov-Tian-Todorov theorem the Kuranishi space $\mathrm{Def}(X)$ of an Enriques manifold X of index d is smooth. Moreover Oguiso and Schröer verified in [OS11b, Proposition 1.2] that, after possibly shrinking $\mathrm{Def}(X)$, every point in it parametrizes an index d Enriques manifold.

If $\mathscr{X} \to B$ is a flat family of marked Enriques manifolds, then, as remarked in [OS11b, Section 2] the universal covering $\widetilde{\mathscr{X}} \to \mathscr{X}$ of the family is also the fiberwise universal covering. The period map of the family $\mathscr{X} \to B$ is then defined as

$$\mathfrak{p}_B : B \to \Omega_M^{\rho,\lambda},\ b \mapsto \eta^{-1}(H^{2,0}(\widetilde{X}_b)). \tag{3}$$

By [OS11b, Theorem 2.4] the local Torelli theorem holds, namely the period map $\mathfrak{p}_B$ is a local isomorphism. Hence, as for IHS manifolds, we can patch together the Kuranishi spaces via the Oguiso-Schröer local Torelli theorem, to construct a (non-separated) moduli space of (marked) Enriques manifolds in terms of (ρ, M)-polarized IHS manifolds.

Let us now briefly discuss the case when the order of G is not necessarily a prime, but the action of G is purely non–symplectic (recall that it means that g acts on the holomorphic two-form by multiplication by a primitive root of unity of the same order as g). So let $|G| = d$, where d is not necessarily a prime number and fix the action as $\rho(g)(\sigma_{\widetilde{X}}) = \lambda\sigma_{\widetilde{X}}$ where λ is a primitive d–root of unity. Then take $T(\lambda)$ to be the eigenspace in $T \otimes \mathbb{C}$ relative to the eigenvalue λ, where we use here the same notations as before, i.e. M is the invariant sublattice in L for the action of $\rho(G)$ and T its orthogonal complement. As before the period $\sigma_{\widetilde{X}}$ of $\widetilde{X}$ belongs to $\Omega_M^{\rho,\lambda}$, but here the Néron–Severi group of a very general $\widetilde{X}$ in the moduli space just contains M and it is not necessarily equal.

We precise this remark in the framework of Enriques manifolds in the next two Lemmas. Recall that a very general Enriques manifold is a point in the corresponding parameter space which lies outside a countable union of proper closed subvarieties.

Proposition 2.3. *For a very general Enriques manifold X of prime index $|G| = p$ with universal cover $\widetilde{X}$, we have that* $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$.

Proof. We argue as in [BCS16, Theorem 3.4]. As in Section 2.4 we set $M := \eta^{-1}(H^2(\widetilde{X}, \mathbb{Z})^G)$ and $T := M^\perp \cap L$. Let $T^+ \subset T\backslash\{0\}$ be the set of $t \in T\backslash\{0\}$ such that $H_t = \{\omega \in \Omega_M^{\rho,\lambda} \mid q(\omega, t) = 0\}$ is a hyperplane section.

Consider $\mathscr{H} := \bigcup_{t\in T^+} H_t$. Each subset $\Omega_M^{\rho,\lambda} \setminus H_t$ is open and dense in $\Omega_M^{\rho,\lambda}$ hence by Baire's Theorem the subset $\Omega^0 := \Omega_M^{\rho,\lambda} \setminus \mathscr{H}$ is dense in $\Omega_M^{\rho,\lambda}$ since $\mathscr{H}$ is a countable union of complex closed subspaces.

We take now a period $\omega \in \Omega^0$, and a marked IHS manifold $(\widetilde{X}, \eta)$ such that $\mathfrak{p}(\widetilde{X}, \eta) = \omega$. To lighten the notation from now on we will omit the marking η.

Then $\mathrm{Pic}(\widetilde{X}) = \{l \in L, q(\omega, l) = 0\}$. Observe that since $\omega \in T(\lambda)$ we have that $H^2(\widetilde{X}, \mathbb{Z})^G \subset \mathrm{Pic}(\widetilde{X})$, as by construction $M := H^2(\widetilde{X}, \mathbb{Z})^G$ is orthogonal to T. Moreover by Lemma 2.2 we have the equality $\mathrm{Pic}(\widetilde{X})^G = H^2(\widetilde{X}, \mathbb{Z})^G$.

We now want to show that

$$(\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X}) = \{0\},$$

from which we deduce that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X},\mathbb{Z})^G$, that is what we want to prove.

Let $0 \neq t \in (\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})$. Then $t^\perp \subset L$ can not determine an hyperplane section of $\Omega_M^{\rho,\lambda}$ as we have taken $\omega \in \Omega^0$. In particular this implies that, for all $t \in (\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X}))$, $t^\perp$ contains the eigenspace $T(\lambda)$ with respect to λ for the action of G on $T \otimes \mathbb{C}$. Hence

$$T(\lambda) \subset [(\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})]^\perp = \cap_{t \in (\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})} \ t^\perp,$$

which implies that

$$T(\lambda) \subset \mathrm{Pic}(\widetilde{X})^G + \mathrm{Pic}(\widetilde{X})^\perp.$$

Now by definition $T(\lambda) \cap \mathrm{Pic}(\widetilde{X})^G = \{0\}$, hence we deduce

$$T(\lambda) \subset \mathrm{Pic}(\widetilde{X})^\perp =: T_{\widetilde{X}}$$

where the orthogonal complement of $\mathrm{Pic}(\widetilde{X})$ is the transcendental lattice $T_{\widetilde{X}}$ of $\widetilde{X}$.

This in particular implies that

$$((\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})) \otimes \mathbb{C}) \cap T(\lambda) = \{0\} \tag{4}$$

otherwise we would find a non-zero element in $\mathrm{Pic}(\widetilde{X}) \cap T_{\widetilde{X}}$, which is not possible.

Now by definition G acts as the identity on $\mathrm{Pic}(\widetilde{X})^G = H^2(\widetilde{X},\mathbb{Z})^G$ while, by the hypothesis $|G| = p$, the eigenvalues of its action on $(\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X}))$ are the primitive roots of unity. Since this is a lattice, the characteristic polynomial is a power of the p-th cyclotomic polynomial (as the characteristic polynomial has integral coefficients). This means that all the primitive roots of unity appear with the same multiplicity, which is different from zero since we are assuming that $t \neq 0$. In particular we deduce that $((\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})) \otimes \mathbb{C}) \cap T(\lambda) \neq \{0\}$, which contradicts (4). So we must have that $(\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X}) = \{0\}$ and this concludes the proof. □

Observe that if m is not a prime number we have that at all points in the moduli space $H^2(\widetilde{X},\mathbb{Z})^G \subset \mathrm{Pic}(\widetilde{X})$ as proved in Lemma 2.2. Now by using the proof of Proposition 2.3 we can show the following result:

Proposition 2.4. *Let $X = \widetilde{X}/G$ be a very general Enriques manifold of index $d = |G|$, $G = \langle g \rangle$ and $\widetilde{X}$ denotes its IHS universal cover. Then $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{C}$ contains all the eigenspaces with eigenvalue a non–primitive root of unity for the action of g on $H^2(\widetilde{X},\mathbb{Z}) \otimes \mathbb{C}$.*

Proof. By [Be83b, Proposition 6] g acts on $T_{\widetilde{X}}$ by primitive d-roots of unity and using a similar argument as in the first part of the proof of Proposition 2.3 we show that $T(\lambda) \subset T_{\widetilde{X}}$ (we use the same notations as in Proposition 2.3).

Now the eigenvalues for the action of G on $(\mathrm{Pic}(\widetilde{X})^G)^\perp \cap \mathrm{Pic}(\widetilde{X})$ are the roots of unity (not necessarily primitive) that are different from 1. Since this is a lattice, the characteristic polynomial is a product of a polynomial with integral coefficients with a power of the d-th cyclotomic polynomial (as the characteristic polynomial has integral coefficients). This means that all the primitive roots of unity appear with the same multiplicity, which is not zero since we are assuming $t \neq 0$ (we keep the same notation as in Proposition 2.3). But since we have shown that $T(\lambda) \subset T_{\widetilde{X}}$ this is not possible. □

Remark 2.5. Observe that the previous Proposition does not exclude that also in the non–prime order case it can happen that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X},\mathbb{Z})^G$ at the very general point. This depends on the action of G on $H^2(\widetilde{X},\mathbb{Z})$, see the discussion of the index 4 examples in Section 4.1.

Throughout this section, X denotes an arbitrary Enriques manifold, $\widetilde{X}$ its IHS universal cover, G its fundamental group, and $\pi : \widetilde{X} \to X = \widetilde{X}/G$ the quotient morphism. First we prove the following.

Lemma 3.1. $H^2(\widetilde{X}, \mathbb{Q}) = \pi^*[H^2(X, \mathbb{Q})] \oplus [\pi^*(H^2(X, \mathbb{Q}))]^{\perp_q}$

Proof. Clearly both spaces $\pi^*[H^2(X, \mathbb{Q})]$ and $[\pi^*(H^2(X, \mathbb{Q}))]^{\perp_q}$ are contained in $H^2(\widetilde{X}, \mathbb{Q})$ we have to show that their sum is direct. Let us take a non-zero class $\beta \in [\pi^*(H^2(X, \mathbb{Q}))]^{\perp_q}$ ($= [H^2(\widetilde{X}, \mathbb{Q})^G]^{\perp_q}$ by Lemma 2.2). Suppose by contradiction that

$$\beta \in \pi^* H^2(X, \mathbb{Q}). \tag{5}$$

Then $q(\beta, \beta) = 0$. Let $A \in \mathrm{Pic}(\widetilde{X})$ an ample line bundle. Without loss of generality we may assume that $A = \pi^* B$ with B ample (so that A is G-invariant). By the choice of β and the G-invariance of A we have that $q(\beta, A) = 0$ which, by (5), contradicts the Hodge-index theorem for q. □

3.1. **The proof of Theorem 1.2.** We assume that we are in the hypothesis of Theorem 1.2 and show that $\widetilde{\tau}^*$ commutes with g^*, for any $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ (respectively $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$)

Lemma 3.2. *Each automorphism $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ (respectively any $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$) acts on $H^2(\widetilde{X}, \mathbb{Z})^G$ and commutes with the action of g on it.*

Proof. By assumption we know that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$. Consider $\eta \in H^2(\widetilde{X}, \mathbb{Z})^G$ and set $e := (\widetilde{\tau})^*(\eta)$. First notice that $e \in H^2(\widetilde{X}, \mathbb{Z})^G$. Indeed automorphisms preserve the Picard group, hence $e = (\widetilde{\tau})^*(\eta) \in \mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$, by hypothesis. Therefore $g^*(e) = e$ from which we deduce

$$g^*((\widetilde{\tau})^*(\eta)) = g^*(e) = e = (\widetilde{\tau})^*(\eta) = (\widetilde{\tau})^*(g^*(\eta))$$

where the last equality holds since $\eta \in H^2(\widetilde{X}, \mathbb{Z})^G$. Notice that the same considerations hold for $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$. □

We then check the commutativity also on $T(X) := (\pi^* H^2(X, \mathbb{Z}))^{\perp_q} \cap H^2(\widetilde{X}, \mathbb{Z})$ under the same assumption.

Lemma 3.3. *Let X be an Enriques manifold, $\widetilde{X}$ its IHS universal cover, G its fundamental group, and $\pi : \widetilde{X} \to X = \widetilde{X}/G$ the quotient morphism. Suppose that $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$. Then any automorphism $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ (respectively any $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$) acts on $T(X)$ and commutes with the action of g on it.*

Proof. Notice that $T(X)$ is the transcendental part of a weight two Hodge structure of $K3$ type. Hence by [Huy16, Corollary 3.4, Chapter 3] the group of all Hodge isometries of $T(X)$ is a finite cyclic group. The automorphisms preserve $\mathrm{Pic}(\widetilde{X})$, which, by hypothesis, equals $H^2(\widetilde{X}, \mathbb{Z})^G$. Therefore any element of $\mathrm{Aut}(\widetilde{X})$ preserves $T(X)$ and induces a Hodge isometry of $T(X)$. Then the restrictions to $T(X)$ of $(\widetilde{\tau})^*$ and g^* are elements of a finite cyclic group and as such commute. Notice again that the same considerations hold for $\widetilde{\tau} \in \mathrm{Bir}(\widetilde{X})$. □

Remark 3.4. Observe that when the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X}, \mathbb{Z}))$ is injective we have shown that in this case $\mathrm{Aut}(\widetilde{X})$ (resp. $\mathrm{Bir}(\widetilde{X})$) can be identified with the quotient by G of the centralizer of G in $\mathrm{Aut}(X)$ (resp. in $\mathrm{Bir}(\widetilde{X})$).

Proof of Theorem 1.2. Consider $\xi \in \mathrm{Nef}^+(X)$. Then it exists $\eta \in \mathrm{Nef}^+(\widetilde{X})$ such that $\xi = \pi_*(\eta)$. By [AV17, AV20] the Cone conjecture holds on $\widetilde{X}$, i.e. there exists a fundamental domain $\widetilde{D}$ for the $\mathrm{Aut}(\widetilde{X})$ action on $\mathrm{Nef}^+(\widetilde{X})$, which is a rational polyhedral convex cone. In particular we have the existence of $\widetilde{\tau} \in \mathrm{Aut}(\widetilde{X})$ and $\widetilde{\delta} \in \widetilde{D} \cap \pi^* N^1(X) =: D$ such that

$$(\widetilde{\tau})^*(\eta) = \widetilde{\delta}. \tag{6}$$

Notice that D is a rational polyhedral convex cone, since $\widetilde{D}$ is and $\pi^* N^1(X)$ is a rational subspace. Since $\mathrm{Pic}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})^G$ then we can apply Lemma 3.2, Lemma 3.3 and Lemma 3.1 to show that the isometry $\widetilde{\tau}^*$ commutes with g^* on $H^2(\widetilde{X}, \mathbb{Z})$. We split now the proof in two cases.

Case 1. If the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X}, \mathbb{Z}))$ is injective, then $\widetilde{\tau}$ and g commute also on $\widetilde{X}$ which by (1) implies that $\widetilde{\tau}$ descends to an automorphism τ on X such that

$$\tau \circ \pi = \pi \circ \widetilde{\tau}$$

So we get

$$\tau^*(\xi) = \tau^*(\pi_*(\eta)) = \pi_*((\widetilde{\tau})^*(\eta)) = \pi_*(\widetilde{\delta}) \in \pi_*(D).$$

This means that $\pi_*(D)$ is a fundamental domain for the $\mathrm{Nef}^+(X)$-action on $\mathrm{Aut}(X)$, which moreover is a rational and polyhedral convex cone as D is.

Case 2. If the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X}, \mathbb{Z}))$ is not injective, then by [Huy99, Proposition 9.1, item (v)] the kernel K is finite. Consider

$$\Gamma := \{\varphi \in \mathrm{Aut}(\widetilde{X}) \mid \varphi g^{-1} \varphi^{-1} g \in K\}$$

and observe that Γ is in fact equal to $\mathrm{Aut}(\widetilde{X})$ as we have shown that each automorphism induces an isometry that commutes on $H^2(\widetilde{X}, \mathbb{Z})$ with g. Consider

$$\Gamma_0 := \{\varphi \in \mathrm{Aut}(\widetilde{X}) \mid \varphi g^{-1} \varphi^{-1} g = id\}$$

and notice that Γ_0 is a subgroup of $\mathrm{Aut}(\widetilde{X})$ which is not necessarily normal. Let us show that it has finite index in $\mathrm{Aut}(\widetilde{X})$. Consider the map

$$\beta : \mathrm{Aut}(\widetilde{X}) \longrightarrow K, \ \varphi \mapsto \varphi g^{-1} \varphi^{-1} g.$$

Then one easily checks that $\beta(\varphi_1) = \beta(\varphi_2)$ if and only if $\varphi_2^{-1}\varphi_1 \in \Gamma_0$. Therefore if we take the set $\mathrm{Aut}(\widetilde{X})/\Gamma_0$ of left cosets we have an injective map $\mathrm{Aut}(\widetilde{X})/\Gamma_0 \longrightarrow K$ showing that Γ_0 has finite index in $\mathrm{Aut}(\widetilde{X})$. Let now $\bar{\gamma}_1, \ldots, \bar{\gamma}_r$ be all the classes in $\mathrm{Aut}(\widetilde{X})/\Gamma_0$ and φ be the automorphism such that $\varphi^*(\pi^*\xi) \in \widetilde{D}$. By considering the class $\bar{\varphi} \in \mathrm{Aut}(\widetilde{X})/\Gamma_0$ then we have that $\bar{\varphi} = \bar{\gamma}_j$ for a certain $j = 1, \ldots, r$ so that $\gamma_j^{-1}\varphi \in \Gamma_0$. We then modify the fundamental domain $\widetilde{D}$ by taking

$$\widetilde{D}' := \widetilde{D} \cup \bigcup_{i=1}^{r} (\gamma_i^{-1})^*(\widetilde{D}).$$

Observe that $\widetilde{D}'$ is obviously still rational and polyhedral and $(\gamma_j^{-1})^*\varphi^*(\pi^*\xi) \in \widetilde{D}'$. Moreover $\gamma_j^{-1}\varphi$ descends to an automorphism on X and, setting $\widetilde{D}' \cap \pi^* N^1(X) =: D'$ we conclude as before that $\pi_*(D')$ is a fundamental domain for the $\mathrm{Nef}^+(X)$-action on $\mathrm{Aut}(X)$.

The theorem is then proved for $\mathrm{Aut}(X)$ in the prime order case at the very general point.

For the existence of a fundamental domain for the $\mathrm{Mov}^+(X)$-action on $\mathrm{Bir}(X)$ the same arguments apply, using (2) instead of (1), the birational cone conjecture for IHS manifolds proved in [Mar11] and noticing that, in Case 2, the kernel of $\mathrm{Bir}(\widetilde{X}) \to O(H^2(\widetilde{X}, \mathbb{Z}))$ is again finite, by [Huy99, Proposition 9.1, items (iv) and (v)]. $\square$

Remark 3.5. Observe that when the map $\mathrm{Aut}(\widetilde{X}) \longrightarrow O(H^2(\widetilde{X},\mathbb{Z}))$ is injective the proof shows that the fundamental domain is a rational and polyhedral *convex* cone.

4. Further remarks and results

In the first subsection we recall from [BNWS11, OS11a] the constructions of all the currently known examples of Enriques manifolds of index 2, 3 and 4 and prove that for these examples the fundamental group G acts as the identity on the Picard group of the IHS universal cover. In the second subsection we check this condition for certain indices d (some of which even non-prime) at all points of the moduli space.

4.1. **The cone conjecture for the known examples.** We start with the index 2 examples.

Index 2: **A quotient of the Hilbert scheme.** Let S be a K3 surface with an Enriques involution ι. For S a generic K3 surface it is well known that $\mathrm{Pic}(S) = H^2(S,\mathbb{Z})^\iota = U(2) \oplus E_8(2)$. The quotient $S^{[n]}/\langle \iota^{[n]} \rangle$, for n odd, is an Enriques manifold of index 2. Recall that $\mathrm{Pic}(S^{[n]}) = \mathrm{Pic}(S) \oplus \mathbb{Z}\delta$, where δ is the half of the class of the exceptional divisor on $S^{[n]}$. Since the automorphism $\iota^{[n]}$ is a natural automorphism, i.e. it comes from an automorphism of S, its action on δ is the identity and the action on $\mathrm{Pic}(S)$ is the same as the action of ι, so that $\mathrm{Pic}(S^{[n]}) = H^2(S^{[n]},\mathbb{Z})^{\iota^{[n]}}$. In other words we have:

Proposition 4.1. *Let $X := S^{[n]}/\langle \iota^{[n]} \rangle$ be the index 2 Enriques manifold recalled above. Then* $\mathrm{Pic}(S^{[n]}) = H^2(S^{[n]},\mathbb{Z})^{\iota^{[n]}}$.

A quotient of a generalized Kummer. Let $A = E \times F$ be the product of two elliptic curves and assume that n is odd, so that we can write $2m = n+1$ for an integer m. For E, F very general elliptic curves we have $\mathrm{rk}\,\mathrm{Pic}(E \times F) = 2$, indeed

$$\mathrm{Pic}(E \times F) = \mathrm{Pic}(E) \times \mathrm{Pic}(F) \times \mathrm{Hom}(\mathrm{Jac(E)}, \mathrm{Jac(F)}), \tag{7}$$

and for the very general choice of E and F we have

$$\mathrm{Hom}(\mathrm{Jac(E)}, \mathrm{Jac(F)}) = \{0\}. \tag{8}$$

Consider now $a := (a_1, a_2)$ where $a_2 \in F$ is a 2-torsion point and $a_1 \in E[n+1]$ is such that $ma_1 \notin \mathbb{Z} \oplus t\mathbb{Z}$ (where $E = \mathbb{C}/\mathbb{Z} \oplus t\mathbb{Z}$). Let t_a be the translation by the point a on $E \times F$ and $h_2 = \mathrm{diag}(-1,1)$ the morphism given by the multiplication by -1 on the first component and the identity on the second. Set $\psi_2 := t_a \circ h_2$. Then $\psi_2^{[n]}$ does not have fixed points on $\mathrm{Kum_n(E \times F)}$ as shown in [OS11a, Section 6].

Proposition 4.2. *Let $X := \mathrm{Kum_n(E \times F)}/\langle \psi_2^{[n]} \rangle$ be the index 2 Enriques manifold recalled above. Then* $\mathrm{Pic}(\mathrm{Kum_n(E \times F)}) = \mathrm{H^2(Kum_n(E \times F), \mathbb{Z})}^{\psi_2^{[n]}}$.

Proof. For $(z,w) \in E \times F$ we have $t_a(h(z,w)) = (-z + a_1, w + a_2)$. Let us now compute the action in cohomology. Recall that $H^2(A,\mathbb{Z}) = U \oplus U \oplus U$ and consider $H^2(A,\mathbb{R})$. If we write $z = z_1 + iz_2$ for the coordinate on E and $w = w_1 + iw_2$ for the coordinate on F then $H^2(A,\mathbb{R})$ is generated by the 2-forms

$$dz_1 \wedge dw_1,\ dz_1 \wedge dw_2,\ dz_2 \wedge dw_1,\ dz_2 \wedge dw_2,\ dz_1 \wedge dz_2,\ dw_1 \wedge dw_2.$$

A translation acts trivially in cohomology and the action of the multiplication by -1 acts sending $z_1 + iz_2$ to $-z_2 - iz_1$ so that the image under ψ_2^* of the previous basis is

$$\psi_2^*(dz_1 \wedge dw_1) = -dz_1 \wedge dw_1,\ \psi_2^*(dz_1 \wedge dw_2) = -dz_1 \wedge dw_2,\ \psi_2^*(dz_2 \wedge dw_1) = -dz_2 \wedge dw_1,$$
$$\psi_2^*(dz_2 \wedge dw_2) = -dz_2 \wedge dw_2,\ \psi_2^*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2,\ \psi_2^*(dw_1 \wedge dw_2) = dw_1 \wedge dw_2.$$

Hence the matrix of ψ_2^* in this basis is

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are -1 with multiplicity 4 and then 1 with multiplicity 2. This means that the invariant part for the action of ν on the cohomology of A is precisely $\mathrm{Pic}(E \times F)$. Consider the induced natural automorphism $\psi_2^{[n]}$ on $\mathrm{Kum_n(E \times F)}$. By the choice of a_1 and a_2 the induced automorphism $\psi_2^{[n]}$ acts freely on $\mathrm{Kum_n(E \times F)}$ (see [OS11a, Theorem 6.4]). From the above description we have that the invariant part for the action on the second cohomology of $\mathrm{Kum_n(E \times F)}$ with integral coefficients of the induced automorphism $\psi_2^{[n]}$ is exactly the Picard group (since the automorphism $\psi_2^{[n]}$ is natural it acts as the identity on the exceptional divisor). Therefore we can invoke Theorem 1.2 to conclude. □

Index 3: **A quotient of a generalized Kummer.** Take integers m, n such that $3m = (n+1)$. Let E_ω be the elliptic curve with complex multiplication by ω, which is a primitive 3d root of unity, and let F another elliptic curve. Set $A := E_\omega \times F$. For F a very general elliptic curve we have $\mathrm{rk}\,\mathrm{Pic}(E_\omega \times F) = 2$, by (7) and (8). Consider now $a := (a_1, a_2)$ where $a_2 \in F$ is a 3-torsion point and $a_1 \in E_\omega[n+1]$ is such that $ma_1(2+\omega) \notin \mathbb{Z} + \omega\mathbb{Z}$. Let t_a be the translation by the point a on $E_\omega \times F$ and $h_3 = \mathrm{diag}(\omega\, 1)$ the morphism given by the multiplication by ω on the first component and the identity on the second. Set $\psi_3 := t_a \circ h_3$. Then $\psi_3^{(n]}$ does not have fixed points on $\mathrm{Kum_n(E_\omega \times F)}$ as shown in [BNWS11, Section 4.2] and [OS11a, Section 6].

Proposition 4.3. *Let* $X_3 := \mathrm{Kum_n(E_\omega \times F)}/\langle \psi_3^{[\mathrm{n}]} \rangle$ *be the index 3 Enriques manifold recalled above. Then* $\mathrm{Pic(Kum_n(E_\omega \times F)) = H^2(Kum_n(E_omega \times F), \mathbb{Z})^{\psi_3^{[n]}}}$.

Proof. To show that $\psi_3^{[n]}$ acts as the identity on $\mathrm{Kum_n(E_\omega \times F)}$ we argue here in a different way than in the index 2 example. Observe that $E_\omega \times F$ is projective, so that it contains an invariant ample class. If the eigenspace relative to ω were in $\mathrm{Pic}(E_\omega \times F)$ then the same would be true for $\bar{\omega}$. Since $\mathrm{Pic}(E_\omega \times F) = 2$ this would lead to a contradiction and so ψ_3 acts as the identity on the Picard group. Now $\psi_3^{[n]}$ acts as the identity on the exceptional divisor on $\mathrm{Kum_n(E_\omega \times F)}$. This means that the action of $\psi_3^{[n]}$ on $\mathrm{Pic(Kum_n(E_\omega \times F))}$ is the identity. □

Index 4: **A quotient of a generalized Kummer.** Take integers m, n such that $4m = (n+1)$. Let E_i be the elliptic curve with complex multiplication by i and F is another elliptic curve. Set $A := E_i \times F$. For F a very general elliptic curve we have $\mathrm{rk}\,\mathrm{Pic}(E_i \times F) = 2$, by (7) and (8). Consider now $a := (a_1, a_2)$ where $a_2 \in F$ is a 4-torsion point and $a_1 \in E_i[n+1]$ is such that $2ma_1(1+i) \notin \mathbb{Z} + i\mathbb{Z}$. Let t_a be the translation by the point a on $E_i \times F$ and $h_4 = \mathrm{diag}(i\, 1)$ the morphism given by the multiplication by i on the first component and the identity on the second. Set $\psi_4 := t_a \circ h_4$. Then $\psi_4^{[n]}$ does not have fixed points on $\mathrm{Kum}_n(E_i \times F)$ as shown in [BNWS11, Section 4.2] and [OS11a, Section 6].

Proposition 4.4. *Let* $X_4 := \mathrm{Kum_n(E_i \times F)}/\langle \psi_4^{[\mathrm{n}]} \rangle$ *be the index 4 Enriques manifold recalled above. Then* $\mathrm{Pic(Kum_n(E_i \times F)) = H^2(Kum_n(E_i \times F), \mathbb{Z})^{\psi_4^{[n]}}}$.

Proof. For $(z, w) \in E_i \times F$ we have $t_a(h(z, w)) = (iz + a_1, w + a_2)$. Let us now compute the action in cohomology. We use the same set of generators of $H^2(A, \mathbb{R})$ as in the index 2 case.

A translation acts trivially in cohomology and the action of the multiplication by i acts sending $z_1 + iz_2$ to $-z_2 + iz_1$ so that the image under ψ_4^* of the previous basis is

$$\psi_4^*(dz_1 \wedge dw_1) = -dz_2 \wedge dw_1,\ \psi_4^*(dz_1 \wedge dw_2) = -dz_2 \wedge dw_2,\ \psi_4^*(dz_2 \wedge dw_1) = dz_1 \wedge dw_1,$$
$$\psi_4^*(dz_2 \wedge dw_2) = dz_1 \wedge dw_2,\ \psi_4^*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2,\ \psi_4^*(dw_1 \wedge dw_2) = dw_1 \wedge dw_2.$$

Hence the matrix of ψ_4^* in this basis is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are i and $-i$ with multiplicity 2 and then 1 with multiplicity 2. This means that the invariant part for the action of ψ_4 on the cohomology of A is precisely $\mathrm{Pic}(E_i \times F)$. Consider the induced natural automorphism $\psi_4^{[n]}$ on $\mathrm{Kum_n(E_i \times F)}$. By the choice of a_1 and a_2 the induced automorphism $\psi_4^{[n]}$ acts freely on $\mathrm{Kum_n(E_i \times F)}$ (see [OS11a, Theorem 6.4]). From the above description we have that the invariant part for the action on the second cohomology of $\mathrm{Kum_n(E_i \times F)}$ with integral coefficients of the induced automorphism $\psi_4^{[n]}$ is exactly the Picard group (since the automorphism $\psi_4^{[n]}$ is natural it acts as the identity on the exceptional divisor). Therefore we can invoke Theorem 1.2 to conclude. □

Remark 4.5. Observe that the argument that we used for the index 3 case does not apply for the index 2 and the index 4 case, because -1 could be an eigenvalue and so we can not deduce immediately that the action of the automorphism is the identity on $\mathrm{Pic}(A)$.

4.2. **The cone conjecture for other possible cases.** We start by recalling a general result stated in [OS11a, Proposition 2.4].

Proposition 4.6. *Let X be an Enriques manifold of dimension $\dim(X) = 2n$, then the index d of X divides $n + 1$.*

We now discuss Enriques manifolds that may arise from all the known examples of IHS manifolds.

$K3^{[n]}$ **and** $\mathbf{Kum}_n$**.** Consider now an Enriques manifold $\widetilde{X} \to X = \widetilde{X}/G$ such that $\widetilde{X}$ is a deformation of $K3^{[n]}$ or of Kum_n. By Proposition 4.6 the order d of the group $G = \langle g \rangle$ divides $n + 1$. Recall that the respective second Betti numbers of $K3^{[n]}$ and of Kum_n manifolds are, respectively, 23 and 7. Hence, since a primitive root of unity is an eigenvalue of the action of g^* on $H^2(\widetilde{X}, \mathbb{Z}) \otimes \mathbb{C}$, we must have for the Euler totient function $\varphi(d) \leq 22$, respectively $\varphi(d) \leq 6$ (in fact in general for an IHS manifold $\widetilde{X}$ we have $\varphi(d) \leq b_2(\widetilde{X}) - 1$). A list of possible d is given in [OS11a, Proposition 2.9], however notice that the authors missed the cases $d = 48, 60$ for $K3^{[n]}$ and they erroneously included $d = 24$ for Kum_n. Indeed this is not possible since $\varphi(24) = 8$ which is bigger than $b_2(\widetilde{X}) = 7$. For convenience we recall in the next Proposition all the value of d and explain when the cone conjecture holds.

Proposition 4.7. *Let $\widetilde{X} \to X = \widetilde{X}/G$ be an Enriques manifold quotient of an IHS manifold $\widetilde{X}$.*

a) *If $\widetilde{X}$ is of $K3^{[n]}$-type, then, see* [OS11a, Proposition 2.9], *$d := |G| \leq 66$ and*
$$d \in \{2, 3, 4, 5, 6, \ldots, 27, 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 46, 48, 50, 54, 60, 66\}.$$
For any such Enriques manifold of index $d \in \{13, 17, 19, 23, 46\}$ we have that G acts as the identity on $\mathrm{Pic}(\widetilde{X})$.

b) *If $\widetilde{X}$ is of* Kum_n*-type, then, see* [OS11a, Proposition 2.9], $d := |G| \leq 18$ *and*

$$d \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}.$$

For any such Enriques manifold of index $d \in \{5, 7, 9, 14, 18\}$ we have that G acts as the identity on $\mathrm{Pic}(\widetilde{X})$.

Proof. a) For the order $d \in \{13, 17, 19, 23\}$ observe that g^* cannot act on $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{C}$ with primitive roots of unity since the rank is too small: we discuss here the case $d = 13$ in details, the other cases being similar. Since the eigenvalues of the automorphism g^* on $T_{\widetilde{X}} \otimes \mathbb{C}$ are the primitive roots of unity we have that $\mathrm{rk}(T_{\widetilde{X}}) \geq 12$ (recall that if a primitive root of unity is an eigenvalue for the action of g^* on $T_{\widetilde{X}} \otimes \mathbb{C}$ then all the others primitive roots are eigenvalues too, with the same multiplicity). This implies that $\mathrm{rk}\,\mathrm{Pic}(\widetilde{X}) \leq 11$. Now if g^* had an eigenvalue which is a primitive root of unity also on $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{C}$ then $\mathrm{rk}\,\mathrm{Pic}(\widetilde{X}) \geq 13$ (since $\mathrm{Pic}(\widetilde{X})$ also contains an invariant ample class), but this is in contradiction with the previous inequality.

For the order 46, we have $\varphi(46) = 22$ and by [Be83b, Proposition 6, item (ii)] we have that the Picard number of $\widetilde{X}$ is one. Now we know that $\widetilde{X}$ always contains an ample invariant class so that the action of G is trivial on $\mathrm{Pic}(\widetilde{X})$ and one concludes with Theorem 1.2.
b) For the order $d \in \{5, 7\}$ observe that g can not act on $\mathrm{Pic}(\widetilde{X}) \otimes \mathbb{C}$ with some primitive roots of unity since the rank is too small, the argument is the same as in part a). For the orders 9, 14 and 18, $\varphi(9) = 6$, $\varphi(14) = 6$ and $\varphi(18) = 6$. Again by [Be83b, Proposition 6, item (ii)] we remark that the rank of the Picard group is forced to be equal to one and we conclude again with Theorem 1.2. □

Remark 4.8. In the case of $d = 3$ and $\widetilde{X}$ of $K3^{[2]}$–type notice that by [BCS16, Table 1] a non–symplectic automorphism of order 3 on a $K3^{[2]}$–type manifold has never empty fixed locus. So that one can not use $K3^{[2]}$–type manifolds to construct Enriques manifolds of index 3. Nevertheless as soon as we consider $K3^{[n]}$–type manifold, with bigger n so that $n + 1$ is divisible by 3 then this may be possible.

The O'Grady examples. The two examples of O'Grady $OG10$ and $OG6$ are 10 and 6–dimensional so by Proposition 4.6 to produce Enriques manifolds we have to take the quotient by a fixed point free automorphism of order 2, 3 or 6, respectively of order 2 or 4. It is an interesting open question to understand whether such automorphisms exist. Observe that in both cases if the order of the automorphism is prime the cone conjecture (respectively the weaker one with the fundamental domain which is a finite union of rational polyhedral convex cones) would be true by Theorem 1.3 under the genericity assumption.

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