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Komplexe Analysis

Organised by
Jean-Pierre Demailly, Grenoble
Klaus Hulek, Hannover
Thomas Peternell, Bayreuth

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Abstract. The aim of this workshop was to discuss recent developments in several complex variables and complex geometry. Special emphasis was put on the interaction of analytic and algebraic methods. Topics included Kähler geometry, Ricci-flat manifolds, moduli theory and themes related to the minimal model program.

Mathematics Subject Classification (2000): 32xx, 14xx.

Introduction by the Organisers

The meeting *Komplexe Analysis* attracted 52 mathematicians from 10 countries. It was the aim of the conference to cover a wide spectrum, thus enabling in particular the younger mathematicians to get an overview of the most recent important developments in the subject.

The main topics were

- Kähler geometry. Presentations were delivered by Dinh, Eyssidieux and Paun, covering complex dynamics, Kähler-Einstein metrics and Kähler manifolds with nef anticanonical classes.
- holomorphic symplectic varieties; covered by talks of Hwang, Kirschner and Sarti. The main themes were Lagrangian tori, singular symplectic varieties and automorphisms.
- moduli spaces (of curves, abelian varieties and sheaves). Talks were given by Casalaina-Martin, Farkas, Grushevsky, Möller, Toma and Wandel, dealing with various features of moduli spaces.

  Calabi-Yau varieties were considered in talks by Diverio and Laza.

Various other talks covered a broad spectrum of questions in complex geometry. Some of them were of algebraic nature such as Mukai's talk on Enriques surfaces, of Gongyo around the minimal model program and of Graf on the Bogomolov-Sommese vanishing theorem. Of more analytic nature were the talks of Klingler relating topology and symmetric differentials, and of Verbitsky on non-Kähler twistor spaces. Contributions with a geometric focus were presented by Pereira on foliations of uniruled manifolds, and by Catanese on uniformization.

Finally Grivaux gave an account of the Grothendieck-Riemann-Roch theorem on complex manifolds and Huckleberry presented results on the hyperbolicity of cycle spaces.

Workshop: Komplexe Analysis

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Abstracts

Hyperbolicity of cycle spaces and automorphism groups of flag domains

Alan Huckleberry

Our goal here is to explain the results in [H], in particular the following fact.

Theorem 1. *The cycle space $\mathcal{C}_q(D)$ of a flag domain D is Kobayashi hyperbolic.*

One application, which is stated precisely below, is the complete description of the connected component $\mathrm{Aut}^0(D)$ of the group of holomorphic automorphisms of the flag domain.

1. Background

Let G be a connected, complex Lie group. A compact complex G-homogeneous space $Z = G/P$ which can be realized as G-orbit in some projective space is referred to as a *flag manifold*. We assume that G acts (almost) effectively on Z and consequently that G is semisimple. The study of such manifolds, e.g., from the point of view of representation theory, requires algebraic group methods and considerations of combinatorial geometry. Borel subgroups B, which by definition are maximal connected solvable subgroups, play a key role in this study of such manifolds. For example, such a subgroup has only finitely many orbits in Z and each such orbit $\mathcal{O} = B.z$ is algebraically equivalent to some affine space $\mathbb{C}^{m(\mathcal{O}}$. In particular, the closures S, *Schubert varieties*, of these *Schubert cells* freely generate the homology of Z. We explain below how certain special Schubert varieties play a role in the proof of the above theorem.

A *real form* G_0 of G is the connected component of the fixed point locus of an involutive antiholomorphic automorphism $\tau : G \to G$. For example, the mixed signature unitary and orthogonal groups, $\mathrm{SU}(p, q)$ and $\mathrm{SO}(p, q)$, are examples of such groups. As a subgroup of G, a real form G_0 acts on every G-flag manifold Z and it is a basic fact that it has only finitely many orbits. In particular, G_0 has at least one (and usually many) open orbit. We denote such an open orbit by D and refer to it as a *flag domain*.

We consider here non-compact real forms G_0 with (unique up to conjugation) maximal compact subgroups K_0. Among the basic facts proved in [W] it is shown that in every flag domain K_0 has exactly one orbit which is a complex submanifold. We view this as a point in the cycle space $\mathcal{C}_q(Z)$ of q-dimensional cycles. It is known that $\mathcal{C}_q(Z)$ is smooth at C_0 (see Part III of [FHW]) and therefore it makes sense to define $\mathcal{C}_q(D)$ as the irreducible component which contains C_0 of the full cycle space of D. This cycle space is not to be confused with the $\mathcal{M}_D$ which is the connected component containing C_0 of the intersection of the orbit $G.C_0$ with $\mathcal{C}_q(D)$ and which has been thoroughly studied (see, e.g., [FHW]). Remarkably, with very few exceptions $\mathcal{M}_D$ only depends on G_0, i.e., neither on D nor Z. On the other hand, the full cycle component $\mathcal{C}_q(D)$ is strongly dependent on both D and Z.

2. Two methods for the proof

The first main method used for the proof of the above theorem is to use special Schubert varieties to parameterize the cycles in $\mathcal{C}_q(D)$. Given K_0 and an Iwasawa-decomposition $G_0 = K_0A_0N_0$ we consider Schubert varieties S of a Borel subgroup B which contains A_0N_0 and which are of dimension $n-q$. A strong duality theorem can be proved: If S is the closure of $\mathcal{O}$ with $S = \mathcal{O}\dot{\cup}E$, then either S has empty intersection with D or S has finite transversal intersection with every $C \in \mathcal{C}_q(D)$. In the latter case E is contained in the complement of D and therefore the incidence hypersurface $H_S := \{C \in \mathcal{C}_q(Z) : E \cap C \neq \emptyset\}$ is in the complement of $\mathcal{C}_q(D)$ in $\mathcal{C}_q(Z)$.

The second method amounts to showing that if enough of the above mentioned incidence hypersurfaces are used (they are $\mathbb{Q}$-Cartier), one can build a G_0-bundle L with the associated projective hyperplanes being realized in the complement of the image of $\mathcal{C}_q(D)$ (the restricted map is finite) being contained in the complement. Moving these hyperplanes with G_0, we obtain sufficiently many complementary hyperplanes to prove the hyperbolicity.

3. Description of the automorphism group

If D is pseudoconvex, which in the case of flag domains is equivalent to it being holomorphically convex, then the Remmert reduction $D \to D_{red}$ has base which is a Hermitian symmetric space of noncompact type. Since such symmetric spaces are contractible Stein manifolds, this fibration is the trivial bundle. Unless $D = D_{red}$, in which case the automorphism group of D is well-understood, $\mathrm{Aut}^0(D)$ is of course infinite-dimensional. Nevertheless it can be easily described. If D is not pseudoconvex, then the following is the key result for the description of $\widehat{G}_0 := \mathrm{Aut}^0(D)$.

Theorem 2. *If D is not pseudoconvex, then the induced action of $\widehat{G}_0$ on $\mathcal{C}_q(D)$ is effective.*

The hyperbolicity of $\mathcal{C}_q(D)$ implies that its automorphism group is a Lie group acting properly in the compact-open topology. One shows that the imbedding of $\widehat{G}_0$ guaranteed by the above theorem realizes it as a closed subgroup of $\mathrm{Aut}\mathcal{C}_q(D)$ and with a bit more work one proves the following.

Theorem 3. *The group $\mathrm{Aut}(D)$ is a Lie group acting properly on D in the compact-open topology.*

In fact one can show that the action of connected component $\widehat{G}_0$ on D can be extended to Z so that, with the exception of a few examples which occur in the classification of Onishchik, $\widehat{G}_0 = G_0$.

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Singular Kähler-Einstein metrics on Fano varieties

Philippe Eyssidieux

(joint work with Robert Berman, Sébastien Boucksom, Vincent Guedj)

The talk was a report on [2]. Kähler-Einstein metrics on pairs (X, D) with klt singularities such that $K_X + D > 0$ or $K_X + D = 0$ have been constructed in [4] and the present work lays the fundations of the study of their counterpart on log-Fano pairs, generalizing to the singular setting a part of the basic theory in the smooth case as presented in [7].

Definition 1. Let Y be a normal complex projective variety, $\dim(Y) = n$, and $D = \sum_i a_i E_i$ be an effective $\mathbb{Q}$-divisor. Say (Y, D) is log-Fano if:

(1) $(K_Y + D)$ is $\mathbb{Q}$-Cartier,
(2) (Y, D) is klt,
(3) $-(K_Y + D)$ is ample.

Let (Y, D) be log-Fano. Fix a smooth metric h on the $\mathbb{Q}$-line bundle $L = K_Y + D$. Then h defines canonical a volume form of Y^{reg} which has finite volume hence extends trivially to a Radon measure $v(h)$ on Y.

A plurisubharmonic metric $\Phi = e^{-\phi}h$ on $-(K_Y + D)$ (i.e.: $\phi - \log |s|_h^2$ is plurisubharmonic for all s a holomorphic multisection of $-(K_Y + D)$) defines a Radon measure on Y by the prescription $m_\Phi \overset{not.}{=} e^{-\Phi} = e^{-\phi}v(h)$ and also a Monge-Ampère probability measure $MA(\Phi) = (dd^c\Phi)^n/V$.

Definition 2. The plurisubharmonic metric Φ is Kähler-Einstein if $MA(\Phi) = e^{-\Phi}/\int_Y e^{-\Phi}$.

The usual regularity theory of degenerate Monge-Ampère equations yields:

Theorem 1. *If Φ is Kähler-Einstein, then $\Phi \in C^0(Y) \cap C^\infty(Y^{reg} - D)$.*

Hence the restriction of $dd^c\Phi$ to $Y^{reg} - D$ is a Kähler Einstein metric in the usual sense.

Next, we generalize in the singular setting the classical Ding-Tian and Mabuchi functionals to plurisubharmonic metrics on $-(K_Y + D)$ so that the classical variational characterisation holds:

Theorem 2. *The infimum of the Ding-Tian and Mabuchi functionals coincide. Moreover, the following are equivalent:*

- Φ *minimizes the Mabuchi functional,*
- Φ *minimizes the Ding-Tian functional,*
- Φ *is Kähler-Einstein.*

Using this, we prove (using a method of Berman-Berndtsson in the smooth case) a Bando-Mabuchi theorem:

Theorem 3. *Kähler-Einstein metrics on the log-Fano pair* (Y, D) *form a single* $Aut^0(Y, D)$ *orbit.*

A compactness theorem is formulated and used to generalize a theorem of Tian claiming that a Kähler-Einstein metric exists provided the Ding-Tian (or Mabuchi) functional tends to infinity when the Aubin-Yau energy does. This enables, using a nice trick due to [1] in the smooth case, to produce examples with non-quotient singularities, such as a double covering of $\mathbb{P}^n$, $n \geq 3$, ramified over a degree d hypersurface with only lc singularities (e.g.: nodes) provided d is even and $n + 2 \leq d \leq 2n + 1$.

The theory also yields a generalization of the convergence of Ricci iteration [5, 6] and of the weak convergence of the Kähler-Ricci flow along the lines of Berman's simplification of this consequence of Perelman's estimates.

To conclude, it is appropriate to quote the work [3] which enables to realize these singular Kähler Einstein metrics on Fano varieties as Gromov Hausdorff limits of smooth Fano Kähler-Einstein metrics.

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Log canonical models and variation of GIT for genus four canonical curves

Sebastian Casalaina-Martin

(joint work with David Jensen, Radu Laza)

The Hassett–Keel program aims to give modular interpretations of certain log canonical models of $\overline{M}_g$, the moduli space of stable curves of fixed genus g, with the ultimate goal of giving a modular interpretation of the canonical model for large g. Recall that for $\alpha \in [0,1]$ the log canonical models of $\overline{M}_g$ considered in the program are the projective varieties

$$\overline{M}_g(\alpha) := \mathbb{P}\left(\bigoplus_{n=0}^{\infty} H^0\left(n(K_{\overline{M}_g} + \alpha\delta)\right)\right),$$

where δ is the boundary divisor in $\overline{M}_g$. The program, while relatively new, has attracted the attention of a number of researchers, and we point out that Hassett and Hyeon have explicitly constructed the spaces $\overline{M}_g(\alpha)$ for $\alpha \geq \frac{7}{10} - \epsilon$ (see Hassett–Hyeon [9, 8]).

For large genus, completing the program in its entirety still seems somewhat out of reach. On the other hand, the case of low genus curves affords a gateway to the general case, providing motivation and corroboration of expected behavior. The genus 2 and 3 cases were completed recently (Hassett [7], Hyeon–Lee [11]).

In genus four, the known spaces in the program are (see Hassett–Hyeon [9, 8], and Hyeon–Lee [10]):

$$\begin{tikzcd}
\overline{M}_4 = \overline{M}_4[1,\tfrac{9}{11}) \arrow[d, Rightarrow] & & & \\
\overline{M}_4^{ps} = \overline{M}_4[\tfrac{9}{11},\tfrac{7}{10}) \arrow[rr, dashed] \arrow[dr] & & \overline{M}_4^{hs} = \overline{M}_4(\tfrac{7}{10},\tfrac{2}{3}) \arrow[dl] \arrow[dr] & \\
& \overline{M}_4^{cs} = \overline{M}_4(\tfrac{7}{10}) & & \overline{M}_4(\tfrac{2}{3})
\end{tikzcd} \tag{1}$$

where the notation $\overline{M}_g(I)$ for an interval I means $\overline{M}_g(\alpha) \cong \overline{M}_g(\beta)$ for all $\alpha, \beta \in I$. The double arrows correspond to divisorial contractions, the single arrows to small contractions, and the dashed arrows to flips. Fedorchuk recently constructed the final space in the program $\overline{M}_4(\frac{8}{17} + \epsilon)$ via GIT for $(3,3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ [6].

In recent work with Jensen and Laza [5], we have further investigated the remaining steps in the program in genus 4. The results are obtained by a variation of GIT construction on a projective bundle $\mathbb{P}E$ parameterizing intersections of quadrics and cubics in $\mathbb{P}^3$. This completes the program in genus 4 outside of the interval $\alpha \in (\frac{5}{9}, \frac{2}{3})$, where using the predictions of Alper–Fedorchuk–Smyth [2], we expect that there are exactly two more critical values: $\alpha = \frac{19}{29}$ and $\alpha = \frac{49}{83}$.

Theorem 1 (Casalain-Martin–Jensen–Laza [5]). *For $\alpha \leq \frac{5}{9}$, the log minimal models $\overline{M}_4(\alpha)$ arise as GIT quotients of the parameter space $\mathbb{P}E$. Moreover, the VGIT problem gives us the following diagram:*

(2)

$$\begin{tikzcd}[column sep=small]
 & \overline{M}_4(\frac{5}{9},\frac{23}{44}) \arrow[dl] \arrow[dr] \arrow[rr, dashed] & & \overline{M}_4(\frac{23}{44},\frac{1}{2}) \arrow[dl] \arrow[dr] \arrow[rr, dashed] & & \overline{M}_4(\frac{1}{2},\frac{29}{60}) \arrow[dl] \arrow[dr, Rightarrow] & \\
\overline{M}_4(\frac{5}{9}) & & \overline{M}_4(\frac{23}{44}) & & \overline{M}_4(\frac{1}{2}) & & \overline{M}_4[\frac{29}{60},\frac{8}{17}) \arrow[d] \\
 & & & & & & \overline{M}_4(\frac{8}{17}) = \{*\}
\end{tikzcd}$$

More specifically,

i) *the end point* $\overline{M}_4(\frac{8}{17}+\epsilon)$ *is obtained via GIT for* $(3,3)$ *curves on* $\mathbb{P}^1 \times \mathbb{P}^1$ *as discussed in Fedorchuk* [6]*;*

ii) *the other end point* $\overline{M}_4(\frac{5}{9})$ *is obtained via GIT for the Chow variety of genus 4 canonical curves (discussed further below);*

iii) *the remaining spaces* $\overline{M}_4(\alpha)$ *for* α *in the range* $\frac{8}{17} < \alpha < \frac{5}{9}$ *are obtained via appropriate* $\mathrm{Hilb}^m_{4,1}$ *quotients, with the exception of* $\alpha = \frac{23}{44}$*.*

In low genus there is an interesting connection between some of the spaces arising in this program, and ball quotient compactifications arising from work of Kondo [12, 13]. Recall that while arithmetic quotients $\mathcal{D}/\Gamma$ admit Bailly–Borel compactifications $(\mathcal{D}/\Gamma)^*$, in general, it is difficult to determine what geometric objects should correspond to the boundary points. For ball quotients and quotients of Type IV domains, a now standard approach to this type of problem is to use a comparison with a moduli space constructed via GIT, and there is a well developed theory that covers this (see Looijenga [14, 15] and Looijenga–Swierstra [17]).

Returning to the case of curves, Kondo [12, 13] has constructed ball quotient compactifications $(\mathcal{B}_6/\Gamma_6)^*$ and $(\mathcal{B}_9/\Gamma_9)^*$ of the moduli space of non-hyperelliptic genus three and four curves respectively. In the case of genus 3 curves, where the space of plane quartics provides a natural GIT compactification, the problem of ascribing geometric meaning to the boundary points was completed by Looijenga [16] and Artebani [3].

In joint work with Jensen and Laza, we discuss the relationship between the space $(\mathcal{B}_9/\Gamma_9)^*$ and a GIT model of $\overline{M}_4$. To be precise, we construct a GIT quotient $\overline{M}_4^{GIT}$ of canonically embedded genus four curves via a related GIT problem for cubic threefolds. Results for cubic threefolds due to Allcock [1] allow us to completely describe the stability conditions for $\overline{M}_4^{GIT}$. With this, we can employ general results of Looijenga [14] to give an explicit resolution of the period map $\overline{M}_4^{GIT} \dashrightarrow (\mathcal{B}_9/\Gamma)^*$. The main result is the following:

Theorem 2 (Casalaina-Martin–Jensen–Laza [4])**.** *The period map* $\overline{M}_4^{GIT} \dashrightarrow (\mathcal{B}_9/\Gamma)^*$ *is resolved by blowing up a single point, which corresponds to a genus four ribbon curve in* $\mathbb{P}^3$*. The GIT quotient* $\overline{M}_4^{GIT}$ *is isomorphic to a GIT quotient of a Chow variety of canonically embedded genus four curves, as well as to the space* $\overline{M}_4(5/9)$ *in the Hassett–Keel program.*

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The Grothendieck-Riemann-Roch theorem for complex manifolds

Julien Grivaux

The Grothendieck-Riemann-Roch (GRR) theorem for abstract complex manifolds has been the object of many researches since the end of the sixties. These can be divided in three main classes :

1. Holomorphic GRR theorem (Hodge cohomology)

This project has been developed by O'Brian, Toledo and Tong, starting from the index theorem [14], [12], and ending ten years later with a proof of the GRR theorem in Hodge cohomology for arbitrary proper morphisms [11]. In 1992, Kashiwara initiated a strategy to give a conceptual proof of this result (see [9, Chap. 5] and [6]). The central tool in his approach is the use of analytic Hochschild homology, and more precisely the construction of a dual Hochschild-Kostant-Rosenberg isomorphism. Kashiwara's strategy has been completely carried out in [7].

2. Topological GRR theorem (De Rham cohomology)

The first step in this direction is Atiyah-Hirzebruch's result for immersions [1]. Much later, Levy [10] proved the GRR theorem in De Rham cohomology in full generality. This proof is very technical and has remained mostly unknown even among specialists, and no other approach of this problem has been found till now.

3. Metric GRR theorem (Bott-Chern cohomology)

The explicit computation of curvature forms for direct images of hermitian holomorphic vector bundles started with Quillen's fundamental paper [13], and was carried out for locally-Kähler fibrations in [3] and [4]. The Kählerianity assumption has recently been removed in [2], this yields the GRR theorem in Bott-Chern cohomology provided that the derived direct images of the bundle are locally free.

One of the finest known cohomology theory for abstract complex manifolds in which Chern classes can be constructed is Deligne-Beĭlinson cohomology. The Grothendieck-Riemann-Roch theorem in this cohomology remains an open question for general proper holomorphic morphisms between complex manifolds. For projective morphisms, the result has been proved in [5]. In [8], it is shown that the degree two component of the GRR formula in Deligne cohomology is invariant by deformation. This is a first step towards the understanding of the GRR theorem in this setting. The result has applications in the theory of non-Kählerian complex surfaces.

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Singularities of theta divisors and the geometry of $\mathcal{A}_5$

Gavril Farkas

This is a report on joint work with Sam Grushevsky, Riccardo Salvati-Manni and Alessandro Verra. We recall that theta divisor Θ of a generic principally polarized abelian variety (ppav) is smooth. The ppav (A, Θ) with a singular theta divisor form the Andreotti-Mayer divisor N_0 in the moduli space $\mathcal{A}_g$. The divisor N_0 has two irreducible components, denoted $\theta_{\rm null}$ and N_0': here $\theta_{\rm null}$ denotes the locus of ppav for which the theta divisor has a singularity at a two-torsion point, and N_0' is the closure of the locus of ppav for which the theta divisor has a singularity not at a two-torsion point. The theta divisor Θ of a generic ppav $(A, \Theta) \in \theta_{\rm null}$ has a unique singular point, which is a double point. Similarly, the theta divisor of a generic element of N_0' has two distinct double singular points x and $-x$. One can naturally assign multiplicities to both components of N_0 such the following equality of cycles holds:

$$N_0 = \theta_{\rm null} + 2N_0'. \tag{1}$$

Generically for both components the double point is an ordinary double point (that is, the quadratic tangent cone to the theta divisor at such a point has maximal rank g — equivalently, the Hessian matrix of the theta function at such a point is non-degenerate). Considering the sublocus $\theta_{\rm null}^{g-1} \subset \theta_{\rm null}$ parameterizing ppav (A, Θ) with a singularity at a two-torsion point, that is not an ordinary double point of Θ, it can be proved that

$$\theta_{\rm null}^{g-1} \subset \theta_{\rm null} \cap N_0'. \tag{2}$$

It is natural to investigate the non-ordinary double points on the other component N_0' of the Andreotti-Mayer divisor. Similarly to $\theta_{\rm null}^{g-1}$, we define $N_0'^{g-1}$, or, to simplify notation, H, to be the closure in N_0' of the locus of ppav whose theta divisor has a non-ordinary double point singularity. Thus we consider the cycle

$$N_0^{g-1} := \theta_{\rm null}^{g-1} + 2N_0'^{g-1} = \theta_{\rm null}^{g-1} + 2H. \tag{3}$$

To understand the geometric situation, especially in low genus, we compute the class:

Theorem 1. *The class of the cycle H inside $\mathcal{A}_g$ is equal to*

$$[H] = \left(\frac{g!}{16}(g^3 + 7g^2 + 18g + 24) - (g+4)2^{g-4}(2^g + 1) \right) \lambda_1^2 \in CH^2(\mathcal{A}_g).$$

As usual, $\lambda_1 := c_1(\mathbb{E})$ denotes the first Chern class of the Hodge bundle and CH^i denotes the $\mathbf{Q}$-vector space parameterizing algebraic cycles of codimension i with rational coefficients modulo rational equivalence. Comparing classes and considering the cycle-theoretic inclusion $3\theta_{\mathrm{null}}^3 \subset H$, we get the following result, see Section 4 for details:

Theorem 2. *In genus* 4 *we have the set-theoretic equality $\theta_{\mathrm{null}}^3 = H$.*

We then turn to genus 5 with the aim of obtaining a geometric description of $H \subset \mathcal{A}_5$ via the dominant Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$. A key role in the study of the Prym map is played by its branch divisor, which in this case equals $N_0' \subset \mathcal{A}_5$, and its ramification divisor $\mathcal{Q} \subset \mathcal{R}_6$. We introduce the *antiramification* divisor $\mathcal{U} \subset \mathcal{R}_6$ defined cycle-theoretically by the equality

$$P^*(N_0') = 2\mathcal{Q} + \mathcal{U}.$$

Using the geometry of the Prym map, we describe both $\mathcal{Q}$ and $\mathcal{U}$ explicitly in terms of Prym-Brill-Noether theory. Our result is the following:

Theorem 3. *The ramification divisor $\mathcal{Q}$ of the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ equals the Prym-Brill-Noether divisor in $\mathcal{R}_6$, that is,*

$$\mathcal{Q} = \{(C,\eta) \in \mathcal{R}_6 : V_3(C,\eta) \neq 0\}.$$

The antiramification divisor is the pull-back of the Gieseker-Petri divisor from $\mathcal{M}_6$, that is, $\mathcal{U} = \pi^*(\mathcal{GP}_{6,4}^1)$. The divisor $\mathcal{Q}$ is irreducible and reduced.

The rich geometry of $\mathcal{Q}$ enables us to (i) compute the classes of the closures $\overline{\mathcal{Q}}$ and $\overline{\mathcal{U}}$ inside the Deligne-Mumford compactification $\overline{\mathcal{R}}_6$, then (ii) determine explicit codimension two cycles in $\mathcal{R}_6$ that dominate the irreducible components of H. In this way we find a complete geometric characterization of 5-dimensional ppav whose theta divisor has a non-ordinary double point. First we characterize θ_{null}^4 as the image under P of a certain component of the intersection $\mathcal{Q} \cap P^*(\theta_{\mathrm{null}})$:

Theorem 4. *A ppav $(A,\Theta) \in \mathcal{A}_5$ belongs to θ_{null}^4 if an only if it is lies in the closure of the locus of Prym varieties $P(C,\eta)$, where $(C,\eta) \in \mathcal{R}_6$ is a curve with two vanishing theta characteristics θ_1 and θ_2, such that*

$$\eta = \theta_1 \otimes \theta_2^\vee.$$

Furthermore, θ_{null}^4 is unirational and $[\theta_{\mathrm{null}}^4] = 27 \cdot 44\, \lambda_1^2 \in CH^2(\mathcal{A}_5)$.

Observing that $[H] \neq [\theta_{\mathrm{null}}^4]$ in $CH^2(\mathcal{A}_5)$, the locus H must have extra irreducible components corresponding to ppav with a non-ordinary singularity that occurs generically not at a two-torsion point. We denote by $H_1 \subset \mathcal{A}_5$ the union of these components, so that at the level of cycles

$$H = \theta_{\mathrm{null}}^4 + H_1,$$

where $[H_1] = 27 \cdot 49\, \lambda_1^2$. We have the following characterization of H_1:

Theorem 5. *The locus H_1 is unirational and its general point corresponds to a Prym variety $P(C, \eta)$, where $(C, \eta) \in \mathcal{R}_6$ is a Prym curve such that $\eta \in W_4(C) - W_4^1(C)$ and $K_C \otimes \eta$ is very ample.*

As an application of this circle of ideas, we determine the slope of $\overline{\mathcal{A}}_5$. Let $\overline{\mathcal{A}}_g$ be the perfect cone (first Voronoi) compactification of $\mathcal{A}_g$. The slope of an effective divisor $E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)$ is defined as the quantity

$$s(E) := \inf \left\{ \frac{a}{b} : a, b > 0,\ a\lambda_1 - b[D] - [E] = c[D],\ c > 0 \right\},$$

where $D := \overline{\mathcal{A}}_g - \mathcal{A}_g$ is the boundary. If E is an effective divisor on $\overline{\mathcal{A}}_g$ with no component supported on the boundary and $[E] = a\lambda_1 - bD$, then $s(E) := \frac{a}{b} \geq 0$. One then defines the slope (of the effective cone) of the moduli space as $s(\overline{\mathcal{A}}_g) := \inf_{E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)} s(E)$. For $g = 5$, the class of the closure of the Andreotti-Mayer divisor is $[\overline{N_0'}] = 108\lambda_1 - 14D$, giving the upper bound $s(\overline{\mathcal{A}}_5) \leq \frac{54}{7}$. Our result is that this divisor actually determines the slope of the moduli space:

Theorem 6. *The slope of $\overline{\mathcal{A}}_5$ is computed by $\overline{N_0'}$, that is, $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, $\kappa(\overline{\mathcal{A}}_5, \overline{N_0'}) = 0$, that is, the only effective divisors on $\overline{\mathcal{A}}_5$ having minimal slope are the multiples of $\overline{N_0'}$.*

Density of positive closed currents and dynamics of Hénon maps

Tien-Cuong Dinh
(joint work with Nessim Sibony)

We introduce a new method to prove equidistribution properties in complex dynamics of several variables. We obtain the equidistribution for saddle periodic points of Hénon-type maps on $\mathbf{C}^k$. A key point of the method is a notion of density which extends both the notion of Lelong number and the theory of intersection for positive closed currents on Kähler manifolds. This is a joint work with Nessim Sibony.

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Rational curves on Calabi-Yau threefolds and a conjecture of Oguiso

Simone Diverio

Let X be a compact projective manifold over $\mathbb{C}$, ω a Kähler metric on X, and consider the following statements:

(1) The holomorphic sectional curvature of ω is strictly negative.
(2) The manifold X has non-degenerate negative k-jet curvature.
(3) The manifold X is Kobayashi hyperbolic.
(4) The manifold X is measure hyperbolic.
(5) The manifold X is of general type.
(6) The manifold X does not contain any rational curve.
(7) The canonical bundle K_X of X is nef.
(8) The canonical bundle K_X of X is ample.

Perhaps only (2) needs some more explanations, which will be given subsequently. We have the following diagram of conjectural and actual implications:

$$\begin{tikzcd}
(7) & (4) \arrow[r, bend right, dashed, "?"'] & (5) \arrow[l, bend right] \arrow[r, bend right, dashed, "(6)"'] & (8) \arrow[lll, bend right] \arrow[l, bend right] \\
(6) \arrow[u] & (3) \arrow[l] \arrow[u] \arrow[ur, dashed, "?"] \arrow[r, bend right, dashed, "?"'] & (2) \arrow[l, bend right] \arrow[u, dashed, "?"] & (1) \arrow[l] \arrow[u, dashed, "?"]
\end{tikzcd}$$

As the diagram shows, the central conjecture here is the equivalence between (4) and (5) (*i.e.* that measure hyperbolic implies general type), which is known to hold true whenever X is a projective surface. In spite of this, in the sequel we will mostly concentrate ourselves on the conjecture *a latere*, known as Kobayashi's conjecture, which states that (3) should imply (5) (and hence (8)). We shall give some hints and recent results both from a differential-geometric and algebro-geometric viewpoints; moreover, we shall fix our attention on threefolds, since it is the first unknown case.

To begin with, observe that several powerful machineries from birational geometry —such as the characterization of uniruledness in terms of negativity of the Kodaira dimension, the Iitaka fibration, the abundance conjecture (which is actually a theorem in dimension three)— permit to reduce this conjecture to the following statement: a projective threefold X of Kodaira dimension $\kappa(X) = 0$ cannot be hyperbolic. By the Beauville-Bogomolov decomposition theorem and elementary properties of hyperbolic manifolds, in dimension three *it suffices to show that a Calabi-Yau threefold is not hyperbolic.* Here, by a Calabi-Yau threefold we mean a simply connected compact projective threefold with trivial canonical class $K_X \simeq \mathcal{O}_X$ and $h^i(X, \mathcal{O}_X) = 0$, $i = 1, 2$.

Differential-geometric viewpoint. A weaker form of (3) $\Rightarrow$ (5) and (8), more differential-geometric in flavor, is to show that negative holomorphic sectional curvature implies ampleness of the canonical bundle. This is known up to dimension three, by the work of [4]. Again, the core here is to show that the negativity of the holomorphic sectional curvature of (X, ω) forces the real first Chern class $c_1(X)_{\mathbb{R}}$ to be non-zero.

A possible proof of this fact goes as follows. First, observe that an averaging argument shows that negative holomorphic sectional curvature implies negative scalar curvature. Now, suppose that $c_1(X)_{\mathbb{R}} = 0$. Then, there exists a smooth function $f\colon X \to \mathbb{R}$ such that $\operatorname{Ricci}(\omega) = i\partial\overline{\partial} f$. Now, take the traces with respect to ω of both sides: modulo non-zero multiplicative constants, on the left we find the scalar curvature and on the right the ω-Laplacian of f which must therefore be always non-zero. Hence, f is constant and the scalar curvature must be zero, contradiction.

Unfortunately, having negative holomorphic sectional curvature is much stronger than being hyperbolic. In [1], it is conjectured that a weaker notion of negative curvature, namely non-degenerate negative k-jet curvature, should be instead equivalent (and it is proved there that it actually implies hyperbolicity). Let us explain briefly what this notion is, referring to [1] for more details.

Let $J_k X \to X$ be the holomorphic fiber bundle of k-jets of germs of holomorphic curves $\gamma\colon (\mathbb{C}, 0) \to X$ and $J_k X^{\mathrm{reg}}$ its subset of regular ones, *i.e.* such that $\gamma'(0) \neq 0$. There is a natural action of the group $\mathbb{G}_k$ of k-jets of biholomorphisms of $(\mathbb{C}, 0)$ on $J_k X$, and the quotient $J_k X^{\mathrm{reg}}/\mathbb{G}_k$ admits a nice geometric relative compactification $J_k X^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow X_k$. Here, X_k is a tower of projective bundles over X. In particular, it is naturally endowed with a tautological line bundle $\mathcal{O}_{X_k}(-1)$, as well as a holomorphic subbundle $V_k \subset T_{X_k}$ of its tangent bundle.

Definition. The manifold X is said to have *non-degenerate negative k-jet curvature* if there exists a singular hermitian metric on $\mathcal{O}_{X_k}(-1)$ whose Chern curvature current is negative definite along V_k and whose degeneration set is contained in $X_k \setminus (J_k X^{\mathrm{reg}}/\mathbb{G}_k)$.

Observe that if X has negative holomorphic sectional curvature, then it naturally has a non-degenerate negative 1-jet curvature. The following question seems therefore particularly appropriate.

Question. *Is it true that if X has non-degenerate negative k-jet curvature then $c_1(X)_{\mathbb{R}} \neq 0$?*

Algebro-geometric viewpoint. Algebraic geometers expect more than non-hyperbolicity of Calabi-Yau's: a *folklore* conjecture states that every Calabi-Yau manifold should contain a rational curve. For threefolds, let us cite a couple of results in this direction:

- The article [3] is the culmination of a series of papers by Wilson in which he studies in a systematic way the geometry of Calabi-Yau threefolds; among many other things, it is shown there that if the Picard number $\rho(X) > 13$, then there always exists a rational curve on X.
- Following somehow the same circle of ideas, it was proven in [6] (see also [5]) that a Calabi-Yau threefold X has a rational curve provided there exists on X a non-zero effective non-ample line bundle on X.

By the Cone Theorem, if there exists on a Calabi-Yau manifold X a non-zero effective non-nef line bundle, then there exists on X a rational curve (generating an extremal ray). Therefore, we can always suppose that such an effective line bundle is nef. Remark, on the other hand, that in Peternell's result, the effectivity hypothesis is crucial (regarding it in a more modern way) in order to make the machinery of the logMMP work. In this spirit, Oguiso asked in [5] the following question: *is it true that if a Calabi-Yau threefold X possesses a non-zero nef non-ample line bundle, then there exists a rational curve on X?*

Here is a positive answer, under a mild condition on the Picard number of X.

Theorem (Diverio, Ferretti [2]). *Let X be a Calabi-Yau threefold and $L \to X$ a non-zero nef non-ample line bundle. Then, X has a rational curve provided $\rho(X) > 4$.*

In order to give a rough idea of the techniques involved in this kind of business, let us state (a special case of) the Kawamata-Morrison Cone conjecture and explain how it would almost imply the Kobayashi conjecture.

Conjecture (Kawamata-Morrison). *Let X be a Calabi-Yau manifold. Then, the action of* $\mathrm{Aut}(X)$ *on the nef-effective cone of X has a rational polyhedral fundamental domain.*

Proposition. *Suppose that the Kawamata-Morrison conjecture holds. Then, the Kobayashi conjecture is true in dimension three, except possibly if there exists a Calabi-Yau threefold of Picard number one which is hyperbolic.*

Proof. We shall suppose that the Kawamata-Morrison conjecture holds true and that there exists a hyperbolic Calabi-Yau threefold X with $\rho(X) \geq 2$ and derive a contradiction.

Since X is supposed to be hyperbolic, it does not contain any rational curve and $\mathrm{Aut}(X)$ is finite. The Kawamata-Morrison conjecture implies therefore that the nef cone of X is rational polyhedral.

Now, since it is rational polyhedral, rational points are dense on each face of the nef boundary. Moreover, at most one of these faces (which are at least in number of $\rho(X) \geq 2$) can be contained in the hyperplane given by $(c_2(X) \cdot D) = 0$: in fact this is a "true" hyperplane since if $c_1(X) = c_2(X) = 0$, then X would be a finite étale quotient of a complex torus, so that X would not be hyperbolic (nor a Calabi-Yau manifold in our strict sense).

Therefore, there exists on X a (in fact plenty of) nef $\mathbb{Q}$-divisor D such that $c_2(X) \cdot D > 0$. Computing its Euler characteristic and using Kawamata–Viehweg vanishing, D can be shown to be effective. But then, since there exists on X a non-zero effective non-ample divisor, there exists a rational curve on X, contradicting its hyperbolicity. ∎

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Symmetric differentials and the fundamental group

Bruno Klingler
(joint work with Yohan Brunebarbe, Burt Totaro)

We discussed the following result proven in [1]: let X be a compact connected Kähler manifold with no global holomorphic symmetric differentials. Then any finite dimensional representation of $\pi_1(X)$ over any field has finite image.

Notice that the class of such manifolds is large, including projective varieties of general type.

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Cusps of plane curves, a conjecture of Chisini, and meromorphic differentials with real periods

Samuel Grushevsky
(joint work with Igor Krichever)

In this talk we presented our joint work in progress with Igor Krichever, on bounding the number of cusps of plane curves using real-normalized differentials. The detailed results will appear as [GK12].

Real-normalized differentials are meromorphic differentials on Riemann surfaces, with no residues, and with all periods real. In [GK09] we applied real-normalized differentials to obtain a new proof of Diaz' theorem on dimension of complete subvarieties of the moduli space of curves $\mathcal{M}_g$; in [GK11] we applied them to prove the vanishing of certain tautological homology classes, predicted by Faber's conjecture. In [Kr11] Krichever used real-normalized differentials to prove Arbarello's conjecture on complete subvarieties of $\mathcal{M}_g$, and in [Kr12] Krichever used real-normalized differentials to obtain a strong upper bound on dimension of complete subvarieties of the moduli space of stable curves of compact type.

The main result of our work will be as follows.

Theorem. *There exists an explicit function $N(g,d)$ such that for any g sufficiently large with respect to d, for any smooth genus g curve with d marked points, $(X,p_1,\ldots,p_d)\in\mathcal{M}_{g,d}$, the number of common zeroes of any pair of real-normalized differentials on X, with double poles at each p_i with no residue (and with $\mathbb{R}$-linearly independent singular parts at each p_i) is at most $N(g,d)$.*

This theorem has an application to bounding the number of cusps of plane curves.

Corollary. *There exists an explicit function $M(d)$ such that a degree d plane curve $C\subset\mathbb{P}^2$ that has only nodes and simple cusps as singularities has at most $M(d)$ cusps.*

It appears that the bound $M(d)$ we obtain above will be better than the previously known upper bounds for the number of cusps of plane curves.

To obtain the corollary from the theorem, note that the two coordinate differentials dx and dy, for the non-homogeneous coordinates on $\mathbb{P}^2$, when restricted to C, vanish simultaneously precisely at the cusps, and have double poles with no nodes at the d intersection points of C with the line at infinity. Suppose C has k cusps and n nodes; then its normalization $\tilde{C}$ has genus $(d-1)(d-2)/2-k-n$, and the pullbacks of dx and dy to $\tilde{C}$ give two meromorphic differentials with d double poles and zero periods. The above theorem then applies to give $k\leq N((d-1)(d-2)/2-k-n,d)$, from which the corollary follows.

We prove the main theorem by degeneration. To this end, we recall that local real-analytic coordinates at any point of $\mathcal{M}_{g,d}$ are given by the set of absolute periods (integrals over elements of $H_1(X,\mathbb{Z})$) and relative periods (integrals from one zero to another) of one real-normalized differential. As explained in [GK11], this implies that the locus where the two real-normalized differentials have a prescribed number of common zeroes, if non-empty, has expected dimension. Moreover, two real-normalized differentials can have n common zeroes, then this can happen for arbitrary values of absolute periods. We further study possible degenerations: doing this for real-normalized differentials in general as opposed to only the case of plane curves allows us to exclude the case of non-separating degenerations, by requiring the period of the real-normalized differential to be non-zero over every cycle — and thus ensuring that a non-zero-homologous loop cannot be contracted. We refer to [GK12] for the details of the argument.

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Characterizations of varieties whose universal cover is a bounded symmetric domain

Fabrizio Catanese
(joint work with Antonio José Di Scala)

A central problem in the theory of complex manifolds has been the one of determining the compact complex manifolds X whose universal covering $\tilde{X}$ is biholomorphic to a bounded domain $\Omega \subset \mathbb{C}^n$.
A first important restriction is given by theorems by Siegel and Kodaira, extending to several variables a result of Poincaré, and asserting that necessarily such a manifold X is projective and has ample canonical divisor K_X.
A restriction on Ω is given by another theorem of Siegel, asserting that Ω must be holomorphically convex.
The question concerning which domains occur was partly answered by Borel who showed that, given a bounded symmetric domain $\Omega \subset \mathbb{C}^n$, there exists a properly discontinuous group $\Gamma \subset \mathrm{Aut}(\Omega)$ which acts freely on Ω and is cocompact (i.e., is such that $X =: \Omega/\Gamma$ is a compact complex manifold with universal cover $\cong \Omega$).

Consider the following question: given a bounded domain $\Omega \subset \mathbb{C}^n$, how can we tell when a projective manifold X with ample canonical divisor K_X has Ω as universal covering ?
The question was solved by Yau in the case of a ball, using the theorem of Aubin and Yau asserting the existence of Kähler Einstein metrics for varieties with ample canonical bundle. The existence of such metrics, joint to some deep knowledge of the differential geometry of bounded symmetric domains, allows to obtain more general results.
Together with Franciosi ([3]) we took up the question for the case of a polydisk, and a fully satisfactory answer was found in [1] for the special case where the bounded symmetric domain has all factors of tube type, i.e., when these are biholomorphic, via the Cayley transform, to some **tube domain**

$$\Omega = V + i\mathfrak{C},$$

where V is a real vector space and $\mathfrak{C} \subset V$ is an open self dual cone containing no lines.

Theorem 1.([1]) Let X be a compact complex manifold of dimension n. Then the following two conditions:

(1) K_X is ample
(2) X admits a semi special tensor $\psi \neq 0 \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$, where η is a 2-torsion invertible sheaf.

hold if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type with the special property

(*) Ω is a product of irreducible bounded symmetric domains D_j of tube type whose rank r_j divides the dimension n_j of D_j,

and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ acting freely.

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.

In particular the polydisk case is as follows: X admits a semi special tensor $\psi \in H^0(S^n(\Omega^1_X)(-K_X) \otimes \eta)$ such that, given any point $p \in X$, the corresponding hypersurface $F_p =: \{\psi_p = 0\} \subset \mathbb{P}(TX_p)$ is reduced $\Leftrightarrow X \cong (\mathbb{H}^n)/\Gamma$.

Theorem 2.([1]) Let X be a compact complex manifold of dimension n. Then the following two conditions:

(1) K_X is ample
(2) X admits a slope zero tensor $\psi \in H^0(S^{mn}(\Omega^1_X)(-mK_X))$, (here m is a positive integer);

hold if and only if $X \cong \Omega/\Gamma$, where Ω is a bounded symmetric domain of tube type and Γ is a cocompact discrete subgroup of $\mathrm{Aut}(\Omega)$ acting freely.

Moreover, the degrees and the multiplicities of the irreducible factors of the polynomial ψ_p determine uniquely the universal covering $\widetilde{X} = \Omega$.

The above characterizations are important in order to obtain precise formulations of some results of Kazhdan.

Corollary. Assume that X is a projective manifold with K_X ample, and that the universal covering $\tilde{X}$ is a bounded symmetric domain of tube type.

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ be an automorphism of $\mathbb{C}$.

Then the conjugate variety X^σ has universal covering $\tilde{X^\sigma} \cong \tilde{X}$.

In our joint work we have been able to give other, perhaps less elegant, but more general formulations which hold for all bounded symmetric domains. For instance, in the case where there are no ball factors, we got the following result, using the algebraic curvature tensor.

Theorem 3.([2]) Let X be a compact complex manifold of dimension n with K_X ample.

Then the universal covering $\tilde{X}$ is a bounded symmetric domain without factors of ball type if and only if there is a holomorphic tensor $s \in H^0(End(T_X \otimes T_X^\vee))$ enjoying the following properties:

1) given any point $p \in X$, there is a splitting of the tangent space $T = T_{X,p}$

$$T = T'_1 \oplus ... \oplus T'_m$$

such that the Mok characteristic cone $\mathcal{CS}$ splits into m irreducible components $\mathcal{CS}'(j)$ with

2) $\mathcal{CS}'(j) = T'_1 \times \mathcal{CS}'_j ... \times T'_m$

3) $\mathcal{CS}'_j \subset T'_j$ fulfills: if $W_j := \{w \in T'_j | w + \mathcal{CS}'_j \subset \mathcal{CS}'_j\}$, then $W_j = 0$

4) $\mathcal{CS}'_j$ is contained in a linear subspace (this is equivalent to $\mathcal{CS}'(j)$ being contained in a linear subspace) if and only if $\mathcal{CS}'_j = 0$ and dim $(T'_j) = 1$.

Recall that the Mok characteristic cone $\mathcal{CS} \subset T_X$ is defined as the projection on the first factor of the intersection $ker(s) \cap \{t \in (T_X \otimes T_X^\vee) Rank(t) = 1\}$.

In work in progress we are pursuing more precise and less tautological characterizations which allow to determine the universal covering explicitly.

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Enriques surfaces whose automorphism groups are virtually abelian

SHIGERU MUKAI

A (minimal) Enriques surface S is the quotient of a $K3$ surface X by a fixed-point-free involution $\varepsilon : X \to X$. We consider it over the complex number field. Its Picard group is isomorphic to the second cohomology group $H^2(S, \mathbb{Z})$. The torsion part of $H^2(S, \mathbb{Z})$ is generated by the first Chern class $c_1(S)$. The torsion free part is an even integral unimodular lattice of rank 10. An Enriques surface S is algebraic and the automorphism group $\mathrm{Aut}\, S$ is discrete. Its number of moduli is 10.

If an Enriques surface S is (moduli theoretically) very general, then S does not contain a smooth rational curve on it. Every positive isotropic divisor class on such an S defines an elliptic fibrtation whose Mordell-Weil rank[1]is 8. In particular, $\mathrm{Aut}\, S$ is infinite if S is very general. The automorphism group $\mathrm{Aut}\, S$ shrinks when S becomes special and has smooth rational curves on it. As the most extreme case, Nikulin[9] and Kondo[5] classified the Enriques surfaces with only finitely many automorphisms into seven types I, –, VII. In this talk we explain our recent result (and its proof) on the next case. Recall that a group is called *virtually abelian* if it contains a finitely generated abelian group as normal subgroup of finite index.

Theorem A. *The automorphism group* $\mathrm{Aut}\, S$ *of an Enriques surface* S *is virtually abelian if and only if either* $\mathrm{Aut}\, S$ *is finite or* S *is of (lattice) type* E_8*, that is, the twisted Picard group* $\mathrm{Pic}^\omega S$ *contains the (negative definite) root lattice of type* E_8 *as sublattice.*

The following lists the lattice type of Enriques surfaces with virtually abelian automorphism groups.

[1]For an elliptic fibration $\Phi : S \to \mathbb{P}^1$, the rank of the Mordell-Weil group of its Jacobian fibration is called the *Mordell-Weil rank* of Φ for short.

No.	I'	II	III	IV	V
Lattice type	E_8	D_9	$(D_8 + A_1 + A_1)^+_W$	$(D_5 + D_5)_W$	$(E_7 + A_2 + A_1)_W$
			Lieberman type (see Example C)		Kondo-Mukai type (cf. [6])

VI	VII
$E_6 + A_4$	$(A_9 + A_1)^+$
Hessian (cf. [4])	Fano[3]

Here L^+ denotes an odd (integral) lattice which contains L as sublattice of index 2. See Definition B below for *lattice type* L or L_W, which is the key of our proof of the theorem. Our lattice type is a refinement of the root invariants of Nikulin[9] in terms of the twisted cohomology group.

1. Twisted cohomology

The kernel of the Gysin map $\pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 12, which we call the *twisted cohomology group* of an Enriques surface S and denote by $H^\omega(S, \mathbb{Z})$, where $\pi : X \to S$ is the canonical (or universal) covering. $H^\omega(S, \mathbb{Z})$ is isomorphic to the second cohomology group $H^2(S, \mathbb{Z}^\omega_S)$ of S with coefficients in the unique non-trivial local system $\mathbb{Z}^\omega_S$. Hence, the natural non-degenerate pairing $\mathbb{Z}^\omega_S \times \mathbb{Z}^\omega_S \to \mathbb{Z}_S$ induces a $\mathbb{Z}$-valued bilinear form on $H^\omega(S, \mathbb{Z})$, for which the following holds (cf. [2]):

- $H^\omega(S, \mathbb{Z})$ is an odd unimodular lattice $I_{2,10}$ of signaure $(2, 10)$.
- $H^\omega(S, \mathbb{Z})$ carries a polarized Hodge structure $H^\omega(S)$ of weight 2 with Hodge number $(1, 10, 1)$.
- The $(1,1)$-part of $H^\omega(S)$ is the kernel of the pushforward map $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} S$. We call it the *twisted Picard lattice* and denote by $\operatorname{Pic}^\omega S$.
- The modulo 2 reduction $H^\omega(S, \mathbb{Z}) \otimes \mathbb{Z}/2$ is canonically isomorphic to the usual cohomology $H^2(S, \mathbb{Z}/2)$ with $\mathbb{Z}/2$ coefficient.

The twisted Picard lattice $\operatorname{Pic}^\omega S$ is negative definite, and does not contain a (-1)-element by Riemann-Roch. Let L be such a lattice, that is, negative definite and $\not\ni (-1)$-element.

Definition B. An Enriques surface S is *of lattice type* L (resp. L_W) if the twisted Picard lattice $\operatorname{Pic}^\omega S$ contains L as primitive sublattice and if the orthogonal complement of $L \hookrightarrow H^\omega(S, \mathbb{Z})$ is odd (resp. even).

The orthogonal complement is even if and only if $L \otimes \mathbb{Z}/2$ contains Wu's class, that is, $c_1(S)$ modulo 2. The number of moduli of Enriques surfaces of (lattice) type L or L_W is equal to $10 - \operatorname{rank} L$.

Example C. An Enriques surface is called *Lieberman type* if it is isomorphic to the quotient of a Kummer surface $\operatorname{Km}(E_1 \times E_2)$ of product type by ε_+, where E_i, $i = 1, 2$, is an elliptic curve and ε_+ is the composite of $(-1_E, 1_E)$ and the translation by a 2-torsion (a_1, a_2) with $0 \neq a_i \in (E_i)_{(2)}$. An Enriques surface S is of Lieberman type if and only if it is of type $D_{8,W}$, that is, $\operatorname{Pic}^\omega S$ contains D_8 primitively and the orthogonal complement of $D_8 \hookrightarrow H^\omega(S, \mathbb{Z})$ is isomorphic to

$U+U(2)$, where U (resp. $U(2)$) denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$). (The orthogonal complement is isomorphic to $\langle 1 \rangle + \langle -1 \rangle + U(2)$ if S is of type D_8.) The IIIrd Enriques surface in the above table is of Lieberman type with $E_1 = E_2 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$.

If C is a smooth rational curve on an Enriques surface, then its pullback splits into two disjoint rational curves C_0 and C_1. Their difference $[C_0] - [C_1]$ defines a (-2)-element in $\mathrm{Pic}^{\omega} S$ up to sign, which is called the *twisted fundamental class* of C. In particular, a tree of smooth rational curves on S of ADE-type defines a negative definite sublattice of the same type in $\mathrm{Pic}^{\omega} S$.

2. Outline of proof of Theorem A

Assume that $\mathrm{Aut}\, S$ is not finite but virtually abelian. Then S has an elliptic fibration $\Phi_0 : S \to \mathbb{P}^1$ of positive Mordell-Weil rank, and all other elliptic fibrations $\Phi \neq \Phi_0$ have Mordell-Weil rank zero. In particular S has one and only one elliptic fibration of positive Mordell-Weil rank modulo $\mathrm{Aut}\, S$. By an argument similar to [9], S is of type either $E_8, A_9, E_7 + A_2, (A_5 + A_5)^+$ or $(D_6 + A_3 + A_1)^+_W$. Except for E_8, S has more than one elliptic fibrations of positive Mordell-Weil rank. For example, in the cases of type A_9 and $(A_5 + A_5)^+$, it is deduced from the action of the alternating group $\mathfrak{A}_5$. In the last case, it is deduced from the fact that S is the normalization of the diagonal Enriques sextic

$$\overline{S} : (x_0^2 + x_1^2 + x_2^2 + x_3^2) + \sqrt{-1} \left(\frac{1}{x_0^2} + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \right) x_0 x_1 x_2 x_3 = 0$$

in $\mathbb{P}^3$. (This equation was found in [8] as the octahedral Enriques sextic.) In the case of type E_8, $\mathrm{Aut}\, S$ is virtually abelian by Barth-Peters[1, §4].

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Foliations on Uniruled Manifolds

Jorge Vitório Pereira
(joint work with Frank Loray, Frédéric Touzet)

Bogomolov and McQuillan proved that the ampleness of the tangent sheaf of a foliation along a curve imposes strong restriction on the leaves intersecting such curve, see [1]. It is natural to ask if less positivity along a curve also imposes constraints on the geometry of the foliation. Although we do not have a general answer to such question we do have some positive results after imposing that the curve is a free rational curve.

Proposition. *Let $\mathcal{F}$ be a foliation on an uniruled projective manifold X. If there exists a general free morphism $f : \mathbb{P}^1 \to X$ such that $h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) > 0$ then through a general point $x \in X$ there exists a rational curve contained in a leaf of $\mathcal{F}$.*

By *general free morphism* we mean general in some irreducible component of the space of morphisms.

If we further impose that $\mathcal{F}$ has codimension one then even the weakest kind of positivity that one can imagine already has implications to the structure of the foliation.

Theorem. *Let $\mathcal{F}$ be a codimension one foliation on an n-dimensional uniruled projective manifold X. If $f : \mathbb{P}^1 \to X$ is a general free morphism, $\delta_0 = h^0(\mathbb{P}^1, f^*T\mathcal{F})$, and $\delta_{-1} = h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ then at least one of the following assertions holds true.*

(a) *The foliation $\mathcal{F}$ is transversely projective.*
(b) *The foliation $\mathcal{F}$ is the pull-back by a rational map of a foliation $\mathcal{G}$ on a projective manifold of dimension $\leq n - \delta_0 + \delta_{-1}$.*

The results above appear in [2] as Proposition 6.13, and Theorem 5 respectively.

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Singular irreducible symplectic spaces

Tim Kirschner

1. Period mappings for (nonproper) families of manifolds

Let $g\colon \mathcal{Y} \to S$ be a submersive, yet not necessarily proper(!), morphism of complex manifolds. Fix an integer n and denote

$$\mathscr{H}^n(g) := \mathrm{R}^n g_*(\Omega^\bullet_{\mathcal{Y}/S})$$

the n-th relative algebraic de Rham module associated to g. Then, in the very spirit of [4], we define (in a canonical way) a sheaf map

$$\nabla^n_{\mathrm{GM}}(g)\colon \mathscr{H}^n(g) \to \Omega^1_S \otimes_{\mathscr{O}_S} \mathscr{H}^n(g),$$

which turns out to be a flat connection on $\mathscr{H}^n(g)$.

Thus, when $\mathscr{H} := \mathscr{H}^n(g)$ is a locally finite free module (i.e., a vector bundle) on S, we see that

$$H := \ker(\nabla^n_{\mathrm{GM}}(g))$$

is a locally constant sheaf of $\mathbb{C}_S$-modules on S with the property that the evident map

$$\mathscr{O}_S \otimes_{\mathbb{C}_S} H \to \mathscr{H}$$

is an isomorphism. In particular, for any $s \in S$, we obtain an isomorphism of complex vector spaces

$$H_s \to \mathscr{H}^n(\mathcal{Y}_s)$$

if we assume the base change map for algebraic de Rham cohomology

$$\mathbb{C} \otimes_{\mathscr{O}_{S,s}} \mathscr{H}_s \to \mathscr{H}^n(\mathcal{Y}_s)$$

to be one.

Furthermore, in case S is simply connected, fixing a base point $t \in S$, we dispose of a family of ismorphisms

$$\phi_{s,t}\colon \mathscr{H}^n(\mathcal{Y}_s) \to H_s \to H(S) \to H_t \to \mathscr{H}^n(\mathcal{Y}_t),$$

where the point s varies through S. Therefore, for any integer p, we are in the position to define a *period mapping*

$$\mathcal{P}^{p,n}_t(g)\colon S \to \mathrm{Gr}(\mathscr{H}^n(\mathcal{Y}_t))$$

by setting

$$(\mathcal{P}^{p,n}_t(g))(s) := \phi_{s,t}\left[\mathrm{F}^p\mathscr{H}^n(\mathcal{Y}_s)\right],$$

where

$$\mathrm{F}^p\mathscr{H}^n(\mathcal{Y}_s) := \mathrm{im}\left(\mathrm{H}^n(\mathcal{Y}_s, \sigma^{\geq p}\Omega^\bullet_{\mathcal{Y}_s}) \to \mathrm{H}^n(\mathcal{Y}_s, \Omega^\bullet_{\mathcal{Y}_s})\right).$$

Now an easy argument shows that when $\mathrm{F}^p\mathscr{H}^n(g) \subset \mathscr{H}^n(g)$ is a vector subbundle and, for all $s \in S$, the base change map

$$\mathbb{C} \otimes_{\mathscr{O}_{S,s}} (\mathrm{F}^p\mathscr{H}^n(g))_s \to \mathrm{F}^p\mathscr{H}^n(\mathcal{Y}_s)$$

is an isomorphism, then the period mapping $\mathcal{P}^{p,n}_t(g)$ is holomorphic.

2. The local Torelli theorem

Following the example set by [1, Definition 1.1], we make the

Definition. A complex space X is called *symplectic* when X is normal and there exists an element $\sigma \in \Omega^2_X(X_{\mathrm{reg}})$ such that

(1) $\mathrm{d}\sigma = 0$,
(2) σ is nondegenerate on X_{reg} in the sense that the contraction with σ gives rise to an isomorphism of sheaves on X_{reg}
$$\Theta_{X_{\mathrm{reg}}} \to \Omega^1_{X_{\mathrm{reg}}};$$
(3) for all resolutions of singularities $f\colon \widetilde{X} \to X$, there exists an element $\widetilde{\sigma} \in \Omega^2_{\widetilde{X}}(\widetilde{X})$ such that we have
$$\widetilde{\sigma}|f^{-1}(X_{\mathrm{reg}}) = f^*(\sigma) \in \Omega^2_{\widetilde{X}}(f^{-1}(X_{\mathrm{reg}})).$$

In the realm of [5], we develop a theory, in parts generalizing classical results from [3], for the period mappings introduced in Section 1. As an application of this theory we prove

Theorem 1 ([5]). *Let $f\colon \mathcal{X} \to S$ be a proper, flat morphism of complex spaces and $t \in S$. Assume that the family f is semi-universal in t, that S is smooth and simply connected, and that the fibers of f are Kähler, have rational singularities and singular loci of codimension ≥ 4.*

Moreover, assume that $\mathcal{X}_t$ is symplectic with
$$\dim_{\mathbb{C}} \left(\Omega^2_{\mathcal{X}_t}((\mathcal{X}_t)_{\mathrm{reg}})\right) = 1.$$

Set
$$\mathcal{Y} := \{x \in \mathcal{X} : f \text{ is submersive in } x\}$$
and $g := f|\mathcal{Y}\colon \mathcal{Y} \to S$. Then the period mapping
$$\mathcal{P}^{2,2}_t(g)\colon S \to \mathrm{Gr}(1, \mathscr{H}^2(\mathcal{Y}_t))$$
is (well-defined, holomorphic, and) a 1-codimensional local immersion at t.

Remark 1. Theorem 1 implicitly claims that the assumptions listed in Section 1 needed to define $\mathcal{P}^{2,2}_t(g)$ (in the holomorphic sense) are fulfilled. As a matter of fact, we show:

Given a proper, flat morphism $f\colon \mathcal{X} \to S$ of complex spaces such that S is smooth and the fibers of f are Kähler, have rational singularities and singular loci of codimension ≥ 4, then the following assertions hold:

(1) For all pairs (p, q) of integers such that $p+q \leq 2$, the module $\mathrm{R}^q g_*(\Omega^p_{\mathcal{Y}/S})$ is locally finite free on S and compatible with base change.
(2) The relative Frölicher spectral sequence for the morphism g degenerates in the corresponding entries.

Remark 2. Theorem 1 generalizes [8, Theorem 8].

Corollary. *In the situation of Theorem 1, when* $\mathrm{R}^2 f_*(\mathbb{C}_{\mathcal{X}})$ *is a locally constant sheaf on* S*, we have:*

(1) *The period mapping of mixed Hodge structures*

$$\mathcal{P}_t^{2,2}(f)_{\mathrm{MHS}} \colon S \to Q_{\mathcal{X}_t} \subset \mathrm{Gr}(1, \mathrm{H}^2(\mathcal{X}_t, \mathbb{C}))$$

is a local biholomorphism at t*, where* $Q_{\mathcal{X}_t}$ *denotes the zero set cut out by the Beauville-Bogomolov form* $q_{\mathcal{X}_t}$ *(see Theorem 2 below).*

(2) *The restriction mapping*

$$\mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_{\mathrm{reg}}, \mathbb{C})$$

is an isomorphism.

3. The Fujiki relation

[7, Theorem (2.5)] says that the Kuranishi space of a projective, symplectic complex space X with singular locus of codimension ≥ 4 is smooth. A slight alteration of the proof in *loc. cit.* shows that the statemant holds true for all compact, Kähler (instead of merely projective) complex spaces.

As a consequence, employing crucially the results of Section 2, one derives

Theorem 2 ([5])**.** *Let* X *be a compact, Kähler type, connected, symplectic complex space such that*

$$\dim_{\mathbb{C}}(\Omega_X^2(X_{\mathrm{reg}})) = 1$$

and

$$\mathrm{codim}(\mathrm{Sing}(X), X) \geq 4.$$

Then the Fujiki relation holds for X*, i.e., for all* $a \in \mathrm{H}^2(X, \mathbb{C})$ *we have*

$$\int_X a^{2r} = \binom{2r}{r} q_X(a)^r,$$

where $r := \frac{1}{2}\dim(X)$ *and*

$$q_X(a) := \frac{r}{2}\int_X (w^{r-1}\overline{w}^{r-1}a^2) + (r-1)\int_X (w^{r-1}\overline{w}^r a)\int_X (w^r\overline{w}^{r-1}a),$$

where $w \in \mathrm{F}^2\mathrm{H}^2(X) \subset \mathrm{H}^2(X, \mathbb{C})$ *such that* $\int_X w^r\overline{w}^r = 1$*.*

Remark. Theorem 2 is classical for X an irreducible holomorphic symplectic manifold (cf. [2, Theorem 4.7]); note that the corresponding proof relies heavily on differential geometric means. Our proof for Theorem 2, which is in a sense of purely algebraic nature, is inspired by [6].

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The abundance conjecture for slc pairs and its applications

Yoshinori Gongyo
(joint work with Osamu Fujino)

The following theorem is one of the main results of this talk. It is a solution of the conjecture raised in [F] (see [F, Conjecture 3.2]). For the definition of the *log pluricanonical representation* ρ_m, see Definitions 1 below.

Theorem 1 (cf. [F, Section 3], [G1, Theorem B]). *Let* (X, Δ) *be a projective log canonical pair. Suppose that* $m(K_X+\Delta)$ *is Cartier and that* $K_X+\Delta$ *is semi-ample. Then* $\rho_m(\mathrm{Bir}(X, \Delta))$ *is a finite group.*

Definition 1 ([F, Definition 3.1]). Let (X, Δ) (resp. (Y, Γ)) be a pair such that X (resp. Y) is a normal scheme with a $\mathbb{Q}$-divisor Δ (resp. Γ) such that $K_X+\Delta$ (resp. $K_Y+\Gamma$) is $\mathbb{Q}$-Cartier. We say that a proper birational map $f : (X, \Delta) \dashrightarrow (Y, \Gamma)$ is *B-birational* if there exist a common resolution $\alpha : W \to X$ and $\beta : W \to Y$ such that $\alpha^*(K_X+\Delta) = \beta^*(K_Y+\Gamma)$. This means that it holds that $E = F$ when we put $K_W = \alpha^*(K_X+\Delta)+E$ and $K_W = \beta^*(K_Y+\Gamma)+F$. We put $\mathrm{Bir}(X, \Delta) = \{\sigma | \sigma : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}$.

Then, we consider *log pluricanonical representation*

$$\rho_m : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X+\Delta)))$$

In Theorem 1, we do not have to assume that $K_X+\Delta$ is semi-ample when $K_X+\Delta$ is big. As a corollary of this fact, we obtain the finiteness of $\mathrm{Bir}(X, \Delta)$ when $K_X+\Delta$ is big. It is an answer to the question raised by Cacciola and Tasin.

Theorem 2. *Let* (X, Δ) *be a projective log canonical pair such that* $K_X+\Delta$ *is big. Then* $\mathrm{Bir}(X, \Delta)$ *is a finite group.*

In the framework of [F], Theorem 1 will play important roles in the study of Conjecture 1 (see [Ft], [AFKM], [Ka], [KMM], [F], [G1], and so on).

Conjecture 1 ((Log) abundance conjecture). *Let* (X, Δ) *be a projective semi log canonical pair such that* Δ *is a* $\mathbb{Q}$*-divisor. Suppose that* $K_X+\Delta$ *is nef. Then* $K_X+\Delta$ *is semi-ample.*

Theorem 1 was settled for surfaces in [F, Section 3] and for the case where $K_X+\Delta \sim_{\mathbb{Q}} 0$ by [G1, Theorem B]. To carry out the proof of Theorem 1, we introduce the notion of $\widetilde{B}$*-birational maps* and $\widetilde{B}$*-birational representations* for sub kawamata log terminal pairs, which is new and is indispensable for generalizing the arguments in [F, Section 3] for higher dimensional log canonical pairs.

By Theorem 1, we obtain a key result.

Theorem 3. *Let (X, Δ) be a projective semi log canonical pair. Let $\nu : X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is semi-ample. Then $K_X + \Delta$ is semi-ample.*

By Theorem 3, Conjecture 1 is reduced to the problem for log canonical pairs. After we circulated the paper [FG], Hacon and Xu proved a relative version of Theorem 3 by using Kollár's gluing theory (cf. [HX]).

Let X be a smooth projective n-fold. By our experience on the low-dimensional abundance conjecture, we think that we need the abundance theorem for projective semi log canonical pairs in dimension $\leq n-1$ in order to prove the abundance conjecture for X. Therefore, Theorem 3 seems to be an important step for the inductive approach to the abundance conjecture. The general strategy for proving the abundance conjecture is explained in the introduction of [F]. Theorem 3 is a complete solution of Step (v) in [F, 0. Introduction].

We will discuss the relationship among the various conjectures in the minimal model program. Let us recall the following two important conjectures.

Conjecture 2 (Non-vanishing conjecture)**.** *Let (X, Δ) be a projective log canonical pair such that Δ is an $\mathbb{R}$-divisor. Assume that $K_X + \Delta$ is pseudo-effective. Then there exists an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D$.*

By [DHP, Section 8] and [G2], Conjecture 2 can be reduced to the case when X is a smooth projective variety and $\Delta = 0$ by using the global ACC conjecture and the ACC for log canonical thresholds (see [DHP, Conjecture 8.2 and Conjecture 8.4]).

Conjecture 3 (Extension conjecture for divisorial log terminal pairs (cf. [DHP, Conjecture 1.3]))**.** *Let (X, Δ) be an n-dimensional projective divisorial log terminal pair such that Δ is a $\mathbb{Q}$-divisor, $\llcorner \Delta \lrcorner = S$, $K_X + \Delta$ is nef, and $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \operatorname{Supp} D$. Then*

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

Note that Conjecture 3 holds true when $K_X + \Delta$ is semi-ample. It is an easy consequence of a cohomology injectivity theorem. We also note that Conjecture 3 is true if (X, Δ) is purely log terminal (cf. [DHP, Corollary 1.8]). The following theorem is one of the main results of this tlk. It is a generalization of [DHP, Theorem 1.4].

Theorem 4 (cf. [DHP, Theorem 1.4])**.** *Assume that* Conjecture 2 *and* Conjecture 3 *hold true in dimension $\leq n$. Let (X, Δ) be an n-dimensional projective divisorial log terminal pair such that $K_X + \Delta$ is pseudo-effective. Then (X, Δ) has a good minimal model. In particular, if $K_X + \Delta$ is nef, then $K_X + \Delta$ is semi-ample.*

By our inductive treatment of Theorem 4, Theorem 3 plays a crucial role. Therefore, Theorem 1 is indispensable for Theorem 4.

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Recent results on automorphisms of irreducible holomorphic symplectic manifolds

Alessandra Sarti

Definition. An irreducible holomorphic symplectic (IHS) manifold X is a compact, complex, Kähler manifold which is simply connected and admits a unique (up to scalar multiplication) everywhere non-degenerate holomorphic 2-form.

From the definition follows that the dimension of X is even and the canonical divisor is trivial.
Examples. **1)** If the dimension of X is 2 then IHS manifolds are K3 surfaces.
2) If $\dim X = 2n$, $n \geq 2$, there are two families of examples studied by Beauville [3], in 1983: the Hilbert scheme of points $\mathrm{Hilb}^{[n]}(\Sigma)$ of a K3 surface of dimension $2n$ and of second Betti number $b_2 = 23$; the generalized Kummer variety $\mathrm{Km}^{[n]}(A)$ of dimension $2n$ and, if $n > 2$, of second Betti number $b_2 = 7$. If $n = 2$ then $\dim(\mathrm{Km}^{[1]}(A)) = 2$ and in this case the generalized Kummer variety is a Kummer surface. There are moreover two examples of O'Grady [13, 12] one in dimension 10 ($b_2 = 24$) and the other in dimension 6 ($b_2 = 8$). Since their Betti numbers are different these examples are not deformation equivalent to each other and until now, up to deformation, these are all the known examples of IHS manifolds.

The aim of the talk is to study automorphisms of prime order on an IHS manifold X. Denote by ω_X the everywhere non-degenerate holomorphic 2-form. We call an automorphism ψ of X *symplectic* if its action on ω_X is trivial otherwise it is called *non-symplectic*. In the case of K3 surfaces the study of automorphisms was started by Nikulin in [10] and then many of its results were generalized to IHS manifolds by Beauville [2]. If X is deformation equivalent to $\mathrm{Hilb}^{[2]}(\Sigma)$ there is a very recent description of the fixed locus of ψ in the case that ψ is a (non)-symplectic involution, by Beauville, Camere, Mongardi [4, 8, 9]. Here we study the case of non-symplectic automorphisms of prime order $p \geq 3$.

Examples of non symplectic automorphisms and their fixed locus.

1) Let Σ be a K3 surface and φ a non-symplectic automorphism of Σ. In a natural way φ induces an automorphism (a *natural automorphism*) $\varphi^{[n]}$ on the Hilbert scheme $\mathrm{Hilb}^{[n]}(\Sigma)$. Assume that the order of φ is 3 and $n = 2$. The fixed locus of φ is always of the form

$$\Sigma^{\varphi} = C_g \cup R_1 \cup \ldots \cup R_k \cup \{p_1, \ldots, p_N\}$$

with C_g a smooth curve of genus $g \geq 0$, R_i a smooth rational curve, $i = 1, \ldots, k$, and p_j, $j = 1, \ldots, N$ an isolated fixed point (cf. [1, Theorem 2.2]), then by using the local action of φ at a fixed point and the properties of the Hilbert scheme, one computes the fixed locus on $\mathrm{Hilb}^{[2]}(\Sigma)$. Up to isomorphism this consists of $(N + Nk + k)$ copies of $\mathbb{P}^1$, $N(N-1)/2$ isolated fixed points, $k(k-1)/2$ copies of $\mathbb{P}^1 \times \mathbb{P}^1$, k copies of $\mathbb{P}^2$, $N + 1$ curves C_g, k copies of $\mathbb{P}^1 \times C_g$ and one $\mathrm{Hilb}^{[2]}(C_g)$ (cf. [6]).

2) Let V be a cubic hypersurface in $\mathbb{P}^5$, then

$$F(V) = \{l \in \mathrm{Gr}(1,5) \,|\, l \subset V\}$$

is the *Fano variety* of lines on V. Beauville and Donagi [5] in 1984 have shown that $F(V)$ is deformation equivalent to $\mathrm{Hilb}^{[2]}(\Sigma)$. Consider the automorphism of $\mathbb{P}^5$ given by:

$$\psi_0 : (x_0 : \ldots : x_5) \mapsto (x_0 : \ldots : x_4 : \xi x_5), \qquad \xi = e^{\frac{2\pi i}{3}}.$$

The family of invariant cubics is

$$V : L_3(x_0, \ldots, x_4) + x_5^3 = 0$$

where L_3 is a homogeneous polynomial of degree 3. One can see that ψ_0 acts non-symplectically on $F(V)$ and the fixed locus of ψ_0 on V consists of a cubic 3-fold $\mathcal{C} = \{L_3(x_0, \ldots, x_4) = 0,\, x_5 = 0\}$. The induced fixed locus on $F(V)$ is the Fano surface of $\mathcal{C}$ which is a surface of general type with Hodge diamond: $h^0 = 1$, $h^{1,0} = h^{0,1} = 5$, $h^{2,0} = h^{0,2} = 10$, $h^{1,1} = 25$.

When studying the fixed locus of automorphisms on manifolds there are two important formulas that one can apply to get information: the topological and the holomorphic Lefschetz fixed point formulas. In the case of non-symplectic automorphisms of prime order acting on K3 surfaces, in [1] the authors uses Smith exact sequences to give a formula for $h^*(\Sigma^{\varphi}, \mathbb{F}_p) = \sum_{i \geq 0} h^i(\Sigma^{\varphi}, \mathbb{F}_p)$, where Σ is a K3 surface, φ is of order p and acts non-symplectically on Σ. A generalization of this formula to IHS manifolds is given by Boissière, Nieper-Wisskirchen and the author in [7, Corollary 5.12]. In order to give it in the case of IHS manifolds that are deformation equivalent to $\mathrm{Hilb}^{[2]}(\Sigma)$ we introduce some notations. A non-symplectic automorphism ψ of prime order p on an IHS manifold X deformation equivalent to $\mathrm{Hilb}^{[2]}(\Sigma)$ induces an action on the $\mathbb{Z}$-module $H^2(X,\mathbb{Z})$, which together with the Beauville-Bogomolov quadratic form is a lattice isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$, where U is the unique unimodular hyperbolic lattice of signature $(1,1)$. The action of ψ on $H^2(X,\mathbb{Z})$ determines an invariant sublattice $T(X)$ and we denote by $S(X)$ its orthogonal complement. It turns out that the transcendental lattice $T_X \subset S(X)$ and the Néron-Severi lattice $NS(X) \subset T(X)$. In the paper [7, Lemma 5.3] it is shown that the lattice $T(X) \oplus S(X)$ has finite index p^a in $H^2(X,\mathbb{Z})$ for a certain integer $a \geq 0$ and the lattice $S(X)$ is p-elementary, more precisely its discriminant group is the sum of a copies of $\mathbb{Z}/p\mathbb{Z}$, [7, Lemma 6.5]. Finally the rank of $S(X)$ is equal to $(p-1)m$ for a certain integer $m > 0$. Recall also that for a non-symplectic automorphism of prime order p acting non-symplectically on X we have $2 \leq p \leq 23$.

Theorem ([7, Corollary 6.15]). *Let X be deformation equivalent to* $\mathrm{Hilb}^{[2]}(\Sigma)$*, G be a group of non-symplectic automorphisms of prime order p on X with $3 \leq p \leq 19$ and $p \neq 5$. Then:*

$$\begin{aligned} h^*(X^G,\mathbb{F}_p) &= 324 - 2a(25-a) - (p-2)m(25-2a) + \frac{1}{2}m((p-2)^2m - p) \\ \textit{with} \qquad & 2 \leq m(p-1) \leq 23, \qquad 0 \leq a \leq \min\{m(p-1),\, 23 - m(p-1)\}. \end{aligned}$$

Remarks. 1) We have to exclude the case $p = 5$ in the Theorem since in the proof one needs the isomorphism $\mathrm{Sym}^2(H^2(X,\mathbb{F}_p)) \cong H^4(X,\mathbb{F}_p)$. By a result of Verbitsky [14], $\mathrm{Sym}^2(H^2(X,\mathbb{Q})) \cong H^4(X,\mathbb{Q})$ and we compute in [7, Proposition 6.6] that the index of $\mathrm{Sym}^2(H^2(X,\mathbb{Q}))$ in $H^4(X,\mathbb{Q})$ is $2^{23} \cdot 5$. So the first isomorphism holds only for $p \neq 2, 5$.

2) An immediate consequence of the formula is that a non-symplectic automorphism of order 3 has always fixed points on X ([7, Proposition 6.17]).

Applications. In the recent work [6] we combine the formula of the Theorem, the topological Lefschetz formula and results of Nikulin on lattices [11] to get tables for any $3 \leq p \leq 19$, $p \neq 5$ of the invariants m and a of the invariant lattice and its orthogonal complement. In fact we show that these two lattices are uniquely determined by m and a, this generalizes an analogous result for K3 surfaces [1, Proposition 3.2]. We show also that all the cases of our table are possible. For example in the case of the **Example 2**, that we gave at the beginning of the report, the topological Euler characteristic of the fixed locus is $\chi = 27$, $h^*(X^{\psi_0},\mathbb{F}_3) = 67$ and one computes that $m = 11$ and $a = 1$. Using lattice theory one shows that $T(X) = \langle 6 \rangle$ and $S(X) = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$.

This example answer also the question: *Does there exist non-natural non-symplectic automorphisms of prime order $p \geq 3$ on X?* Here we say that an automorphism ψ on X is natural if there exists a K3 surface Σ and an automorphism φ on Σ

such that the couple (X, ψ) is deformation equivalent to $(\mathrm{Hilb}^{[2]}(\Sigma), \varphi^{[2]})$ (see [9, Definition 1.1]). The previous example answer positively this question, since for a natural automorphism we must have that the rank of $T(X)$ is at least 2. In fact, in [2], Beauville shows that X is algebraic so we can find an invariant ample class and the class of the exceptional set on $\mathrm{Hilb}^{[2]}(\Sigma)$ is invariant too. This gives two classes in $T(X)$.

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Semi-algebraic horizontal subvarieties of Calabi-Yau type

Radu Laza

(joint work with Robert Friedman)

Alongside curves, two classes of varieties for which we have a reasonably good understanding of the moduli spaces are the $K3$ surfaces and the principally polarized abelian varieties. For these two cases the main tool for studying their moduli is the period map. Except these cases the use of period maps towards understanding moduli spaces has been limited so far. One of the main reasons for this is that, except for weight 1 Hodge structures or weight 2 HS with $h^{2,0} = 1$, the periods of algebraic varieties satisfy non-trivial relations (the Griffiths transversality relations, $\nabla F^p \subseteq F^{p-1}$). The purpose of this note is to discuss the image of the period map in some cases (beyond abelian varieties and $K3$s), with special focus on Calabi-Yau threefolds.

Specifically, let $\mathbb{D}$ be a period domain, i.e. a classifying space for polarized Hodge structures of weight n. We consider $Z \subset \mathbb{D}$ a closed horizontal subvariety (i.e. the Griffiths transversality relations are satisfied for Z). If Z is coming from algebraic geometry, then Z is stabilized by a big monodromy group Γ. More precisely, if Γ is the stabilizer of Z in the natural arithmetic group $\mathbb{G}(\mathbb{Z})$ acting on $\mathbb{D}$, then Γ acts properly discontinuously on Z, the image $\Gamma\backslash Z \subseteq \mathbb{G}(\mathbb{Z})\backslash\mathbb{D}$ is a closed subvariety, and it is the image of a quasi-projective variety under a proper holomorphic map. Thus it is reasonable in general to look at closed horizontal subvarieties Z of $\mathbb{D}$ such that $\Gamma\backslash Z$ is quasi-projective. Actually, we will need a mild technical strengthening of this hypothesis which we call *strongly quasi-proj! ective* (i.e. $\Gamma'\backslash Z$ is quasi-projective for all finite index subgroups Γ' in Γ). On the other hand, since $\mathbb{D}$ is an open subset of its compact dual $\check{\mathbb{D}}$, and $\check{\mathbb{D}}$ is a projective variety, it is natural to look at those Z which can be defined algebraically, i.e. such that Z is a connected component of $\hat{Z} \cap \mathbb{D}$, where $\hat{Z}$ is a closed algebraic subvariety of $\check{\mathbb{D}}$. We will refer to such Z as *semi-algebraic in* $\mathbb{D}$. Under this assumptions (big monodromy and semi-algebraic), we obtain the following theorem:

Theorem 1. *Let Z be a closed horizontal subvariety of a classifying space $\mathbb{D}$ for Hodge structures and let Γ be the stabilizer of Z in the appropriate arithmetic group $\mathbb{G}(\mathbb{Z})$. Assume that*

(i) *$S = \Gamma\backslash Z$ is strongly quasi-projective;*
(ii) *Z is semi-algebraic in $\mathbb{D}$.*

Then Z is a Hermitian symmetric domain whose embedding in $\mathbb{D}$ is an equivariant, holomorphic, horizontal embedding.

The main ingredients used in the proof are the theorem of the fixed part as proved by Schmid for variations of Hodge structure over quasi-projective varieties, Deligne's characterization of Hermitian symmetric domains [Del79], and the recent theory of Mumford–Tate domains as developed by Green–Griffiths–Kerr [GGK12]. Theorem 1, in the case where $\mathbb{D}$ itself is Hermitian symmetric (and thus S is a subvariety of a Shimura variety), has been proved independently by Ullmo–Yafaev [UY11], using similar methods. This result is related in spirit, but in a somewhat different direction, to a conjecture of Kollár [KP11], which says roughly that, if Z is simply connected and semi-algebraic, and $S = \Gamma\backslash Z$ is projective for some discrete group Γ of biholomorphisms of Z, then Z is the product of a Hermitian symmetric space and a simply connected projective variety.

Theorem 1 is a general result for arbitrary Hodge numbers. We now focus on the classification of the possibilities occurring in the previous theorem for the case of Hodge structures of *Calabi–Yau type* (i.e. weight n Hodge structures with $h^{n,0} = 1$). Partial classification results in this direction were previously obtained by Gross [Gro94] and Sheng–Zuo [SZ10]. Specifically, for Hermitian symmetric spaces of tube type, Gross [Gro94] has constructed certain natural variations of Hodge structure of Calabi–Yau type. This construction was extended by Sheng–Zuo

[SZ10] to the non-tube case to construct complex variations of Hodge structure. Generalizing, these results we obtain a complete classification of the Hermitian VHS of Calabi-Yau type:

Theorem 2. *For every irreducible Hermitian symmetric domain of non-compact type $\mathcal{D} = G(\mathbb{R})/K$, there exists a canonical $\mathbb{R}$-variation of Hodge structure $\mathcal{V}$ of Calabi–Yau type. Any other irreducible equivariant $\mathbb{R}$-variation of Hodge structure of Calabi–Yau type on $\mathcal{D}$ can be obtained as a summand of $\mathrm{Sym}^n\mathcal{V}$ or $\mathrm{Sym}^n\mathcal{V}\left\{-\frac{a}{2}\right\}$ (if $\mathcal{D}$ is not a tube domain), where $\{\ \}$ denotes the half-twist operation (cf.* [vG01]*).*

We note that the periods of many families of Calabi-Yau threefolds (e.g. quintic threefolds, mirror quintics, etc.) do not satisfy the semi-algebraic assumption of Theorem 1. However, there are interesting geometric examples (such as examples constructed by Borcea [Bor97], Voisin [Voi93], Rohde [Roh09], van Geemen [GvG10]) that fit in with Theorems 1 and 2. We also mention here that one of main open questions related to Theorem 2 is the geometric realization of VHS of Calabi-Yau threefold type over the exceptional domain of type E_7 given by the theorem (e.g. see [Bai00]).

Finally, in the weight three case, it is possible to explicitly describe the embedding $Z \hookrightarrow \mathbb{D}$ from the perspective of Griffiths transversality and to relate this description to the Harish-Chandra realization of $\mathcal{D}$ and to the Korányi–Wolf tube domain description. There are further connections to homogeneous Legendrian varieties (e.g. [BG83], [LM07]) and the four Severi varieties of Zak.

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Bogomolov–Sommese vanishing on log canonical pairs

Patrick Graf

In his famous paper [2], Bogomolov proved the following theorem.

Theorem 1 (Bogomolov–Sommese vanishing)**.** *Let X be a complex projective manifold and $D \subset X$ a divisor with simple normal crossings. For any invertible subsheaf $\mathscr{L} \subset \Omega^p_X(\log D)$, we have $\kappa(\mathscr{L}) \le p$, where $\kappa(\mathscr{L})$ denotes the Kodaira–Iitaka dimension of $\mathscr{L}$.*

Theorem 1 is an important ingredient in the proof of the Bogomolov–Miyaoka–Yau inequality.

Building on the Extension Theorem of Greb–Kebekus–Kovács–Peternell [4, Thm. 1.5], I generalized Theorem 1 to the setting of reflexive differential forms on log canonical pairs as follows.

Theorem 2 (Bogomolov–Sommese vanishing on log canonical pairs)**.** *Let (X, D) be a complex projective log canonical pair. If $\mathscr{A} \subset \Omega^{[p]}_X(\log\lfloor D\rfloor) := \big(\Omega^p_X(\log\lfloor D\rfloor)\big)^{**}$ is a Weil divisorial subsheaf, then its Kodaira–Iitaka dimension $\kappa(\mathscr{A}) \le p$.*

A Weil divisorial sheaf is a reflexive sheaf of rank 1. The Kodaira–Iitaka dimension of a Weil divisorial sheaf $\mathscr{A}$ measures the growth of sections of reflexive tensor powers of $\mathscr{A}$, analogously to the case of line bundles.

There is also a slightly more general version of Theorem 2. It uses Campana's language of *orbifoldes géométriques* [3], called $\mathcal{C}$-pairs in [5].

In the course of the proof of Theorem 1, I showed the following generalization of the well-known Negativity lemma of birational geometry (see [1, Lem. 3.6.2] or [6, Lem. 3.39]).

Proposition 1 (Negativity lemma for bigness)**.** *Let $\pi\colon Y \to X$ be a proper birational morphism between normal quasi-projective varieties. Then for any nonzero effective π-exceptional $\mathbb{Q}$-Cartier divisor E, there is a component $E_0 \subset E$ such that $-E|_{E_0}$ is $\pi|_{E_0}$-big.*

Furthermore, I showed that on dlt $\mathcal{C}$-pairs, there is a version of the adjunction formula as well as a residue map for symmetric differential forms, and that these two are compatible with each other in the following sense.

Theorem 3 (Residues of symmetric differentials)**.** *Let* (X, D) *be a dlt* $\mathcal{C}$*-pair and* $D_0 \subset \lfloor D \rfloor$ *a component of the reduced boundary. Set* $D_0^c := \mathrm{Diff}_{D_0}(D - D_0)$*, such that* $(K_X + D)|_{D_0} = K_{D_0} + D_0^c$*. Then the pair* (D_0, D_0^c) *is also a dlt* $\mathcal{C}$*-pair, and for any integer* $p \geq 1$*, there is a map*

$$\mathrm{res}^k_{D_0} : \mathrm{Sym}^{[k]}_{\mathcal{C}}\, \Omega^p_X(\log D) \to \mathrm{Sym}^{[k]}_{\mathcal{C}}\, \Omega^{p-1}_{D_0}(\log D_0^c)$$

which on the snc locus of $(X, \lceil D \rceil)$ *coincides with the* k*-th symmetric power of the usual residue map for snc pairs.*

The main idea for the proof of Theorem 1 is best explained in the simple case where X is a cone over an elliptic curve and $D = 0$. Let $\mathscr{A} \subset \Omega^{[1]}_X$ be a Weil divisorial sheaf. We want to show that $\kappa(\mathscr{A}) \leq 1$.

Let $f : Z \to X$ be the log resolution of X given by blowing up the vertex, and let $E = \mathrm{Exc}(f)$. By the Extension Theorem of [4, Thm. 1.5], there is an embedding

$$f^{[*]}\mathscr{A} := (f^*\mathscr{A})^{**} \hookrightarrow \Omega^1_Z(\log E),$$

and $\kappa(f^{[*]}\mathscr{A}) \leq 1$ by Theorem 1. However, since reflexive pullback does not commute with reflexive tensor powers, in this situation in general we only have the inequality $\kappa(f^{[*]}\mathscr{A}) \leq \kappa(\mathscr{A})$. Therefore we enlarge the sheaf $f^{[*]}\mathscr{A}$ by taking its saturation $\mathscr{B} \subset \Omega^1_Z(\log E)$. We prove that sections of $\mathscr{A}^{[k]} := (\mathscr{A}^{\otimes k})^{**}$ extend to sections of $\mathscr{B}^{[k]}$. Then $\kappa(\mathscr{A}) = \kappa(\mathscr{B})$, and we are done.

The proof is by contradiction: Assuming that some section of $\mathscr{A}^{[k]}$ acquires a pole when being pulled back, we use the fact that $E \subset Z$ has negative self-intersection to deduce that $\mathscr{B}|_E$ is ample. On the other hand, the residue sequence for the pair (Z, E) shows that $\mathscr{B}|_E$ injects into the trivial line bundle. This yields the desired contradiction.

For the general proof, the fact that contractible curves have negative self-intersection is replaced by Proposition 1. A more serious issue is that we cannot really work on a log resolution, because it "extracts too many divisors". Instead, we have to pass to a *minimal dlt model* $(Z, D_Z) \to (X, D)$, a partial resolution of (X, D) which extracts only divisors of discrepancy exactly -1. Minimal dlt models exist by [1]. However, (Z, D_Z) is not an snc pair, but only a dlt pair, which makes the proof technically rather involved. In particular, we have to use Theorem 3 on (Z, D_Z).

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Rational curves and special metrics on twistor spaces

Misha Verbitsky

1. Special Hermitian metrics on complex manifolds

The twistor spaces, as shown by Hitchin, are never Kähler, except two examples: $\mathbb{C}P^3$, being a twistor space of S^4, and the flag space, being a twistor space of $\mathbb{C}P^2$ ([Hi]). A way of weaken the Kähler condition is to consider the equation $dd^c(\omega^k) = 0$, where $d^c = -IdI$; this equation is non-trivial for all $0 < k < n$. When $k = 1$, a metric, satisfying $dd^c\omega = 0$, is called **pluriclosed**, or **strong Kähler torsion**; such metrics are quite important in physics and in generalized complex geometry. The main result of this talk is the following theorem, similar to Hitchin's result.

Theorem 1. *Let M be a twistor space of a compact 4-dimensional anti-selfdual Riemannian manifold. Assume that M admits a pluriclosed Hermitian form ω. Then M is Kähler.*

2. Symplectic Hermitian metrics

When a pluriclosed Hermitian form ω is (1,1)-part of a closed (and hence symplectic) form $\tilde{\omega}$, ω is called **taming** or **Hermitian symplectic**.

In the paper [ST] Streets and Tian have constructed a parabolic flow for Hermitian symplectic metric, analoguous to the Kähler-Ricci flow. They asked whether there exists a compact complex Hermitian symplectic manifold not admitting a Kähler structure. This question was considered in [EFG] for complex nilmanifolds, but the results were mostly negative. However, the pluriclosed metrics exists on many complex nilmanifolds.

The present talk grew as an attempt to answer the Streets-Tian's question for twistor spaces. However, it was found that the twistor spaces are not only never Hermitian symplectic, they never admit a pluriclosed metric unless they are Kähler.

3. Rational curves and pluriclosed metrics

Unlike many complex non-algebraic manifolds, the twistor spaces are very rich in curves: there exists a smooth rational curve passing through any finite subset of a twistor space.

For an almost complex structure I equipped with a taming symplectic form, all components of the space of complex curves are compact, by Gromov's compactness theorem ([Gr, AL]). I will show that the same is true for pluriclosed metrics, if I

is integrable (1). This is used to show that a twistor space admitting a pluriclosed metrics is actually Moishezon (3).

However, Moishezon varieties satisfy the dd^c-lemma. This is used to show that any pluriclosed metric is in fact Hermitian symplectic.

Finally, by using the Peternell's theorem characterizing Moishezon manifolds in terms of currents, I prove that no Moishezon manifold can be Hermitian symplectic (1).

4. Rational curves on $\mathrm{Tw}(M)$

Definition 1. An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$.

Theorem 2 (Gromov)**.** *Let M be a compact Hermitian almost complex manifold, $\mathfrak{X}$ the space of all complex curves on M, and $\mathfrak{X} \xrightarrow{\mathrm{Vol}} \mathbb{R}^{>0}$ the volume function. Then* Vol *is proper (that is, preimage of a compact set is compact).*

Proof: [Gr], [AL]. ■

Corollary 1. *Let M be a complex manifold, equipped with a pluriclosed Hermitian form ω, and X a component of the moduli of complex curves. Then the function* $\mathrm{Vol}: X \longrightarrow \mathbb{R}^{>0}$ *is constant, and X is compact.*

Proof: Since $\mathrm{Vol} \geq 0$, the set $\mathrm{Vol}^{-1}(]-\infty, C])$ is compact for all $C \in \mathbb{R}$, hence Vol has a minimum somewhere in X. However, a pluriharmonic function which has a minimum is necessarily constant (E. Hopf's strong maximum principle). Therefore, Vol is constant: $\mathrm{Vol} = A$. Now, compactness of $X = \mathrm{Vol}^{-1}(A)$ follows from Gromov's theorem. ■

5. Quasilines and Moishezon manifolds

Theorem 3 (Campana)**.** *Let M be a complex manifold, $S \subset M$ a ample line, and W its deformation space. Assume that W is compact. Then M is Moishezon.* ■

Corollary 2. *Let M be a twistor space admitting a pluriclosed (or plurinegative) Hermitian metric. Then M is Moishezon.* ■

6. Pluriclosed and Hermitian symplectic metrics on twistor spaces

The following theorem is similar to results from [HL], and proven in the same fashion.

Theorem 4. *Let M be a compact, complex n-manifold. Then*

- **(a):** *M admits no pluriclosed metrics $\Leftrightarrow$ M admits a positive, dd^c-exact $(n-1, n-1)$-current.*
- **(b):** *M admits no Hermitian symplectic metrics $\Leftrightarrow$ M admits a positive, exact $(n-1, n-1)$-current.*

■

This leads to the following proposition.

Proposition 1. *Any twistor space M which admits a pluriclosed metric also admits a Hermitian symplectic structure.*

Proof: By 2, M is Moishezon. Then, [DGMS] implies that M satisfies dd^c-lemma. Therefore, any exact (2,2)-current is dd^c-exact. Applying 4, we obtain that M is Hermitian symplectic. ■

Corollary 3. *Let M be a twistor space admitting a pluriclosed (or Hermitian symplectic) metric. Then M is Kähler.*

Proof: Th. Peternell has shown that any non-Kähler Moishezon n-manifold admits an exact, positive $(n-1, n-1)$-current. Therefore, it is never Hermitian symplectic (4). Therefore, by 1, M cannot be pluriclosed. ■

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Moduli Spaces of Tautological Sheaves on Hilbert Squares of K3 Surfaces

Malte Wandel

Introduction. Recall that the list of known examples of irreducible holomorphic symplectic manifolds is very short. Up to deformation they are:

- moduli spaces of sheaves on K3 surfaces (e.g. K3 surfaces, Hilbert schemes of points on K3 surfaces),
- generalized Kummer varieties,
- two sporadic examples of O'Grady (one of which is constructed from moduli spaces of sheaves on K3 surfaces).

Note that many of them are directly connected to moduli spaces of sheaves on K3 surfaces which are, themselves, irreducible holomorphic symplectic manifolds. This leads to the following question:

Question. Let X be an IHS manifold and M a moduli space of sheaves on X. Is X again an IHS manifold?
Since this question posed as above seems far to general let us formulate the following first aim:

Aim. Find examples of stable sheaves on higher (i.e. greater than 2) dimensional IHS manifolds and study their moduli spaces!

Tautological sheaves. Let X be an algebraic K3 surfaces. Denote by $X^{[n]}$ the Hilbert scheme of n points on X. There is a universal family $\Xi \subseteq X \times X^{[n]}$ which leads to the so-called **tautological functor** $(-)^{[n]} := \Phi_{\mathcal{O}_\Xi} \colon D^b(X) \to D^b(X^{[n]})$. It takes sheaves (bundles of rank r) to sheaves (bundles of rank nr). Furthermore the McKay correspondence gives an equivalence of derived categories $D^b(X^{[n]}) \tilde{\to} D^b_{S_n}(X^n)$ ([BKR], [Hai]). In his thesis Scala ([Sca]) gave a description of the image of tautological objects in $D^b_{S_n}(X^n)$ under the BKRH-correspondence. Using these results Krug ([Kru]) computed the extension groups of tautological objects. These computations, among other things, show that the tautological functor is faithful.

Stability. (From now on we will restrict to the case $n = 2$.)
In his thesis Schlickewei ([Schl]) gave a first proof of the stability of a certain class of tautological vector bundles on $X^{[2]}$ associated with line bundles. When we talk about stability we have to fix a polarization first. So let us recall that $\mathrm{Pic}(X^{[2]}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}D$, where $D \subset X^{[2]}$ parametrizes length two subschemes of X having support only in one point. Now let $H \in \mathrm{Pic}(X)$ be an ample line bundle. Then $H_N := H - ND$ is ample for all $N \gg 0$. We have the following result:

Theorem. *(*[Wan]*) Let $\mathcal{F}$ be a rank one torsion-free sheaf or a rank two μ_H-stable vector bundle on X satisfying $c_1(\mathcal{F}) \neq 0$. Then for sufficiently large N, $\mathcal{F}^{[2]}$ is a rank two μ_{H_N}-stable sheaf (rank four μ_{H_N}-stable vector bundle resp.) on $X^{[2]}$.*

Deformations of tautological sheaves. By Krug the infinitessimal deformations of a tautological sheaf $\mathcal{F}^{[2]}$ are given as follows:

$$\mathrm{Ext}^1(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \bigoplus \mathrm{H}^1(\mathcal{F})^\vee \otimes \mathrm{H}^0(\mathcal{F}).$$

(Here we assumed $h^2(\mathcal{F}) = 0$.) Let us call the first summand on the right hand side the *surface deformations* of $\mathcal{F}^{[2]}$ and the second one the *additional deformations.* Since $\mathcal{F}^{[2]}$ is built from $\mathcal{F}$ by means of a Fourier−Mukai transformation and the deformation theory of $\mathcal{F}$ is unobstructed, we can deduce:

Proposition. *Let $\mathcal{F}$ be a stable sheaf on X s.t. $\mathcal{F}^{[2]}$ is again stable. Then the surface deformations of $\mathcal{F}^{[2]}$ are unobstructed.*

Corollary. *We have a local immersion of the corresponding moduli spaces.*

Remark. If $\mathrm{h}^1(\mathcal{F}) = 0$ we have a local isomorphism of the moduli spaces.

Caution. In general these moduli spaces of sheaves on $X^{[2]}$ are not smooth in the point corresponding to $\mathcal{F}^{[2]}$. This will be illustrated in the following example.

Example. Let X be an elliptically fibred K3 and denote by E the fibre class and by C the class of a section. Consider $\mathcal{F} := \mathcal{O}(C + kE) \otimes \mathcal{I}_p$, for $k \geq 2$ and some $p \in X$. The surface deformations in this case correspond exactly to deformations of the point p inside X. Now the line bundle $\mathcal{O}(C + kE)$ has base locus C. So if $p \in C$ we deduce $\mathrm{H}^0(\mathcal{F}) \cong \mathrm{H}^0(\mathcal{O}(C + kE))$. But if we deform p in a direction $e \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ which is transversal to C, some section $s \in \mathrm{H}^0(\mathcal{F})$ will not deform with it. Now with a bit of deformation theory one can deduce that the deformation of $\mathcal{F}^{[2]}$ corresponding to (e, s) in Krug's formula has nonvanishing obstruction.

Open Questions.

- So far it is not known if the additional deformations may be realized as real deformation of the tautological sheaves and may produce interesting new examples of moduli spaces.
- A good description of the Atiyah class of the tautological objects would be very helpful to discuss many interesting problems concerning the deformation theory of these sheaves.

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Connected components of strata of the cotangent bundle to the moduli space of curves

Martin Möller

(joint work with Dawei Chen)

The cotangent bundle to the moduli space of curves parameterizes pairs (X, q) of a Riemman surface togehter with a quadratic differential. It is stratified according to the number and types of zeros, the strata thus being indexed by partitions of $4g - 4$. The topology of the strata is of fundamental interest for understanding the Teichmüller geodesic flow. Here we complete the classification of connected components. This is motivated by the investigation of non-varying phenomena for slopes of one-dimensional Hurwitz spaces ([1]).

A slightly easier problem is the classification of connected components of strata of the Hodge bundle. This parameterizes Riemann sufaces together with abelian differentials. We denote the strata by

$$\Omega M_g(m_1, \ldots, m_k) = \left\{ (X, \omega) : g(X) = g, \operatorname{div}(\omega) = \sum_{i=1}^{k} m_i z_i \right\},$$

where $\sum m_i = 2g-2$. We call a component of such a stratum hyperelliptic, if it parameterizes exclusively curves with abelian (resp. quadratic) differentials that arise as pullback of quadratic differentials on the projective line via a double covering.

Theorem 1 ([2]). *Besides hyperelliptic strata, the stata $\Omega M_g(m_1,\ldots,m_k)$ are connected except when all m_i are even. In this case there are two components, distinguished by the parity of the spin structure $H^0(X,\mathcal{O}_X(\sum \frac{m_i}{2}z_i))$.*

For quadratic differentials we address a slightly more general problem that the one alluded to in the introduction. That is, we allow simple poles, so that the volume of the Riemann surface X in the euclidian metric $|q|$ is still finite. For $d_i \geq 1$ with $\sum d_i = 4g-4$ we let

$$\mathcal{Q}_g(d_1,\ldots,d_k) = \left\{(X,q) : g(X) = g, \operatorname{div}(q) = \sum_{i=1}^{k} d_i z_i\right\}.$$

This problem was addressed in [3], but no algebraic invariant was found distinguishing the exceptional strata. Probably for this reason, moreover, some of the exceptional strata were missing. In the following statement we give the algebraic invariant and the complete list.

Theorem 2 ([3], [1]). *Besides hyperelliptic strata, the stata $\mathcal{Q}_g(d_1,\ldots,d_k)$ are connected except for the strata $\mathcal{Q}_3(9,-1)$, $\mathcal{Q}_3(6,3,-1)$, $\mathcal{Q}_3(3,3,3,-1)$ in genus three and the strata $\mathcal{Q}_4(12)$, $\mathcal{Q}_4(9,3)$, $\mathcal{Q}_4(6,6)$, $\mathcal{Q}_4(6,3,3)$ and $\mathcal{Q}_4(3,3,3,3)$ in genus four. In this case there are two components, distinguished by the parity of $H^0(X,\mathcal{O}_X(\operatorname{div}_o(q)/3)$.*

The interesting observation is that the dimension distinguishes connected components rather than jumping at loci of codimension one, as expected from semicontinuity. We prove that for the exceptional strata the dimension statement is equivalent to a bundle on an elliptic curve canonically associated with the pair (X,q) being trivial resp. three-torsion but not trivial. Torsion orders, on the other hand, are well-known to be deformation invariants.

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Webs of Lagrangian tori in projective symplectic manifolds

Jun-Muk Hwang
(joint work with Richard M. Weiss)

A simply-connected compact Kähler manifold M with a holomorphic symplectic form ω is called a compact hyperkähler manifold if $H^0(M, \Omega^2_M) = \mathbf{C}\omega$ (cf. [5]). One central problem in compact hyperkähler manifolds is to find a good condition for the existence of holomorphic or almost holomorphic fibrations on a compact hyperkähler manifold. In the survey [1] of problems in hyperkähler geometry, Beauville asked whether the existence of a Lagrangian torus in M gives rise to such a fibration (Question 6 in [1]).

In a joint-work [6] with R. Weiss, we prove the following partial answer to Beauville's question.

Theorem 1 *Let $A \subset M$ be a Lagrangian torus in a compact hyperkähler manifold. Then there exists a meromorphic map $f : M \dashrightarrow B$ dominant over a projective variety B, such that on a nonempty Zariski open subset $M^o \subset M$ with $A \subset M^o$, the restriction $f|_{M^o}$ is a proper holomorphic submersion onto a Zariski open subset $B^o \subset B$ and A is a fiber of f.*

In [7], Matsushita showed that if a compact hyperkähler manifold admits an almost holomorphic fibration, like the map f in Theorem 1, then it admits a regular fibration morphism. Thus Theorem 1 together with Matsushita's result gives a complete answer to Beauville's question.

The outline of the proof of Theorem 1 is as follows. By the result of Ran and Voisin (Theorem 8.7 in [2]), deformations of a Lagrangian torus $A \subset M$ give rise to a multi-valued holomorphic foliation on a Zariski open subset in M. If this foliation is univalent, Theorem 1 is easily obtainable. Thus the main issue is how to deal with the multi-valuedness. Let d be the number of sheets of this multi-valued foliation. Assuming $d > 1$, we want to derive a contradiction. Let $\mu : \tilde{M} \to M$ be the corresponding generically finite covering of degree d. By fixing a general point $x \in M$, we have the finite monodromy group G acting on $\mu^{-1}(x)$. This group G has distinguished subgroups $H_1, \ldots, H_d$ arising from the loops based at x that lie on the leaves of the multi-valued foliation through x. By a result of [3], one can assume that the subgroup $\langle H_1, \ldots, H_d \rangle$ of G acts transitively on $\mu^{-1}(x)$. From this transitivity, a purely group-theoretic argument shows that we have a pair $H_i \neq H_j$ such that the action of the subgroup $\langle H_i, H_j \rangle$ moves the i-th sheet to the j-th sheet. The proof of the last statement uses Wielandt's work on subnormal subgroups in [8]. To get a contradiction from this, we need a geometric result on the integrability of the local distribution given by a pair of sheets of multi-valued foliation. This geometric result is established by means of the theory of action-angle variables (cf.[4], Section 44) for completely integrable Hamiltonian systems.

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Compact moduli spaces for slope-semistable sheaves

Matei Toma
(joint work with Daniel Greb)

In this talk we present the construction of a compactification of the moduli spaces of slope-stable reflexive sheaves with respect to *real* ample polarizations over higher dimensional projective manifolds. In general even with respect to rational polarizations no complex analytic compactifications were previously known for moduli spaces of slope-stable vector bundles over n-dimensional projective manifolds X when $n \geq 3$. Gauge-theoretical compactifications of these moduli spaces have been constructed over base manifolds of arbitrary dimension n, cf. [5], but the existence of a complex analytic structure on them was established only for $n = 2$, [1], [2].

Let X be a projective manifold of dimension n and Picard number ρ and fix topological invariants $r \in \mathbb{Z}_{>0}$, $c_i \in H^{2i}(X, \mathbb{Z})$, $1 \leq i \leq n$ of the torsion free sheaves on X we wish to parameterize. We start by introducing a locally finite chamber structure on the ample cone $\mathrm{Amp}_{\mathbb{R}}(X) \subset \mathrm{NS}_{\mathbb{R}}(X)$ accounting for the slope-stability variation for our sheaves in dependence of the chosen polarization. When $n \geq 3$ and $\rho \geq 3$ this chamber structure differs from that introduced in [3], which is not locally finite. For $n \geq 3$ and $\rho = 2$ it had been observed in [4] that the corresponding walls may be irrational and the question of existence of complex analytic moduli spaces for real polarizations was asked. We show that slope stability with respect to any real ample polarization $H \in \mathrm{Amp}_{\mathbb{R}}(X)$ is equivalent to slope-stability with respect to some *rational multi-polarization* $(H_1, ..., H_{n-1})$, where $H_i \in \mathrm{Amp}_{\mathbb{Q}}(X)$; here we say that a torsion free sheaf E on X is slope-stable (resp. slope-semistable) with respect to $(H_1, ..., H_{n-1})$ if for any coherent subsheaf F of E of intermediate rank we have

$$\frac{c_1(F)\cdot H_1\cdot \ldots \cdot H_{n-1}}{\operatorname{rank}(F)} < \text{ (resp. } \leq) \ \frac{c_1(E)\cdot H_1\cdot \ldots \cdot H_{n-1}}{\operatorname{rank}(E)}.$$

Our main result is

Theorem 1. *Let X be an n-dimensional complex projective manifold, $H_1, \ldots, H_{n-1}$ rational ample classes on X, $c_i \in H^{2i}\big(X, \mathbb{Z}\big)$, $1 \leq i \leq n-1$ integer cohomology classes, r a positive integer, $c \in K(X)_{num}$ a class with rank r, and Chern classes $c_i(c) = c_i$, and Q a line bundle on X with $c_1(Q) = c_1 \in H^2(X, \mathbb{Z})$. Denote by $\underline{M}^{\mu ss}$ the functor that to each weakly normal scheme of finite type over $\mathbb{C}$ associates the set of isomorphism classes (modulo twists by line bundles coming from S) of S-flat families of $(H_1, \ldots, H_{n-1})$-semistable torsion-free coherent sheaves of class c and determinant Q on X. Then, there exists a class $u_{n-1} \in K(X)_{num}$, a positive integer N, a weakly normal projective variety $M^{\mu ss}$ with ample line bundle $\mathcal{O}_{M^{\mu ss}}(1)$, and a natural transformation*

$$\underline{M}^{\mu ss} \to \underline{Hom}(\cdot, M^{\mu ss})$$

with the following properties:

(1) *For any S-flat family $\mathcal{F}$ of slope-semistable sheaves of class c and determinant Q with induced classifying morphism $\Phi_{\mathcal{F}} : S \to M^{\mu ss}$ we have*

$$\Phi^*_{\mathcal{F}}(\mathcal{O}_{M^{\mu ss}}(1)) = \lambda_{\mathcal{F}}(u_{n-1})^N,$$

where $\lambda_{\mathcal{F}}(u_{n-1})$ is the determinant line bundle on S induced by $\mathcal{F}$ and u_{n-1}.

(2) *For any other polarised variety $(M', \mathcal{O}_{M'}(1))$ fulfilling the conditions spelled out in (1), there exists some positive integer d, and a uniquely determined morphism $\psi : M^{\mu ss} \to M'$ such that $\psi^*(\mathcal{O}_{M'}(1)) = \mathcal{O}_{M^{\mu ss}}(d)$.*

The triple $(M^{\mu ss}, \mathcal{O}_{M^{\mu ss}}(1), N)$ is characterized by these properties up to unique isomorphism, and up to replacing $(\mathcal{O}_{M^{\mu ss}}(1), N)$ by $(\mathcal{O}_{M^{\mu ss}}(d), dN)$.

The constructed moduli space $M^{\mu ss}$ is a compactification of the weak normalization of the moduli space of reflexive $(H_1, \ldots, H_{n-1})$-stable sheaves. More precisely we have:

Proposition 1. *There exists a natural morphism*

$$\phi : \big(M^{\mu s}_{refl}\big)^{wn} \to M^{\mu ss}$$

from the weak normalization of the moduli space of $(H_1, \ldots, H_{n-1})$-stable, reflexive sheaves $M^{\mu s}_{refl}$ to $M^{\mu ss}$ that embeds $\big(M^{\mu s}_{refl}\big)^{wn}$ as a Zariski-open subset of $M^{\mu ss}$.

Moreover when $H_1 = \ldots = H_{n-1} =: H$ there is a natural morphism $(M^{Gss})^{wn} \to M^{\mu ss}$ from the weak normalization of the moduli space of Gieseker-semistable sheaves with the given invariants on X to our moduli space of slope-semistable sheaves.

The proof of our main result generalizes Le Potier's from the two-dimensional case. In [1] Le Potier mentions that his results could be extended to higher dimensions if a restriction theorem of Mehta-Ramanathan type were available for Gieseker-H-semistable sheaves. Indeed such a result would be needed if one proceeded by restrictions to hyperplane sections on X. We avoid the Gieseker-semistability issue and instead restrict our families directly to the corresponding complete intersection curves. Here slope-semistability and Gieseker-semistability are the same. The price to pay is some loss of flatness of the restricted families. In order to overcome this difficulty we need to pass to weak normalizations for our family bases.

We expect that the constructed spaces $M^{\mu ss}$ provide complex analytic structures on the gauge-theoretical compactifications of moduli spaces of slope-stable vector bundles also in dimensions larger than two.

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Twisted Fiberwise Kähler-Einstein Metrics and Albanese Map of Manifolds with Nef Anticanonical Class

Mihai Păun

We will give here an account of our recent work [33], in which we establish the surjectivity of the Albanese map associated to a compact Kähler manifold whose anticanonical bundle is nef. This result was conjectured by J.-P. Demailly, Th. Peternell and M. Schneider in [12]. We obtain it as a consequence of a theorem concerning the variation of twisted fiberwise Kähler-Einstein metrics, which is our main technical result in [33].

Let $p : X \to Y$ be a holomorphic surjective map, where X and Y are compact Kähler manifolds. We denote by $W \subset Y$ an analytic set containing the singular values of p, and let $X_0 := p^{-1}(Y \setminus W)$. Let $\{\beta\} \in H^{1,1}(X, \mathbb{R})$ be a real cohomology class of $(1,1)$-type, which contains a non-singular, semi-positive definite representative β.

Our primary goal in [33] is to investigate the positivity properties of the class

$$c_1(K_{X/Y}) + \{\beta\},$$

which are inherited from similar *fiberwise* properties.

In this perspective, the main statement we obtain is as follows.

Theorem 1. *Let $p : X \to Y$ be a surjective map. We consider a semi-positive class $\{\beta\} \in H^{1,1}(X, \mathbb{R})$, such that the adjoint class $c_1(K_{X_y}) + \{\beta\}|_{X_y}$ is Kähler for any $y \in Y \setminus W$. Then the relative adjoint class*

$$c_1(K_{X/Y}) + \{\beta\}$$

contains a closed positive current Θ, which equals a (non-singular) semi-positive definite form on X_0.

As a consequence of the proof of the previous result, the current Θ will be greater than a Kähler metric when restricted to any relatively compact open subset of X_0, provided that β is a Kähler metric. Also, if $\beta \geq p^\star(\gamma)$ for some (1,1)-form γ on Y, then we have

$$\Theta \geq p^\star(\gamma).$$

We remark that if the class β is the first Chern class of a holomorphic $\mathbb{Q}$-line bundle L, that is to say, if

$$\{\beta\} \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q}),$$

then there are many results concerning the positivity of the twisted relative canonical bundle, cf. [2], [3], [4], [6], [14], [17], [18], [21], [22], [23], [24], [25], [28], [35], [36], [40], [41], [42] to quote only a few.

The references [35], [37] are particularly important for us; indeed, a large part of the arguments presented by G. Schumacher in [35], [36] will be used in our proof, as they rely on the complex Monge-Ampère equation as *substitute* for the theory of linear bundles used in the other works quoted above (see section 3.2 of this paper).

Before stating a few consequences of our main result, we recall the following metric version of the usual notion of *nef* line bundle in algebraic geometry, as it was introduced in [10].

Definition 1. Let (X, ω) be a compact complex manifold endowed with a hermitian metric, and let $\{\rho\}$ be a real (1,1) class on X. We say that $\{\rho\}$ is nef (in metric sense) if for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in \mathcal{C}^\infty(X)$ such that

$$\rho + \sqrt{-1}\partial\overline{\partial} f_\varepsilon \geq -\varepsilon\omega. \tag{1}$$

Thus the class $\{\rho\}$ is nef if it admits non-singular representatives with arbitrary small negative part. It was established in [11] that if X is projective and if $\{\rho\}$ is the first Chern class of a line bundle L, then L is nef in the algebro-geometric sense if and only if L is nef in metric sense.

Let $\mathcal{X} \to \mathbb{D}$ be a non-singular Kähler family over the unit disk. Then we have the following (direct) consequence of Theorem 1.1.

Corollary 1. *We assume that the bundle $K_{\mathcal{X}_t}$ is nef, for any $t \in \mathbb{D}$. Then $K_{\mathcal{X}/\mathbb{D}}$ is nef.*

We remark that in the context of the previous corollary, much more is expected to be true. For example, if the Kähler version of the *invariance of plurigenera* turns out to be true, then it would be enough to assume in Corollary 1.3 that $K_{\mathcal{X}_0}$ is pseudo-effective in order to derive the conclusion that $K_{\mathcal{X}/\mathbb{D}}$ is pseudo-effective.

The second application of Theorem 1.1 concerns the Albanese morphism associated to a compact Kähler manifold X. We denote by $q := h^0(X, T_X^\star)$ the irregularity of X, and let

$$\mathrm{Alb}(X) := H^0(X, T_X^\star)^\star / H_1(X, \mathbb{Z})$$

be the Albanese torus of associated to X. We recall that the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is defined as follows

$$\alpha_X(p)(\gamma) := \int_{p_0}^{p} \gamma$$

modulo the group $H_1(X, \mathbb{Z})$, i.e. modulo the integral of γ along loops at p_0.

We assume that $-K_X$ is nef, in the sense of the definition above. It was conjectured by J.-P. Demailly, Th. Peternell and M. Schneider in [12] that α_X is surjective; some particular cases of this problem are established in [12], [32], [7]. If X is assumed to be *projective*, then the surjectivity of the Albanese map was established by Q. Zhang in [45], by using in an essential manner the *char p* methods. More recently, in the article [46], the same author provides an alternative proof of this result, based on the semi-positivity of direct images.

We settle here the conjecture in full generality.

Theorem 2. *Let X be a compact Kähler manifold such that $-K_X$ is nef. Then its Albanese morphism $\alpha_X : X \to \mathrm{Alb}(X)$ is surjective.*

Besides Theorem 0.1, our proof is using some ideas from [12] and [5]; we sketch next an argument in which we still need to assume that X is projectuve, but which can be adapted to the Kähler case via 0.1.

To begin with, as a consequence of the semi-positivity results in [4], [38] we infer the next statement.

Corollary 2. *Let $p : X \to Y$ be a surjective map between non-singular projective manifolds. We consider $L \to X$ a nef line bundle, such that $H^0\big(X_y, K_{X_y} + L|_{X_y}\big) \neq 0$. Then the bundle $K_{X/Y} + L$ is pseudo-effective.*

We refer e.g. to [33] for the proof. □

Let X be a non-singular manifold such that $-K_X$ is nef, and let

$$\alpha_X : X \to \mathrm{Alb}(X)$$

be its Albanese morphism. We assume that α_X is not surjective; let

$$Y \subsetneq \mathrm{Alb}(X)$$

be the image of α_X. We denote by $\pi_Y : \widehat{Y} \to Y$ the desingularization of Y, and let $p : \widehat{X} \to \widehat{Y}$ be the map obtained by resolving the indeterminacy of the rational map $X \dashrightarrow \widehat{Y}$.

The idea is to apply Corollary 0.5 with the following data

$$X := \widehat{X}, \quad Y := \widehat{Y}$$

and $L := \pi_X^\star(-K_X)$; here we denote by $\pi_X : \widehat{X} \to X$ the modification of X, so that we have

$$\pi_Y \circ p = \alpha_X \circ \pi_X.$$

The hypothesis required by Corollary 0.5 are quickly seen to be verified: indeed, the nefness of the bundle L is due to the fact that $-K_X$ is nef, and if we denote by E the effective divisor such that

$$K_{\widehat{X}} = \pi^\star(K_X) + E \tag{2}$$

then we see that $K_{\widehat{X}_y} + L$ is simply equal to $E|_{\widehat{X}_y}$. This bundle is clearly effective.

Hence we infer that the bundle

$$K_{\widehat{X}/\widehat{Y}} + \pi_X^\star(-K_X) \tag{3}$$

is pseudo-effective. But this bundle equals $E - p^\star(K_{\widehat{Y}})$; let Λ be a closed positive current in the class corresponding to $E - p^\star(K_{\widehat{Y}})$. Since the Kodaira dimension of $K_{\widehat{Y}}$ is at least 1 (we refer to [20] for a justification of this property), we obtain two $\mathbb{Q}$-effective divisors say $W_1 \neq W_2$ linearly equivalent with $K_{\widehat{Y}}$. As a conclusion, we obtain two different closed positive currents belonging to the class of the exceptional divisor E, namely $\Lambda + p^\star(W_j)$ for $j = 1, 2$. This is of course absurd. □

Remark 1. The proof above shows that Theorem 0.4 still holds true if X is projective, and if we replace the hypothesis $-K_X$ *nef* with the hypothesis $-K_X$ *pseudo-effective, and the multiplier ideal sheaf associated to some of its positively curved metrics is equal to the structural sheaf*. The arguments used are absolutely similar.

Remark 2. Let X be a Fano manifold, and let $p : X \to Y$ be a submersion onto a non-singular manifold Y. Then it follows that Y is Fano as well (see [26], [16]). This result can be obtained *via* the following elegant argument, very recently found and explained to us by S. Boucksom cf. [5]. By the results e.g. in [2], the direct image of the bundle

$$K_{X/Y} + L$$

is positive provided that L is an ample line bundle. We take $L = -K_X$ and we are done. A similar idea, ε-close to our arguments in this section can be found in the article [12] by J.-P. Demailly, Th. Peternell and M.Schneider (cf. the proof of their Theorem 2.4). □

In order to prove Theorem 0.1, we show that the so-called *fiberwise twisted Kähler-Einstein* metric endows the bundle $K_{X/Y}|_{X_0}$ with a metric whose curvature is bounded from below by $-\beta$. Thus, the twisted version of the psh variation of the Kähler-Einstein metric established in [35] holds true. Finally, we show that the local weights of the metric constructed in this way are bounded near the analytic set $X \setminus X_0$. In order to establish this crucial fact we combine the approximation theorem in [13], together with a precise version of the Ohsawa-Takegoshi extension theorem, [4]. The difficulty steams from the fact that in order to establish the estimates for the said weights we cannot rely on the geometry of the manifold X_y, as y is approaching a singular value of the map p.

Once this result is proved, the general case of Theorem 0.4 is obtained along the lines of the projective case discussed above.

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