

NIKULIN INVOLUTIONS ON K3 SURFACES

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ABSTRACT. We study the maps induced on cohomology by a Nikulin (i.e. a symplectic) involution on a K3 surface. We parametrize the eleven dimensional irreducible components of the moduli space of algebraic K3 surfaces with a Nikulin involution and we give examples of the general K3 surface in various components. We conclude with some remarks on Morrison-Nikulin involutions, these are Nikulin involutions which interchange two copies of $E_8(-1)$ in the Néron Severi group.

In his paper [Ni1] Nikulin started the study of finite groups of automorphisms on K3 surfaces, in particular those leaving the holomorphic two form invariant, these are called *symplectic*. He proves that when the group G is cyclic and acts symplectically, then $G \cong \mathbf{Z}/n\mathbf{Z}$, $1 \leq n \leq 8$. Symplectic automorphisms of K3 surfaces of orders three, five and seven are investigated in the paper [GS]. Here we consider the case of $G \cong \mathbf{Z}/2\mathbf{Z}$, generated by a symplectic involution ι . Such involutions are called *Nikulin involutions* (cf.[Mo, Definition 5.1]). A Nikulin involution on the K3 surface X has eight fixed points, hence the quotient $\bar{Y} = X/\iota$ has eight nodes, by blowing them up one obtains a K3 surface Y .

In the paper [Mo] Morrison studies such involutions on algebraic K3 surfaces with Picard number $\rho \geq 17$ and in particular on those surfaces whose Néron Severi group contains two copies of $E_8(-1)$. These K3 surfaces always admit a Nikulin involution which interchanges the two copies of $E_8(-1)$. We call such involutions *Morrison-Nikulin involutions*.

The paper of Morrison motivated us to investigate Nikulin involutions in general. After a study of the maps on the cohomology induced by the quotient map, in the second section we show that an algebraic K3 surface with a Nikulin involution has $\rho \geq 9$ and that the Néron Severi group contains a primitive sublattice isomorphic with $E_8(-2)$. Moreover if $\rho = 9$ (the minimal possible) then the following two propositions are the central results in the paper:

Proposition 2.2. Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group $NS(X)$ of X has rank nine. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:

- (1) in case $L^2 \equiv 2 \pmod{4}$ we have $\Lambda_{2d} = NS(X)$;

The second author is supported by DFG Research Grant SA 1380/1-1.

2000 *Mathematics Subject Classification*: 14J28, 14J10.

Key words: K3 surfaces, automorphisms, moduli.

- (2) in case $L^2 \equiv 0 \pmod{4}$ we have that either $NS(X) \cong \Lambda_{2d}$ or $NS(X) \cong \Lambda_{\widetilde{2d}}$ where $\Lambda_{\widetilde{2d}}$ is the unique even lattice containing Λ_{2d} with $\Lambda_{\widetilde{2d}}/\Lambda_{2d} \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\Lambda_{\widetilde{2d}}$.

Proposition 2.3. Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X, \mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

The coarse moduli space of Γ -polarized K3 surfaces has dimension 11 and will be denoted by \mathcal{M}_{2d} if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Thus we classified all the algebraic K3 surfaces with Picard number nine with a Nikulin involution. For the proofs we use lattice theory and the surjectivity of the period map for K3 surfaces. We also study the ι^* -invariant line bundle L on the general member of each family, for example in Proposition 2.7 we decompose the space $\mathbf{P}H^0(X, L)^*$ into ι^* -eigenspaces. This result is fundamental for the description of the ι -equivariant map $X \rightarrow \mathbf{P}H^0(X, L)^*$. In section three we discuss various examples of the general K3 surface in these moduli spaces, recovering well-known classical geometry in a few cases. We also describe the quotient surface \bar{Y} .

In the last section we give examples of K3 surfaces with an elliptic fibration and a Nikulin involution which is induced by translation by a section of order two in the Mordell-Weil group of the fibration. Such a family has only ten moduli, and the minimal resolution of the quotient K3 surface Y is again a member of the same family. By using elliptic fibrations we also give an example of K3 surfaces with a Morrison-Nikulin involution. These surfaces with involution are parametrized by three dimensional moduli spaces. The Morrison-Nikulin involutions have interesting applications towards the Hodge conjecture for products of K3 surfaces (cf. [Mo], [GL]). In section 2.4 we briefly discuss possible applications of the more general Nikulin involutions.

1. GENERAL RESULTS ON NIKULIN INVOLUTIONS

1.1. Nikulin's uniqueness result. A Nikulin involution ι of a K3 surface X is an automorphism of order two such that $\iota^*\omega = \omega$ for all $\omega \in H^{2,0}(X)$. That is, ι preserves the holomorphic two form and thus it is a symplectic involution. Nikulin, [Ni1, Theorem 4.7], proved that any abelian group G which acts symplectically on a K3 surface, has a unique, up to isometry, action on $H^2(X, \mathbf{Z})$.

1.2. Action on cohomology. D. Morrison ([Mo, proof of Theorem 5.7],) observed that there exist K3 surfaces with a Nikulin involution which acts in the following way on the second cohomology group:

$$\iota^* : H^2(X, \mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \longrightarrow H^2(X, \mathbf{Z}), \quad (u, x, y) \longmapsto (u, y, x).$$

Thus for *any* K3 surface X with a Nikulin involution ι there is an isomorphism $H^2(X, \mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)$ such that ι^* acts as above.

Given a free \mathbf{Z} -module M with an involution g , there is an isomorphism

$$(M, g) \cong M_1^s \oplus M_{-1}^t \oplus M_p^r,$$

for unique integers r, s, t (cf. [R]), where:

$$M_1 := (\mathbf{Z}, \iota_1 = 1), \quad M_{-1} := (\mathbf{Z}, \iota_{-1} = -1), \quad M_p := \left(\mathbf{Z}^2, \iota_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Thus for a Nikulin involution acting on $H^2(X, \mathbf{Z})$ the invariants are $(s, t, r) = (6, 0, 8)$.

1.3. The invariant lattice. The invariant sublattice is:

$$H^2(X, \mathbf{Z})^\iota \cong \{(u, x, x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong U^3 \oplus E_8(-2).$$

The anti-invariant lattice is the lattice perpendicular to the invariant sublattice:

$$(H^2(X, \mathbf{Z})^\iota)^\perp \cong \{(0, x, -x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong E_8(-2).$$

The sublattices $H^2(X, \mathbf{Z})^\iota$ and $(H^2(X, \mathbf{Z})^\iota)^\perp$ are obviously primitive sublattices of $H^2(X, \mathbf{Z})$.

1.4. The standard diagram. The fixed point set of a Nikulin involution consists of exactly eight points ([Ni1, section 5]). Let $\beta : \tilde{X} \rightarrow X$ be the blow-up of X in the eight fixed points of ι . We denote by $\tilde{\iota}$ the involution on \tilde{X} induced by ι . Moreover, let $\bar{Y} = X/\iota$ be the eight-nodal quotient of X , and let $Y = \tilde{X}/\tilde{\iota}$ be the minimal model of \bar{Y} , so Y is a K3 surface. This gives the ‘standard diagram’:

$$\begin{array}{ccc} X & \xleftarrow{\beta} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ \bar{Y} & \longleftarrow & Y. \end{array}$$

We denote by $E_i, i = 1, \dots, 8$ the exceptional divisors in \tilde{X} over the fixed points of ι in X , and by $N_i = \pi(E_i)$ their images in Y , these are (-2) -curves.

1.5. The Nikulin lattice. The minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ containing the N_i is called the Nikulin lattice N (cf. [Mo, section 5]). As $N_i^2 = -2, N_i N_j = 0$ for $i \neq j$, the Nikulin lattice contains the lattice $\langle -2 \rangle^8$. The lattice N has rank eight and is spanned by the N_i and a class \hat{N} :

$$N = \langle N_1, \dots, N_8, \hat{N} \rangle, \quad \hat{N} := (N_1 + \dots + N_8)/2.$$

A set of 8 rational curves on a K3 surface whose sum is divisible by 2 in the Néron Severi group is called an even set, see [B] and section 3 for examples.

1.6. The cohomology of \tilde{X} . It is well-known that

$$H^2(\tilde{X}, \mathbf{Z}) \cong H^2(X, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z}E_i) \cong U^3 \oplus E_8(-1)^2 \oplus \langle -1 \rangle^8.$$

For a smooth surface S with torsion free $H^2(S, \mathbf{Z})$, the intersection pairing, given by the cup product to $H^4(S, \mathbf{Z}) = \mathbf{Z}$, gives an isomorphism $H^2(S, \mathbf{Z}) \rightarrow \text{Hom}_{\mathbf{Z}}(H^2(S, \mathbf{Z}), \mathbf{Z})$.

The map β^* is:

$$\beta^* : H^2(X, \mathbf{Z}) \longrightarrow H^2(\tilde{X}, \mathbf{Z}) = H^2(Y, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z}E_i), \quad x \longmapsto (x, 0),$$

and its dual $\beta_* : H^2(\tilde{X}, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z})$ is $(x, e) \mapsto x$.

Let $\pi : \tilde{X} \rightarrow Y$ be the quotient map, let $\pi^* : H^2(Y, \mathbf{Z}) \rightarrow H^2(\tilde{X}, \mathbf{Z})$ be the induced map on the cohomology and let $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \rightarrow H^2(Y, \mathbf{Z})$ be its dual, so:

$$\pi_* a \cdot b = a \cdot \pi^* b \quad (a \in H^2(\tilde{X}, \mathbf{Z}), b \in H^2(Y, \mathbf{Z})).$$

Moreover, as π^* is compatible with cup product we have:

$$\pi^* b \cdot \pi^* c = 2(b \cdot c) \quad (b, c \in H^2(Y, \mathbf{Z})).$$

1.7. Lattices. For a lattice $M := (M, b)$, where b is a \mathbf{Z} -valued bilinear form on a free \mathbf{Z} -module M , and an integer n we let $M(n) := (M, nb)$. In particular, M and $M(n)$ have the same underlying \mathbf{Z} -module, but the identity map $M \rightarrow M(n)$ is *not* an isometry unless $n = 1$ or $M = 0$.

1.8. Proposition. Using the notations and conventions as above, the map $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \rightarrow H^2(Y, \mathbf{Z})$ is given by

$$\pi_* : U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -1 \rangle^8 \rightarrow U(2)^3 \oplus N \oplus E_8(-1) \hookrightarrow H^2(Y, \mathbf{Z}),$$

$$\pi_* : (u, x, y, z) \mapsto (u, z, x + y).$$

The map π^* , on the sublattice $U(2)^3 \oplus N \oplus E_8(-1)$ of $H^2(Y, \mathbf{Z})$ is given by:

$$\pi^* : U(2)^3 \oplus N \oplus E_8(-1) \hookrightarrow H^2(\tilde{X}, \mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -1 \rangle^8,$$

$$\pi^* : (u, n, x) \mapsto (2u, x, x, 2\tilde{n}),$$

here if $n = \sum n_i N_i$, $\tilde{n} = \sum n_i E_i$.

Proof. This follows easily from the results of Morrison. In the proof of [Mo, Theorem 5.7], it is shown that the image of each copy of $E_8(-1)$ under π_* is isomorphic to $E_8(-1)$. As $E_8(-1)$ is unimodular, it is a direct summand of the image of π_* . As $\pi_* \iota^* = \pi_*$, we get that $\pi_*(0, x, 0, 0) = \pi_*(0, 0, y, 0) \in E_8(-1)$. The $\langle -1 \rangle^8$ maps into N (the image has index two). As U^3 is a direct summand of $H^2(X, \mathbf{Z})^\iota$, [Mo, Proposition 3.2] gives the first component.

As π_* and π^* are dual maps, $\pi^* a = b$ if for all $c \in H^2(\tilde{X}, \mathbf{Z})$ one has $(b \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y$. In particular, if $a \in U(2)^3$ and $c \in U^3$ we get $(\pi^* a \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y = 2(a \cdot c)_{\tilde{X}}$ since we compute in $U(2)^3$, hence $\pi^* a = 2a$. Similarly, $(\pi^* N_i \cdot E_j)_{\tilde{X}} = (N_i \cdot \pi_* E_j)_Y = -2\delta_{ij}$, so $\pi^* N_i = 2E_i$ (this also follows from the fact that the N_i are classes of the branch curves, so $\pi^* N_i$ is twice the class of $\pi^{-1}(N_i) = E_i$). Finally for $x \in E_8(-1)$ and $(y, 0) \in E_8(-1)^2$ we have $(\pi^* x \cdot (y, 0))_{\tilde{X}} = (x \cdot \pi_*(y, 0))_Y = (x \cdot y)_Y$ and also $(\pi^* x \cdot (0, y))_{\tilde{X}} = (x \cdot y)_Y$, so $\pi^* x = (x, x) \in E_8(-1)^2$. \square

1.9. Extending π^* . To determine the homomorphism $\pi^* : H^2(Y, \mathbf{Z}) \rightarrow H^2(\tilde{X}, \mathbf{Z})$ on all of $H^2(Y, \mathbf{Z})$, and not just on the sublattice of finite index $U(2)^3 \oplus N \oplus E_8(-1)$ we need to study the embedding $U(2)^3 \oplus N \hookrightarrow U^3 \oplus E_8(-1)$. This is done below. For any $x \in U^3 \oplus E_8(-1)$, one has $2x \in U(2)^3 \oplus N$ and $\pi^*(2x)$ determined as in Proposition 1.8. As π^* is a homomorphism and lattices are torsion free, one finds $\pi^* x$ as $\pi^* x = (\pi^*(2x))/2$.

1.10. **Lemma.** The sublattice of $(U(2)^3 \oplus N) \otimes \mathbf{Q}$ generated by $U(2)^3 \oplus N$ and the following six elements, each divided by two, is isomorphic to $U^3 \oplus E_8(-1)$:

$$\begin{aligned} e_1 + (N_1 + N_2 + N_3 + N_8), & \quad e_2 + (N_1 + N_5 + N_6 + N_8), & \quad e_3 + (N_2 + N_6 + N_7 + N_8), \\ f_1 + (N_1 + N_2 + N_4 + N_8), & \quad f_2 + (N_1 + N_5 + N_7 + N_8), & \quad f_3 + (N_3 + N_4 + N_5 + N_8), \end{aligned}$$

here e_i, f_i are the standard basis of the i -th copy of $U(2)$ in $U(2)^3$. Any embedding of $U(2)^3 \oplus N$ into $U^3 \oplus E_8(-1)$ such that the image of N is primitive in $U^3 \oplus E_8(-1)$ is isometric to this embedding.

Proof. The theory of embeddings of lattices can be found in [Ni2, section 1]. The dual lattice M^* of a lattice $M = (M, b)$ is

$$M^* = \text{Hom}(M, \mathbf{Z}) = \{x \in M \otimes \mathbf{Q} : b(x, m) \in \mathbf{Z} \ \forall m \in M\}.$$

Note that $M \hookrightarrow M^*$, intrinsically by $m \mapsto b(m, -)$ and concretely by $m \mapsto m \otimes 1$. If (M, b_M) and (L, b_L) are lattices such that $M \hookrightarrow L$, that is $b_M(m, m') = b_L(m, m')$ for $m, m' \in M$, then we have a map $L \rightarrow M^*$ by $l \mapsto b_L(l, -)$. In case M has finite index in L , so $M \otimes \mathbf{Q} \cong L \otimes \mathbf{Q}$, we get inclusions:

$$M \hookrightarrow L \hookrightarrow L^* \hookrightarrow M^*.$$

Therefore L is determined by the image of L/M in the finite group $A_M := M^*/M$, the discriminant group of M .

Since $b = b_M$ extends to a \mathbf{Z} -valued bilinear form on $L \subset M^*$ we get $q(l) := b_L(l, l) \in \mathbf{Z}$ for $l \in L$. If L is an even lattice, the discriminant form

$$q_M : A_M \longrightarrow \mathbf{Q}/2\mathbf{Z}, \quad m^* \longmapsto b_L(m^*, m^*)$$

is identically zero on the subgroup $L/M \subset A_M$. In this way one gets a bijection between even overlattices of M and isotropic subgroups of A_M . In our case $M = K \oplus N$, with $K = U(2)^3$, so $A_M = A_K \oplus A_N$ and an isotropic subgroup of A_M is the direct sum of an isotropic subgroup of A_K and one isotropic subgroup of A_N . We will see that $(A_K, q_K) \cong (A_N, -q_N)$, hence the even unimodular overlattices L of M , with N primitive in L , correspond to isomorphisms $\gamma : A_N \rightarrow A_K$ with $q_N = -q_K \circ \gamma$. Then one has that

$$L/M = \{(\gamma(\bar{n}), \bar{n}) \in A_M = A_K \oplus A_N : \bar{n} \in A_N\}.$$

The overlattice L_γ corresponding to γ is:

$$L_\gamma := \{(u, n) \in K^* \oplus N^* : \gamma(\bar{n}) = \bar{u}\}.$$

We will show that the isomorphism γ is unique up to isometries of K and N .

Let e, f be the standard basis of U , so $e^2 = f^2 = 0, ef = 1$, then $U(2)$ has the same basis with $e^2 = f^2 = 0, ef = 2$. Thus $U(2)^*$ has basis $e/2, f/2$ with $(e/2)^2 = (f/2)^2 = 0, (e/2)(f/2) = 2/4 = 1/2$. Thus $A_K = (U(2)^*/U(2))^3 \cong (\mathbf{Z}/2\mathbf{Z})^6$, and the discriminant form q_K on A_K is given by

$$q_K : A_K = (\mathbf{Z}/2\mathbf{Z})^6 \longrightarrow \mathbf{Z}/2\mathbf{Z}, \quad q_K(x) = x_1x_2 + x_3x_4 + x_5x_6.$$

The Nikulin lattice N contains $\bigoplus \mathbf{Z}N_i$ with $N_i^2 = -2$, hence $N^* \subset \mathbf{Z}(N_i/2)$. As $N = \langle N_i, (\sum N_i)/2 \rangle$ we find that $n^* \in \mathbf{Z}(N_i/2)$ is in N^* iff $n^* \cdot (\sum N_i)/2 \in \mathbf{Z}$, that is, $n^* = \sum x_i(N_i/2)$ with $\sum x_i \equiv 0 \pmod{2}$. Thus we obtain an identification:

$$A_N = N^*/N = \{(x_1, \dots, x_8) \in (\mathbf{Z}/2\mathbf{Z})^8 : \sum x_i = 0\} / \langle (1, \dots, 1) \rangle \cong (\mathbf{Z}/2\mathbf{Z})^6,$$

where $(1, \dots, 1)$ is the image of $(\sum N_i)/2$. Any element in A_N has a unique representative which is either 0, $(N_i + N_j)/2$, with $i \neq j$ and $((N_i + N_j)/2)^2 = 1 \pmod{2\mathbf{Z}}$, or $(N_1 + N_i + N_j + N_k)/2 = (N_l + N_m + N_n + N_r)/2$, with distinct indices and with $\{i, \dots, r\} = \{2, \dots, 8\}$ and $((N_1 + N_i + N_j + N_k)/2)^2 = 0 \pmod{2}$. The quadratic spaces, over the field $\mathbf{Z}/2\mathbf{Z}$, $((\mathbf{Z}/2\mathbf{Z})^6, q_K)$ and $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ are isomorphic, an explicit isomorphism is defined by

$$\gamma : A_N \longrightarrow A_K, \quad \gamma((N_1 + N_2 + N_3 + N_8)/2) = e_1/2,$$

etc. where we use the six elements listed in the lemma.

The orthogonal group of the quadratic space $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ obviously contains S_8 , induced by permutations of the basis vectors in $(\mathbf{Z}/2\mathbf{Z})^8$, and these groups are actually equal cf. [Co]. Thus any two isomorphisms $A_N \rightarrow A_K$ preserving the quadratic forms differ by an isometry of A_N which is induced by a permutation of the nodal classes N_1, \dots, N_8 . A permutation of the 8 nodal curves N_i in N obviously extends to an isometry of N .

This shows that such an even unimodular overlattice of $U(2)^3 \oplus N$ is essentially unique. As these are classified by their rank and signature, the only possible one is $U^3 \oplus E_8(-1)$. Using the isomorphism γ , one obtains the lattice L_γ , which is described in the lemma. \square

1.11. The lattices $N \oplus N$ and Γ_{16} . Using the methods of the proof of Lemma 1.10 we show that any even unimodular overlattice L of $N \oplus N$ such that $N \oplus \{0\}$ is primitive in L , is isomorphic to the Barnes-Wall lattice $\Gamma_{16}(-1)$ (cf. [Se, Chapter V, 1.4.3]). The lattice $\Gamma_{16}(-1)$ is the unique even unimodular negative definite lattice which is not generated by its roots, i.e. by vectors v with $v^2 = -2$.

The discriminant form q_N of the lattice N has values in $\mathbf{Z}/2\mathbf{Z}$, hence $q_N = -q_N$. Therefore isomorphisms $\gamma : N \rightarrow N$ correspond to the even unimodular overlattices L_γ of $N \oplus N$ with $N \oplus \{0\}$ primitive in L_γ . Since $N \oplus N$ is negative definite, so is L_γ . The uniqueness of the overlattice follows, as before, from the fact $O(q_N) \cong S_8$. To see that this overlattice is $\Gamma_{16}(-1)$, recall that

$$\Gamma_{16} = \{x = (x_1, \dots, x_{16}) \in \mathbf{Q}^{16} : 2x_i \in \mathbf{Z}, x_i - x_j \in \mathbf{Z}, \sum x_i \in 2\mathbf{Z}\},$$

and the bilinear form on Γ_{16} is given by $\sum x_i y_i$. Let e_i be the standard basis vectors of \mathbf{Q}^{16} . As

$$N \oplus N \hookrightarrow \Gamma_{16}(-1), \quad (N_i, 0) \mapsto e_i + e_{i+8}, \quad (0, N_i) \mapsto e_i - e_{i+8},$$

is a primitive embedding $N \oplus N$ into $\Gamma_{16}(-1)$ (note $(\hat{N}, 0) \mapsto (\sum e_i)/2 \in \Gamma_{16}$, $(0, \hat{N}) \mapsto ((\sum_{i=1}^8 e_i) - (\sum_{i=9}^{16} e_i))/2 \in \Gamma_{16}$) the claim follows.

2. ELEVEN DIMENSIONAL FAMILIES OF K3 SURFACES WITH A NIKULIN INVOLUTION

2.1. **Néron Severi groups.** As X is a K3 surface it has $H^{1,0}(X) = 0$ and

$$\text{Pic}(X) = NS(X) = H^{1,1}(X) \cap H^2(X, \mathbf{Z}) = \{x \in H^2(X, \mathbf{Z}) : x \cdot \omega = 0 \forall \omega \in H^{2,0}(X)\}.$$

For $x \in (H^2(X, \mathbf{Z})^\iota)^\perp$ we have $\iota^*x = -x$. As $\iota^*\omega = \omega$ for $\omega \in H^{2,0}(X)$ we get:

$$\omega \cdot x = \iota^*\omega \cdot \iota^*x = -\omega \cdot x \quad \text{hence} \quad (H^2(X, \mathbf{Z})^\iota)^\perp \subset NS(X).$$

As we assume X to be algebraic, there is a very ample line bundle M on X , so $M \in NS(X)$ and $M^2 > 0$. Therefore the Néron Severi group of X contains $E_8(-2) \cong (H^2(X, \mathbf{Z})^\iota)^\perp$ as a primitive sublattice and has rank at least 9.

The following proposition gives all even, rank 9, lattices of signature $(1+, 8-)$ which contain $E_8(-2)$ as a primitive sublattice. We will show in Proposition 2.3 that any of these lattices is the Néron Severi group of a K3 surface with a Nikulin involution. Moreover, the moduli space of K3 surfaces, which contain such a lattice in the Néron Severi group, is an 11-dimensional complex variety.

2.2. **Proposition.** Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group of X has rank 9. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda = \Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:

- (1) in case $L^2 \equiv 2 \pmod{4}$ we have $\Lambda = NS(X)$;
- (2) in case $L^2 \equiv 0 \pmod{4}$ we have that either $NS(X) = \Lambda$ or $NS(X) \cong \tilde{\Lambda}$ where $\tilde{\Lambda} = \Lambda_{\tilde{2d}}$ is the unique even lattice containing Λ with $\tilde{\Lambda}/\Lambda \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$.

Proof. As $L^2 > 0$, either L or $-L$ is effective, so may assume that L is effective. As there are no (-2) -curves in $L^\perp = E_8(-2)$, any (-2) -curve N has class $aL + e$ with $a \in \mathbf{Z}_{>0}$ and $e \in E_8(-2)$. Thus $NL = aL^2 > 0$ and therefore L is ample.

From the definition of L and the description of the action of ι on $H^2(X, \mathbf{Z})$ it follows that $\mathbf{Z}L$ and $E_8(-2)$ respectively are primitive sublattices of $NS(X)$. The discriminant group of $\langle L \rangle$ is $A_L := \langle L \rangle^* / \langle L \rangle \cong \mathbf{Z}/2d\mathbf{Z}$ with generator $(1/2d)L$ where $L^2 = 2d$ and thus $q_L((1/2d)L) = 1/2d$. The discriminant group of $E_8(-2)$ is $A_E \cong (1/2)E_8(-2)/E_8(-2) \cong (\mathbf{Z}/2\mathbf{Z})^8$, as the quadratic form on $E_8(-2)$ takes values in $4\mathbf{Z}$, the discriminant form q_E takes values in $\mathbf{Z}/2\mathbf{Z}$.

The even lattices $\tilde{\Lambda}$ which have Λ as sublattice of finite index correspond to isotropic subgroups H of $A_L \oplus A_E$ where $A_L := \langle L \rangle^* / \langle L \rangle \cong \mathbf{Z}/2d\mathbf{Z}$. If $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$, H must have trivial intersection with both A_L and A_E . Since A_E is two-torsion, it follows that H is generated by $((1/2)L, v/2)$ for some $v \in E_8(-2)$. As $((1/2)L)^2 = d/2 \pmod{2\mathbf{Z}}$ and $(v/2)^2 \in \mathbf{Z}/2\mathbf{Z}$, for H to be isotropic, d must be even. Moreover, if $d = 4m + 2$ we must have $v^2 = 8k + 4$ for some k and if $d = 4m$ we must have $v^2 = 8k$. Conversely, such a $v \in E_8(-2)$ defines an isotropic subgroup $\langle (L/2, v/2) \rangle \subset A_L \oplus A_E$ which corresponds to an overlattice $\tilde{\Lambda}$.

The group $O(E_8(-2))$ contains $W(E_8)$ (cf. [Co]) which maps onto $O(q_E)$. As $O(q_E)$ has three orbits on A_E , they are $\{0\}$, $\{v/2 : (v/2)^2 \equiv 0 \pmod{2}\}$ and $\{v/2 : (v/2)^2 \equiv 1 \pmod{2}\}$, the overlattice is unique up to isometry. \square

2.3. Proposition. Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X, \mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

The coarse moduli space of Γ -polarized K3 surfaces has dimension 11 and will be denoted by \mathcal{M}_{2d} if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Proof. We show that there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Lambda_{\widetilde{2d}}$ and under this isomorphism $(H^2(X, \mathbf{Z})^\iota)^\perp \cong E_8(-2)$. The case $NS(X) \cong \Lambda_{2d}$ is similar but easier and is left to the reader.

The primitive embedding of $\Lambda_{\widetilde{2d}}$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1], and we will identify $\Lambda_{\widetilde{2d}}$ with a primitive sublattice of $U^3 \oplus E_8(-1)^2$ from now on. We choose an $\omega \in \Lambda_{\widetilde{2d}}^\perp \otimes_{\mathbf{Z}} \mathbf{C}$ with $\omega^2 = 0$, $\omega\bar{\omega} > 0$ and general with these properties, hence $\omega^\perp \cap (U^3 \oplus E_8(-1)^2) = \Lambda_{\widetilde{2d}}$. By the ‘surjectivity of the period map’, there exists a K3 surface X with an isomorphism $H^2(X, \mathbf{Z}) \cong U^3 \oplus E_8(-1)^2$ such that $NS(X) \cong \Lambda_{\widetilde{2d}}$.

The involution of $\Lambda = \mathbf{Z}L \oplus E_8(-2)$ which is trivial on L and -1 on $E_8(-2)$, extends to an involution of $\Lambda_{\widetilde{2d}} = \Lambda + \mathbf{Z}(L/2, v/2)$. The involution is trivial on the discriminant group of $\Lambda_{\widetilde{2d}}$ which is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^6$. Therefore it extends to an involution ι_0 of $U^3 \oplus E_8(-1)^2$ which is trivial on $\Lambda_{\widetilde{2d}}^\perp$. As $((U^3 \oplus E_8(-1)^2)^{\iota_0})^\perp = E_8(-2)$ is negative definite, contains no (-2) -classes and is contained in $NS(X)$, results of Nikulin ([Ni1, Theorems 4.3, 4.7, 4.15]) show that X has a Nikulin involution ι such that $\iota^* = \iota_0$ up to conjugation by an element of the Weyl group of X . Since we assume L to be ample and the ample cone is a fundamental domain for the Weyl group action, we do get $\iota^* = \iota_0$, hence $(H^2(X, \mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

For the precise definition of Γ -polarized K3 surfaces we refer to [Do]. We just observe that each point of the moduli space corresponds to a K3 surface X with a primitive embedding $\Gamma \hookrightarrow NS(X)$. The moduli space is a quotient of the 11-dimensional domain

$$\mathcal{D}_\Gamma = \{\omega \in \mathbf{P}(\Gamma^\perp \otimes_{\mathbf{Z}} \mathbf{C}) : \omega^2 = 0, \omega\bar{\omega} > 0\}$$

by an arithmetic subgroup of $O(\Gamma)$. \square

2.4. Note on the Hodge conjecture. For a smooth projective surface S with torsion free $H^2(S, \mathbf{Z})$, let $T_S := NS(S)^\perp \subset H^2(S, \mathbf{Z})$ and let $T_{S, \mathbf{Q}} = T_S \otimes_{\mathbf{Z}} \mathbf{Q}$. Then T_S , the transcendental lattice of S , is an (integral, polarized) weight two Hodge structure.

The results in section 1 show that $\pi_* \circ \beta^*$ induces an isomorphism of rational Hodge structures:

$$\phi_\iota : T_{X, \mathbf{Q}} \xrightarrow{\cong} T_{Y, \mathbf{Q}},$$

in fact, both are isomorphic to $T_{\widetilde{X}, \mathbf{Q}}$. Any homomorphism of rational Hodge structures $\phi : T_{X, \mathbf{Q}} \rightarrow T_{Y, \mathbf{Q}}$ defines, using projection and inclusion, a map of Hodge structures $H^2(X, \mathbf{Q}) \rightarrow T_{X, \mathbf{Q}} \rightarrow T_{Y, \mathbf{Q}} \hookrightarrow H^2(Y, \mathbf{Q})$ and thus it gives a Hodge (2,2)-class

$$\phi \in H^2(X, \mathbf{Q})^* \otimes H^2(Y, \mathbf{Q}) \cong H^2(X, \mathbf{Q}) \otimes H^2(Y, \mathbf{Q}) \hookrightarrow H^4(X \times Y, \mathbf{Q}),$$

where we use Poincaré duality and the Künneth formula. Obviously, the isomorphism $\phi_\iota : T_{X,\mathbf{Q}} \rightarrow T_{Y,\mathbf{Q}}$ corresponds to the class of the codimension two cycle which is the image of \tilde{X} in $X \times Y$ under (β, π) .

Mukai showed that any homomorphism between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ where S and Z are K3 surfaces which is moreover an isometry (w.r.t. the quadratic forms induced by the intersection forms) is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \leq 11$ ([Mu, Corollary 1.10]). Nikulin, [Ni3, Theorem 3], strengthened this result and showed that it suffices that $NS(X)$ contains a class e with $e^2 = 0$. In particular, this implies that any Hodge isometry $T_{S,\mathbf{Q}} \rightarrow T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \leq 18$ (cf. [Ni3, proof of Theorem 3]).

The Hodge conjecture predicts that any homomorphism of Hodge structures between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle, *without* requiring that it is an isometry. There are few results in this direction, it is therefore maybe worth noticing that ϕ_ι is not an isometry if T_X has odd rank, see the proposition below. In [GL] a similar result of D. Morrison in a more special case is used to obtain new results on the Hodge conjecture. In Proposition 4.2 we show that there exists a K3 surface X with Nikulin involution where $T_{X,\mathbf{Q}}$ has even rank and $T_{X,\mathbf{Q}}$ is isometric to $T_{Y,\mathbf{Q}}$.

2.5. Proposition. Let $\phi_\iota : T_{X,\mathbf{Q}} \xrightarrow{\cong} T_{Y,\mathbf{Q}}$ be the isomorphism of Hodge structures induced by the Nikulin involution ι on X and assume that $\dim T_{X,\mathbf{Q}}$ is an odd integer. Then ϕ_ι is not an isometry.

Proof. Let $Q : \mathbf{Q}^n \rightarrow \mathbf{Q}$ be a quadratic form, then Q is defined by an $n \times n$ symmetric matrix, which we also denote by Q : $Q(x) := {}^t x Q x$. An isomorphism $A : \mathbf{Q}^n \rightarrow \mathbf{Q}^n$ gives an isometry between (\mathbf{Q}^n, Q) and (\mathbf{Q}^n, Q') iff $Q' = {}^t A^{-1} Q A^{-1}$. In particular, if $(\mathbf{Q}^n, Q) \cong (\mathbf{Q}^n, Q')$ the quotient $\det(Q)/\det(Q')$ must be a square in \mathbf{Q}^* .

For a \mathbf{Z} -module M we let $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q}$. Let V_X be the orthogonal complement of $E_8(-2)_{\mathbf{Q}} \subset NS(X)_{\mathbf{Q}}$, then $\det(NS(X)_{\mathbf{Q}}) = 2^8 \det(V_X)$ up to squares. Let V_Y be the orthogonal complement of $N_{\mathbf{Q}} \subset NS(Y)_{\mathbf{Q}}$ then $\det(NS(Y)_{\mathbf{Q}}) = 2^6 \det(V_Y)$ up to squares. Now $\beta_* \pi^* : H^2(Y, \mathbf{Q}) \rightarrow H^2(X, \mathbf{Q})$ induces an isomorphism $V_X \rightarrow V_Y$ which satisfies $(\beta_* \pi^* x)(\beta_* \pi^* y) = 2xy$ for $x, y \in V_Y$, hence $\det(V_X) = 2^d \det(V_Y)$ where $d = \dim V_X = 22 - 8 - \dim T_{X,\mathbf{Q}}$, so d is odd by assumption.

For a K3 surface S , $\det(T_{S,\mathbf{Q}}) = -\det(NS(S)_{\mathbf{Q}})$ and thus $\det(T_{X,\mathbf{Q}})/\det(T_{Y,\mathbf{Q}}) = 2^{d+2}$ up to squares. As d is odd and 2 is not a square in the multiplicative group of \mathbf{Q} , it follows that there exists no isometry between $T_{X,\mathbf{Q}}$ and $T_{Y,\mathbf{Q}}$. \square

2.6. The bundle L . In case $NS(X)$ has rank 9, the ample generator L of $E_8(-2)^\perp \subset NS(X)$ defines a natural map

$$\phi_L : X \longrightarrow \mathbf{P}^g, \quad g = h^0(L) - 1 = L^2/2 + 1$$

which we will use to study X and Y . As $\iota^* L \cong L$, the involution ι acts as an involution on $\mathbf{P}^g = |L|^*$ and thus it has two fixed spaces $\mathbf{P}^a, \mathbf{P}^b$ with $(a+1) + (b+1) = g+1$. The fixed points of ι map to these fixed spaces. Even though L is ι -invariant, it is not the case in general that on \tilde{X} we have $\beta^* L = \pi^* M$ for some line bundle $M \in NS(Y)$. In fact, $\beta^* L = \pi^* M$ implies

$L^2 = (\beta^*L)^2 = (\pi^*M)^2 = 2M^2$ and as M^2 is even we get $L^2 \in 4\mathbf{Z}$. Thus if $L^2 \notin 4\mathbf{Z}$, the ι -invariant line bundle L cannot be obtained by pull-back from Y . On the other hand, if for example $|L|$ contains a reduced ι -invariant divisor D which does not pass through the fixed points, then $\beta^*D = \beta^{-1}D$ is invariant under $\tilde{\iota}$ on \tilde{X} and does not contain any of the E_i as a component. Then $\beta^*D = \pi^*D'$ where $D' \subset Y$ is the reduced divisor with support $\pi(\beta^{-1}D)$.

The following lemma collects the basic facts on L and the splitting of $\mathbf{P}^g = \mathbf{P}H^0(X, L)^*$.

2.7. Proposition.

- (1) Assume that $NS(X) = \mathbf{Z}L \oplus E_8(-2)$. Let E_1, \dots, E_8 be the exceptional divisors on \tilde{X} .

In case $L^2 = 4n + 2$, there exist line bundles $M_1, M_2 \in NS(Y)$ such that for a suitable numbering of these E_i we have:

$$\beta^*L - E_1 - E_2 = \pi^*M_1, \quad \beta^*L - E_3 - \dots - E_8 = \pi^*M_2.$$

The decomposition of $H^0(X, L)$ into ι^* -eigenspaces is:

$$H^0(X, L) \cong \pi^*H^0(Y, M_1) \oplus \pi^*H^0(Y, M_2), \quad (h^0(M_1) = n + 2, h^0(M_2) = n + 1).$$

and the eigenspaces $\mathbf{P}^{n+1}, \mathbf{P}^n$ contain six, respectively two, fixed points.

In case $L^2 = 4n$, for a suitable numbering of the E_i we have:

$$\beta^*L - E_1 - E_2 - E_3 - E_4 = \pi^*M_1, \quad \beta^*L - E_5 - E_6 - E_7 - E_8 = \pi^*M_2$$

with $M_1, M_2 \in NS(Y)$. The decomposition of $H^0(X, L)$ into ι^* -eigenspaces is:

$$H^0(X, L) \cong \pi^*H^0(Y, M_1) \oplus \pi^*H^0(Y, M_2), \quad (h^0(M_1) = h^0(M_2) = n + 1).$$

and each of the eigenspaces \mathbf{P}^n contains four fixed points.

- (2) Assume that $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$. Then there is a line bundle $M \in NS(Y)$ such that:

$$\beta^*L \cong \pi^*M, \quad H^0(X, L) \cong H^0(Y, M) \oplus H^0(Y, M - \hat{N}),$$

where $\hat{N} = (\sum_{i=1}^8 N_i)/2 \in NS(Y)$ and this is the decomposition of $H^0(X, L)$ into ι^* -eigenspaces. One has $h^0(M) = n + 2$, $h^0(M - \hat{N}) = n$, and all fixed points map to the eigenspace $\mathbf{P}^{n+1} \subset \mathbf{P}^{2n+1} = \mathbf{P}^g$.

Proof. The primitive embedding of $\mathbf{Z}L \oplus E_8(-2)$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1]. Therefore if $L^2 = 2r$ we may assume that $L = e_1 + rf_1 \in U \subset U^3 \oplus E_8(-1)^3$ where e_1, f_1 are the standard basis of the first copy of U .

In case $r = 2n + 1$, it follows from Lemma 1.10 that $(e_1 + (2n + 1)f_1 + N_3 + N_4)/2 \in NS(Y)$. By Proposition 1.8, $M_1 := (e_1 + (2n + 1)f_1 + N_3 + N_4)/2 - N_3 - N_4$ satisfies $\pi^*M_1 = \beta^*L - E_3 - E_4$. Similarly, let $M_2 = (e_1 + (2n + 1)f_1 + N_3 + N_4)/2 - \hat{N} \in NS(Y)$, then $\pi^*M_2 = \beta^*L - (E_1 + E_2 + E_5 + \dots + E_8)$.

Any two sections $s, t \in H^0(X, L)$ lie in the same ι^* -eigenspace iff the rational function $f = s/t$ is ι -invariant. Thus $s, t \in \pi^*H^0(Y, M_i)$ are ι^* -invariant, hence each of these two spaces is contained in an eigenspace of ι^* in $H^0(X, L)$. If both are in the same eigenspace, then this eigenspace would have a section with no zeroes in the 8 fixed points of ι . But a ι -invariant divisor on X which doesn't pass through any fixed point is the pull back of divisor

on Y , which contradicts that L^2 is not a multiple of 4. Thus the $\pi^*H^0(Y, M_i)$ are in distinct eigenspaces. A dimension count shows that $h^0(L) = h^0(M_1) + h^0(M_2)$, hence the $\pi^*H^0(Y, M_i)$ are the eigenspaces.

In case $r = 2n$, again by Lemma 1.10 we have $(e_1 + N_1 + N_2 + N_3 + N_8)/2 \in NS(Y)$. Let $M_1 := nf_1 + (e_1 + N_1 + N_2 + N_3 + N_8)/2 - (N_1 + N_2 + N_3 + N_8)$ then $\pi^*M_1 = \beta^*L - (E_1 + E_2 + E_3 + E_8)$. Put $M_2 = M_1 + \hat{N} - (N_4 + N_5 + N_6 + N_7)$, then $\pi^*M_2 = \beta^*L - (E_4 + E_5 + E_6 + E_7)$. As above, the $\pi^*H^0(Y, M_i)$, $i = 1, 2$, are contained in distinct eigenspaces and a dimension count again shows that $h^0(L) = h^0(M_1) + h^0(M_2)$.

If $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$, the (primitive) embedding of $NS(X)$ into $U^3 \oplus E_8(-1)$ is still unique up to isometry. Let $L^2 = 4n$. Choose an $\alpha \in E_8(-1)$ with $\alpha^2 = -2$ if n is odd, and $\alpha^2 = -4$ if n is even. Let $v = (0, \alpha, -\alpha) \in E_8(-2) \subset U^3 \oplus E_8(-1)^2$ and let $L = (2u, \alpha, \alpha) \in U^3 \oplus E_8(-1)^2$ where $u = e_1 + (n+1)/2f_1$ if n is odd and $u = e_1 + (n/2+1)f_1$ if n is even. Note that $L^2 = 4u^2 + 2\alpha^2 = 4n$ and that $(L+v)/2 = (u, \alpha, 0) \in U^3 \oplus E_8(-1)^2$. Thus we get a primitive embedding of $NS(X) \hookrightarrow U^3 \oplus E_8(-1)^2$ which extends the standard one of $E_8(-2) \subset NS(X)$. Proposition 1.8 shows that $\beta^*L = \pi^*M$ with $M = (u, 0, \alpha) \in U^3(2) \oplus N \oplus E_8(-1) \subset H^2(Y, \mathbf{Z})$. For the double cover $\pi : \tilde{X} \rightarrow Y$ branched along $2\hat{N} = \sum N_i$ we have as usual: $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_Y \oplus \mathcal{O}_Y(-\hat{N})$ hence, using the projection formula:

$$H^0(\tilde{X}, \pi^*M) \cong H^0(Y, \pi_*(\pi^*M \otimes \mathcal{O}_{\tilde{X}})) \cong H^0(Y, M) \oplus H^0(Y, M - \hat{N}).$$

Note that the sections in $\pi^*H^0(Y, M - \hat{N})$ vanish on all the exceptional divisors, hence the fixed points of ι map to a \mathbf{P}^{n+1} . \square

3. EXAMPLES

3.1. In Proposition 2.3 we showed that K3 surfaces with a Nikulin involution are parametrized by eleven dimensional moduli spaces \mathcal{M}_{2d} and $\mathcal{M}_{\tilde{4e}}$ with $d, e \in \mathbf{Z}_{>0}$. For some values of d, e we will now work out the geometry of the corresponding K3 surfaces. We will also indicate how to verify that the moduli spaces are indeed eleven dimensional.

3.2. **The case \mathcal{M}_2 .** Let X be a K3 surface with Nikulin involution ι and $NS(X) \cong \mathbf{Z}L \oplus E_8(-2)$ with $L^2 = 2$ and $\iota^*L \cong L$ (cf. Proposition 2.3). The map $\phi_L : X \rightarrow \mathbf{P}^2$ is a double cover of \mathbf{P}^2 branched over a sextic curve C , which is smooth since there are no (-2) -curves in L^\perp . The covering involution will be denoted by $i : X \rightarrow X$. The fixed point locus of i is isomorphic to C .

As i^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and -1 on T_X , whereas ι^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and $+1$ on T_X , these two involutions commute. Thus ι induces an involution $\bar{\iota}_{\mathbf{P}^2}$ on \mathbf{P}^2 (which is ι^* acting on $\mathbf{P}H^0(X, L)^*$) and in suitable coordinates:

$$\bar{\iota}_{\mathbf{P}^2} : (x_0 : x_1 : x_2) \longmapsto (-x_0 : x_1 : x_2).$$

We have a commutative diagram

$$\begin{array}{ccccc} C & \hookrightarrow & X & \xrightarrow{\iota} & X \\ \cong \downarrow & & \downarrow \phi & & \downarrow \phi \\ C & \hookrightarrow & \mathbf{P}^2 & \xrightarrow{\bar{\iota}_{\mathbf{P}^2}} & \mathbf{P}^2 = X/i. \end{array}$$

The fixed points of $\bar{\iota}_{\mathbf{P}^2}$ are:

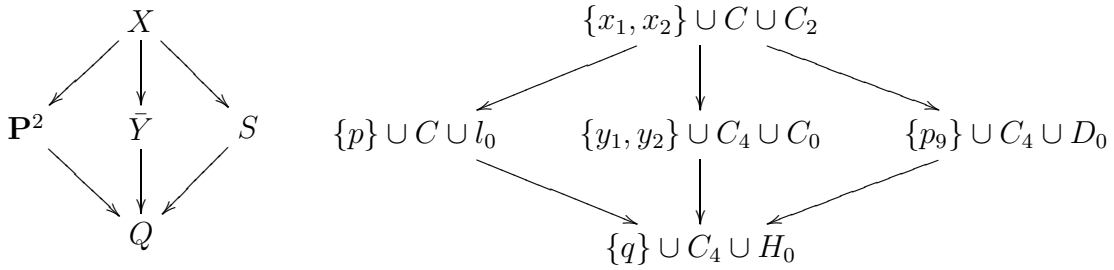
$$(\mathbf{P}^2)\bar{\iota}_{\mathbf{P}^2} = l_0 \cup \{p\}, \quad l_0 : x_0 = 0, \quad p = (1 : 0 : 0).$$

The line l_0 intersects the curve C in six points, which are the images of six fixed points x_3, \dots, x_8 of ι on X . Thus the involution ι induces an involution on $C \subset X$ with six fixed points. The other two fixed points x_1, x_2 of ι map to the point p , so i permutes these two fixed points of ι . In particular, these two points are not contained in C so $p \notin C (\subset \mathbf{P}^2)$, which will be important in the moduli count below. The inverse image $C_2 = \phi^{-1}(l_0)$ is a genus two curve in the system $|L|$. Both ι and i induce the hyperelliptic involution on C_2 . By doing then the quotient by ι , since this has six fixed points on C_2 we obtain a rational curve C_0 .

To describe the eight nodal surface $\bar{Y} = X/\iota$, we use the involution $\bar{i}_{\bar{Y}}$ of \bar{Y} which is induced by $i \in \text{Aut}(X)$. Then we have:

$$Q := \bar{Y}/\bar{i}_{\bar{Y}} \cong X/\langle \iota, i \rangle \cong \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2}.$$

This leads to the following diagrams of double covers and fixed point sets:



The quotient of $\mathbf{P}^2 = X/i$ by $\bar{\iota}_{\mathbf{P}^2}$ is isomorphic to a quadric cone Q in \mathbf{P}^3 whose vertex q is the image of the fixed point $(1 : 0 : 0)$. In coordinates, the quotient map is:

$$\mathbf{P}^2 \longrightarrow Q = \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2} \subset \mathbf{P}^3, \quad (x_0 : x_1 : x_2) \longmapsto (y_0 : \dots : y_3) = (x_0^2 : x_1^2 : x_1 x_2 : x_2^2)$$

and Q is defined by $y_1 y_3 - y_2^2 = 0$.

The sextic curve $C \subset \mathbf{P}^2$, which has genus 10, is mapped 2:1 to a curve C_4 on the cone. The double cover $C \rightarrow C_4$ ramifies in the six points where C intersects the line $x_0 = 0$. Thus the curve C_4 is smooth, has genus four and degree six (the plane sections of C_4 are the images of certain conic sections of the branch sextic) and does not lie in a plane (so C_4 spans \mathbf{P}^3). The only divisor class D of degree $2g - 2$ with $h^0(D) \geq g$ on a smooth curve of genus g is the canonical class, hence C_4 is a canonically embedded curve. The image of the line l_0 is the plane section $H_0 \subset Q$ defined by $y_0 = 0$.

The branch locus in Q of the double cover

$$\bar{Y} \longrightarrow Q = \bar{Y}/\bar{i}_{\bar{Y}}$$

is the union of two curves, C_4 and the plane section H_0 , these curves intersect in six points, and the vertex q of Q .

To complete the diagram, we consider the involution

$$j := \iota \circ i : X \longrightarrow X, \quad S := X/j.$$

The fixed point set of j is the (smooth) genus two curve C_2 lying over the line l_0 in \mathbf{P}^2 (use $j(p) = p$ iff $\iota(p) = i(p)$ and consider the image of p in \mathbf{P}^2). Thus the quotient surface S is a smooth surface. The Riemann-Hurwitz formula implies that the image of C_2 in S is a curve $D_0 \in |-2K_S|$, note that $D_0 \cong C_2$.

The double cover $S \rightarrow Q$ branches over the curve $C_4 \subset Q$ and the vertex $q \in Q$. It is well-known that such a double cover is a Del Pezzo surface of degree 1 ([Dem], [DoO]) and the map $S \rightarrow Q \subset \mathbf{P}^3$ is given by ϕ_{-2K} , which verifies that the image of D_0 is a plane section.

On the other hand, any Del Pezzo surface of degree 1 is isomorphic to the blow up of \mathbf{P}^2 in eight points. The linear system $|-K_S|$ corresponds to the pencil of elliptic curves on the eight points, the ninth base point in \mathbf{P}^2 corresponds to the unique base point p_9 of $|-K_S|$ in S . The point p_9 maps to the vertex $q \in Q$ under the 2:1 map ϕ_{-2K} ([DoO, p. 125]). The Néron Severi group of S is thus isomorphic to

$$NS(S) \cong \mathbf{Z}e_0 \oplus \mathbf{Z}e_1 \oplus \dots \oplus \mathbf{Z}e_8, \quad e_0^2 = 1, \quad e_i^2 = -1 \quad (1 \leq i \leq 8)$$

and $e_i e_j = 0$ if $i \neq j$. The canonical class is $K_S = -3e_0 + e_1 + \dots + e_8$. Since $K_S^2 = 1$, we get a direct sum decomposition:

$$NS(S) \cong \mathbf{Z}K_S \oplus K_S^\perp \cong \mathbf{Z}K_S \oplus E_8(-1)$$

(cf. [DoO, VII.5]). The surface S has 240 exceptional curves (smooth rational curves E with $E^2 = -1$), cf. [DoO, p.125]. The adjunction formula shows that $EK_S = -1$ and the map $E \mapsto E + K$ gives a bijection between these exceptional curves and the roots of $E_8(-1)$, i.e. the $x \in E_8(-1)$ with $x^2 = -2$. An exceptional divisor $E \subset S$ meets the branch curve $D_0 (\in |-2K_S|)$ of $X \rightarrow S$ in two points, hence the inverse image of E in X is a (-2) -curve. Thus we get 240 such (-2) -curves. Actually,

$$j^* : NS(S) = \mathbf{Z}K_S \oplus E_8(-1) \longrightarrow NS(X) = \mathbf{Z}L \oplus E_8(-2)$$

is the identity on the \mathbf{Z} -modules and $NS(X) \cong NS(S)(2)$. The class of such a (-2) -curve is $L + x$, with $x \in L^\perp = E_8(-2)$, $x^2 = -2$. As $i^*(L + x) = L - x \neq L + x$, these (-2) -curves map pairwise to conics in \mathbf{P}^2 , which must thus be tangent to the sextic C . As also $\iota(L + x) = L - x$, these conics are invariant under $\bar{\iota}_{\mathbf{P}^2}$ and thus they correspond to plane sections of $Q \subset \mathbf{P}^3$, tangent to C_4 , that is tritangent planes. This last incarnation of exceptional curves in S as tritangent planes (or equivalently, odd theta characteristics of C_4) is of course very classical.

Finally we compute the moduli. A $\bar{\iota}_{\mathbf{P}^2}$ -invariant plane sextic which does not pass through $p = (1 : 0 : 0)$ has equation

$$\sum a_{ijk} x_0^{2i} x_1^j x_2^k \quad (2i + j + k = 6, \quad a_{000} \neq 0).$$

The vector space spanned by such polynomials is 16-dimensional. The subgroup of $GL(3)$ of elements commuting with $\bar{\iota}_{\mathbf{P}^2}$ (which thus preserve the eigenspaces) is isomorphic to $\mathbf{C}^* \times GL(2)$, hence the number of moduli is $16 - (1 + 4) = 11$ as expected.

Alternatively, the genus four curves whose canonical image lies on a cone have $9 - 1 = 8$ moduli (they have one vanishing even theta characteristic), next one has to specify a plane in \mathbf{P}^3 , this gives again $8 + 3 = 11$ moduli.

3.3. The case \mathcal{M}_6 . The map ϕ_L identifies X with a complete intersection of a cubic and a quadric in \mathbf{P}^4 . According to Proposition 2.7, in suitable coordinates the Nikulin involution is induced by

$$\iota_{\mathbf{P}^4} : \mathbf{P}^4 \longrightarrow \mathbf{P}^4, \quad (x_0 : x_1 : x_2 : x_3 : x_4) \longmapsto (-x_0 : -x_1 : x_2 : x_3 : x_4).$$

The fixed locus in \mathbf{P}^4 is:

$$(\mathbf{P}^4)^{\iota_{\mathbf{P}^4}} = l \cup H, \quad l : x_2 = x_3 = x_4 = 0, \quad H : x_0 = x_1 = 0.$$

The points $X \cap l$ and $X \cap H$ are fixed points of ι on X and Proposition 2.7 shows that $\sharp(X \cap l) = 2$, $\sharp(X \cap H) = 6$. In particular, the plane H meets the quadric and cubic defining X in a conic and a cubic curve which intersect transversely. Moreover, the quadric is unique, so must be invariant under $\iota_{\mathbf{P}^4}$, and, by considering the action of $\iota_{\mathbf{P}^4}$ on the cubics in the ideal of X , we may assume that the cubic is invariant as well.

$$\begin{aligned} l_{00}(x_2, x_3, x_4)x_0^2 + l_{11}(x_2, x_3, x_4)x_1^2 + l_{01}(x_2, x_3, x_4)x_0x_1 + f_3(x_2, x_3, x_4) &= 0 \\ \alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1 + f_2(x_2, x_3, x_4) &= 0 \end{aligned}$$

where the α_{ij} are constants, the l_{ij} are linear forms, and f_2, f_3 are homogeneous polynomials of degree two and three respectively. Note that the cubic contains the line $l : x_2 = x_3 = x_4 = 0$.

The projection from \mathbf{P}^4 to the product of the eigenspaces $\mathbf{P}^1 \times \mathbf{P}^2$ maps X to a surface defined by an equation of bidegree $(2, 3)$. In fact, the equations imply that $(\sum l_{ij}x_ix_j)/f_3 = (\sum \alpha_{ij}x_ix_j)/f_2$ hence the image of X is defined by the polynomial: $(\sum l_{ij}x_ix_j)f_2 - (\sum \alpha_{ij}x_ix_j)f_3$. Adjunction shows that a smooth surface of bidegree $(2, 3)$ is a K3 surface, so the equation defines \check{Y} . The space of invariant quadrics is $3 + 6 = 9$ dimensional and the space of cubics is $3 \cdot 3 + 10 = 19$ dimensional. Multiplying the quadric by a linear form $a_2x_2 + a_3x_3 + a_4x_4$ gives an invariant cubic. The automorphisms of \mathbf{P}^4 commuting with ι form a subgroup which is isomorphic with $GL(2) \times GL(3)$ which has dimension $4 + 9 = 13$. So the moduli space of such K3 surfaces has dimension:

$$(9 - 1) + (19 - 1) - 3 - (13 - 1) = 11$$

as expected.

3.4. The case \mathcal{M}_4 . The map $\phi_L : X \rightarrow \mathbf{P}^3$ is an embedding whose image is a smooth quartic surface. From Proposition 2.7 the Nikulin involution ι on $X \subset \mathbf{P}^3 \cong \mathbf{P}(\mathbf{C}^4)$ is induced by

$$\tilde{\iota} : \mathbf{C}^4 \longrightarrow \mathbf{C}^4, \quad (x_0, x_1, x_2, x_3) \longmapsto (-x_0, -x_1, x_2, x_3)$$

for suitable coordinates. The eight fixed points of the involution are the points of intersection of these lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ with the quartic surface X .

A quartic surface which is invariant under $\tilde{\iota}$ and which does not contain the lines has an equation which is a sum of monomials $x_0^a x_1^b x_2^c x_3^d$ with $a + b = 0, 2, 4$ and $c + d = 4 - a - b$.

The quadratic polynomials invariant under $\tilde{\iota}$ define a map:

$$\mathbf{P}^3 \longrightarrow \mathbf{P}^5, \quad (x_0 : \dots : x_3) \longmapsto (z_0 : z_1 : \dots : z_5) = (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 : x_2x_3)$$

which factors over $\mathbf{P}^3/\tilde{\iota}$. Note that any quartic invariant monomial is a monomial of degree two in the z_i . Thus if $f = 0$ is the equation of X , then $f(x_0, \dots, x_3) = q(z_0, \dots, z_5)$ for a quadratic

form q . This implies that

$$\bar{Y} : \quad q(z_0, \dots, z_5) = 0, \quad z_0 z_1 - z_4^2 = 0, \quad z_2 z_3 - z_5^2 = 0$$

is the intersection of three quadrics.

The invariant quartics span a $5 + 9 + 5 = 19$ -dimensional vector space. On this space the subgroup H of $GL(4)$ of elements which commute with $\iota_{\mathbf{P}^3}$ acts, it is easy to see that $H \cong GL(2) \times GL(2)$ (in block form). Thus $\dim H = 8$ and we get an $19 - 8 = 11$ dimensional family of quartic surfaces in \mathbf{P}^3 , as desired. See [I] for some interesting sub-families.

3.5. The case $\mathcal{M}_{\bar{4}}$. In this case $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$. Choose a $v \in E_8(-2)$ with $v^2 = -4$. Then we may assume that $NS(X)$ is generated by $L, E_8(-2)$ and $E_1 := (L+v)/2$, cf. (the proof of) Proposition 2.2. Let $E_2 := (L - v)/2$, then $E_i^2 = L^2/4 + v^2/4 = 1 - 1 = 0$. By Riemann-Roch we have:

$$\chi(\pm E_i) = E_i^2/2 + 2 = 2$$

and so $h^0(\pm E_i) \geq 2$ so E_i or $-E_i$ is effective. Now $L \cdot E_i = L^2/2 + v/2 \cdot L = 2$, hence E_i is effective. As $p_a(E_i) = 1$ and $E_i N \geq 0$ for all (-2) -curves N , each E_i is the class of an elliptic fibration. As $L = E_1 + E_2$, by [SD, Theorem 5.2] the map ϕ_L is a 2:1 map to a quadric Q in \mathbf{P}^3 and it is ramified on a curve B of bi-degree $(4, 4)$. The quadric is smooth, hence isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, because there are no (-2) -curves in $NS(X)$ perpendicular to L .

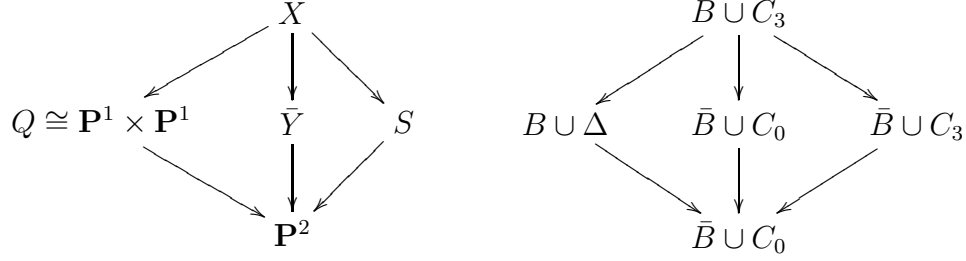
Let $i : X \rightarrow X$ be the covering involution of $X \rightarrow Q$. Then i and the Nikulin-involution ι commute. The elliptic pencils E_1 and E_2 are permuted by ι because $\iota^*L = L, \iota^*v = -v$. This means that the involution $\bar{\iota}_Q$ on $Q = \mathbf{P}^1 \times \mathbf{P}^1$ induced by ι acts as $((s : t), (u : v)) \mapsto ((u : v), (s : t))$. The quotient of $Q/\bar{\iota}_Q$ is well known to be isomorphic to \mathbf{P}^2 .

The fixed point set of $\bar{\iota}_Q$ in $\mathbf{P}^1 \times \mathbf{P}^1$ is the diagonal Δ . Thus Δ intersects the branch curve B in eight points. The inverse image of these points in X are the eight fixed points of the Nikulin involution.

The diagonal maps to a conic C_0 in $\mathbf{P}^2 = Q/\bar{\iota}_Q$, which gives the representation of a smooth quadric as double cover of \mathbf{P}^2 branched along a conic (in equations: $t^2 = q(x, y, z)$). The curve B maps to a plane curve isomorphic to $\bar{B} = B/\iota$. As ι has 8 fixed points on the genus 9 curve B , the genus of \bar{B} is 3 and $\bar{B} \subset \mathbf{P}^2$ is a quartic curve.

Let $j = i\iota = \iota i \in \text{Aut}(X)$. The fixed point set of j is easily seen to be the inverse image C_3 of the diagonal $\Delta \subset Q$. As $C_3 \rightarrow \Delta$ branches over the 8 points in $B \cap \Delta$, C_3 is a smooth (hyperelliptic) genus three curve. Thus the surface $S := X/j$ is smooth and the image of C_3 in S lies in the linear system $| -2K_S |$. The double cover $S \rightarrow \mathbf{P}^2$ is branched over the plane quartic $\bar{B} \subset \mathbf{P}^2$. This implies that S is a Del Pezzo surface of degree 2, cf. [Dem], [DoO].

This leads to the following diagrams of double covers and fixed point sets:



In particular the eight nodal surface \bar{Y} is the double cover of \mathbf{P}^2 branched over the reducible sextic with components the conic C_0 and the quartic \bar{B} . The nodes of \bar{Y} map to the intersection points of C_0 and \bar{B} .

To count the moduli we note that the homogeneous polynomials of degree two and four in three variables span vector spaces of dimension 6 and 15, as $\dim GL(3) = 9$ we get: $(6 - 1) + (15 - 1) - (9 - 1) = 11$ moduli.

3.6. The case \mathcal{M}_8 . We have $H^0(X, L) \cong \pi^*H^0(Y, M_1) \oplus \pi^*H^0(Y, M_2)$ and $L^2 = 8$, $M_i^2 = 2$ so $h^0(L) = 6$, $h^0(M_i) = 3$ for $i = 1, 2$. The image of X under ϕ_L is the intersection of three quadrics in \mathbf{P}^5 and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \quad (x_0, x_1, x_2, y_0, y_1, y_2) \longmapsto (x_0, x_1, x_2, -y_0, -y_1, -y_2).$$

The multiplication map maps the 21-dimensional space $S^2H^0(X, L)$ onto the 18-dimensional space $H^0(X, 2L)$. Using ι we can get some more information on the kernel of this map, which are the quadrics defining $X \subset \mathbf{P}^5$. We have:

$$S^2H^0(X, L) \cong (S^2H^0(Y, M_1) \oplus S^2H^0(Y, M_2)) \oplus (H^0(Y, M_1) \otimes H^0(Y, M_2)),$$

Moreover, as

$$\beta^*(2L) = \pi^*M, \quad \text{with } M = 2M_1 + N_1 + \dots + N_4 = 2M_2 + N_5 + \dots + N_8,$$

(cf. Proposition 2.7) we have the decomposition

$$H^0(X, 2L) \cong \pi^*H^0(Y, M) \oplus \pi^*H^0(Y, M - \hat{N}), \quad (h^0(M) = (M^2)/2 + 2 = 10, h^0(M - \hat{N}) = 8).$$

In particular, the multiplication maps splits as:

$$H^0(M_1) \otimes H^0(M_2) \longrightarrow H^0(Y, M - \hat{N})$$

(vector spaces with dimensions with $3 \cdot 3 = 9$ and 8 resp.) and

$$S^2H^0(Y, M_1) \oplus S^2H^0(Y, M_2) \longrightarrow H^0(Y, M)$$

(with dimensions $6 + 6 = 12$ and 10 resp.). Each of these two maps is surjective, and as $S^2H^0(Y, M_1) \rightarrow H^0(Y, M)$ is injective (ϕ_{M_1} maps Y onto \mathbf{P}^2), the quadrics in the ideal of X can be written as:

$$Q_1(x) - Q_2(y) = 0, \quad Q_3(x) - Q_4(y) = 0, \quad B(x, y) = 0$$

with Q_i homogeneous of degree two in three variables, and B of bidegree $(1, 1)$. Note that each eigenspace intersects X in $2 \cdot 2 = 4$ points.

The surface \bar{Y} maps to $\mathbf{P}^2 \times \mathbf{P}^2$ with the map $\phi_{M_1} \times \phi_{M_2}$, its image is the image of X under the projections to the eigenspaces $\mathbf{P}^5 \rightarrow \mathbf{P}^2 \times \mathbf{P}^2$. As $(x_0 : \dots : y_2) \mapsto Q_1(x)/Q_2(y)$ is a constant rational function on X and similarly for $Q_3(x)/Q_4(y)$, there is a $c \in \mathbf{C}$ such that the image of X is contained in the complete intersection of type $(2, 2)$, $(1, 1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ defined by

$$Q_1(x)Q_4(y) - cQ_3(x)Q_2(y) = 0, \quad B(x, y) = 0.$$

By adjunction, smooth complete intersections of this type are K3 surfaces.

To count the moduli, note that the first two equations come from a $6 + 6 = 12$ -dimensional vector space and the third comes from a $3 \cdot 3 = 9$ -dimensional space. The Grassmanian of 2-dimensional subspaces of a 12 dimensional space has dimension $2(12 - 2) = 20$. The subgroup of $GL(6)$ which commutes with $\iota_{\mathbf{P}^5}$ is isomorphic to $GL(3) \times GL(3)$ and has dimension $9 + 9 = 18$. Thus we get $20 + (9 - 1) - (18 - 1) = 11$ moduli, as expected.

3.7. The case $\mathcal{M}_{\bar{g}}$. We have $H^0(X, L) \cong \pi^*H^0(Y, M) \oplus \pi^*H^0(Y, M - \hat{N})$ and $L^2 = 8$, $M^2 = 4$ so $h^0(M) = 4$, $h^0(M - N) = 2$. The image of X under ϕ_L is the intersection of three quadrics in \mathbf{P}^5 and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \quad (x_0, x_1, x_2, x_3, y_0, y_1) \longmapsto (x_0, x_1, x_2, x_3, -y_0, -y_1).$$

To study the quadrics defining X , that is the kernel of the multiplication map $S^2H^0(X, L) \rightarrow H^0(X, 2L)$ we again split these spaces into ι^* -eigenspaces:

$$S^2H^0(X, L) \cong \left(S^2H^0(Y, M) \oplus S^2H^0(Y, M - \hat{N}) \right) \oplus \left(H^0(Y, M) \otimes H^0(Y, M - \hat{N}) \right),$$

(with dimensions $21 = (10 + 3) + 8$) and

$$H^0(X, 2L) \cong \pi^*H^0(Y, 2M) \oplus \pi^*H^0(Y, 2M - \hat{N})$$

(with dimensions $h^0(2M) = 10$, $h^0(2M - \hat{N}) = 8$).

This implies that there are no quadratic relations in the 8 dimensional space $H^0(Y, M) \otimes H^0(Y, M - \hat{N})$. As ϕ_M maps Y onto a quartic surface in \mathbf{P}^3 and $M - \hat{N}$ is a map of Y onto \mathbf{P}^1 , the quadrics in the ideal of X are of the form:

$$y_0^2 = Q_1(x), \quad y_0y_1 = Q_2(x), \quad y_1^2 = Q_3(x).$$

The fixed points of the involution are the eight points in the intersection of X with the \mathbf{P}^3 defined by $y_0 = y_1 = 0$.

The image of Y by ϕ_M is the image of the projection of X from the invariant line to the invariant \mathbf{P}^3 , which is defined by $y_0 = y_1 = 0$. The image is the quartic surface defined by $Q_1Q_3 - Q_2^2 = 0$ which can be identified with \bar{Y} . The equation is the determinant of a symmetric 2×2 matrix, which also implies that this surface has 8 nodes, (cf. [Ca, Theorem 2.2], [B, section 3]), the nodes form an even set (cf. [Ca, Proposition 2.6]).

We compute the number of moduli. Quadrics of this type span a space U of dimension $3 + 10 = 13$. The dimension of the Grassmanian of three dimensional subspaces of U is $3(13 - 3) = 30$. The group of automorphisms of \mathbf{C}^6 which commute with $\iota_{\mathbf{P}^5}$ is $GL(2) \times GL(4)$. So we have a $30 - (4 + 16 - 1) = 11$ dimensional space of such K3-surfaces in \mathbf{P}^5 , as expected.

3.8. The case \mathcal{M}_{12} . We have $H^0(X, L) \cong \pi^*H^0(Y, M_1) \oplus \pi^*H^0(Y, M_2)$ and $L^2 = 12$, $M_i^2 = 4$ so $h^0(L) = 8$, $h^0(M_i) = 4$ for $i = 1, 2$. The image of X under ϕ_L is the intersection of ten quadrics in \mathbf{P}^7 .

Following Example 3.6, we use ι^* to split the multiplication map from the $36 = (10+10)+16$ -dimensional space $S^2H^0(X, L)$ onto the $26 = 14 + 12$ -dimensional space $H^0(X, 2L)$, again $\beta^*(2L) = \pi^*M$ for an $M \in NS(Y)$ with $M^2 = 24$. Thus we find $20 - 14 = 6$ quadrics of the type $Q_1(x) - Q_2(y)$ with Q_i quadratic forms in 4 variables, and $16 - 12 = 4$ quadratic forms $B_i(x, y)$, $i = 1, \dots, 4$ where x, y are coordinates on the two eigenspaces in $H^0(X, L)$.

In particular, the projection from \mathbf{P}^7 to the product of the eigenspaces $\mathbf{P}^3 \times \mathbf{P}^3$ maps X onto a surface defined by 4 equations of bidegree $(1, 1)$. Adjunction shows that a complete intersection of this type is a K3 surface, so the four B_i 's define $\bar{Y} \subset \mathbf{P}^3 \times \mathbf{P}^3$.

Each B_i can be written as: $B_i(x, y) = \sum_j l_{ij}(x)y_j$ with linear forms l_{ij} in $x = (x_0, \dots, x_3)$. The image of $\bar{Y} \subset \mathbf{P}^3 \times \mathbf{P}^3$ under the projection to the first factor is then defined by $\det(l_{ij}(x)) = 0$, which is a quartic surface in \mathbf{P}^3 as expected. In fact, a point $x \in \mathbf{P}^3$ has a non-trivial counter image $(x, y) \in X \subset \mathbf{P}^3 \times \mathbf{P}^3$ iff the matrix equation $(l_{ij})y = 0$ has a non-trivial solution.

As X is not a complete intersection, we omit the moduli count.

3.9. The case $\mathcal{M}_{\tilde{12}}$. In this case $\beta^*L \cong \pi^*M$, $h^0(L) = 8 = 5 + 3 = h^0(M) + h^0(M - \hat{N})$. We consider again the quadrics in the ideal of X in Example 3.7. The space $S^2H^0(X, L)$ of quadrics on \mathbf{P}^7 decomposes as:

$$S^2H^0(X, L) \cong \left(S^2H^0(Y, M) + S^2H^0(Y, M - \hat{N}) \right) \oplus \left(H^0(Y, M) \otimes H^0(Y, M - \hat{N}) \right),$$

with dimensions $36 = (15 + 6) + 15$, whereas the sections of $2L$ decompose as:

$$h^0(2L) = (4L^2)/2 + 2 = 26 = 14 + 12 = h^0(2M) \oplus h^0(2M - \hat{N}).$$

Thus there are $(15 + 6) - 14 = 7$ independent quadrics in the ideal of $X \subset \mathbf{P}^7$ which are invariant and there are $15 - 12 = 3$ quadrics which are anti-invariant under the map

$$\tilde{\iota} : \mathbf{C}^8 \longrightarrow \mathbf{C}^8, \quad (x_0, \dots, x_4, y_0, y_1, y_2) \longmapsto (x_0, \dots, x_4, -y_0, -y_1, -y_2).$$

An invariant quadratic polynomial looks like $q_0(x_0, \dots, x_4) + q_1(y_0, y_1, y_2)$, and since the space of quadrics in three variables is only 6 dimensional, there is one non-zero quadric q in the ideal of the form $q = q(x_0, \dots, x_4)$. An anti-invariant quadratic polynomial is of bidegree $(1, 1)$ in x and y . In particular, the image of the projection of X to the product of the eigenspaces $\mathbf{P}^4 \times \mathbf{P}^2$ is contained in one hypersurface of bidegree $(2, 0)$ and in three hypersurfaces of bidegree $(1, 1)$. The complete intersection of four general such hypersurfaces is a K3 surface (use adjunction and $(2 + 3 \cdot 1, 3 \cdot 1) = (5, 3)$).

The three anti-invariant quadratic forms can be written as $\sum_j l_{ij}(x)y_j$, $i = 1, 2, 3$. The determinant of the 3×3 matrix of linear forms $(l_{ij}(x))$, defines a cubic form which is an equation for the image of X in \mathbf{P}^4 (cf. Example 3.8). Thus the projection \bar{Y} of X to \mathbf{P}^4 is the intersection of the quadric defined by $q(x) = 0$ and a cubic.

The projection to \mathbf{P}^2 is 2:1, as it should be, since for general $y \in \mathbf{P}^2$ the three linear forms in x given by $\sum_j l_{ij}(x)y_j$ define a line in \mathbf{P}^4 which cuts the quadric $q(x) = 0$ in two points.

4. ELLIPTIC FIBRATIONS WITH A SECTION OF ORDER TWO

4.1. Elliptic fibrations and Nikulin involutions. Let X be a K3 surface which has an elliptic fibration $f : X \rightarrow \mathbf{P}^1$ with a section σ . The set of sections of f is a group, the Mordell-Weil group MW_f , with identity element σ . This group acts on X by translations and these translations preserve the holomorphic two form on X . In particular, if there is an element $\tau \in MW_f$ of order two, then translation by τ defines a Nikulin involution ι .

In that case the Weierstrass equation of X can be put in the form:

$$X : \quad y^2 = x(x^2 + a(t)x + b(t))$$

the sections σ, τ are given by the section at infinity and $\tau(t) = (x(t), y(t)) = (0, 0)$. For the general fibration on a K3 surface X , the degrees of a and b are 4 and 8 respectively.

4.2. Proposition. Let $X \rightarrow \mathbf{P}^1$ be a general elliptic fibration with sections σ, τ as above in section 4.1. and let ι be the corresponding Nikulin involution on X . These fibrations form a 10-dimensional family.

The quotient K3 surface Y also has an elliptic fibration:

$$Y : \quad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4b(t))),$$

We have:

$$NS(X) \cong NS(Y) \cong U \oplus N, \quad T_X \cong T_Y \cong U^2 \oplus N.$$

The bad fibers of $X \rightarrow \mathbf{P}^1$ are eight fibers of type I_1 (which are rational curves with a node) over the zeroes of $a^2 - 4b$ and eight fibers of type I_2 (these fibers are the union of two \mathbf{P}^1 's meeting in two points) over the zeroes of b . The bad fibers of $Y \rightarrow \mathbf{P}^1$ are eight fibers of type I_2 over the zeroes of $a^2 - 4b$ and eight fibers of type I_1 over the zeroes of b .

Proof. Since X has an elliptic fibration with a section, $NS(X)$ contains a copy of the hyperbolic plane U (with standard basis the class of a fiber f and $f + \sigma$). The discriminant of the Weierstrass model of X is $\Delta_X = b^2(a^2 - 4b)$ and the fibers of the Weierstrass model over the zeroes of Δ_X are nodal curves. Thus $f : X \rightarrow \mathbf{P}^1$ has eight fibers of type I_1 (which are rational curves with a node) over the zeroes of $a^2 - 4b$ and 8 fibers of type I_2 (these fibers are the union of two \mathbf{P}^1 's meeting in two points) over the zeroes of b .

The components of the singular fibers which do not meet the zero section σ , give a sublattice $\langle -2 \rangle^8$ perpendicular to U . If there are no sections of infinite order, the lattice $U \oplus \langle -2 \rangle^8$ has finite index in the Néron Severi group of X . Hence X has $22 - 2 - 10 = 10$ moduli. One can also appeal to [Shim] where the Néron Severi group of the general elliptic K3 fibration with a section of order two is determined. To find the moduli from the Weierstrass model, note that a and b depend on $5 + 9 = 14$ parameters. Using transformations of the type $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ (and dividing the equation by λ^6) and the automorphism group $\mathbf{PGL}(2)$ of \mathbf{P}^1 we get $14 - 1 - 3 = 10$ moduli.

The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) shows that the discriminant of the Néron Severi group is $2^8/n^2$ where n is the order of the torsion subgroup of MW_f . The curve defined by $x^2 + a(t)x + b(t) = 0$ cuts out the remaining pair of points of order two on each smooth fiber. As it is irreducible in general, MW_f must be cyclic. If there were a section σ of

order four, it would have to satisfy $2\sigma = \tau$. But in a fiber of type I_2 the complement of the singular points is the group $G = \mathbf{C}^* \times (\mathbf{Z}/2\mathbf{Z})$ and the specialization $MW_f \rightarrow G$ is an injective homomorphism. Now τ specializes to $(\pm 1, \bar{1})$ (the sign doesn't matter) since τ specializes to the node in the Weierstrass model. But there is no $g \in G$ with $2g = (\pm 1, \bar{1})$. We conclude that for general X we have $MW_f = \{\sigma, \tau\} \cong \mathbf{Z}/2\mathbf{Z}$ and that the discriminant of the Néron Severi group of X is 2^6 .

The Néron Severi group has \mathbf{Q} basis $\sigma, f, N_1, \dots, N_8$ where the N_i are the components of the I_2 fibers not meeting σ . As $\tau \cdot \sigma = 0$, $\tau \cdot f = 1$ and $\tau \cdot N_i = 1$, we get:

$$\tau = \sigma + 2f - \hat{N}, \quad \hat{N} = (N_1 + \dots + N_8)/2.$$

Thus the smallest primitive sublattice containing the N_i is the Nikulin lattice. Comparing discriminants we conclude that:

$$NS(X) = \langle s, f \rangle \oplus \langle N_1, \dots, N_8, \hat{N} \rangle \cong U \oplus N.$$

The transcendental lattice T_X of X can be determined as follows. It is a lattice of signature $(2+, 10-)$ and its discriminant form is the opposite of the one of N , but note that $q_N = -q_N$ since q_N takes values in $\mathbf{Z}/2\mathbf{Z}$. Moreover, $T_X^*/T_X \cong N^*/N \cong (\mathbf{Z}/2\mathbf{Z})^6$. Using [Ni2, Corollary 1.13.3], we find that T_X is uniquely determined by the signature and the discriminant form. The lattice $U^2 \oplus N$ has these invariants, so

$$T_X \cong U^2 \oplus N.$$

As the Nikulin involution preserves the fibers of the elliptic fibration on X , the desingularisation Y of the quotient X/ι has an elliptic fibration $g : Y \rightarrow \mathbf{P}^1$, with a section $\bar{\sigma}$, (the image of σ). Observe that given any elliptic curve in Weierstrass form, as explained in [ST, Section 4, p.76 and Proposition p.79] there is a straightforward way to write down the Weierstrass form of its quotient by a translation by a point of order two, and so one can immediately write down the Weierstrass equation of Y .

The discriminant of the Weierstrass model of Y is $\Delta_Y = 4b(a^2 - 4b)^2$ and, reasoning as before, we find the bad fibers of $g : Y \rightarrow \mathbf{P}^1$. In particular, the I_1 and I_2 fibers of X and Y are indeed 'interchanged'.

Geometrically, the reason for this is as follows. The fixed points of translation by τ are the eight nodes in the I_1 -fibers, blowing them up gives I_2 -type fibers which map to I_2 -type fibers in Y . The exceptional curves lie in the ramification locus of the quotient map, the other components, which meet σ , map 2:1 to components of the I_2 -fibers which meet $\bar{\sigma}$. The two components of an I_2 -fiber in X are interchanged and also the two singular points of the fiber are permuted, so in the quotient this gives an I_1 -type fiber. \square

4.3. Remark. Note that $NS(X) \oplus T_X \cong U^3 \oplus N^2$, however, there is *no* embedding of N^2 into $E_8(-1)^2$, such that $N \oplus \{0\} (\subset NS(X))$ is primitive in $E_8(-1)^2$. However, $N^2 \subset \Gamma_{16}(-1)$ (cf. section 1.11), an even, negative definite, unimodular lattice of rank 16 and $U^3 \oplus \Gamma_{16}(-1) \cong U^3 \oplus E_8(-1)^2$ by the classification of even indefinite unimodular quadratic forms.

4.4. Morrison-Nikulin involutions. D. Morrison observed that a K3 surface X having two perpendicular copies of $E_8(-1)$ in the Néron Severi group has a Nikulin involution which exchanges the two copies of $E_8(-1)$, cf. [Mo, Theorem 5.7]. We will call such an involution a Morrison-Nikulin involution. This involution then has the further property that $T_Y \cong T_X(2)$ where Y is the quotient K3 surface and we have a Shioda-Inose structure on Y (cf. [Mo, Theorem 6.3])

4.5. Moduli. As $E_8(-1)$ has rank eight and is negative definite, a projective K3 surface with a Morrison-Nikulin involution has a Néron Severi group of rank at least 17 and hence has at most three moduli. In case the Néron Severi group has rank exactly 17, we get

$$NS(X) \cong \langle 2n \rangle \oplus E_8(-1) \oplus E_8(-1)$$

since the sublattice $E_8(-1)^2$ is unimodular. Results of Kneser and Nikulin, [Ni2, Corollary 1.13.3], guarantee that the transcendental lattice $T_X := NS(X)^\perp$ is uniquely determined by its signature and discriminant form. As the discriminant form of T_X is the opposite of the one on $NS(X)$ we get

$$T_X \cong \langle -2n \rangle \oplus U^2.$$

In case $n = 1$ such a three dimensional family can be obtained from the double covers of \mathbf{P}^2 branched along a sextic curve with two singularities which are locally isomorphic to $y^3 = x^5$. The double cover then has two singular points of type E_8 , that is, each of these can be resolved by eight rational curves with incidence graph E_8 . As the explicit computations are somewhat lengthy and involved, we omit the details. See [GL, Appendix], [P] and [Deg] for more on double covers of \mathbf{P}^2 along singular sextics.

4.6. Morrison-Nikulin involutions on elliptic fibrations. We consider a family of K3 surfaces with an elliptic fibration with a Morrison-Nikulin involution induced by translation by a section of order two. It corresponds to the family with $n = 2$ from section 4.5.

Note that in the proposition below we describe a K3 surface Y with a Nikulin involution and quotient K3 surface X such that $T_Y = T_X(2)$, which is the ‘opposite’ of what would happen if the involution of Y was a Morrison-Nikulin involution. It is not hard to see that there is no primitive embedding $T_Y \hookrightarrow U^3$, so Y does not have a Morrison-Nikulin involution at all (cf. [Mo, Theorem 6.3]).

4.7. Proposition. Let $X \rightarrow \mathbf{P}^1$ be a general elliptic fibration defined by the Weierstrass equation

$$X : \quad y^2 = x(x^2 + a(t)x + 1), \quad a(t) = a_0 + a_1t + a_2t^2 + t^4 \in \mathbf{C}[t].$$

The K3 surface X has a Morrison-Nikulin involution defined by translation by the section, of order two, $t \mapsto (x(t), y(t)) = (0, 0)$. Then:

$$NS(X) = \langle 4 \rangle \oplus E_8(-1) \oplus E_8(-1), \quad T_X = \langle -4 \rangle \oplus U^2.$$

The bad fibers of the fibration are nodal cubics (type I_1) over the eight zeroes of $a^2(t) - 4$ and one fiber of type I_{16} over $t = \infty$.

The quotient K3 surface Y has an elliptic fibration defined by the Weierstrass model:

$$Y : \quad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4)), \quad T_Y \cong \langle -8 \rangle \oplus U(2)^2.$$

This K3 surface has a Nikulin involution defined by translation by the section $t \mapsto (x(t), y(t)) = (0, 0)$ and the quotient surface is X . For general X , the bad fibers of Y are 8 fibers of type I_2 over the same points in \mathbf{P}^1 where X has fibers of type I_1 and at infinity Y has a fiber of type I_8 .

Proof. As we observed in section 4.1, translation by the section of order two defines a Nikulin involution.

Let $\hat{a}(s) := s^4 a(s^{-1})$, it is a polynomial of degree at most four and $\hat{a}(0) \neq 0$. Then on $\mathbf{P}^1 - \{0\}$, with coordinate $s = t^{-1}$, the Weierstrass model is

$$v^2 = u(u^2 + \hat{a}(s)u + s^8), \quad \Delta = s^{16}(\hat{a}(s)^2 - 4s^8), \quad u = s^4 x, \quad v = s^6 y,$$

where Δ is the discriminant. The fiber over $s = 0$ is a stable (nodal) curve, so the corresponding fiber X_∞ is of type I_m where m is the order of vanishing of the discriminant in $s = 0$ (equivalently, it is the order of the pole of the j -invariant in $s = 0$). Thus X_∞ is an I_{16} fiber. As the section of order two specializes to the singular point $(u, v, s) = (0, 0, 0)$, after blow up it will not meet the component of the fiber which meets the zero section.

The group structure of the elliptic fibration induces a Lie group structure on the smooth part of the I_{16} fiber. Taking out the 16 singular points in this fiber, we get the group $\mathbf{C}^* \times \mathbf{Z}/16\mathbf{Z}$. The zero section meets the component C_0 , where

$$C_n := \mathbf{P}^1 \times \{\bar{n}\} \hookrightarrow X_\infty,$$

and the section of order two must meet C_8 . Translation by the section of order two induces the permutation $C_n \mapsto C_{n+8}$ of the 16 components of the fiber. The classes of the components C_n , with $n = -2, \dots, 4$, generate a lattice of type $A_7(-1)$ which together with the zero section gives an $E_8(-1)$. The Nikulin involution maps this $E_8(-1)$ to the one whose components are the C_n , $n = 6, \dots, 12$, and the section of order two. Thus the Nikulin involution permutes two perpendicular copies of $E_8(-1)$ and hence it is a Morrison-Nikulin involution.

The bad fibers over $\mathbf{P}^1 - \{\infty\}$ correspond to the zeroes of $\Delta = a^2(t) - 4$. For general a , Δ has eight simple zeroes and the fibers are nodal, so we have eight fibers of type I_1 in $\mathbf{P}^1 - \{\infty\}$.

By considering the points on \mathbf{P}^1 where there are bad fibers it is not hard to see that we do get a three dimensional family of elliptic K3 surfaces with a Morrison-Nikulin involution. Thus the general member of this three dimensional family has a Néron Severi group S of rank 17.

As we constructed a unimodular sublattice $E_8(-1)^2 \subset S$, we get $S \cong \langle -d \rangle \oplus E_8(-1)^2$ and $d (> 0)$ is the discriminant of S . The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) gives that $d = 16/n^2$ where n is the order of the group of torsion sections. As n is a multiple of 2 and d must be even it follows that $d = 4$. As the embedding of $NS(X)$ into $U^3 \oplus E_8(-1)^2$ is unique up to isometry it is easy to determine $T_X = NS(X)^\perp$. Finally $T_Y \cong T_X(2)$ by the results of [Mo].

The Weierstrass model of the quotient elliptic fibration Y can be computed with the standard formula cf. [ST, p.79], the bad fibers can be found from the discriminant $\Delta = -4(a^2 - 4)^2$ (and

j -invariant). Alternatively, fixed points of the involution on X are the nodes in the I_1 -fibers. Since these are blown up, we get 8 fibers of type I_2 over the same points in \mathbf{P}^1 where X has fibers of type I_1 . At infinity Y has a fiber of type I_8 because the involution on X permutes of the 16 components of the I_{16} -fiber ($C_n \leftrightarrow C_{n+8}$). Now the minimal model of the quotient surface of Y by the translation of order two is again X , cf. [ST, Section 4, p.76]. Indeed consider the generic fiber, this is an elliptic curve C in Weierstrass form, its quotient by a point of order two is an elliptic curve \overline{C} . The kernel of the multiplication by two:

$$[2] : C \longrightarrow C$$

contains all points of order two. Hence this map factorizes as

$$C \xrightarrow{2:1} \overline{C} \xrightarrow{2:1} C.$$

□

4.8. Remark. The Weierstrass model we used to define X , $y^2 = x(x^2 + a(t)x + 1)$, exhibits X as the minimal model of the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$, with affine coordinates x and t . The branch curve consists of the the lines $x = 0$, $x = \infty$ and the curve of bidegree $(2, 4)$ defined by $x^2 + a(t)x + 1 = 0$. Special examples of such double covers are studied in section V.23 of [BPV]. In particular, on p.185 the 16-gon appears with the two sections attached and the E_8 's are pointed out in the text. Note however that our involution is *not* among those studied there.

4.9. Remark. Note that if $X \longrightarrow \mathbf{P}^1$ is an elliptic fibration with a section, multiplication by n on each smooth fiber gives a fiber preserving rational map $X \dashrightarrow X$ of degree n^2 . In the Proposition 4.7 we give an example with $n = 2$ (such self-maps are rather rare for non-rational surfaces).

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