CONTRACTION OF EXCESS FIBRES BETWEEN THE MCKAY CORRESPONDENCES IN DIMENSIONS TWO AND THREE

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ABSTRACT. The quotient singularities of dimensions two and three obtained from polyhedral groups and the corresponding binary polyhedral groups admit natural resolutions of singularities as Hilbert schemes of regular orbits whose exceptional fibres over the origin reveal similar properties. We construct a morphism between these two resolutions, contracting exactly the excess part of the exceptional fibre. This construction is motivated by the study of some pencils of K3-surfaces arising as minimal resolutions of quotients of nodal surfaces with high symmetries.

1. INTRODUCTION

Consider a binary polyhedral group $\widetilde{G} \subset \mathrm{SU}(2)$ corresponding to a polyhedral group $G \subset \mathrm{SO}(3,\mathbb{R})$ through the double-covering $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$. The group $\widetilde{G}$ acts freely on $\mathbb{C}^2 - \{0\}$ and the quotient $\mathbb{C}^2/\widetilde{G}$ is a surface singularity with an isolated singular point at the origin. The exceptional divisor of its minimal resolution of singularities $\mathcal{X} \to \mathbb{C}^2/\widetilde{G}$ is a tree of smooth rational curves of self-intersection -2, intersecting transversely, whose intersection graph is an A-D-E Dynkin diagram. The classical McKay correspondence ([23]) relates this intersection graph to the representations of the group $\widetilde{G}$, associating bijectively each exceptional curve to a non-trivial irreducible representation of the group: the correspondence in fact identifies the intersection graph with the McKay quiver of the action of $\widetilde{G}$ on $\mathbb{C}^2$. Among these irreducible representations we find all irreducible representations of the group G: we call them *pure* and the remaining ones *binary*. Since $\widetilde{G}/G \cong \{\pm 1\}$, one can produce a G-invariant cone $\mathbb{C}^2/\{\pm 1\} \xrightarrow{\sim} K \hookrightarrow \mathbb{C}^3$ whose quotient K/G is isomorphic to $\mathbb{C}^2/\widetilde{G}$. In this note, we prove the following result, conjectured by W. P. Barth:

Theorem 1.1. *There exists a crepant resolution of singularities of $\mathbb{C}^3/G$ containing a partial resolution $\mathcal{Y} \to K/G$ with the property that the intersection graph of its exceptional locus is precisely the McKay quiver of the action of G on $\mathbb{C}^3$, together with a resolution map $\mathcal{X} \to \mathcal{Y}$ mapping isomorphically the exceptional curves corresponding to pure representations and contracting those associated with binary representations to ordinary nodes.*

We make this construction in the framework of the Hilbert schemes of regular orbits of Nakamura ([25]) providing, thanks to the Bridgeland-King-Reid theorem ([5]), the natural candidates for the resolutions of singularities in dimensions two and three. We produce a morphism $\mathscr{S}$ between these two resolutions of singularities, define our partial resolution

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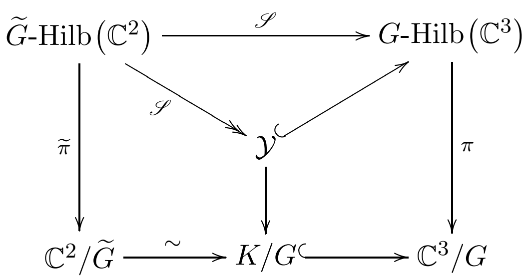
$\mathcal{Y}$ as the image of this map and study the effect of $\mathscr{S}$ on the exceptional fibres:



Although the exceptional fibres can be described very explicitly in all cases (see [19]), by principle our proof avoids any case-by-case analysis. Therefore, the key point consists in a systematic modular interpretation of the objects at issue.
From the strict point of view of the McKay correspondence, this construction shows some new properties revealing again the fertility of the geometric construction of the McKay correspondence following Gonzales-Sprinberg and Verdier [14], Ito-Nakamura [19], Ito-Najakima [18] and Reid [26]. The beginning of the story was devoted to the study of all situations in dimension two and three, in general by a case-by-case analysis. Then efforts were made to understand how to get all these cases by one general geometric construction ([18, 5]). The development followed then the cohomological direction in great dimensions in a symplectic setup ([21, 11]), leading to an explicit study of a family of examples of increasing dimension for the specific symmetric group problem ([3]). The new point of view in the present paper consists in working between two situations of different dimensions for different - but related - groups and construct a relation between them. This may be considered as a concrete application of some significant results in this area coming again at the beginning of the story, dealing with a now quite classical material approached by natural transformations between moduli spaces.
This study is motivated by previous works of Sarti [27] and Barth-Sarti [2] studying special pencils of surfaces in $\mathbb{P}_3$ with bipolyhedral symmetries. The minimal resolutions of the associated quotient surfaces are K3-surfaces with maximal Picard numbers. For some special fibres of these pencils, the resolution looks locally like the quotient of a cone by a polyhedral group, and our result gives a local interpretation of the exceptional locus in these cases.
The structure of the paper is as follows: in Section 2 we introduce the notations and we recall some basic facts about clusters and in Section 3 we recall the construction of the Hilbert schemes of points and clusters. The Sections 4, 5 and 6 give a brief survey on polyhedral, binary polyhedral and bipolyhedral groups, their representations and the classical Mckay correspondences in dimensions two and three. In Section 7 we start the study of the map $\mathscr{S}$. First we show that it is well defined (lemma 7.1) and then that it is a regular projective map, which induces a map between the exceptional fibres (proposition 7.2). In Section 8 the theorem 8.1 is the fundamental step for proving the main theorem 1.1: we show that the map $\mathscr{S}$ contracts the curves corresponding to the binary representations and maps the curves corresponding to the pure representations isomorphically to the exceptional curves downstairs. In Section 9, as an example we describe in details the case when $\widetilde{G}$ is a cyclic group. Finally the Section 10 is devoted to an application to resolutions of pencils of K3-surfaces.

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2. Clusters

In the sequel, we aim to study a link between the two-dimensional and the three-dimensional McKay correspondences. In order to avoid confusion, we shall use different sets of letters for the corresponding algebraic objects at issue in both situations. In this section, we fix the notations and the terminology.

2.1. **General setup.** Let V be a n-dimensional complex vector space and $\mathfrak{G}$ a finite subgroup of $\mathrm{SL}(V)$. We denote by $\mathcal{O}(V) := \mathrm{S}^*(V^\vee)$ the algebra of polynomial functions on V, with the induced left action $g \cdot f := f \circ g^{-1}$ for $f \in \mathcal{O}(V)$ and $g \in \mathfrak{G}$.
We choose a basis $X_1, \ldots, X_n$ of linear forms on V, denote the ring of polynomials in n indeterminates by $S := \mathbb{C}[X_1, \ldots, X_n]$ and identify $\mathcal{O}(V) \cong S$. The ring S is given a graduation by the total degree of a polynomial, where each indeterminate X_i has degree 1. In particular, the action of the group $\mathfrak{G}$ on S preserves the degree.
Let $\mathfrak{m}_S := \langle X_1, \ldots, X_n \rangle$ be the maximal ideal of S at the origin. We denote by $S^{\mathfrak{G}}$ the subring of $\mathfrak{G}$-invariant polynomials, by $\mathfrak{m}_{S^{\mathfrak{G}}}$ its maximal ideal at the origin and by $\mathfrak{n}_{\mathfrak{G}} := \mathfrak{m}_{S^{\mathfrak{G}}} \cdot S$ the ideal of S generated by the non-constant $\mathfrak{G}$-invariant polynomials vanishing at the origin. The quotient ring of *coinvariants* is by definition $S_{\mathfrak{G}} := S/\,\mathfrak{n}_{\mathfrak{G}}$.
An ideal $\mathfrak{I} \subset S$ is called a $\mathfrak{G}$-*cluster* if it is globally invariant under the action of $\mathfrak{G}$ and the quotient $S/\mathfrak{I}$ is isomorphic, as a $\mathfrak{G}$-module, to the regular representation of $\mathfrak{G}$: $S/\mathfrak{I} \cong \mathbb{C}[\mathfrak{G}]$. A closed subscheme $Z \subset \mathbb{C}^n$ is called a $\mathfrak{G}$-*cluster* if its defining ideal $\mathfrak{I}(Z)$ is a $\mathfrak{G}$-cluster. Such a subscheme is then zero-dimensional and has length $|\mathfrak{G}|$. For instance, a free $\mathfrak{G}$-orbit defines a $\mathfrak{G}$-cluster. In particular, a $\mathfrak{G}$-cluster contains only one orbit: the support of a cluster is a union of orbits, and any function constant on one orbit and vanishing on another one would induce a different copy of the trivial representation in the quotient $S/\mathfrak{I}$.
We are particularly interested in $\mathfrak{G}$-clusters supported at the origin. Then $\mathfrak{I} \subset \mathfrak{m}_S$ and in fact this condition is enough to assert that the cluster is supported at the origin: else, the support of the cluster would consist in more than one orbit. Furthermore, one has automatically $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I}$, since any non-constant function $f \in \mathfrak{n}_{\mathfrak{G}}$ not contained in $\mathfrak{I}$ would induce a new copy of the trivial representation in the quotient $S/\mathfrak{I}$, different from the one already given by the constant functions. Hence we wish to understand the structure of the $\mathfrak{G}$-clusters $\mathfrak{I}$ such that $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I} \subset \mathfrak{m}_S$, equivalent to the study of the quotient ideals $\mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \subset S/\mathfrak{n}_{\mathfrak{G}} = S_{\mathfrak{G}}$, with the exact sequence:

$$0 \longrightarrow \mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow S_{\mathfrak{G}} \longrightarrow S/\mathfrak{I} \longrightarrow 0. \tag{1}$$

From now on, we assume that the group $\mathfrak{G}$ is a subgroup of index 2 of a group $\mathfrak{R} \in \mathrm{GL}(V)$ generated by *reflections* (we follow here the terminology of [7]), *i.e.* elements $g \in \mathfrak{R}$ such that $\mathrm{rk}(g - \mathrm{Id}_V) = 1$.
The structure of the action of $\mathfrak{R}$ on S has the following properties (see [7]):

- The algebra of invariants $S^{\mathfrak{R}}$ is a polynomial algebra generated by exactly n algebraically independent homogeneous polynomials $f_1, \ldots, f_n$ of degrees d_i.
- $|\mathfrak{R}| = d_1 \cdot \ldots \cdot d_n$.
- The set of degrees $\{d_1, \ldots, d_n\}$ is independent of the choice of the homogeneous generators.

- The algebra of coinvariants is isomorphic to the regular representation: $S_{\mathfrak{R}} \cong \mathbb{C}[\mathfrak{R}]$.

As a byproduct, we get that the algebra of coinvariants $S_{\mathfrak{R}}$ is a graded finite-dimensional algebra.
From this and the fact that $\mathfrak{G} = \mathfrak{R} \cap \mathrm{SL}(V)$, one deduces the structure of the action of $\mathfrak{G}$ on S (see [4, 12, 13]):

- There exists a homogeneous $\mathfrak{R}$-skew-invariant polynomial $f_{n+1} \in S$, *i.e.* such that $g \cdot f_{n+1} = \det(g).f_{n+1}$ for all $g \in \mathfrak{R}$, unique up to a multiplicative constant, dividing any $\mathfrak{R}$-skew-invariant polynomial: hence the set $f_{n+1} \cdot S^{\mathfrak{R}}$ is precisely the set of $\mathfrak{R}$-skew-invariants. A natural choice for this element is $f_{n+1} = \mathrm{Jac}(f_1, \dots, f_n)$.
- $S^{\mathfrak{G}} = \mathbb{C}[f_1, \dots, f_n, f_{n+1}]$.
- $\mathfrak{n}_{\mathfrak{G}} = \mathfrak{n}_{\mathfrak{R}} \oplus \mathbb{C} f_{n+1}$.
- $S_{\mathfrak{R}} = S_{\mathfrak{G}} \oplus \mathbb{C} f_{n+1}$.

Note that, as a $\mathfrak{G}$-module, $\mathbb{C}[\mathfrak{R}]$ is isomorphic to two copies of $\mathbb{C}[\mathfrak{G}]$. It follows that $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$ is a graded finite-dimensional algebra which, as a $\mathfrak{G}$-module, consists exactly of each non-trivial representation ρ of $\mathfrak{G}$ repeated $2\dim\rho$ times: one can denote the occurrences of each representation ρ by $V^{(1)}(\rho), \dots, V^{(2\dim\rho)}(\rho)$ where each $V^{(i)}(\rho)$ is given by homogeneous polynomials modulo $\mathfrak{n}_{\mathfrak{G}}$.
Thanks to the exact sequence (1), giving a $\mathfrak{G}$-cluster supported at the origin consists in choosing, for each non-trivial representation ρ of $\mathfrak{G}$, $\dim\rho$ copies of ρ in $\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}}$. But this gives many choices since any linear combination of some $V^{(i)}(\rho)$ and $V^{(j)}(\rho)$ provides such a copy. The ground idea is that one does not have to make all these choices in order to define $\mathfrak{I}$ (see §9 for an explicit example).
For such an ideal $\mathfrak{I}$ with $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{I} \subset \mathfrak{m}_S$, we consider the finite-dimensional $\mathfrak{G}$-modules $W \subset S$ generating $\mathfrak{I}$ in the sense that $\mathfrak{I} = W \cdot S + \mathfrak{n}_{\mathfrak{G}}$. Such modules do exist thanks to the preceding construction. Among these choices, we consider the minimal ones, *i.e.* such that no strict $\mathfrak{G}$-submodule of them generate $\mathfrak{I}$ in the preceding sense.
If W is a generator in this sense, then

$$\mathfrak{I} = W \cdot S + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot W + \mathfrak{n}_{\mathfrak{G}} = W + \mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}.$$

This means that the $\mathbb{C}$-linear map $W \to \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$ is surjective. Also, since W is a $\mathfrak{G}$-module and since $\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}$ is $\mathfrak{G}$-stable, this map is $\mathfrak{G}$-linear. If W is a minimal set of generators, it satisfies in particular $W \cap (\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}}) = \{0\}$ since this intersection would provide a $\mathfrak{G}$-submodule whose complementary in W is a smaller $\mathfrak{G}$-submodule generating $\mathfrak{I}$. Hence, for W minimal one gets an isomorphism of $\mathfrak{G}$-modules $W \cong \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$. We set then $V(\mathfrak{I}) := \mathfrak{I}/(\mathfrak{m}_S \cdot \mathfrak{I} + \mathfrak{n}_{\mathfrak{G}})$. The set of generators of $V(\mathfrak{I})$ may not be uniquely determined, but its structure as a $\mathfrak{G}$-module is unique. The important issue, that will be the core of the classification, will be to determine whether $V(\mathfrak{I})$ is irreducible or not, although it is a minimal set of generators.

2.2. **Notations for the two- and three-dimensional cases.** When applying the preceding constructions in dimensions two or three, we fix the following notations:

- For $n = 2$, the polynomial ring is denoted by $A := \mathbb{C}[x, y]$, the group by $\widetilde{G}$ and any ideal by I.
- For $n = 3$, the polynomial ring is denoted by $B := \mathbb{C}[a, b, c]$, the group by G and any ideal by J.

3. Moduli space of clusters

We recall here the constructions of the Hilbert schemes of points or clusters.

3.1. **Hilbert scheme of points.** Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a quasi-projective scheme and N a positive integer. Consider the contravariant functor $\mathcal{H}ilb^N_X$ from the category of schemes to the category of sets

$$\mathcal{H}ilb^N_X : (Schemes) \to (Sets)$$

which is given by

$$\mathcal{H}ilb^N_X(T) := \left\{ Z \subset T \times X \;\middle|\; \begin{array}{ll} (a) & Z \text{ is a closed subscheme} \\ (b) & \text{the morphism } Z \hookrightarrow T \times X \xrightarrow{p} T \text{ is flat} \\ (c) & \forall t \in T, Z_t \subset X \text{ is a closed subscheme} \\ & \text{of dimension 0 and length } N \end{array} \right\}$$

By a theorem of Grothendieck ([15]), this functor is representable by a quasi-projective scheme $\mathrm{Hilb}^N(X)$ equipped with a *universal family* $\Xi^X_N \subset \mathrm{Hilb}^N(X) \times X$. In the sequel, we shall always denote by p the projection to the moduli space (here $\mathrm{Hilb}^N(X)$) and by q the projection to the base (here X). When X is projective, the scheme $\mathrm{Hilb}^N(X)$ is projective and comes with a very ample line bundle (for $\ell \gg 0$):

$$\det\left(p_*\left(\mathcal{O}_{\Xi^X_N} \otimes q^*\mathcal{O}_X(\ell)\right)\right).$$

When $X = \mathbb{C}^n$, one gets an open immersion $\mathrm{Hilb}^N(\mathbb{C}^n) \hookrightarrow \mathrm{Hilb}^N(\mathbb{P}^n_{\mathbb{C}})$ corresponding to the restriction of the universal family. The induced restriction of the preceding determinant line bundle provides us the very ample line bundle $\det\left(p_*\mathcal{O}_{\Xi^{\mathbb{C}^n}_N}\right)$ on $\mathrm{Hilb}^N(\mathbb{C}^n)$.

There exists a natural projective morphism from $\mathrm{Hilb}^N(X)$ to the symmetric product $\mathrm{S}^N(X)$ sending a closed subscheme to the corresponding 0-cycle describing its support, called the *Hilbert-Chow* morphism:

$$\mathscr{H} : \mathrm{Hilb}^N(X) \longrightarrow \mathrm{S}^N(X).$$

By a theorem of Fogarty ([10]), the scheme $\mathrm{Hilb}^N(X)$ is connected. For $\dim X = 2$, it is reduced, smooth and the morphism $\mathscr{H}$ is a resolution of singularities.

3.2. **Hilbert scheme of regular orbits.** We consider the sub-functor $\mathfrak{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^n}$ of $\mathcal{H}ilb^{|\mathfrak{G}|}_{\mathbb{C}^n}$ given by

$$\mathfrak{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^n}(T) := \left\{ Z \in \mathcal{H}ilb^{|\mathfrak{G}|}_{\mathbb{C}^n}(T) \,|\, \forall t \in T, Z_t \subset \mathbb{C}^n \text{ is a } \mathfrak{G}\text{-cluster} \right\}.$$

This functor is representable by a quasi-projective scheme $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ called the *Hilbert scheme of* $\mathfrak{G}$*-regular orbits*, which is a union of some connected components of the subscheme of $\mathfrak{G}$-fixed points $\left(\mathrm{Hilb}^{|\mathfrak{G}|}(\mathbb{C}^n)\right)^{\mathfrak{G}}$. Furthermore, the quotient $\mathbb{C}^n/\mathfrak{G}$ can be identified with a closed subscheme of $\mathrm{S}^{|\mathfrak{G}|}(\mathbb{C}^n)$ and since the support of a $\mathfrak{G}$-cluster consists exactly of one orbit through $\mathfrak{G}$, the restriction of the Hilbert-Chow morphism factorizes through a projective morphism (see [5, 18, 28]):

$$\mathscr{H} : \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}.$$

There is a unique irreducible component of $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ containing the free $\mathfrak{G}$-orbits and mapping birationally onto $\mathbb{C}^n/\mathfrak{G}$. This component is taken as the definition of the Hilbert scheme of $\mathfrak{G}$-regular orbits in [25]. By the theorem of Bridgeland-King-Reid [5], if $n \leq 3$, then $\mathfrak{G}\text{-Hilb}(\mathbb{C}^n)$ is already irreducible, reduced, smooth and the map $\mathscr{H}$ a crepant resolution of singularities of the quotient $\mathbb{C}^n/\mathfrak{G}$. Moreover, $\mathscr{H}$ is an isomorphism over the open subset of free $\mathfrak{G}$-orbits. As a byproduct, the two definitions coincide.

As before, the scheme $\mathfrak{G}$-Hilb$(\mathbb{C}^n)$ is equipped with a universal family $\mathcal{Z}_{\mathfrak{G}}$ which is the restriction of the universal family $\Xi^{\mathbb{C}^n}_{|\mathfrak{G}|}$ corresponding to the closed immersion $\mathfrak{G}$-Hilb$(\mathbb{C}^n) \hookrightarrow$ Hilb$^{|\mathfrak{G}|}(\mathbb{C}^n)$. The induced restriction of the determinant line bundle provides us, by naturality of the construction of the determinant of a family (see [17, §8.1]), the very ample line bundle $\det(p_*\mathcal{O}_{\mathcal{Z}_{\mathfrak{G}}})$ on $\mathfrak{G}$-Hilb$(\mathbb{C}^n)$.

4. Rotation groups

4.1. **Polyhedral groups.** Let $\mathrm{SO}(3,\mathbb{R})$ be the group of rotations in $\mathbb{R}^3$. Up to conjugation, there are five different types of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$, called *polyhedral groups*:

- the cyclic groups $C_n \cong \mathbb{Z}/n\mathbb{Z}$ of order $n \geq 1$;
- the dihedral groups $D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of order $2n$, $n \geq 1$;
- the group $\mathcal{T}$ of positive isometries of a regular tetrahedra, isomorphic to the alternate group $\mathfrak{A}_4$ of order 12;
- the group $\mathcal{O}$ of positive isometries of a regular octahedra or a cube, isomorphic to the symmetric group $\mathfrak{S}_4$ of order 24;
- the group $\mathcal{I}$ of positive isometries of a regular icosahedra or a regular dodecahedra, isomorphic to the alternate group $\mathfrak{A}_5$ of order 60.

4.2. **Binary polyhedral groups.** Let $\mathbb{H}$ be the real algebra of quaternions, with basis $(1, i, j, k)$. The *norm* of a quaternion $q = a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$ is $N(q) := a^2 + b^2 + c^2 + d^2$, $a, b, c, d \in \mathbb{R}$. Let $\mathbb{S}$ be the three-dimensional sphere of quaternions of length 1 and H the three-dimensional vector subspace of *pure* quaternions (*i.e.* $a = 0$). For $q \in \mathbb{S}$, the action by conjugation $\phi(q) : H \to H$, $x \mapsto q\cdot x\cdot q^{-1}$ is an isometry. Since the group $\mathbb{S}$ is isomorphic to $\mathrm{SU}(2)$ by the identification

$$q = \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix},$$

one gets an exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \xrightarrow{\phi} \mathrm{SO}(3,\mathbb{R}) \longrightarrow 0.$$

For any finite subgroup $G \subset \mathrm{SO}(3,\mathbb{R})$, the inverse image $\widetilde{G} := \phi^{-1}G$ is called a *binary polyhedral group*. It is a finite subgroup of $\mathrm{SU}(2)$ or equivalently, up to conjugation, of $\mathrm{SL}(2,\mathbb{C})$:

- the binary cyclic groups $\widetilde{C_n} \cong C_{2n}$ have order $2n$;
- the binary dihedral groups $\widetilde{D_n}$ have order $4n$;
- the binary tetrahedral group $\widetilde{\mathcal{T}}$ has order 24;
- the binary octahedral group $\widetilde{\mathcal{O}}$ has order 48;
- the binary icosahedral group $\widetilde{\mathcal{I}}$ has order 120.

4.3. **Representations of polyhedral groups.** Consider a binary polyhedral group $\widetilde{G}$, the associated polyhedral group G and set $\tau := \{\pm 1\}$:

$$0 \longrightarrow \tau \longrightarrow \widetilde{G} \xrightarrow{\phi} G \longrightarrow 0.$$

This exact sequence induces an injection of the set of irreducible representations of G in the set of irreducible representations of $\widetilde{G}$: if $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation of G, it induces by composition a representation of $\widetilde{G}$ which is τ-invariant, *i.e.* such that $\rho(-g) = \rho(g)$ for all $g \in \widetilde{G}$. Thanks to this property, if the representation ρ would admit a non-trivial $\widetilde{G}$-submodule, it would also be a non-trivial G-submodule after going to the quotient $\widetilde{G}/\tau \cong G$. This shows also that the image of the injection (since G is a quotient of $\widetilde{G}$):

$$\mathrm{Irr}(G) \hookrightarrow \mathrm{Irr}(\widetilde{G})$$

consists precisely on those irreducible representations which are τ-invariant. These representations are called *pure* and the remaining representations are called *binary*. More precisely, if $\rho : \widetilde{G} \to \mathrm{GL}(V)$ is an irreducible representation of $\widetilde{G}$, the subspace

$$V^\tau := \{v \in V \mid v = \rho(-1)v\}$$

is a $\widetilde{G}$-submodule of V. Hence either $V^\tau = V$ and the representation ρ is pure, or ρ is binary and $V^\tau = \{0\}$.
For each type of binary polyhedral group, we draw the list of the irreducible representations with their dimension. The binary representations are labelled by a "$\sim$" and the trivial representation is denoted by χ_0 in all cases:

- binary cyclic group $\widetilde{C}_n$, $n \geq 1$:

representation	χ_0	$\{\chi_j\}_{j=1,\ldots,n-1}$	$\{\widetilde{\chi}_j\}_{j=1,\ldots,n}$
dimension	1	1	1

- binary dihedral group $\widetilde{D}_n$ for $n = 2\ell + 1$, $\ell \geq 1$:

representation	χ_0	χ_1	$\{\tau_j\}_{j=1,\ldots,\ell}$	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\{\widetilde{\sigma}_j\}_{j=1,\ldots,\ell}$
dimension	1	1	2	1	1	2

- binary dihedral group $\widetilde{D}_n$ for $n = 2\ell$, $\ell \geq 1$:

representation	χ_0	χ_1	χ_2	χ_3	$\{\tau_j\}_{j=1,\ldots,\ell-1}$	$\{\widetilde{\sigma}_j\}_{j=1,\ldots,\ell}$
dimension	1	1	1	1	2	2

- binary tetrahedral group $\widetilde{\mathcal{T}}$:

representation	χ_0	χ_1	χ_2	χ_3	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$
dimension	1	1	1	3	2	2	2

- binary octahedral group $\widetilde{\mathcal{O}}$:

representation	χ_0	χ_1	χ_2	χ_3	χ_4	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$
dimension	1	1	2	3	3	2	2	4

- binary icosahedral group $\widetilde{\mathcal{I}}$:

representation	χ_0	χ_1	χ_2	χ_3	χ_4	$\widetilde{\chi}_1$	$\widetilde{\chi}_2$	$\widetilde{\chi}_3$	$\widetilde{\chi}_4$
dimension	1	3	3	4	5	2	2	4	6

4.4. **Bipolyhedral groups.** For $p, q \in \mathbb{S}$, the action $\sigma(p,q) : \mathbb{H} \to \mathbb{H}, x \mapsto p \cdot x \cdot q^{-1}$ is an isometry and one gets an exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\sigma} \mathrm{SO}(4, \mathbb{R}) \longrightarrow 0.$$

For any binary polyhedral group $\widetilde{G}$, the direct image $\sigma(\widetilde{G} \times \widetilde{G}) \subset \mathrm{SO}(4,\mathbb{R})$ is called a *bipolyhedral group*. In §10, we shall make use of the following particular groups:

- $G_6 = \sigma(\widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}})$ of order 288;
- $G_8 = \sigma(\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}})$ of order 1152;
- $G_{12} = \sigma(\widetilde{\mathcal{I}} \times \widetilde{\mathcal{I}})$ of order 7200.

5. Graph-theoretic intuition

5.1. **McKay quivers.** If $\mathfrak{G} \subset \mathrm{SL}(n, \mathbb{C})$ is a finite subgroup, it defines a natural faithful representation $\mathcal{Q}$ of $\mathfrak{G}$. Let $\{V_0, \ldots, V_k\}$ be a complete set of irreducible representations of $\mathfrak{G}$, where V_0 denotes the trivial one. For each such representation, one may decompose the tensor products

$$\mathcal{Q} \otimes V_i \cong \bigoplus_{j=0}^{k} V_j^{\oplus a_{i,j}}$$

for some non-negative integers $a_{i,j}$. If the character of the representation $\mathcal{Q}$ is real-valued, then $a_{i,j} = a_{j,i}$ for all i, j. One defines the *McKay quiver* as the graph with vertices $V_0, V_1, \ldots, V_k$ and $a_{i,j}$ edges between the vertices V_i and V_j. In particular, this quiver may contain some loops. For our purpose, we only consider the *reduced* McKay quiver with vertices $V_1, \ldots, V_k$ and one edge between V_i and V_j if $i \neq j$ and $a_{i,j} \neq 0$: this means that we remove from the McKay quiver the vertex V_0, all edges starting from it, all loops and all multiple edges. When there is an edge joining V_i and V_j, the vertices are called *adjacent*.

One may check that all finite subgroups of SL(2) or $\mathrm{SO}(3,\mathbb{R})$ enter in this context since their natural representation $\mathcal{Q}$ is real-valued.

5.2. **McKay quivers for the polyhedral groups.** For each binary polyhedral group $\widetilde{G} \subset \mathrm{SU}(2)$ and its corresponding polyhedral group $G \subset \mathrm{SO}(3,\mathbb{R})$, we draw the reduced McKay quiver with our conventions. For the binary polyhedral groups, we denote by a white vertex the pure representations and by a black vertex the binary ones. We get (see for example [14, 12, 13]) the graphs of figure 1.

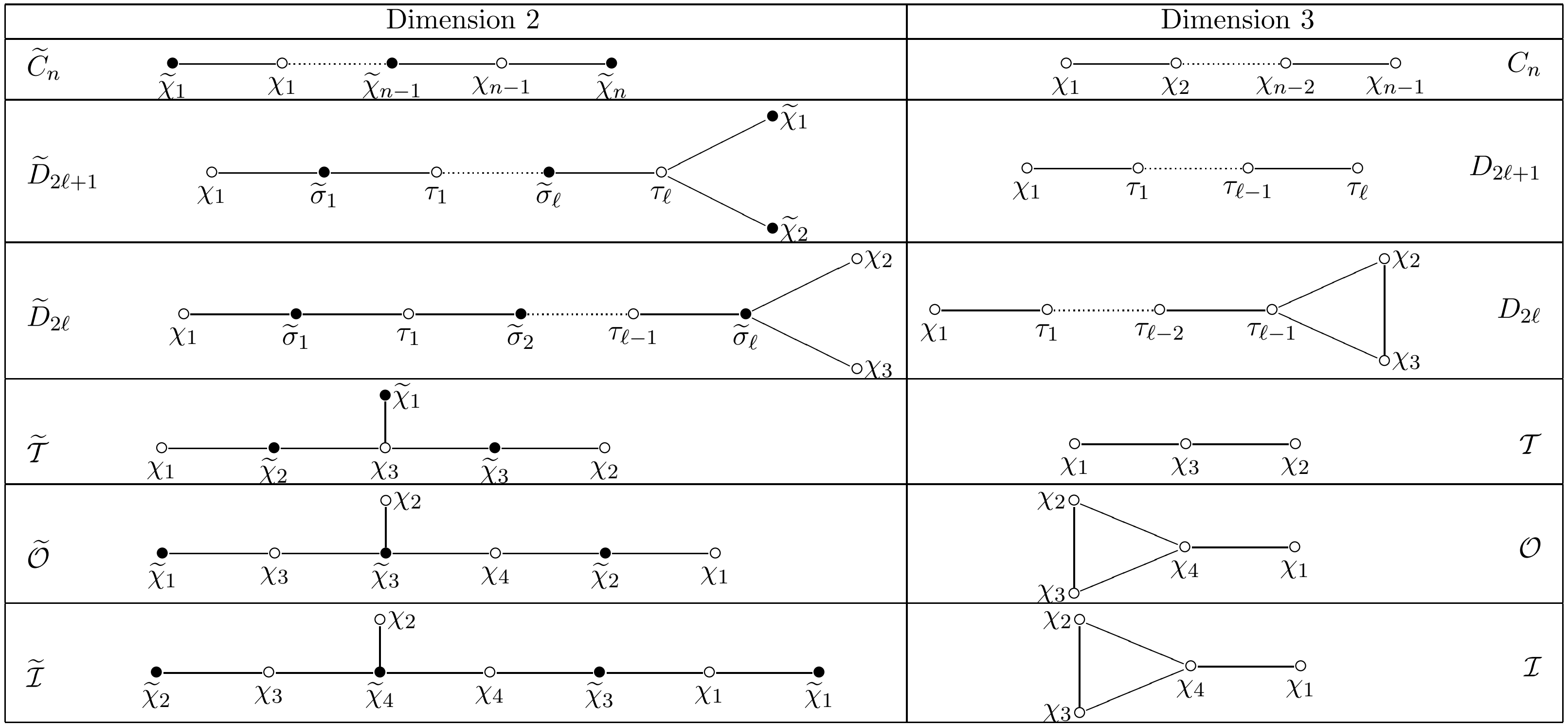

FIGURE 1. Reduced McKay quivers

In the sequel, we shall interpret these graphs as the intersection graphs of a family of smooth rational curves meeting transversally. One may then get the following intuition: looking at the two-dimensional graphs, if one contracts the curves associated to a binary representation (black nodes), then one gets as intersection graph precisely the corresponding graph in dimension three!
Another property of the two-dimensional quivers is that no two pure representations and no two binary representations are adjacent. This means that the preceding idea of contraction contracts only one curve each time.

6. Exceptional fibres in dimensions two and three

Considering the Hilbert-Chow morphism $\mathscr{H} : \mathfrak{G}\text{-Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/\mathfrak{G}$, our purpose is to describe the exceptional fibre $\mathscr{H}^{-1}(O)$ over the origin $O \in \mathbb{C}^n/\mathfrak{G}$ in the two- and three-dimensional cases. Note that all finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SO}(3,\mathbb{R})$ enter in the context of §2 since they are subgroups of index 2 of a reflection group (see [13, §2.7]). Hence we may apply the general procedure for the study of the clusters supported at the origin.
The understanding of the exceptional fibre in these cases was achieved by Ito-Nakamura [19, 20] in dimension two and by Gomi-Nakamura-Shinoda [12, 13] in dimension three, by a case-by-case analysis. For the two-dimensional case, there is another proof by Crawley-Boevey [8] avoiding this case-by-case analysis. We recall the results.
For any finite group $\mathfrak{G}$, $\mathrm{Irr}^*(\mathfrak{G})$ denotes the set of irreducible representations but the trivial one.

6.1. **Structure of the exceptional fibre in dimension two.** Let $\widetilde{G} \subset \mathrm{SL}(2,\mathbb{C})$ be a binary polyhedral group and denote the Hilbert-Chow morphism by

$$\widetilde{\pi} : \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \longrightarrow \mathbb{C}^2/\widetilde{G}.$$

For each non-trivial irreducible representation ρ of $\widetilde{G}$, set

$$E(\rho) := \{I \in \widetilde{\pi}^{-1}(O)_{\mathrm{red}} \,|\, V(I) \supset \rho\}.$$

Theorem 6.1. ([19, Theorem 3.1]

- *Each $E(\rho)$ is a smooth rational curve of self-intersection -2.*
- $\widetilde{\pi}^{-1}(O)_{red} = \bigcup_\rho E(\rho)$ *and* $\widetilde{\pi}^{-1}(O) = \sum_\rho \dim\rho \cdot E(\rho)$ *as a Cartier-divisor,* $\rho \in \mathrm{Irr}^*(\widetilde{G})$.
- *If $I \in E(\rho)$ and $I \notin E(\rho')$ for all $\rho \neq \rho'$, then $V(I) \cong \rho$.*
- *If $I \subset E(\rho) \cap E(\rho')$, then $V(I) \cong \rho \oplus \rho'$ and the curves $E(\rho)$ and $E(\rho')$ intersect transversally at I.*
- *The intersection graph of these curves is the reduced McKay quiver of the group $\widetilde{G}$.*

In particular, a generator $V(I)$ does not contain more than one copy of any irreducible representation, and $E(\rho)\cap E(\rho') \neq \emptyset$ if and only if the representations ρ and ρ' are adjacent.

6.2. **Structure of the exceptional fibre in dimension three.** Let $G \subset \mathrm{SO}(3,\mathbb{R})$ be a polyhedral group and denote the Hilbert-Chow morphism by

$$\pi : G\text{-Hilb}\big(\mathbb{C}^3\big) \longrightarrow \mathbb{C}^3/G.$$

For each non-trivial irreducible representation ρ of G, set

$$C(\rho) := \{J \in \pi^{-1}(O)_{\mathrm{red}} \,|\, V(J) \supset \rho\}.$$

Theorem 6.2. ([13, Theorem 3.1])

- *Each* $C(\rho)$ *is a smooth rational curve.*
- $\pi^{-1}(O)_{red} = \bigcup_\rho C(\rho)$, $\rho \in \mathrm{Irr}^*(G)$.
- *If* $J \in C(\rho)$ *and* $J \notin C(\rho')$ *for all* $\rho \neq \rho'$, *then* $V(J) \cong \rho$.
- *The intersection graph of these curves is the reduced McKay quiver of the group* G.

6.3. **Explicit parameterizations.** Let us explain briefly the explicit parameterizations of the exceptional curves obtained in *loc.cit.* This description holds both in dimensions two and three so we do it with our general notations. The example of the cyclic group is treated in §9. As we explained in §2,

$$\mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \cong \bigoplus_{\substack{\rho\in\mathrm{Irr}(\mathfrak{G})\\ \rho\neq\rho_0}} \bigoplus_{i=1}^{2\dim\rho} V^{(i)}(\rho)$$

where ρ_0 denotes the trivial representation. Thanks to the exact sequence

$$0 \longrightarrow \mathfrak{I}/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{n}_{\mathfrak{G}} \longrightarrow \mathfrak{m}_S/\mathfrak{I} \longrightarrow 0,$$

if one wants to parameterize a flat family of clusters over $\mathbb{P}_1$, one has to choose, in the trivial sheaf:

$$\mathcal{O}_{\mathbb{P}^1} \otimes \bigoplus_{\substack{\rho\in\mathrm{Irr}(\mathfrak{G})\\ \rho\neq\rho_0}} \bigoplus_{i=1}^{2\dim\rho} V^{(i)}(\rho),$$

a locally free $\mathfrak{G}$-equivariant sheaf affording the regular representation on each fibre whose quotient is also locally free. The parameterizations are then produced as follows: one chooses *one* non trivial subbundle

$$\mathcal{O}_{\mathbb{P}_1}(-1) \otimes \rho \hookrightarrow \mathcal{O}_{\mathbb{P}_1} \otimes (V^{(i)}(\rho) \oplus V^{(j)}(\rho))$$

for some appropriate choice of the indices, and shows that this gives the required family whose points $\mathfrak{I}$ are characterized by their generator

$$V(\mathfrak{I}) \subset \mathbb{P}(V^{(i)}(\rho) \oplus V^{(j)}(\rho)).$$

That is: once one choice has been made, the other choices are automatic, and we shall see that they always correspond to a trivial subbundle (see 8.4).

7. Geometric construction

Let $\widetilde{G}$ be a binary polyhedral group acting on $A = \mathbb{C}[x,y]$. Set $\tau := \langle \pm 1 \rangle \subset \widetilde{G}$ and $G := \widetilde{G}/\tau$ the associated polyhedral group as before. It is important for the sequel to begin so, and not to choose the group G with its action on some coordinates first, as we shall see. We aim to define a regular map

$$\mathscr{S} : \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \longrightarrow G\text{-Hilb}\big(\mathbb{C}^3\big)$$

inducing a map between the exceptional fibres over the origin.
Since $A^\tau = \mathbb{C}[x^2, y^2, xy]$, we consider the following composition of ring morphisms, with $B = \mathbb{C}[a,b,c]$:

$$\sigma : B \twoheadrightarrow B \,/\langle ab - c^2\rangle \xrightarrow{\sim} A^\tau \longrightarrow A \tag{2}$$

where the identification is defined by $a = x^2, b = y^2, c = xy$. The action of $\widetilde{G}$ on A induces an action of G on A^τ. Using the identification, we can define an action of G on the coordinates a, b, c, inducing an action on B with the property that the cone $K = \langle ab - c^2 \rangle$ is G-invariant. This is the reason why we did not fix the action of G at first: another choice of identification would induce another action of G.
Let I be an ideal of A and $J := \sigma^{-1}(I)$ the corresponding ideal of B. Observe the following property of the map σ:

Lemma 7.1. *If I is a $\widetilde{G}$-cluster in A, then J is a G-cluster in B. Furthermore, if I is supported at the origin, then so is J.*

Proof. If I is a $\widetilde{G}$-cluster, then $A/I \cong \mathbb{C}[\widetilde{G}]$. Since the group τ is finite, we have isomorphisms:

$$B/J \cong A^\tau / I^\tau \cong (A/I)^\tau \cong \mathbb{C}[\widetilde{G}]^\tau \cong \mathbb{C}[G],$$

hence J is a G-cluster in B. Furthermore, note that $\sigma^{-1}\mathfrak{m}_A = \mathfrak{m}_B$ hence if I is a $\widetilde{G}$-cluster supported at the origin, one has $I \subset \mathfrak{m}_A$ and then $J \subset \mathfrak{m}_B$, which implies that J is also supported at the origin (see §2.1). □

Therefore, this construction defines set-theoretically a map between the two moduli spaces of clusters $\mathscr{S} : \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow G\text{-Hilb}(\mathbb{C}^3)$ by $\mathscr{S}(I) \overset{\text{Def}}{=} J$. It remains to see that this map is a regular morphism.

Proposition 7.2. *The map $\mathscr{S}$ is regular, projective, and induces a map between the exceptional fibres.*

Proof.
⋄ In order to get that the map $\mathscr{S}$ is regular, we show that it is induced by a natural transformation between the two functors of points

$$\widetilde{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^2}(\cdot) \Longrightarrow G\text{-}\mathcal{H}ilb_{\mathbb{C}^3}(\cdot).$$

Let T be a scheme and $Z \in \widetilde{G}\text{-}\mathcal{H}ilb_{\mathbb{C}^2}(T)$. Then $Z \subset T \times \mathbb{C}^2$ is a flat family of $\widetilde{G}$-clusters over T and the map $Z \hookrightarrow T \times \mathbb{C}^2$ is τ-equivariant (for a trivial action on T). It induces a family

$$Z/\tau \hookrightarrow T \times (\mathbb{C}^2/\tau) \hookrightarrow T \times \mathbb{C}^3$$

where the quotient $\mathbb{C}^2/\tau$ is considered as the cone $\langle ab - c^2 \rangle$ in $\mathbb{C}^3$. If T is a point, this is precisely our set-theoretic construction since then if Z is given by an ideal I, Z/τ is given by the ideal I^τ.
In order to show that $Z/\tau \in G\text{-}\mathcal{H}ilb_{\mathbb{C}^3}(T)$, we have to prove that this family is flat over T. Since this problem is local in T, we may assume that T is an affine scheme, say $T = \operatorname{Spec} R$. Then the family Z is given by a τ-equivariant quotient $R \otimes A \twoheadrightarrow Q$ so that the composition $R \hookrightarrow R \otimes_{\mathbb{C}} A \twoheadrightarrow Q$ makes Q a flat R-module. The family Z/τ is then given by the quotient

$$R \hookrightarrow R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^\tau \twoheadrightarrow Q^\tau,$$

where the quotient $R \otimes_{\mathbb{C}} B \twoheadrightarrow R \otimes_{\mathbb{C}} A^\tau$ is induced by tensorization of the quotient $B \twoheadrightarrow A^\tau$. We have to show that this makes Q^τ a flat R-module. By hypothesis, the functor $Q \otimes_R -$ in the category of R-modules is exact. Since τ is finite, the functor $(-)^\tau$ is also exact in this category, and we note that the functor $Q^\tau \otimes_R -$ is the composition of this two functors since

$$Q^\tau \otimes_R N = (Q \otimes_R N)^\tau$$

for any R-module N. Hence the functor $Q^\tau \otimes_R -$ is exact, which means that the family is flat.

$\diamond$ The composition of ring morphisms (2) gives an equivariant ring morphism

$$\begin{array}{ccc} \mathbb{C}[a,b,c] & \xrightarrow{\sigma} & \mathbb{C}[x,y] \\ \circlearrowright_{G} & & \circlearrowright_{\widetilde{G}} \end{array}$$

inducing a surjective map at the level of the invariants: $\mathbb{C}[a,b,c]^G \twoheadrightarrow \mathbb{C}[x,y]^{\widetilde{G}}$, hence a closed immersion

$$\eta : \mathbb{C}^2/\widetilde{G} = \operatorname{Spec} \mathbb{C}[x,y]^{\widetilde{G}} \longrightarrow \operatorname{Spec} \mathbb{C}[a,b,c]^G = \mathbb{C}^3/G.$$

Taking more care of the cone $K = \mathbb{C}^2/\tau$ (in the notations of the introduction), the equivariant map

$$\begin{array}{ccccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\tau & \longrightarrow & \mathbb{C}^3 \\ \circlearrowright_{\widetilde{G}} & & \circlearrowright_{G} & & \circlearrowright_{G} \end{array}$$

induces the η map between the quotients:

$$\eta : \mathbb{C}^2/\widetilde{G} \xrightarrow{\sim} \left(\mathbb{C}^2/\tau\right)/\,G \longrightarrow \mathbb{C}^3/G$$

sending the origin $O \in \mathbb{C}^2/\widetilde{G}$ to the origin $O \in \mathbb{C}^3/G$ and by definition of $\mathscr{S}$ the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) & \xrightarrow{\mathscr{S}} & G\text{-Hilb}\left(\mathbb{C}^3\right) \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{C}^2/\widetilde{G} & \xrightarrow{\quad\eta\quad} & \mathbb{C}^3/G \end{array}$$

This implies that $\mathscr{S}$ induces a map between the exceptional fibres

$$\widetilde{\pi}^{-1}(O) \xrightarrow{\mathscr{S}} \pi^{-1}(O).$$

$\diamond$ We prove that the map $\mathscr{S}$ is proper by applying the valuative criterion of properness. Let K be any field over $\mathbb{C}$ and $R \subset K$ any valuation ring with quotient field K. Consider a commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\phi} & \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle\mathscr{S}} \\ \operatorname{Spec} R & \xrightarrow{\psi} & G\text{-Hilb}\left(\mathbb{C}^3\right) \end{array}$$

We have to show that there exists a unique factorization

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\phi} & \widetilde{G}\text{-Hilb}\left(\mathbb{C}^2\right) \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle\tilde{\phi}} & \downarrow{\scriptstyle\mathscr{S}} \\ \operatorname{Spec} R & \xrightarrow{\psi} & G\text{-Hilb}\left(\mathbb{C}^3\right) \end{array}$$

making the whole diagram commute.

By modular interpretation, the data of the map ϕ consists in an ideal $I \subset K[x,y]$ such that $K[x,y]/I \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$ and $K[x,y]/I$ is K-flat (it is here trivial since K is a field). Similarly, the data of the map ψ consists in an ideal $J \subset R[a,b,c]$ such that $R[a,b,c]/J \cong$

$\mathbb{C}[G] \otimes_{\mathbb{C}} R$ and $R[a,b,c]/J$ is R-flat. The commutativity $\mathscr{S} \circ \phi = \psi \circ i : \operatorname{Spec} K \to G\text{-Hilb}(\mathbb{C}^3)$ means the following. Consider the diagram of ring morphisms induced by natural extension of scalars and base-change from the map σ:

$$\begin{array}{ccc} R[a,b,c] & \xrightarrow{\sigma_R} & R[x,y] \\ \downarrow & & \downarrow \\ K[a,b,c] & \xrightarrow{\sigma_K} & K[x,y] \end{array}$$

Then the commutativity condition means that $\sigma_K^{-1}(I) = J \cdot K[a,b,c]$.
We are looking for a map $\tilde{\phi}$ such that $\tilde{\phi} \circ i = \phi$ and $\mathscr{S} \circ \tilde{\phi} = \psi$, *i.e.* for an ideal $\widetilde{I} \subset R[x,y]$ such that $R[x,y]/\widetilde{I} \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} R$ and $R[x,y]/\widetilde{I}$ is R-flat, satisfying the conditions $\widetilde{I} \cdot K[x,y] = I$ and $\sigma_R^{-1}(\widetilde{I}) = J$.
A natural candidate is $\widetilde{I} \overset{\text{Def}}{=} I \cap R[x,y]$. We have to prove that it satisfies all the conditions and that it is unique for these properties. Denote by $\nu : K - \{0\} \to H$ the valuation with values in a totally ordered group H, satisfying the properties:

$$\nu(x \cdot y) = \nu(x) + \nu(y) \text{ and } \nu(x+y) \geq \min(\nu(x), \nu(y)) \quad \text{for } x, y \in K - \{0\}$$

and such that $R = \{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$. Recall that R is by definition integral and that a R-module is flat if and only if it is torsion-free (see for instance [1, 16]).

- It is already clear that $\widetilde{I} \cdot K[x,y] \subset I$. Conversely, Let $P = \sum_{i,j} p_{i,j} x^i y^j \in I$ and $p \in \{p_{i,j}\}$ an element of minimal valuation. If $\nu(p) \geq 0$, then $P \in \widetilde{I}$. Else all coefficients of $p^{-1}P$ have positive valuation and so $p^{-1}P \in \widetilde{I}$. So $P = p \cdot (p^{-1}P) \in \widetilde{I} \cdot K[x,y]$, hence the equality.
- By commutativity of the above diagram,

$$\begin{aligned} \sigma_R^{-1}(\widetilde{I}) &= \sigma_R^{-1}(I \cap R[x,y]) \\ &= \sigma_K^{-1}(I) \cap R[a,b,c] \\ &= (J \cdot K[a,b,c]) \cap R[a,b,c]. \end{aligned}$$

  It is already clear that $J \subset (J \cdot K[a,b,c]) \cap R[a,b,c]$. Conversely, let $P \in (J \cdot K[a,b,c]) \cap R[a,b,c]$, decomposed as $P = \sum_\ell U_\ell \cdot V_\ell$ with $U_\ell \in J$ and $V_\ell \in K[a,b,c]$. As before, there exists a coefficient q in all V_ℓ's of minimal valuation, and we assume $\nu(q) < 0$ (else there is no problem). Then $q^{-1}P \in J$. By assumption, the R-module $R[a,b,c]/J$ is torsion-free, so the multiplication by $q^{-1} \in R$ is injective. This means that $P \in J$.
- By definition, we have an R-linear inclusion $R[x,y]/\widetilde{I} \hookrightarrow K[x,y]/I$, which shows that $R[x,y]/\widetilde{I}$ is torsion-free, hence flat. It inherits an action of $\widetilde{G}$ and since $K[x,y]/I \cong \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$, there exists a subrepresentation V of $\mathbb{C}[\widetilde{G}]$ such that $R[x,y]/\widetilde{I} \cong V \otimes_{\mathbb{C}} R$ (this uses the flatness, see [20, lemma 9.4]). By the isomorphism of R-modules $R[x,y]/\widetilde{I} \otimes_R K \cong K[x,y]/I$, the representation V is such that $V \otimes_R K = \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}} K$, which forces $V \cong \mathbb{C}[\widetilde{G}]$.
- The uniqueness of the candidate follows from the condition $\tilde{I} \cdot K[x,y] = I$ since as we already noted:

$$I \cap R[x,y] = (\tilde{I} \cdot K[x,y]) \cap R[x,y] = \tilde{I}$$

  so our natural candidate is the only possibility.

$\diamond$ To finish with, remark that any proper map between to quasi-projective varieties is automatically a projective map. □

8. Contracted versus non-contracted fibres

Theorem 8.1. *Consider the restriction of the map* $\mathscr{S} : \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow G\text{-Hilb}(\mathbb{C}^3)$ *to a reduced curve* $E(\rho)$*. Then:*

(1) *If the representation* ρ *is pure, then* $\mathscr{S}$ *maps isomorphically the curve* $E(\rho)$ *onto the curve* $C(\rho)$*.*
(2) *If the representation* ρ *is binary, then* $\mathscr{S}$ *contracts the curve* $E(\rho)$ *to a point.*

Proof. Let $E(\rho)$ be any exceptional curve. Since the map $\mathscr{S}$ sends this curve to the bunch of curves $\pi^{-1}(O)$, the image lies in some irreducible component C and the restricted morphism $\mathscr{S} : E(\rho) \to C$ is a proper map. We prove that:

- if the representation ρ is binary, then the map $\mathscr{S} : E(\rho) \to C$ contracts the curve to a point;
- if the representation ρ is pure, then $C = C(\rho)$ and the restricted map $\mathscr{S} : E(\rho) \to C(\rho)$ is an isomorphism.

The parameterizations of the two curves $E(\rho)$ and C defines a composite proper map f whose properties reflect those of the restriction of $\mathscr{S}$:

$$\begin{array}{ccc} \mathbb{P}_1 & \xrightarrow[\phi]{\sim} & E(\rho) \subset \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \\ \downarrow f & & \downarrow \mathscr{S} \\ \mathbb{P}_1 & \xrightarrow[\psi]{\sim} & C \subset G\text{-Hilb}(\mathbb{C}^3) \end{array}$$

We know (see [16, II.6.8,II.6.9]) that either the map f contracts the curve to a point, or it is a finite surjective map. The basic idea in order to determine which case occurs is to suppose given an ample line bundle $\mathcal{O}_{\mathbb{P}_1}(a)$ on the target (with $a > 0$): if the map f contracts the curve to a point, then $f^*\mathcal{O}_{\mathbb{P}_1}(a)$ is trivial and else $f^*\mathcal{O}_{\mathbb{P}_1}(a) \cong \mathcal{O}_{\mathbb{P}_1}(\deg(f)\cdot a)$ is ample.
The natural candidate for an ample line bundle over the curve C is the determinant $\det(p_*\mathcal{O}_{Z(C)})$ obtained by restriction of the universal family $Z(C) := \mathcal{Z}_G|_C$.

The parameterization $\mathbb{P}_1 \xrightarrow{\phi} \widetilde{G}\text{-Hilb}(\mathbb{C}^2)$ of the curve $E(\rho)$ corresponds to a flat family $Z_{\widetilde{G}}(\rho) \subset \mathbb{P}_1 \times \mathbb{C}^2$ which is the restriction to $E(\rho)$ of the universal family $\mathcal{Z}_{\widetilde{G}}$ over $\widetilde{G}\text{-Hilb}(\mathbb{C}^2)$. The direct image $p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)}$ is a vector bundle of rank $|\widetilde{G}|$ over $\mathbb{P}_1$ equipped with an action of $\widetilde{G}$ affording the regular representation on each fibre. It admits an isotypical decomposition over the irreducible representation of $\widetilde{G}$ and we recall the well-known explicit decomposition:

Lemma 8.2.

$$p_*\mathcal{O}_{Z_{\widetilde{G}}(\rho)} \cong \left(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim\rho - 1}\right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \mathrm{Irr}(\widetilde{G}) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim\rho'} \otimes \rho'$$

Proof of the lemma. This is an equivalent form of [22, §2.1 lemma] or [18, Proposition 6.2(3)]. We recall briefly the argument. Since this bundle is a quotient of $\mathcal{O}_{\mathbb{P}_1} \otimes A$ (see §6.3), it is generated by its global sections, hence it is a sum of line bundles $\mathcal{O}_{\mathbb{P}_1}(a)$ for

$a \geq 0$. By the classical observation $\deg(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)}) = 1$ (see [14]), all line bundles are trivial but one, of degree one. □

In particular, note that $\det(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)}) \cong \mathcal{O}_{\mathbb{P}_1}(\dim \rho)$ is the ample determinant line bundle in dimension two.

Thanks to the functorial definition of the map $\mathscr{S}$, the composition

$$\mathbb{P}_1 \xrightarrow{\phi} \widetilde{G}\text{-Hilb}\big(\mathbb{C}^2\big) \xrightarrow{\mathscr{S}} G\text{-Hilb}\big(\mathbb{C}^3\big)$$

parameterizes the flat family $Z_{\widetilde{G}}(\rho)/\tau$ whose structural sheaf is $\mathcal{O}_{Z_{\widetilde{G}}(\rho)/\tau} = \left(\mathcal{O}_{Z_{\widetilde{G}}(\rho)}\right)^{\tau}$ and one gets:

$$f^*(\det(p_* \mathcal{O}_{Z(C)})) = \det\left((p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)})^{\tau}\right).$$

Now, as we noticed in §4.3, taking the invariants under τ keeps invariant the pure representations and kills the binary ones. Hence:

- If the representation ρ is binary, then:

$$\left(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)}\right)^{\tau} \cong \bigoplus_{\rho' \in \mathrm{Irr}(G)} \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim \rho'} \otimes \rho'$$

  hence $\det(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)})^{\tau} \cong \mathcal{O}_{\mathbb{P}_1}$ is trivial;
- If the representation ρ is pure, then:

$$\left(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)}\right)^{\tau} \cong \left(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim \rho - 1}\right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \mathrm{Irr}(G) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim \rho'} \otimes \rho'$$

  hence $\det(p_* \mathcal{O}_{Z_{\widetilde{G}}(\rho)})^{\tau} \cong \mathcal{O}_{\mathbb{P}_1}(\dim \rho)$ is ample.

This achieves the first part of the proof. It remains to show that in the case of a pure representation ρ, the target curve is $C = C(\rho)$ and that the finite surjective map f is an isomorphism. We do it by hand. A point $I \in E(\rho)$ is characterized by the choice of $V(I)$ and generically $V(I) \cong \rho$. For a pure representation ρ, the polynomials defining $V(I)$ are even hence:

$$V(I^{\tau}) = V\left((A \cdot V(I) + \mathfrak{n}_A)^{\tau}\right) \supset V(I)$$

so generically $V(I^{\tau}) = V(I)$ (only modified by setting $a = x^2, b = y^2, c = xy$). This means that $C = C(\rho)$ and if $I \neq J \in E(\rho)$, then $V(I) \neq V(J)$ hence the images are also different, so the map is generically injective. This concludes the proof. □

As a byproduct of our argument, we get the following equivalent in dimension three of the lemma 8.2 which, to our knowledge, does not appear explicitly in the literature:

Corollary 8.3. *For any finite subgroup $G \subset \mathrm{SO}(3, \mathbb{R})$ and any non-trivial representation ρ of G, the restriction of the tautological bundle to the exceptional curve $C(\rho)$ decomposes as:*

$$p_* \mathcal{O}_{Z_G(\rho)} \cong \left(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim \rho - 1}\right) \otimes \rho \oplus \bigoplus_{\substack{\rho' \in \mathrm{Irr}(G) \\ \rho' \neq \rho}} \mathcal{O}_{\mathbb{P}_1}^{\oplus \dim \rho'} \otimes \rho'$$

Proof. The same argument as in the proof of lemma 8.2 shows that this bundle in generated by its global sections. The bijectivity of the map f on the curves associated to pure representations (in the notation of the proof of theorem 8.1) implies that $\det(p_* \mathcal{O}_{Z_G(\rho)}) \cong \mathcal{O}_{\mathbb{P}_1}(\dim \rho)$, hence in the isotypical decomposition there is only one non-trivial line bundle, of degree one, and we already know by the explicit parameterizations that the isotypical component corresponding to ρ is not trivial. □

Remark 8.4. *In the decomposition of the lemma 8.2, the unique presence of the $\mathcal{O}_{\mathbb{P}_1}(1)$ corresponds to the choice of the line $V(I)$ in a projective space $\mathbb{P}(\rho \oplus \rho)$ as explicitly described in §6.3. The fact that no other ample bundle occurs reflects the property that once one choice has been made, the other generators of the ideal do not involve the choice any more, as one can easily notice from the explicit computations of* [20, §13,§14] *(see §9 in this paper for an example). In the three-dimensional case, the same situation occurs thanks to the corollary 8.3.*

We get now the theorem 1.1 presented in the introduction as a corollary of the theorem 8.1:

Corollary 8.5. *The image $\mathcal{Y} := \mathscr{S}(\widetilde{G}\text{-Hilb}(\mathbb{C}^2))$ projects onto the quotient K/G, inducing a partial resolution of singularities containing only the exceptional curves corresponding to pure representations. The map $\mathscr{S} : \widetilde{G}\text{-Hilb}(\mathbb{C}^2) \longrightarrow \mathcal{Y}$ is a resolution of singularities contracting the excess exceptional curves to ordinary nodes.*

Proof. The projection $\pi : \mathcal{Y} \longrightarrow \mathbb{C}^3/G$ factors through K/G by construction of $\mathcal{Y}$. The other assertions result from theorem 8.1. The excess curves contract to ordinary nodes since, as one checks with the figure 1, each excess (-2)-curve is contracted to a different point. □

9. Example: the cyclic group case

Let the cyclic group $\widetilde{C}_n \cong \mathbb{Z}/(2n)\mathbb{Z}$ act on $\mathbb{C}^2$ with generator:

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{with } \xi = e^{\frac{2\pi i}{(2n)}}.$$

The choice of coordinates made in §7 implies that the group $C_n \cong \mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^3$ with generator:

$$\begin{pmatrix} \xi^2 & 0 & 0 \\ 0 & \xi^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The irreducible representations of the cyclic group $\widetilde{C}_n$ are given by the matrices (ξ^i), $i = 0, \ldots, 2n-1$. For i even, they are also the irreducible representations of C_n. There are then n pure and n binary representations. With the notations of §4.3, we set $\chi_i := \rho_{2i}$ and $\widetilde{\chi}_i = \rho_{2i+1}$ for $i = 0, \ldots, n-1$. By Theorem 8.1, the exceptional curves on $\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2)$ corresponding to the binary representations are contracted by $\mathscr{S}$ to a node on $\mathscr{S}(\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2))$ whereas the curves corresponding to the pure representations are in $1:1$ correspondence with the exceptional curves downstairs (see figure 2). In this section, we check this by a direct computation.

The ring of invariants $\mathbb{C}[x,y]^{\widetilde{C}_n}$ is generated by x^{2n}, y^{2n}, xy and $\mathbb{C}[a,b,c]^{C_n}$ is generated by c, a^n, b^n, ab. Recall the description of the exceptional curves of $\widetilde{C}_n\text{-Hilb}(\mathbb{C}^2)$ following [20, Theorem 12.3]. We sort the basis of the algebra of coinvariants with respect to each irreducible representation:

$$\{1\}, \{x, y^{2n-1}\}, \ldots, \{x^i, y^{2n-i}\}, \ldots, \{x^{2n-1}, y\}.$$

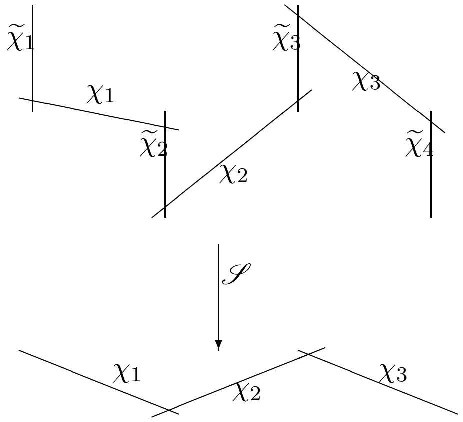

FIGURE 2. Contracted fibres for $\widetilde{C}_4$

To choose a cluster $I/\mathfrak{n}_A$ supported at the origin amounts in choosing one copy of each non-trivial representation, *i.e.* for all $i = 1, \dots, 2n-1$ a point $(p_i : q_i) \in \mathbb{P}_1$ defining the ideal by the generators:

$$\langle p_1 x - q_1 y^{2n-1}, \dots, p_i x^i - q_i y^{2n-i}, \dots, p_{2n-1} x^{2n-1} - q_{2n-1} y \rangle.$$

But the point is that one only needs *one* choice. Suppose there exists an index i such that $p_i q_i \neq 0$, and take the smaller i with this property. Set $p = p_i, q = q_i$ and $v = px^i - qy^{2n-i}$. Then since xy is invariant, $x^{i+1}, \dots, x^{2n-1} \in I/\mathfrak{n}_A$ and $y^{2n-i+1}, \dots, y^{2n-1} \in I/\mathfrak{n}_A$ so all our other choices were trivial, and $V(I) = \mathbb{C} \cdot v$. More formally, we parameterized the exceptional curve $E(\rho_i)$ by a subbundle:

$$\mathcal{O}_{\mathbb{P}_1}(-1) \otimes \rho_i \oplus \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}_1} \otimes \rho_j \hookrightarrow \bigoplus_j (\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}) \otimes \rho_j.$$

If there is no such index, suppose x^i is the minimal power of x in the choice: in order to find once each non-trivial representation one has to choose y^{2n-i+1} and the minimal set of generators $V(I) = \mathbb{C} \cdot x^i \oplus \mathbb{C} \cdot y^{2n-i+1}$ contains two adjacent representations.
Otherwise stated, a $\widetilde{C}_n$-cluster at the origin takes the form:

$$I_j(p:q) := \langle px^j - qy^{2n-j}, xy, x^{j+1}, y^{2n-j+1} \rangle, \\ 1 \leq j \leq 2n-1,\ (p:q) \in \mathbb{P}_1$$

(the above expression contains enough generators to include the two possible cases) and

$$E(\rho_j) = \{I_j(p:q)\}.$$

By the same method, one sees easily that a C_n-cluster at the origin takes the form:

$$J_k(s:t) := \langle sa^k - tb^{n-k}, c, a^{k+1}, b^{n-k+1}, ab \rangle, \\ 1 \leq k \leq n-1,\ (s:t) \in \mathbb{P}_1$$

and

$$C(\chi_k) = \{J_k(s:t)\}.$$

Recall that with the construction (2) we have to compute $\sigma^{-1}(I_j(p:q))$. Denoting by $\bar{\sigma}$ the map $B \big/ \langle ab - c^2 \rangle \longrightarrow A$, it is equivalent to compute $\bar{\sigma}^{-1}(I_j(p:q))$. First we compute $I_j(p:q)^\tau \in A^\tau$. We distinguish two cases:

- j *even, i.e.* $j = 2j',\ j' = 1, \dots, n-1$. In this case we have
$$I_j(p:q)^\tau = I_j(p:q) = \langle p(x^2)^{j'} - q(y^2)^{n-j'}, xy, (x^2)^{j'+1}, (y^2)^{n-j'+1} \rangle$$
expressed in $A^\tau = \mathbb{C}[x^2, y^2, xy]$. Then
$$\begin{aligned} \bar{\sigma}^{-1}(I_j(p:q)) &= \langle pa^{j'} - qb^{n-j'}, c, a^{j'+1}, b^{n-j'+1} \rangle \\ &= J_{j'}(p:q). \end{aligned}$$

- j *odd, i.e.* $j = 2j'+1,\ j' = 0, \dots, n-1$. Observe that $xy \in I_j(p:q)^\tau$ and $(x^2)^{j'+1}$, $y^{n-j'} \in I_j(p:q)^\tau$, but $px^{2j'+1} - qy^{2n-2j'-1} \notin I_j(p:q)^\tau$. So
$$\bar{\sigma}^{-1}(I_j(p:q)) = \langle a^{j'+1}, b^{n-j'}, c \rangle.$$
We observe then that
$$\bar{\sigma}^{-1}(I_j(p:q)) \in C(\rho_{j'}) \cap C(\rho_{j'+1})$$
since
$$\bar{\sigma}^{-1}(I_j(p:q)) = J_{j'}(0:1) = J_{j'+1}(1:0).$$

The curves $E(\rho_j)$ with j even correspond to the pure representations and are not contracted by $\mathscr{S}$ as the previous computation shows, the curves with j odd correspond to the binary representations: these are contracted by $\mathscr{S}$.

10. Application

10.1. Pencils of symmetric surfaces. Let $\mathbb{H}_\mathbb{C} := \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ be the complexification of the space of quaternions. By the choice of the coordinates $q = a\cdot 1 + b\cdot i + c\cdot j + d\cdot k$, $a, b, c, d \in \mathbb{C}$, one gets an isomorphism $\mathbb{P}_3 \cong \mathbb{P}(\mathbb{H}_\mathbb{C})$ such, that for $n = 6, 8, 12$ the bipolyhedral group G_n acts linearly on $\mathbb{P}_3$, leaving invariant the quadratic polynomial $Q := a^2 + b^2 + c^2 + d^2$. In [27] it is shown that the next non-trivial invariant is a homogeneous polynomial S_n of degree n. Consider then the following pencil of G_n-symmetric surfaces in $\mathbb{P}_3$:
$$X_n(\lambda) = \{S_n + \lambda Q^{n/2} = 0\}, \quad \lambda \in \mathbb{C}.$$
In [27] it is proved that the general surface $X_n(\lambda)$ is smooth and that for each n there are precisely four singular surfaces in the corresponding pencil: the singularities of these surfaces are ordinary nodes forming one orbit through G_n.
Consider now the pencil of quotient surfaces in $\mathbb{P}_3/G_n$:
$$\{X_n(\lambda)/G_n\}, \quad \lambda \in \mathbb{C}.$$
In [2] it is proved that these quotient surfaces have only A-D-E singularities and that the minimal resolutions of singularities $Y_n(\lambda) \to X_n(\lambda)/G_n$ are K3-surfaces with Picard number greater than 19. For the four nodal surfaces in each pencil, a careful study of the stabilizers of the nodes shows that, if X denotes one of these nodal surfaces, the image of the node on $X/G_n \subset \mathbb{P}_3/G_n$ is a particular quotient singularity locally isomorphic to $\mathbb{C}^2/\widetilde{G} \subset \mathbb{C}^3/G$ for some polyhedral group G explicitly computed (see [2, §3, Proposition 3.1]):

- for $n = 6$: $C_3, \mathcal{T}$;
- for $n = 8$: $D_2, D_3, D_4, \mathcal{O}$;
- for $n = 12$: $D_3, D_5, \mathcal{T}, \mathcal{I}$.

Therefore, our theorem 1.1 gives locally a group-theoretic interpretation of the exceptional curves of the K3-surfaces $Y_n(\lambda)$ over the particular singularities of the nodal surfaces.

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