LOGARITHMIC ENRIQUES VARIETIES

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Abstract. We introduce logarithmic Enriques varieties as a singular analogue of Enriques manifolds, generalizing the notion of log-Enriques surfaces introduced by Zhang. We focus mainly on the properties of the subfamily of log-Enriques varieties that admit a quasi-étale cover by a singular symplectic variety and we give many examples.

1. Introduction

In the Enriques–Kodaira classification of smooth complex algebraic surfaces, an *Enriques surface* is by definition a minimal surface X of Kodaira dimension $\kappa = 0$, arithmetic genus $p_g(X) \coloneqq h^0(K_X) = 0$ and irregularity $q(X) \coloneqq h^1(\mathcal{O}_X) = 0$. It follows that the canonical divisor K_X is 2-torsion and that the étale double cover defined by the 2-torsion line bundle $\mathcal{O}_X(K_X)$ realizes X as the quotient of a K3 surface Y by a fixed point free nonsymplectic involution. As a consequence, its fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The existence of a nonsymplectic automorphism imposes that Y is projective, hence X too. Conversely, every fixed point free involution on a K3 surface Y is nonsymplectic by the holomorphic Lefschetz fixed point formula. This imposes that Y is projective, and the quotient variety is an Enriques surface.

In a different flavour, starting from a smooth complex algebraic surface X with nontrivial but 2-torsion canonical bundle, the étale degree two cover $Y \to X$ defined by the canonical sheaf is such that $K_Y = 0$, so $p_g(Y) = 1$ and Y is a minimal surface which is either a K3 surface or an Abelian surface. If we further assume that $q(X) = 0$, then the Euler characteristic of X is $\chi(X, \mathcal{O}_X) = 1$, so $\chi(Y, \mathcal{O}_Y) = 2$, hence $q(Y) = 0$ and Y is a K3 surface. We refer to Beauville [8, Chapter VIII], Barth–Hulek–Peters–van de Ven [6] or Cossec–Dolgachev [25] for more characterizations of Enriques surfaces.

These equivalent points of view on Enriques surfaces offer different strategies to extend their definition in higher dimension in the nonsingular setup, that have been explored in Boissière–Nieper-Wisskirchen–Sarti [17], Oguiso–Schröer [51] and Yoshikawa [59]: these are called *Enriques manifolds*. A recent interest for Enriques manifolds occurs in the framework of the investigation of the Morrison–Kawamata cone conjecture by Pacienza and the third author [53]. In light of recent developments on the Beauville–Bogomolov decomposition of singular varieties with trivial first Chern class [29, 35, 39], it is interesting to extend further the definition of Enriques manifolds to the singular setting. In this paper, after reviewing the theory of

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singular symplectic varieties and of singular Calabi–Yau varieties in Section 2, and after discussing some properties of K-torsion varieties in Section 3, we first propose in Definition 4.1 a general definition of Enriques manifolds that interpolates with the previous ones cited above and then we propose in Definition 4.6 an extension of this notion in the singular setup, that we call *logarithmic Enriques varieties*, following the term *logarithmic Enriques surfaces* introduced by Zhang [61]. Then in Section 5 we discuss general properties and characterizations of these varieties in the special case of log-Enriques varieties *of symplectic type*, *i.e.* those log-Enriques varieties admitting a cyclic quasi-étale cover by a symplectic variety. In particular, in Proposition 5.9 we show that if the quasi-étale cover Y is a PSV (respectively an ISV) then the log-Enriques variety X is the quotient of Y by the action of a purely nonsymplectic automorphism. Finally, in Section 6 we give many examples: in particular, in the case of Enriques manifolds, we review the examples already mentioned in the cited references and we also construct some new ones.

Angel David Ríos Ortiz, Francesco Denisi, Nikolaos Tsakanikas and Zhixin Xie [27] have been working independently and simultaneously on a variant of this notion that they called *primitive Enriques varieties*, that is compatible with our definitions, although they explore different developments as ours.

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2. Preliminaries

In this paper, the term "variety" denotes an integral separated noetherian scheme of finite type over the field of complex numbers. Let X be a projective variety. We denote by X_{reg} its regular locus with its open embedding $\iota\colon X_{\mathrm{reg}} \hookrightarrow X$, by $\Omega^1_{X_{\mathrm{reg}}}$ the sheaf of Kähler differentials and by $\Omega^p_{X_{\mathrm{reg}}} := \bigwedge^p \Omega^1_{X_{\mathrm{reg}}}$ the sheaf of p-forms. For any $p \in \mathbb{N}$ such that $0 \leq p \leq \dim X$, we define:

$$\Omega_X^{[p]} := \iota_* \Omega^p_{X_{\mathrm{reg}}}.$$

Assuming that X is normal, $\Omega_X^{[p]}$ is a reflexive sheaf isomorphic to the bidual $(\Omega_X^p)^{**}$, whose sections are called *reflexive forms* (see for instance [33, 38, 56]). We denote in particular the *canonical sheaf* by:

$$\omega_X := \Omega_X^{[\dim X]},$$

it is a divisorial sheaf whose associated linear equivalence class of Weil divisors is the *canonical divisor* K_X of X. Following the definition given in [56, p.282] for the divisorial sheaf associated to a Weil divisor, we have $\omega_X = \mathcal{O}_X(K_X)$. With a slight, but customary abuse of notation, we denote:

$$\omega_X^{[i]} = \mathcal{O}_X(iK_X) = \left(\omega_X^{\otimes i}\right)^{**}, \quad \forall i \geq 0.$$

Recall that a normal variety X has *rational singularities* if for any resolution of singularities $f\colon Z \to X$, one has $R^{>0}f_*\mathcal{O}_Z = 0$. By the extension theorem of Kebekus–Schnell [42, Corollary 1.8], if X is a normal variety with rational singularities, the pullback over X_{reg} of any reflexive form extends regularly on any resolution of singularities. We also recall that by the Elkik–Flenner theorem, rational Gorenstein singularities are exactly canonical singularities of index one [57, §3(C) & p.363].

Using these notions, we reformulate Beauville's definition [9, Definition 1.1] of a projective *symplectic variety* as: a normal projective variety X with rational singularities, whose regular part admits a symplectic holomorphic form. The complex dimension of X is thus necessarily even.

A *finite cover map* is a finite and surjective morphism $f\colon \widetilde{X} \to X$ between normal projective varieties of the same dimension. It is called *quasi-étale* if it is étale in codimension one [34, Definition 3.2&3.3]: this means that there exists a closed subset $Z \subset \widetilde{X}$ of codimension at least two such that the restriction $f\colon \widetilde{X} \setminus Z \to X$ is étale. The *augmented irregularity* [35, Definition 3.1] of a normal projective variety X is the supremum $\tilde{q}(X) \in \mathbb{N} \cup \{\infty\}$ of the irregularities $q(\widetilde{X}) \coloneqq h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ for all finite quasi-étale covers $\widetilde{X} \to X$.

A *Calabi–Yau variety* (CY) is a normal projective variety X such that $\omega_X \cong \mathcal{O}_X$, with rational singularities and such that for any finite quasi-étale cover $f\colon \widetilde{X} \to X$, one has $H^0(\widetilde{X}, \Omega_{\widetilde{X}}^{[q]}) = 0$ for all $0 < q < \dim X$ (see [35, Definition 8.16.1] or [32, Definition 1.3]. By Hodge symmetry, $\tilde{q}(X) = 0$ and X is simply connected if X has even dimension [32, Corollary 13.3]. Nonsingular CY varieties correspond to the CY manifolds as defined in [7]. A weaker variant co-exists in the literature, without the condition on all étale covers, assuming only that X is normal with rational singularities, $\omega_X \cong \mathcal{O}_X$ and $H^0(X, \Omega_X^{[q]}) = 0$ for all $0 < q < \dim X$: to avoid confusion, we call here these later *weak CY varieties*.

An *irreducible symplectic variety* (ISV) is a projective symplectic variety X admitting a symplectic form $\sigma_X \in H^0(X, \Omega_X^{[2]})$ such that for any finite quasi-étale cover $f\colon \widetilde{X} \to X$, the reflexive pullback $f^{[*]}(\sigma_X)$, as defined in [33, §II.4], generates the exterior algebra of global sections of $\Omega_{\widetilde{X}}^{[*]}$ (see [35, Definition 8.16.2] and [3, Definition 1.1]). In particular, we have $H^0(X, \Omega_X^{[2]}) = \mathbb{C}\sigma_X$. Clearly X is even-dimensional, $\tilde{q}(X) = 0$ and X is simply connected by [32, Corollary 13.3], but X_{reg} is not always simply connected. Nonsingular ISV coincide with the *irreducible holomorphic symplectic* (IHS) manifolds introduced by Beauville [7].

A *primitive symplectic variety* (PSV), as considered in particular by Schwald [58] and Bakker–Lehn [5], is a projective symplectic variety X such that $q(X) = 0$ and $H^0(X, \Omega_X^{[2]})$ is one-dimensional. ISV are PSV but the augmented irregularity of a PSV may be nonzero: take for instance the quotient of an abelian surface by ± 1, it is a PSV since its minimal resolution is a Kummer surface but by construction its augmented irregularity is positive. We refer to [4, 54] for a more detailed discussion on these definitions and its possible variants.

Checking directly the vanishing condition of the augmented irregularity on a given variety is a difficult problem in general. We give a sufficient criterion for a variety to be Calabi–Yau or irreducible symplectic, following [18, 55].

Proposition 2.1. *Let X be a normal projective variety with rational singularities and such that $\omega_X \cong \mathcal{O}_X$, and assume that $\pi_1(X_{\mathrm{reg}}) = \{1\}$.*

(i) If there exists a dominant rational map $\varphi\colon Z \dashrightarrow X$, where Z is a Calabi–Yau variety, then X is also a Calabi–Yau variety.
(ii) If X is a symplectic variety and if there exists a dominant rational map $\varphi\colon Z \dashrightarrow X$, where Z is an ISV, then X is also an ISV.

Proof. The argument for ISV is given in [18, Proposition 3.15], the CY case is similar, we write it for completeness. By the Zariski–Nagata purity theorem [49, 60] (see also [62, Theorem 2.4]), every quasi-étale cover is étale over the nonsingular locus. Since X_{reg} is simply connected, every quasi-étale cover of X is an isomorphism by the Zariski Main Theorem, so we only need to show that $H^0(X, \Omega_X^{[q]}) = 0$ for all $0 < q < \dim X$. Consider the following diagram:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, dashed] & X \end{tikzcd}$$

where X' is a resolution of singularities of X and Z' is a nonsingular resolution of indeterminacies of the rational dominant map $Z \dashrightarrow X'$. For all $0 < q < \dim X$, by [43, Proposition 5.8] applied to the dominant morphism $\gamma\colon Z' \to X$, the reflexive pullback morphism $\gamma^{[*]}\colon H^0(X, \Omega_X^{[q]}) \to H^0(Z', \Omega^q_{Z'})$ is injective. By Hodge symmetry [35, Proposition 6.9], we have $h^0(Z', \Omega^q_{Z'}) = h^q(Z', \mathcal{O}'_Z)$. Since the singularities of Z are rational and since $Z' \to Z$ is birational, we have $h^q(Z', \mathcal{O}_{Z'}) = h^q(Z, \mathcal{O}_Z)$. By Hodge symmetry again, since Z has canonical singularities we have $h^q(Z, \mathcal{O}_Z) = h^0(Z, \Omega_Z^{[q]})$, which is zero since Z is CY. We get $h^0(X, \Omega_X^{[q]}) = 0$. □

3. K-torsion varieties

Let X be a normal projective variety. If X has only canonical singularities and if its canonical divisor is numerically trivial, by Kawamata [41, Theorem 8.2] we get that K_X is torsion and that $\kappa(X) = 0$. Assume now more generally that the singularities of X are log terminal (klt). If K_X is numerically effective, the abundance conjecture predicts that some multiple dK_X of K_X is base-point-free. If we further assume that $\kappa(X) = 0$, the morphism induced by the linear system $|dK_X|$ maps to a point, so K_X is torsion. This motivates the following definition:

Definition 3.1. A *K-torsion variety* is a normal projective variety X with klt singularities and torsion (but nontrivial) canonical divisor K_X. The *index* of X is the smallest integer $d \geq 2$ such that $dK_X = 0$.

Under this assumption, the divisorial sheaf ω_X satisfies $\omega_X^{[d]} \cong \mathcal{O}_X$, it is thus a reflexive d-th root of the structural sheaf. We refer to [44, Definition 2.34] for the definition of klt singularities. Here we could simply equivalently require log terminal singularities, but the klt framework is more common.

Following [44, Definition 5.19], if X is a K-torsion variety of index d, the quasi-étale cyclic cover associated to ω_X, with group μ_d (the order d cyclic group), is:

$$Y \coloneqq \operatorname{Spec}\left(\oplus_{i=0}^{d-1} \omega_X^{[-i]}\right) \xrightarrow{p} X.$$

Since ω_X is locally free on X_{reg}, the restriction of p to $p^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is étale, with $p^{-1}(X_{\mathrm{reg}}) \subset Y_{\mathrm{reg}}$. Following Kawamata [41, Section 8], we call Y the *canonical cover* of X. Recall that Y is projective, irreducible, normal (this is a consequence of [56, Proposition 2]), it has canonical singularities [44, Corollary 5.21] and $K_Y = 0$. More generally, if X is klt and $p\colon Y \to X$ is any quasi-étale morphism with Y normal, then Y is also klt by [44, Proposition 5.20].

Proposition 3.2. *Let $p\colon Y \to X$ be a quasi-étale cyclic cover of degree $d \geq 2$ between projective normal varieties. We have the following exact sequence of class groups:*

$$\{0\} \longrightarrow \mu_d \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)^{\mu_d} \longrightarrow \{0\}. \tag{1}$$

Moreover:

(i) *If $q(Y) = 0$, then the group $\mathrm{Cl}(X)$ is finitely generated.*
(ii) *If $K_Y = 0$, then $rK_X = 0$ for some divisor r of d.*

Proof. This statement extends [51, Proposition 2.8] to the singular setup, using similar arguments. By the Zariski–Nagata purity theorem, since p is quasi-étale, it is étale over the nonsingular locus X_{reg} and $Y^\circ := p^{-1}(X_{\mathrm{reg}})$ is an open subset of Y_{reg} whose complementary in Y has codimension at least two. The restriction $p\colon Y^\circ \to X_{\mathrm{reg}}$ is an étale cyclic cover with group μ_d. Following the methods and notation in [37, §5.2], there are two spectral sequences associated to p, applied here to the sheaf $\mathcal{O}^*_{Y^\circ}$:

$$\begin{aligned} {}^aE_2^{i,j} &:= H^i(X_{\mathrm{reg}}, R^j p_*^{\mu_d}\mathcal{O}^*_{Y^\circ}) \Rightarrow R^{i+j}\Gamma^{\mu_d}_{Y^\circ}(\mathcal{O}^*_{Y^\circ}), \\ {}^bE_2^{i,j} &:= H^i(\mu_d, H^j(Y^\circ, \mathcal{O}^*_{Y^\circ})) \Rightarrow R^{i+j}\Gamma^{\mu_d}_{Y^\circ}(\mathcal{O}^*_{Y^\circ}). \end{aligned}$$

Since μ_d acts freely on Y°, the functor $p_*^{\mu_d}$ is exact so the first spectral sequence gives $R^i\Gamma^{\mu_d}_{Y^\circ}(\mathcal{O}_{Y^\circ}) = H^i(X_{\mathrm{reg}}, p_*^{\mu_d}\mathcal{O}_{Y^\circ})$. The five-terms exact sequence associated to the second spectral sequence writes:

$$0 \to {}^bE_2^{1,0} \to H^1(X_{\mathrm{reg}}, p_*^{\mu_d}\mathcal{O}^*_{Y^\circ}) \to {}^bE_2^{0,1} \to {}^bE_2^{2,0} \to H^2(X_{\mathrm{reg}}, p_*^{\mu_d}\mathcal{O}^*_{Y^\circ}).$$

Let us compute the first terms of this exact sequence.

- Since Y is normal and projective, we get:
  $${}^bE_2^{1,0} = H^1(\mu_d, H^0(Y^\circ, \mathcal{O}^*_{Y^\circ})) = H^1(\mu_d, H^0(Y, \mathcal{O}^*_Y)) = H^1(\mu_d, \mathbb{C}^*),$$
  where $\mathbb{C}^*$ is considered as a trivial μ_d-module. An easy computation of the group cohomology of μ_d gives ${}^bE_2^{1,0} \cong \mu_d$.
- We have $p_*^{\mu_d}\mathcal{O}^*_{Y^\circ} = \mathcal{O}^*_{X_{\mathrm{reg}}}$ and since X is normal we get:
  $$H^1(X_{\mathrm{reg}}, p_*^{\mu_d}\mathcal{O}^*_{Y^\circ}) = H^1(X_{\mathrm{reg}}, \mathcal{O}^*_{X_{\mathrm{reg}}}) = \mathrm{Pic}(X_{\mathrm{reg}}) = \mathrm{Cl}(X).$$
- Since Y is normal, we get:
  $${}^bE_2^{0,1} = H^0(\mu_d, H^1(Y^\circ, \mathcal{O}^*_{Y^\circ})) = \mathrm{Pic}(Y^\circ)^{\mu_d} = \mathrm{Cl}(Y)^{\mu_d}.$$
- Similarly,
  $${}^bE_2^{2,0} = H^2(\mu_d, H^0(Y^\circ, \mathcal{O}^*_{Y^\circ})) = H^2(\mu_d, \mathbb{C}^*) = \{0\}.$$

  We thus obtain the exact sequence:
  $$\{0\} \longrightarrow \mu_d \longrightarrow \mathrm{Cl}(X) \xrightarrow{p^*} \mathrm{Cl}(Y)^{\mu_d} \longrightarrow \{0\}.$$

Proof of assertion (i). The subgroup $\mathrm{Cl}_a(Y)$ of $\mathrm{Cl}(Y)$ parametrizing linear equivalence classes of Weil divisors on Y algebraically equivalent to zero admits a structure of abelian variety defined by Lang [45, IV§4] as the Picard variety $\mathrm{P}(Y) \coloneqq \mathrm{Alb}(Y)^\vee$, the dual of the Albanese variety of Y (see for instance [15] for more details). The dimension of the Picard variety is thus $q(Y) = h^1(Y, \mathcal{O}_Y) = 0$ by assumption, so $\mathrm{Cl}_a(Y) = \{0\}$. Since the Néron–Severi group $\mathrm{NS}(Y) \coloneqq \mathrm{Cl}(Y)/\mathrm{Cl}_a(Y)$ is finitely generated by [40, Théorème 3], we obtain that $\mathrm{Cl}(Y)$ is finitely generated. Using the exact sequence (1) we get that $\mathrm{Cl}(X)$ is also finitely generated.

Proof of assertion (ii). To prove that $dK_X = 0$, it is enough to show that $dK_{X_{\mathrm{reg}}} = 0$ since X is normal, so after restricting to the étale cover $p\colon Y^\circ \to X_{\mathrm{reg}}$, we may assume that Y and X are nonsingular. Since p is étale, we have $p^*\mathcal{O}_X(K_X) = \mathcal{O}_Y(K_Y) = \mathcal{O}_Y$ and the result follows using the exact sequence (1). Here is a more direct argument that will be used below. Since p is finite and X is nonsingular, p is flat so $p_*\mathcal{O}_Y$ is locally free of rank d. By projection formula we get:

$$p_*\mathcal{O}_Y = p_*\left(p^*\mathcal{O}_X(K_X)\right) = \mathcal{O}_X(K_X) \otimes p_*\mathcal{O}_Y.$$

Since $p_*\mathcal{O}_Y$ is locally free of rank d, by taking the determinant we get:

$$\det(p_*\mathcal{O}_Y) = \det\left(\mathcal{O}_X(K_X) \otimes p_*\mathcal{O}_Y\right) \cong \mathcal{O}_X(K_X)^{\otimes d} \otimes \det(p_*\mathcal{O}_Y),$$

so finally $\mathcal{O}_X(K_X)^{\otimes d} \cong \mathcal{O}_X$, hence $dK_X = 0$ and $rK_X = 0$ for some divisor r of d. □

4. Enriques manifolds and log-Enriques varieties

Inspired by [17, 51, 59], we give the following general definition of Enriques manifolds. We intend to give a sufficiently flexible definition to ensure that there are enough examples to make the theory interesting and to prepare the generalization given below to a singular setup using a similar point of view.

Definition 4.1. An *Enriques manifold* is a nonsingular K-torsion variety X such that $q(X) = 0$.

The condition of vanishing irregularity is needed to exclude bielliptic surfaces, which are nonsingular K-torsion projective varieties of index $2, 3, 4$ or 6 and positive irregularity.

Let X be an Enriques manifold and consider its canonical cover $Y \to X$, which is a projective manifold with trivial canonical divisor. By the Beauville-Bogomolov decomposition theorem [7], up to an étale cover, the variety Y splits as a product of IHS manifolds, CY manifolds and an abelian manifold. Since there is in general not much control on this étale cover, we focus on special types of Enriques manifolds where the variety Y itself is split, and we are mostly interested in the "atomic" types.

Definition 4.2. An Enriques manifold is of *symplectic type* (*resp. CY type*) if it admits a cyclic étale cover by an IHS (*resp.* CY) manifold.

Our definition is less restrictive than that of [17] since we do not require any condition on the holomorphic Euler characteristic of X. It is also more general than that of [51] that is what we call here Enriques manifolds of symplectic type, since every étale quotient of a projective IHS manifold is cyclic. We will prove below that for both types, it is equivalent to ask that the canonical cover is of the given type. We could *a priori* define a third *abelian type* by the condition of the existence of a cyclic étale cover by an abelian manifold, as a special case inside the subfamily of generalized hyperelliptic varieties called Bagnera – de Franchis varieties (see [26]), however we show below that this does not exist.

Lemma 4.3. *Let X be an Enriques manifold. Then X admits no cyclic étale cover by an abelian variety.*

Proof. We use an argument taken from [26, Remark 2.8], communicated to us by Martina Monti. Let $A = V/\Lambda$ be an abelian variety, where Λ is a lattice in a finite dimensional complex vector space V, and assume that $g \in \mathrm{Aut}(A)$ is an automorphism acting freely such that $A/\langle g\rangle = X$. Decomposing $g(x) = \alpha x + b$ where $\alpha \in \mathrm{GL}(V)$ and $b \in \Lambda$, the freeness is equivalent to the property that there exist no $(x, \ell) \in V \times \Lambda$ such that $(\alpha - \mathrm{id})x = \ell - b$, so $\alpha - \mathrm{id}$ is not invertible. Let z be an eigenvector of α for the eigenvalue 1. Then $g^*dz = dz$ and hence dz descends to an element of $H^0(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)$, so $q(X) > 0$: this is a contradiction. □

Lemma 4.4. *Enriques manifolds of symplectic or CY types are even-dimensional and the two types exclude each other.*

Proof. An Enriques manifold X of IHS type is clearly even-dimensional. If X is of CY type, let $Y \to X$ be a cyclic étale cover where Y is a CY manifold. If Y is odd-dimensional, then its Euler characteristic is $\chi(Y, \mathcal{O}_Y) = 0$, so $\chi(X, \mathcal{O}_X) = 0$ also. But since the quotient is étale, $H^0(X, \Omega^p_X) = H^0(Y, \Omega^p_Y)^{\mathrm{inv}} = 0$ for $0 < p < n$ and $H^0(X, \omega_X) = 0$ since K_X is torsion, so $\chi(X, \mathcal{O}_X) = 1$: this is a contradiction so Y is even-dimensional. Since even dimensional CY manifolds are simply connected, in each type the cyclic étale cover of the definition is the universal cover, so both types exclude each other. □

Corollary 4.5. *An Enriques manifold is of CY type if and only if its canonical cover is a CY manifold, and it admits a unique cyclic étale CY cover. Its index is always two.*

Proof. Let X be an Enriques manifold of CY type and let $Y \to X$ be a cyclic étale cover of degree $d \geq 2$ where Y is a CY manifold. Since Y is even-dimensional by Lemma 4.4, we have $\chi(Y, \mathcal{O}_Y) = 2$ so the relation $d\chi(X, \mathcal{O}_X) = 2$ gives $d = 2$ and $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$. It follows that $2K_X = 0$ (see the proof of Proposition 3.2) and by assumption K_X is non trivial. The canonical cover $Z \to X$ is an étale double cover and since Y is simply connected, the factorization $Y \to Z$ is an isomorphism by Zariski Main Theorem. □

To extend the notion of Enriques manifold to a singular setup, in a similar spirit as above, inspired by [61, Definition 1.1] we give the following definition:

Definition 4.6. A *logarithmic Enriques variety* (*log-Enriques* for short) is a K-torsion variety X such that $q(X) = 0$.

Recall that by assumption (see Definition 3.1) log-Enriques varieties have klt singularities. The term *log* comes from the fact that if $\widehat{X}$ is a resolution of singularities of X with exceptional divisor D, then the pair $(\widehat{X}, D)$ is log-terminal.

Let X be a log-Enriques variety and consider its canonical cover $Y \to X$. By the decomposition theorem of Höring–Peternell [39], up to a quasi-étale cover, the variety Y splits as a product of ISVs, (singular) CY varieties and an abelian manifold. As in the nonsingular setup, we focus on the "atomic" types:

Definition 4.7. A log-Enriques variety is of *symplectic type* (*resp. CY type*, *abelian type*) if it admits a cyclic quasi-étale cover by a symplectic variety (*resp.* a CY variety, an abelian manifold).

In the symplectic setup, if the quasi–étale cover Y is respectively an IHS manifold, an ISV or a PSV, we will make the terminology more precise by saying that X is a log-Enriques variety of respectively *IHS type*, *ISV type* or *PSV type*.

Lemma 4.8. *Log-Enriques varieties of ISV, CY or abelian types exclude each other.*

Proof. The argument has been communicated to us by Nikolaos Tsakanikas. By [32, Lemma 2.19] and [50, Proposition 2.10], the augmented irregularity of a log-Enriques variety X is finite (precisely $\widetilde{q}(X)\leq \dim X$) and it is invariant under quasi-étale covers. If X is of ISV or CY type, then $\widetilde{q}(X)=0$ so log-Enriques varieties of ISV type or of CY type cannot be simultaneously of abelian type. Moreover, by [32, Proposition F] the restricted holonomy of a CY variety is a special unitary group whereas those of an ISV is a symplectic group, and by [32, Lemma 4.7] the restricted holonomy is invariant under quasi-étale covers, so the ISV types and the CY types exclude each other. □

This result is similar to the nonsingular setup, but log-Enriques varieties of PSV type may well be of some other type too. In a different spirit, we give in §6.2.2 an example of a log-Enriques variety of IHS types that is a quasi-étale cover of a weak Calabi–Yau variety.

5. Log-Enriques varieties of symplectic type

Let Y be a PSV and σ_Y be a reflexive symplectic form on Y. Since the space $H^0(Y,\Omega_Y^{[2]})$ is one dimensional, for any automorphism $\varphi\in\mathrm{Aut}(Y)$ there exists $\lambda\in\mathbb{C}^*$ such that $\varphi^{[*]}\sigma_Y=\lambda\sigma_Y$. We call φ *symplectic* if $\lambda=1$, and *nonsymplectic* otherwise. In particular, if φ has order d, we call it *purely nonsymplectic* if λ is a primitive d-th root of unity. A finite order automorphism φ on Y is called *free* if its fixed locus is empty, and *étale* if all its powers are free, or equivalently if the quotient morphism $Y\to Y/\langle\varphi\rangle$ is étale. Clearly every free automorphism of prime order is étale. It is called *quasi-étale* if the quotient morphism $Y\to Y/\langle\varphi\rangle$ is quasi-étale.

Proposition 5.1. *Let $p\colon Y\to X$ be a cyclic étale cover of degree $d\geq 2$ between nonsingular varieties. Then $\chi(\mathcal{O}_Y)=d\chi(\mathcal{O}_X)$. In particular, if Y is an IHS manifold, then d divides $\frac{1}{2}\dim Y+1$.*

Proof. The statement and the argument follow [51, §2]. Since p is a finite projective morphism, we have $\chi(\mathcal{O}_Y)=\chi(p_*\mathcal{O}_Y)$. Since $p\colon Y\to X$ is an étale cyclic cover of order d, there exists a line bundle L on X such that $p_*\mathcal{O}_Y\cong\mathcal{O}_X\oplus L\oplus\cdots\oplus L^{d-1}$ with $L^{\otimes d}\cong\mathcal{O}_X$. We thus have $\chi(\mathcal{O}_Y)=\sum_{i=0}^{d-1}\chi(L^i)$. Since X is nonsingular, by the Hirzebruch–Riemann–Roch theorem, the Euler characteristic of L^i depends only on the rational numerical class of L, which is trivial since $L^d\cong\mathcal{O}_X$. We thus get $\chi(\mathcal{O}_Y)=d\chi(\mathcal{O}_X)$.

By Hodge symmetry, we have:

$$\chi(\mathcal{O}_Y)=\sum_{i=0}^{\dim Y}(-1)^i\dim H^i(Y,\mathcal{O}_Y)=\sum_{i=0}^{\dim Y}(-1)^i\dim H^0(Y,\Omega_Y^i).$$

If Y is an IHS manifold, the spaces $H^0(Y,\Omega_Y^i)$ are zero for odd i and are generated by $\omega_Y^{i/2}$ for even i, so $\chi(\mathcal{O}_Y)=\frac{1}{2}\dim Y+1$. Then the index d divides $\frac{1}{2}\dim Y+1$. $\square$

Proposition 5.2. *Let Y be a PSV and $\varphi\in\mathrm{Aut}(Y)$ be a symplectic automorphism of order $d\geq 2$ and let $X:=Y/\langle\varphi\rangle$. Then $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)$ and $K_X=0$. In particular, a symplectic automorphism of finite order of an IHS manifold is never étale.*

Proof. The argument is inspired by the proof of [51, Lemma 2.3]. The variety X is normal and projective, with klt singularities by [44, Proposition 5.20]. We keep notation as in the proof of Proposition 3.2.

Since φ is symplectic, the regular parts of the fixed loci of its nontrivial powers are symplectic submanifolds so their codimensions are at least two, hence φ is quasi-étale and the quotient p restricts to an étale cyclic cover $p\colon Y^\circ\to X_{\mathrm{reg}}$. The restriction maps $H^0(Y,\Omega_Y^{[i]})\to H^0(Y^\circ,\Omega^i_{Y^\circ})$ are isomorphisms, generated on the right hand side by the restrictions of $\wedge^i\sigma_Y$ to Y° since Y is a PSV. If φ is symplectic, all these spaces are φ-invariant so we get by Hodge symmetry for singular spaces:

$$\chi(\mathcal{O}_Y)=\sum_{i=0}^{\dim Y}(-1)^i\dim H^0(Y^\circ,\Omega^i_{Y^\circ})^{\langle\varphi\rangle}.$$

Since p is étale on Y°, we have $\Omega^1_{Y^\circ}\cong p^*\Omega^1_{X_{\mathrm{reg}}}$, so $\Omega^i_{Y^\circ}\cong p^*\Omega^i_{X_{\mathrm{reg}}}$ for all i and we obtain:

$$\chi(\mathcal{O}_Y)=\sum_{i=0}^{\dim Y}(-1)^i\dim H^0(Y^\circ,p^*\Omega^i_{X_{\mathrm{reg}}})^{\langle\varphi\rangle}=\sum_{i=0}^{\dim Y}(-1)^i\dim H^0(X_{\mathrm{reg}},\Omega^i_{X_{\mathrm{reg}}}),$$

where the last equality is a standard fact on sections of $\langle\varphi\rangle$-linearized sheaves obtained by pullback from a quotient. Using Hodge symmetry as above, this time for X, we get $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)$. Since φ is symplectic, the symplectic form on Y_{reg} goes down to a symplectic form on X_{reg}, so $K_X=0$.

In particular, if Y is an IHS manifold and if φ is symplectic and étale of order $d\geq 2$, then using Proposition 5.1 we get $\frac{1}{2}\dim Y+1=\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)=d\chi(\mathcal{O}_Y)$: this is a contradiction so φ cannot be étale. $\square$

Example 5.3. Let Y be a projective IHS manifold that is deformation equivalent to the Hilbert scheme of two points on a K3 surface. Assume that Y admits a symplectic involution ι. Then the quotient Y/ι has trivial canonical bundle and it admits a partial resolution that is an ISV (more precisely an irreducible symplectic orbifold), called a *Nikulin orbifold* [23, Definition 3.1].

Proposition 5.4. *Let Y be a PSV (resp. ISV) of dimension $2n$ with a quasi-étale purely non symplectic automorphism φ of order d. If d divides n then $K_X=0$, otherwise $X:=Y/\langle\varphi\rangle$ is a log-Enriques variety of PSV type (resp. ISV type), and index $r\geq 2$ given as the smallest integer such that r divides d and d divides rn.*

The quasi-étale assumption in the statement of Proposition 5.4 is automatic whenever $\dim Y>2$, since the singular locus has codimension at least two and the regular part of the fixed loci in Y_{reg} of all the nontrivial powers of φ are isotropic submanifolds. This is well-known to the experts, but due to a lack of reference we state and prove this result below.

Lemma 5.5. *Let Y be a PSV of dimension at least 4 with symplectic form σ_Y and let φ be a nonsymplectic automorphism of finite order on Y. Then any irreducible component of the fixed locus of φ in Y_{reg} is isotropic for σ_Y.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $\varphi^{[*]}\sigma_Y = \lambda\sigma_Y$; by assumption $\lambda \neq 1$. Let $Z := Y_{\mathrm{reg}}$ and $F \subset Z$ be an irreducible component of the fixed locus of φ in Z. Consider the splitting of the restriction of the tangent bundle: $(T_Z)_{|F} = T_F \oplus N_{F|Z}$. Since φ is of finite order, T_F coincides with the $(+1)$-eigenbundle of the action of φ on $(T_Z)_{|F}$. For any $x \in F$ and for any $u, v \in T_{F,x}$ we thus get:

$$\sigma_X(u,v) = \sigma_X(d\varphi_x(u), d\varphi_x(u)) = (\varphi^*\sigma_Y)(u,v) = \lambda\sigma_X(u,v),$$

hence, $\sigma_X(u,v) = 0$ since $\lambda \neq 1$. □

Proof of Proposition 5.4. Denote the quasi-étale quotient by $p\colon Y \to Y/\langle\varphi\rangle := X$. The variety X is normal and projective, with klt singularities by [44, Proposition 5.20]. By Proposition 3.2(ii), the torsion index r of the Weil divisor K_X divides d. Using similar arguments as in the proof of Proposition 5.2, we have:

$$\begin{aligned} H^0(X,\omega_X) &= H^0(X_{\mathrm{reg}}, \omega_{X_{\mathrm{reg}}}) = H^0(Y^\circ, p^*\omega_{X_{\mathrm{reg}}})^{\langle\varphi\rangle} \\ &= H^0(Y^\circ, \omega_{Y^\circ})^{\langle\varphi\rangle} = H^0(Y,\omega_Y)^{\langle\varphi\rangle}. \end{aligned}$$

The space $H^0(Y,\omega_Y)$ is generated by $\wedge^n\sigma_Y$. Since φ is purely nonsymplectic of order d, we have $\varphi^*(\wedge^n\sigma_Y) = \xi^n(\wedge^n\sigma_Y)$, where ξ is a primitive d-th root of unity. If d divides n, the space $H^0(X,\omega_X)$ contains a regular global section that does not vanish, so $K_X = 0$. Otherwise this global section is not φ-invariant so $H^0(X,\omega_X) = 0$ and K_X is not trivial. Since $rK_X = 0$, a similar computation as above gives:

$$\mathbb{C} = H^0(X, \omega_X^{\otimes r}) \cong H^0(Y, \omega_Y^{\otimes r})^{\langle\varphi\rangle}.$$

The space $H^0(Y, \omega_Y^{\otimes r})$ is generated by the nonvanishing section $(\wedge^n\sigma_Y)^{\otimes r}$, that is multiplied by ξ^{rn} under the action of φ^*, so this global section is φ-invariant if and only if d divides rn. We conclude that d divides rn. It follows that r is the smallest integer satisfying both conditions (this argument originates in [51, proof of Proposition 2.8]). □

Remark 5.6. We add two observations to Proposition 5.4:

(1) We have $r = d$ if and only if d and n are coprime. In particular, it is not possible to construct an index six (log)-Enriques variety as a quotient of a four-dimensional IHS manifold by an order 6 automorphism, see [17, Remark 4.1] and [51, Remark 6.6], but see §6.2.3 and §6.2.4 for examples of log-Enriques varieties of index 6 or 12 using higher-dimensional quotients.

(2) If Y is an IHS manifold and φ is étale, by Proposition 5.1 futhermore d divides $n+1$, so we have always $r = d$.

Corollary 5.7. *An Enriques manifold is of symplectic type if and only if its canonical cover is an IHS manifold, and it admits a unique cyclic étale IHS cover.*

Proof. Let X be an Enriques manifold of symplectic type and let $Y \to X$ be a cyclic étale cover of degree $d \geq 2$, where Y is an IHS manifold. Following [51], the cyclic group is generated by a purely nonsymplectic automorphism, otherwise some of its powers would be symplectic, and using the holomorphic Lefschetz formula we see that the quotient map would be ramified. By Remark 5.6, the index of the canonical divisor of X is equal to d. The canonical cover of $Z \to X$ has thus order d and, since Y is simply connected, Y is isomorphic to Z. $\square$

Example 5.8. Let S be a projective K3 surface with a nonsymplectic involution ι. The natural involution $\iota^{[2]}$ induced on the Hilbert square $Y := S^{[2]}$ of S is non-symplectic. Its fixed locus is non empty, even if ι is an Enriques involution. The quotient map $S^{[2]} \to S^{[2]}/\langle \iota^{[2]} \rangle =: X$ is quasi-étale and $K_X = 0$. Similarly as in the proof of Proposition 5.2, we have:

$$\chi(\mathcal{O}_X) = \sum_{i=0}^{2} \dim H^0(S^{[2]}, \Omega_{S^{[2]}}^{[2i]})^{\langle \varphi \rangle}.$$

The symplectic form σ_Y generates $H^0(S^{[2]}, \Omega_{S^{[2]}}^{[2]})$ and is not invariant, but its exterior product $\wedge^2 \sigma_Y$ is φ-invariant and generates $H^0(S^{[2]}, \Omega_{S^{[2]}}^{4})$, so $\chi(\mathcal{O}_X) = 2$ whereas $\chi(\mathcal{O}_Y) = 3$. Camere–Garbagnati–Mongardi [24] prove that X admits a crepant resolution by a weak Calabi–Yau manifold. By Proposition 3.2, we have an exact sequence:

$$\{0\} \longrightarrow \mu_2 \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(S^{[2]})^{\mu_2} \longrightarrow \{0\}$$

and $\mathrm{Pic}(S^{[2]})$ is torsion-free. See §6.1.3 for a variant of this construction producing Enriques manifolds.

Proposition 5.9. *Let X be a log-Enriques variety of PSV (resp. ISV) type of index d, and let $p\colon Y \to X$ be a cyclic quasi-étale cover such that Y is a PSV (resp. ISV). Assume that p has degree d. Then X is the quotient of Y by a purely non symplectic automorphism of order d. If moreover $\mathrm{Cl}(Y)$ is torsion-free, then Y is the canonical cover of X.*

Proof. Let φ be any generator of the cyclic group μ_d acting on Y, that is quasi-étale by assumption. If φ is symplectic, then by Proposition 5.2 we have $K_X = 0$, this is a contradiction. Similarly, if φ^k acts symplectically on Y for some strict divisor k of d we may factorize the quotient as follows:

$$p\colon Y \longrightarrow \overline{Y} := Y/\langle \varphi^k \rangle \xrightarrow{q} \overline{Y}/\langle \overline{\varphi} \rangle = X,$$

where $\overline{\varphi}$ is the class of φ in $\langle \varphi \rangle / \langle \varphi^k \rangle$. Then $K_{\overline{Y}} = 0$ by Proposition 5.2 and a similar computation as in the proof of Proposition 3.2(ii) shows that $\frac{d}{k} K_X = 0$. This is a contradiction, so φ is purely nonsymplectic.

Since p is a cyclic quotient, there exists a reflexive sheaf L on X that is a d-torsion element in $\mathrm{Cl}(X)$ such that $Y = \mathrm{Spec} \oplus_{i=0}^{d-1} L^{[-i]}$. By Proposition 3.2, if $\mathrm{Cl}(Y)$ is torsion-free, the torsion part of $\mathrm{Cl}(X)$ is generated by ω_X. It follows that the $\mathcal{O}_X$-algebras $\oplus_{i=0}^{d-1} L^{[-i]}$ and $\oplus_{i=0}^{d-1} \omega_X^{[-i]}$ are equal, so Y is the canonical cover of X. $\square$

Remark 5.10. By a recent result of Bertini, Grossi, Mauri and Mazzon [11, Proposition 3.16] quotients of ISV by finite order automorphisms acting symplectically are again ISV, so in our proof, if Y is an ISV then $\overline{Y}$ is an ISV. In particular since Y and $\overline{Y}$ are simply connected, the quotient by φ^k is quasi-étale, but certainly not étale. This remark holds for every quotient of an ISV by a finite order automorphism acting symplectically, so that the second assertion of Proposition 5.2 extends to the singular setting: a symplectic automorphism of finite order of an ISV is never étale.

Let us recall some classical basic facts about the topology of a singular quasi-projective complex algebraic variety X. When working with the classical topology, X is Hausdorff, connected and locally path-connected and it has the homotopy type of a finite CW-complex, so it is in particular locally simply connected. Thus it admits a universal cover (unramified) $p\colon \widetilde{X} \to X$ that inherits a complex analytic structure such that p is a local analytic isomorphism, but in general $\widetilde{X}$ is not algebraic. Assuming that X is normal, it is locally irreducible for the classical topology: it follows that $\widetilde{X}$ is irreducible and that the natural map $\pi_1(X_{\mathrm{reg}}) \to \pi_1(X)$ is surjective (see [31, §0.7] and [1, §2] and references therein).

If the cyclic cover $p\colon Y \to X$ defining a log-Enriques variety of ISV type and index d is étale, then clearly $\pi_1(X) \cong \mathbb{Z}/d\mathbb{Z}$. Otherwise, the topological fundamental group of X may be smaller than expected. In particular, if the group μ_d acting on Y has a fixed point, then X is simply connected (this is a special case of a result due to Kaneko–Yoshida, see [30, Lemma 1.2]).

Enriques manifolds are defined in [51] as projective, nonsingular varieties, non-simply connected, whose universal cover is an IHS manifold. In the singular setup, the behavior of the fundamental group and of the universal cover pushed us to use instead the properties of the canonical sheaf to define log-Enriques varieties. However, the following result remains true in the singular setup, but it is not a characterization any more.

Proposition 5.11. *Let X be a K-torsion variety, non simply connected, whose universal cover Y is a PSV (respectively and ISV). If the fundamental group $\pi_1(X)$ acts purely nonsymplectically on Y, then X is a log-Enriques variety of PSV (respectively ISV) type.*

Proof. The proof extends [51, Proposition 2.4] to the singular setup, using similar arguments. The fundamental group of X acts biholomorphically on its universal cover Y, so $\pi_1(X)$ is a subgroup of $\mathrm{Aut}(Y)$. The Lie algebra of $\mathrm{Aut}(Y)$ is isomorphic to $H^0(Y, T_Y)$ (see [46, Lemma 3.4]). Since Y is an ISV, we have $H^0(Y, T_Y) = \{0\}$ by [5, Lemma 4.6], so $\mathrm{Aut}(Y)$ is a discrete group. Since the quotient morphism $p\colon Y \to X = Y/\pi_1(X)$ is étale, the group $\pi_1(X)$ is in bijection with any fiber of p. These fibers are discrete and projective, hence finite, so $\pi_1(X)$ is finite. By assumption, the group $\pi_1(X)$ acts purely nonsymplectically on Y, so the group morphism $\pi_1(X) \to \mathbb{C}^*$ sending an automorphism φ to the complex number ξ such that $\varphi^{[*]}\omega_Y = \xi\omega_Y$ is injective. We deduce that $\pi_1(X)$ is a cyclic group. By Proposition 5.4, since X is assumed to be a K-torsion variety, it is a log-Enriques variety. □

6. Examples

6.1. **Known examples of Enriques manifolds.** There is a wide diversity of geometric constructions of Enriques surfaces, we refer for instance to [25, 28, 48]. However, in higher dimension quite few geometric constructions of nonsingular Enriques varieties exist. We briefly review the constructions given in [17, 51] and give some new examples using similar methods.

For any two-dimensional complex torus A with origin $0 \in A$, we denote by $A^{(n+1)}$ the $n+1$-th symmetric power of A, by $s\colon A^{(n+1)} \to A$ the summation map and by $A^{[n+1]}$ the Hilbert scheme of $n+1$ points on A. Consider the Hilbert–Chow morphism $\rho\colon A^{[n+1]} \to A^{(n+1)}$, the *generalized Kummer variety* (of dimension $2n$) of A is the fiber $\mathrm{K}_n(A) \coloneqq (s \circ \rho)^{-1}(0)$.

6.1.1. *The Hilbert scheme of points on Enriques surfaces.* Let E be an Enriques surface. Then for any $n \geq 2$, the Hilbert scheme $E^{[n]}$ of n points on E is an Enriques manifold of Calabi–Yau type, of dimension $2n$ and index two [51, Theorem 3.1].

6.1.2. *Involutions on generalized Kummer varieties of decomposable abelian surfaces.* Let $A = E \times F$ be the product of two elliptic curves and consider the automorphism ι_A of A defined by $\iota_A(x, y) = (-x + u, y + v)$. For any integer $n \geq 1$, the natural automorphism ${\iota_A}^{[n+1]}$ of $A^{[n+1]}$ respects the fiber $\mathrm{K}_n(A)$ if u and v are $(n+1)$-torsion points of E and F respectively. Moreover ι_A is an involution on A if v is a 2-torsion point. Assuming thus that n is odd, $u \in E[n+1]$ and $v \in F[2]$, it is easy to check that the natural automorphism $\mathrm{K}_n(\iota_A)$ is a fixed point free involution on $\mathrm{K}_n(A)$ if u is not an $\frac{n+1}{2}$-torsion point. Under these conditions, if n is odd, by Propositions 5.1&5.4 the quotient $\mathrm{K}_n(A)/\langle \mathrm{K}_n(\iota_A)\rangle$ is an Enriques manifold of symplectic type, of dimension $2n$ and index two.

6.1.3. *Enriques involutions on K3 surfaces.* Let S be a K3 surface carrying an *Enriques involution* ι_S, *i.e.* a fixed point free involution, and let $n \geq 1$ be an odd integer. Consider the natural involution ${\iota_S}^{[n]}$ induced on the Hilbert scheme $S^{[n]}$. Then, when n is odd, the quotient $S^{[n]}/\langle {\iota_S}^{[n]}\rangle$ is an Enriques manifold of dimension $2n$, of symplectic type and index two. If n is even, then by Proposition 5.4 the quotient has trivial canonical class so it is not an Enriques manifold. See Example 5.8 for a description of the case $n = 2$.

6.1.4. *Index* 2 *Enriques manifolds from moduli spaces of stable sheaves on* $K3$ *surfaces.* Let S be a very general $K3$ surface carrying an Enriques involution ι_S as above By genericity we can assume that $\rho(S) = 10$ and that the Picard lattice $\mathrm{Pic}(S)$ is ι_S-invariant. Given a primitive Mukai vector $v = (r, l, \chi - r) \in H^*(S, \mathbb{Z})$ and a v-general polarization $H \in \mathrm{Pic}(S)$, the induced involution ι_S^* acts on the moduli space $M_{v,H}(S)$ of H-stable sheaves on S with Mukai vector v. If χ is an odd integer, the quotient $M_{v,H}(S)/\langle \iota_S^* \rangle$ is an Enriques manifold of dimension v^2+2, of symplectic type and index two [51, Theorem 5.3]. This is a generalization of the previous construction.

6.1.5. *Complete intersections of quadrics.* For any even integer $n \geq 2$, consider the projective space $\mathbb{P}^{2n+1}$ with homogeneous coordinates $[x_0 : \ldots : x_n : y_0 : \ldots : y_n]$. For generic quadrics $Q_j(x), \widetilde{Q}_j(y)$ for $j = 1, \ldots, n+1$, the complete intersection:

$$Y \coloneqq \{[x : y] \,|\, Q_j(x) = \widetilde{Q}_j(y), \quad \forall j = 1, \ldots, n+1\}$$

is a nonsingular Calabi–Yau variety (we denote here $x \coloneqq (x_0, \ldots, x_n)$ and $y \coloneqq (y_0, \ldots, y_n)$). The involution ι of $\mathbb{P}^{2n+1}$ defined by $\iota(x_i) = -x_i$ and $\iota(y_i) = y_i$ for $i = 0, \ldots, n$ fixes the two n-dimensional projective spaces defined by $\{x_i = 0 \,|\, i = 0, \ldots, n\}$ and $\{y_i \,|\, i = 0, \ldots, n\}$. For generic choices of the quadrics, this fixed locus does not meet Y so the quotient $Y/\langle\iota\rangle$ is an Enriques manifold of Calabi–Yau type, of dimension n and index two. Observe that if n is odd, the same construction gives an involution preserving the volume form so the quotient is a Calabi–Yau manifold.

6.1.6. *Index* 3 *Enriques manifolds from generalized Kummer varieties of special decomposable abelian surfaces.* Let $A = E \times E_\omega$ where E is an elliptic curve, ω is a primitive 3rd root of unity and $E_\omega = \frac{\mathbb{C}}{\mathbb{Z}\oplus\omega\mathbb{Z}}$. Take $u \in E$, $v \in E_\omega$ and consider the automorphism $f_A(x,y) = (x+u, \omega y+v)$. Assume that u is a 3-torsion point, so that f_A has order 3. Assuming that u and v are $(n+1)$-torsion points, the natural automorphism ${f_A}^{[n+1]}$ of $A^{[n+1]}$ respects the fiber $\mathrm{K}_n(A)$ and the necessary condition given by Proposition 5.1 imposes that 3 divides $n+1$, so we write $n+1 = 3m$. If $m(2+\omega)v \neq 0$, the natural automorphism $\mathrm{K}_n(f_A)$ is a fixed point free order three automorphism on $\mathrm{K}_n(A)$ and by Proposition 5.4, under these assumptions, if 3 divides $n+1$ the quotient $\mathrm{K}_n(A)/\langle \mathrm{K}_n(f_A)\rangle$ is an Enriques manifold of symplectic type, of dimension $2n$ and index three.

A variant of this construction, starting from a quotient $A = \frac{E\times E_\omega}{\mathbb{Z}/3\mathbb{Z}}$, produces for an appropriate choice of the automorphism a different family of Enriques manifolds of symplectic type as quotients of generalized Kummer manifolds, see [16, §4.3.7 - Type 6].

6.1.7. *Index* 4 *Enriques manifolds from generalized Kummer manifolds of special decomposable abelian surfaces.* Similarly as above, take $A = E \times E_{\mathrm{i}}$ where $E_{\mathrm{i}} = \frac{\mathbb{C}}{\mathbb{Z}\oplus \mathrm{i}\mathbb{Z}}$. Take $u \in E$, $v \in E_{\mathrm{i}}$ and consider the automorphism $f_A(x,y) = (x+u, \mathrm{i}y+v)$. Assume that u is a 4-torsion point, so that f_A has order 4. Assuming that u and v are $(n+1)$-torsion points, the natural automorphism ${f_A}^{[n+1]}$ of $A^{[n+1]}$ respects the fiber $\mathrm{K}_n(A)$ and the necessary condition given by Proposition 5.1 imposes that 4 divides $n+1$, so we write $n+1 = 4m$. If $2mu \neq 0$ or if $2m(1+\mathrm{i})v \neq 0$, the natural automorphism $\mathrm{K}_n(f_A)$ is a fixed point free order four automorphism on $\mathrm{K}_n(A)$ and by Proposition 5.4, under these assumptions, if 4 divides $n+1$ the quotient $\mathrm{K}_n(A)/\langle \mathrm{K}_n(f_A)\rangle$ is an Enriques manifold of symplectic type, of dimension $2n$ and index four.

6.2. **Log-Enriques varieties of IHS type.** By Proposition 5.4, any quotient of a $2n$-dimensional IHS manifold, with $n \geq 2$, by a purely non-symplectic automorphism is a log-Enriques variety as long as the order of the automorphism does not divide n. Here, we discuss some features of examples which can be constructed from some deformation families of IHS manifolds.

6.2.1. *Log-Enriques varieties of prime index as quotients of IHS manifolds.* Non-symplectic automorphisms of prime order acting on IHS manifolds have now been classified for all known deformation families (see [14, 13, 16, 21, 22, 19] for $K3^{[n]}$ type, [47] for generalized Kummer manifolds, [36] for OG_6 manifolds and [12, 19] for OG_{10} manifolds). Let Y be an IHS manifold of dimension $2n \geq 4$ and let $\varphi \in \mathrm{Aut}(Y)$ be nonsymplectic of prime order p; in particular, the order satisfies $2 \leq p \leq 7$ when Y is either of Kum_n–type or of OG_6–type, respectively $2 \leq p \leq 23$ when Y is either of $K3^{[n]}$–type or of OG_{10}–type. Proposition 5.4 implies that the quotient $\pi\colon Y \to Y/\langle\varphi\rangle =: X$ is log-Enriques of index p when p does not divide n, and in this case the second Betti number $b_2(X)$ coincides with the rank of the invariant sublattice $T_\varphi \subset H^2(Y,\mathbb{Z})$. As a consequence, this construction yields $2n$-dimensional log-Enriques varieties of index p such that $(p,n) = 1$ and with second Betti number $1 \leq b_2(X) \leq b_2(Y) - 2$, $b_2(X) = b_2(Y) \mod p-1$.

As explained in [14] and [24, Lemma 3.4(2)], given a fixed component $Z \subset Y^\varphi$ of dimension s, the action of φ linearizes near a point $q \in Z$ as follows. If $p = 2$

then $s = n$ since Z is Lagrangian by [10, Lemma 1] and the action diagonalizes as:

$$\operatorname{diag}\left(\underbrace{1,\ldots,1}_{\text{s times}},\underbrace{-1,\ldots,-1}_{\text{s times}}\right).$$

If $p \geq 3$ then the diagonalization may be written as:

$$\operatorname{diag}\left(\underbrace{\underbrace{1,\xi_p},\ldots,\underbrace{1,\xi_p}}_{s\text{ times}},\underbrace{\underbrace{\xi_p^{a_1},\xi_p^{p+1-a_1}},\ldots,\underbrace{\xi_p^{a_1},\xi_p^{p+1-a_t}}}_{t\text{ times}},\underbrace{\xi_p^{\frac{p+1}{2}},\ldots,\xi_p^{\frac{p+1}{2}}}_{2n-2s-2t}\right),$$

where ξ_p is a primitive p-th root of unity, with $s \leq n$ by Lemma 5.5 since Z is isotropic and $1 < a_j < p$ for $1 \leq j \leq t$. By the Reid–Sheperd-Barron–Tai criterion for quotient singularities (see [56, Theorem 3.1] and [57, §4.11]), the type of the quotient singularity at $\pi(q)$ is determined by its *age* (see [24]), denoted $\operatorname{age}(\varphi, \pi(q))$, as follows:

$$\begin{cases} \text{klt} & \text{if } \operatorname{age}(\varphi,\pi(q)) > 0,\\ \text{canonical} & \text{if } \operatorname{age}(\varphi,\pi(q)) \geq 1,\\ \text{terminal} & \text{if } \operatorname{age}(\varphi,\pi(q)) > 1,\\ \text{strictly klt} & \text{if } 0 < \operatorname{age}(\varphi,\pi(q)) < 1,\\ \text{Gorenstein} & \text{if } \operatorname{age}(\varphi,\pi(q)) = 1.\end{cases}$$

We compute here:

$$\operatorname{age}(\varphi,\pi(q)) = \begin{cases} \frac{n}{2} & \text{if } p = 2,\\ n - s + \frac{n}{p} & \text{if } p > 2.\end{cases}$$

hence the log-Enriques variety X has terminal singularities along the image of Z if and only if p does not divide n and one of the following conditions holds:

(1) $s = n$ and $n > p$,
(2) $s < n$.

Indeed, if $s = n$, $\operatorname{age}(\varphi) = \frac{n}{p}$, but p cannot be a divisor of n, hence $\operatorname{age}(\varphi) > 1$ if $n > p$; otherwise, if $s < n$ then p must be odd and $\operatorname{age}(\varphi) > n - s > 1$. Canonical nonterminal singularities are never possible on these log-Enriques varieties, since $\operatorname{age}(\varphi) = 1$ if and only if p does not divide n, $s = n$ and $n = p$, this is impossible. In the low dimensional case $n = 2$, all the singularities of X are terminal except for two-dimensional components of X_{sing}, which are not even canonical. Recall that a normal projective variety with numerically trivial canonical divisor is uniruled if and only if it has worse than canonical singularities see e.g. [27, Lemma 2.1] where this property is investigated for several log-Enriques varieties. We give in §6.2.2 an exemple of a log-Enriques variety of IHS type and index three which has strctly klt singularities, showing in this way that it is uniruled.

Remark 6.1. By [61, Lemma 2.3] the maximum prime index for a log-Enriques surface is 19. In higher dimension the prime index may be bigger. In fact in [13] it is shown that it exists and IHS manifold of $K3^{[2]}$-type admitting an automorphism of order 23 acting purely nonsymplectically. By using the holomorphic Lefschetz formula one easily shows that the fixed locus is not empty and by Proposition 5.4 we get a log-Enriques variety of IHS-type and index 23.

6.2.2. *Log-Enriques varieties of IHS type from weak CY varieties.* Let $C \subset \mathbb{P}^5$ be a smooth cubic fourfold defined by an equation of the form

$$X_0^2 L(X_1, \ldots, X_4) + G(X_1, \ldots, X_4) + X_5^3 = 0,$$

where $L \in \mathbb{C}[X_1, \ldots, X_5]_1$ and $G \in \mathbb{C}[X_1, \ldots, X_4]_3$ are *resp.* a linear and a cubic form. Then there exist two commuting automorphisms $\iota, \sigma \in \mathrm{Aut}(C)$, *resp.* of order two and three, given by $\iota([X_0 : \ldots : X_5]) = [-X_0 : X_1 : \ldots : X_5]$ and $\sigma([X_0 : \ldots : X_5]) = [X_0 : \ldots : X_4 : \zeta_3 X_5]$, where ζ_3 is a nontrivial third root of unity.

The induced automorphisms $\iota, \sigma \in \mathrm{Aut}(F(C))$ act nonsymplectically on the Fano variety of lines $F(C)$ (see [20, §7] and [14, Example 6.4]). The quotient $Y := F(C)/\langle \iota \rangle$ is a singular weak Calabi–Yau variety [24] and the order three automorphism σ descends to an automorphism $\overline{\sigma}$ of the quotient Y; moreover, $\overline{\sigma}$ does not preserve the volume form on Y. Hence, the quotient:

$$X := Y/\langle \overline{\sigma} \rangle = F(C)/\langle \iota \circ \sigma \rangle,$$

is a log-Enriques variety of IHS type and index three. On the other hand one can consider first the quotient $F(C)/\langle \sigma \rangle$. The fixed locus of σ on $F(C)$ is a smooth surface (see [14, Example 6.4]), its image will determine the singular locus on $F(C)/\langle \sigma \rangle$. By using the formula of §6.2.1 we compute the age at a singular point to be equal to $2/3$, this means that the singularities are here strictly klt. Now the quotient $F(C)/\langle \iota \circ \sigma \rangle$ has klt singularities by [44, Proposition 5.20] and since by *loc. cit.* discrepancies decrease by doing finite quotients we get that X is a uniruled log-Enriques variety.

6.2.3. *Index* 6 *log-Enriques varieties.* Let $A = E \times E_\omega$ where E is an elliptic curve, and $E_\omega = \frac{\mathbb{C}}{\mathbb{Z} \oplus \omega \mathbb{Z}}$ where ω is a primitive third root of unity. Take $u \in E$, $v \in E_\omega$ and consider the order 6 automorphism $f_A(x, y) = (-x + u, \omega y + v)$. Assuming that u and v are $(n+1)$-torsion points, the natural automorphism ${f_A}^{[n+1]}$ of $A^{[n+1]}$ respects the fiber $\mathrm{K}_n(A)$. By Proposition 5.4, we get that the $2n$-dimensional quotient $\mathrm{K}_n(A)/\langle \mathrm{K}_n(f_A) \rangle$ has trivial canonical bundle if $n \equiv 0 \mod 6$, but it is a log-Enriques variety of IHS type otherwise:

- of index 2 if $n \equiv 3 \mod 6$;
- of index 3 if $n \equiv \pm 2 \mod 6$;
- of index 6 if $n \equiv \pm 1 \mod 6$.

6.2.4. *Index* 12 *log-Enriques varieties.* Let $A = E_{\mathrm{i}} \times E_\omega$ where ω is a primitive 3rd root of unity, $E_{\mathrm{i}} = \frac{\mathbb{C}}{\mathbb{Z} \oplus \mathrm{i} \mathbb{Z}}$ and $E_\omega = \frac{\mathbb{C}}{\mathbb{Z} \oplus \omega \mathbb{Z}}$. Take $u \in E_{\mathrm{i}}$, $v \in E_\omega$ and consider the order 12 automorphism $f_A(x, y) = (\mathrm{i}x + u, \omega y + v)$. Assuming that u and v are $(n+1)$-torsion points, the natural automorphism ${f_A}^{[n+1]}$ of $A^{[n+1]}$ respects the fiber $\mathrm{K}_n(A)$. By Proposition 5.4, we get that the $2n$-dimensional quotient $\mathrm{K}_n(A)/\langle \mathrm{K}_n(f_A) \rangle$ has trivial canonical bundle if $n \equiv 0 \mod 12$, but it is a log-Enriques variety of IHS type otherwise:

- of index 2 if $n \equiv 6 \mod 12$;
- of index 3 if $n \equiv \pm 4 \mod 12$;
- of index 4 if $n \equiv \pm 3 \mod 12$;
- of index 6 if $n \equiv \pm 2 \mod 12$;
- of index 12 if $n \equiv \pm 1 \mod 12$ or $n \equiv \pm 5 \mod 12$.

6.3. **Log-Enriques varieties of ISV and PSV type.** Although the theory of automorphism groups of irreducible symplectic varieties is not yet as well-developed as in the smooth case, we want to list here some examples of log-Enriques varieties which are obtained as quasi-étale cyclic quotients of an ISV or a PSV.

6.3.1. *Index 2 log-Enriques varieties from moduli spaces of semistable sheaves on $K3$ surfaces.* Here, we observe that Oguiso–Schröer's construction of Enriques manifolds illustrated in §6.1.4 generalizes to singular moduli spaces of semistable sheaves on $K3$ surfaces, which are known to be ISV [55]. Indeed, let S be a very general $K3$ surface carrying an Enriques involution ι_S. By genericity we can assume that $\rho(S) = 10$ and that the Picard lattice $\mathrm{Pic}(S)$ is ι_S-invariant. Given a non-primitive Mukai vector $w = (r, l, \chi - r) \in H^*(S, \mathbb{Z})$ such that $w^2 \equiv 0 \mod 4$ and a w-general polarization $H \in \mathrm{Pic}(S)$, the induced involution ι_S^* acts on the moduli space $M_{w,H}(S)$ of H-semistable sheaves on S with Mukai vector w, and the quotient $M_{w,H}(S)/\langle \iota_S^* \rangle$ is a log-Enriques variety of dimension $w^2 + 2$, of ISV type and index two.

Moreover, if χ is odd, ι_S^* acts freely on the regular locus of $M_{w,H}(S)$, which is the locus of H-stable sheaves on S. In order to show this, it is enough to observe that the proof of [51, Theorem 5.3] works: if a stable sheaf $\mathcal{F}$ is fixed, by descent there exists a coherent sheaf $\mathcal{F}'$ on $S/\langle \iota_S \rangle$ such that $\mathcal{F} = p^*\mathcal{F}'$, and thus $\chi = \chi(\mathcal{F}) = 2\chi(\mathcal{F}')$, in contradiction with χ being odd.

If $w = kv$ with $k \in \mathbb{Z}$ and $v \in H^*(S, \mathbb{Z})$ primitive, the condition that χ is odd implies that k is odd, so in particular the freeness of the action on the regular locus of $M_{w,H}(S)$ does not occur in the case that $M_{w,H}(S)$ is birational to an $OG10$ manifold.

6.3.2. *Index 2 log-Enriques varieties from relative Prym varieties.* Let S be a very general $K3$ surface carrying an Enriques involution i_S, as in §6.1.4 and §6.3.1. Let $\pi\colon S \to T := S/\langle i_S \rangle$ be the quotient Enriques surface, let C be a smooth curve of genus g on T with primitive class in $\mathrm{NS}(T)$ and denote $D := \pi^{-1}(C)$. The moduli space $M_{v,D}(S)$ is singular when $v = (0, D, 2 - 2g) \in H^*(S, \mathbb{Z})$ (see [2]) and it is an irreducible symplectic variety (see [18] and references therein). Moreover, since $D^2 = 4g - 4$, it is of dimension $2(2g - 1)$, hence it follows from §6.3.1 that i_S^* acts freely on $(M_{v,D}(S))_{\mathrm{reg}}$. On the other hand, the moduli space is also endowed with the duality j, which is a nonsymplectic involution commuting with i_S^*. The connected component $\mathcal{P}_{v,D}$ of the zero-section inside the fixed locus of the symplectic involution $i_S^* \circ j$ is the so-called relative Prym variety, and it is an irreducible symplectic variety by [2]. The nonsymplectic involution i_S^* restricts to $\mathcal{P}_{v,D}$ and acts freely on $\mathcal{P}_{v,D} \cap (M_{v,D}(S))_{\mathrm{reg}}$. For all genera $g \geq 2$ then $\mathcal{P}_{v,D}/i_S^*$ is a log-Enriques variety of ISV type and of index two.

This construction also works when S is a very general $K3$ surface endowed with a nonsymplectic involution i such that $\mathrm{NS}(S) \simeq U(2) \oplus E_8(-1)^{\oplus 2}$ or $\mathrm{NS}(S) \simeq U \oplus E_8(-1) \oplus D_4(-1)^{\oplus 2}$. In such a case, the quotient surface T is rational. By [18], one can also construct relative Prym varieties $\mathcal{P}_{v,D}$ starting from a curve C on T of genus g and considering the moduli space $M_{v,D}(S)$ of D-semistable sheaves on S with Mukai vector $(0, D, 1 - g(D))$, where $D := \pi^{-1}(C)$ is of genus $g(D)$. Moreover, in this case S also carries an Enriques involution i_S by [52], which commutes with i. As a consequence, whenever D is also invariant for i_S, the involution i_S^* acts on $\mathcal{P}_{v,D}$ and the quotient $\mathcal{P}_{v,D}/\langle i_S^* \rangle$ is a log-Enriques variety of index two, which is of PSV type (*resp.* of ISV type) when C and D satisfy the assumptions of [18, Theorem 1.2] (*resp.* of [18, Theorem 1.3 and 1.4]).

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