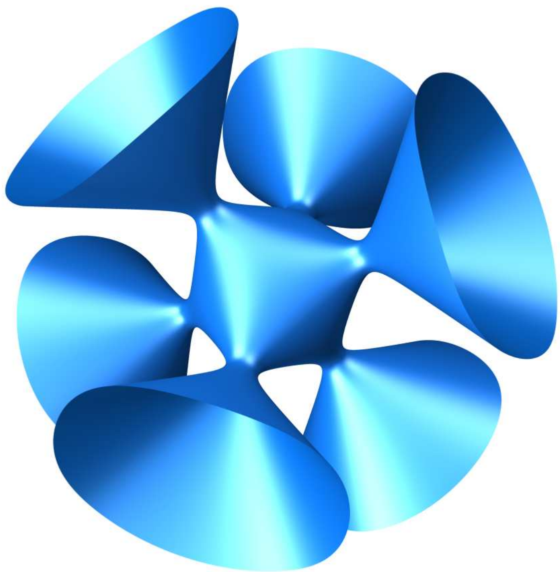
The geometry of some special algebraic varieties



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Contents

1. Zusammenfassung

Das Thema von meiner Habilitation ist *Die Geometrie einiger spezieller algebraischer Varietäten*, insbesondere untersuche ich die K3-Flächen. Die Arbeiten, die ich vorlege, sind die Artikeln:

(1) mit Wolf Barth, *Polyhedral Groups and Pencils of K3-Surfaces with maximal Picard Number*, **Asian J. of Math.** Vol. 7, No. 4, pp. 519–538, 2003.

(2) *Symmetric surfaces with many Singularities*, **Comm. in Algebra** Vol. 32, No. 10, pp. 3745-3770, 2004.

(3) *A geometrical construction for the generators of some reflection group*, **Serdica Math. J.**, 31, pp. 229-242, 2005.

(4) with Andreas Knutsen, Carla Novelli, *On Varieties that are uniruled by lines*, **Compositio Math.** 142, pp. 889–906, 2006.

(5) *Group Actions, cyclic coverings and families of K3 surfaces*, erscheint in **Canadian Math. Bull.**

(6) *Transcendental lattices of some K3 surfaces*, erscheint in **Math. Nachr.**

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(9) with Alice Garbagnati, *Symplectic automorphisms of prime order on K3 surfaces*, Preprint math.AG/0603742, eingereicht.

(10) with Samuel Boissière, *Counting lines on surfaces*, Preprint math.AG/0606100, eingereicht.

(11) with Alice Garbagnati, *Projective models of K3 surfaces with an even set*, Preprint math.AG/0611182, eingereicht.

Ich werde sie im Folgenden kurz beschreiben, man siehe die Literaturangaben für die verwendeten Abkürzungen.

1) Arbeiten über Flächen mit vielen Doppelpunkten: In meiner Promotion habe ich mich mit der Frage beschäftigt, wieviele gewöhnliche Doppelpunkte eine Fläche vom Grad d in $\mathbb{P}_3$ maximal haben kann, und ich habe drei neue eindimensionale Familien von Flächen in $\mathbb{P}_3$ beschrieben. Diese haben Grad 6,8 bzw. 12 und die Symmetrien der sogenannten bi-polyedrischen Tetraedergruppe ($=G_6$), Oktaedergruppe($=G_8$) bzw. Ikosaedergruppe ($=G_{12}$), d.h. die Polynome, die die Familien definieren, sind invariant unten der Operation von $G_d \subset SO(4,\mathbb{R})$, $d = 6$, 8 bzw. 12. Jede Familie enth"alt genau vier Fl"achen mit gew"ohnlichen Doppelpunkten. Insbesondere gibt es in der Familie vom Grad 12 eine Fläche, die 600 gewöhnliche Doppelpunkte hat (s. [Sa1]).
Die Gruppen G_6 und G_{12} sind Untergruppen der Spiegelungsgruppen $[3,4,3]$ und $[3,3,5]$. Mit Hilfe der G_6- bzw. $G_{12}-$invarianten Flächen in $\mathbb{P}_3$ konnte ich in der Arbeit [Sa3] eine

einfache geometrische Konstruktion für die Erzeugenden des Rings der invarianten Polynome vom Grad $2, 6, 8, 12$ bzw. $2, 12, 20, 30$ angeben (diese wurden auf eine andere Weise von Racah beschrieben). In der Arbeit [Sa2] betrachte ich weitere Untergruppen der $SO(4, \mathbb{R})$ und untersuche deren eindimensionalen Familien von invarianten Flächen in $\mathbb{P}_3$. Ich schränke meine Untersuchung auf die Gruppen ein, die die Heisenberggruppe enthalten. Zusammen mit den Grppen G_d ergeben die Gruppen aus [Sa2] eine vollständige Liste von Gruppen, die die Heisenberggruppe enthalten.
2) Flächen mit vielen (disjunkten) Geraden: In der Arbeit [BoSa2] mit Samuel Boissière (Universität Nizza) konstruieren wir Flächen in $\mathbb{P}_3$ mit vielen (disjunkten) Geraden. Für diese Anzahl gibt es Abschätzungen von Segre und Miyaoka. Es ist wohlbekannt, dass eine glatte Kubik 27 Geraden enthält; für Flächen vom Grad vier gibt es Arbeiten u.a. von Segre und Nikulin. Das Problem ist noch offen für den Grad $d \geq 5$. Klassische Beispiele sind die Fermatsche Flächen $x_0^d + x_1^d + x_2^d + x_3^d = 0$, die $3d^2$ Geraden enthalten. Andere Beispiele sind die Flächen der Art: $\phi(x, y) - \psi(z, t) = 0$ wobei ϕ, ψ homogene Polynome vom Grad d sind. In diesem Artikel beschreiben wir Flächen, die gegeben sind durch die Gleichung: $\phi(x, y) - \psi(z, t) = 0$ vollständig und wir geben alle möglichen Anzahlen von Geraden an. Wir studieren außerdem einige Flächen mit vielen Symmetrien und wir geben ein Beispiel einer Fläche vom Grad acht mit 352 Geraden an. Das verbessert ein Ergebnis von Caporaso-Harris-Mazur, die eine Fläche mit 256 Geraden konstruieren. Wir geben auch einige neue Beispiele von Flächen mit vielen disjunkten Geraden an, die ein vorheriges Ergebnis von Rams verbessern.
3) Arbeit über die Klassifikation von geregelten 3-Mannigfaltigkeiten: In [KNS] konnten wir das folgende Ergebnis zeigen: Sei X eine irreduzible Varietät vom Dimension $k \geq 3$, $\mathcal{H}$ ist ein global erzeugter und big Geradenbündel auf X mit $\mathcal{H}^k := d$, $n = dim\ H^0(X, \mathcal{H}) - 1$. Wenn $d < 2(n - k) - 4$, und $(k, d, n) \neq (3, 27, 19)$, dann ist X geregelt von Geraden. Im Fall von 3-Mannigfaltigkeiten ist diese Abschätzung optimal, denn für $d = 2n - 10$ haben wir Beispiele von 3-Mannigfaltigkeiten gefunden, die nicht geregelt von Geraden sind. Unser Ergebnis gilt insbesondere für Varietäten in $\mathbb{P}^n$. Im Fall von Flächen wurde eine solche Abschätzung von M. Reid und Xiao angegeben. Das bis jetzt beste Ergebnis für k-Mannigfaltigkeiten X in $\mathbb{P}^n$, X glatt war von Horowitz. Er zeigte: ist der Grad $d < 3/2(n - k - 1)$, dann ist X geregelt von Geraden. Unser Ergebnis verbessert dieses Resultat, außerdem gilt es für jede Varietät ohne Annahme über die Singularitäten von X. Um unser Ergebnis zu zeigen, verwenden wir die Mori-Theorie und das Minimal-Model-Programm, insbesondere benutzen wir einige Ergebnisse von Mella.
4) Arbeiten über K3-Flächen: Eine K3-Fläche S ist eine glatte, kompakte Fläche über $\mathbb{C}$, die einfach zusammenhängend ist und ein triviales kanonisches Bündel hat. Die K3-Flächen sind von besonderem Interesse wegen ihrer wichtigen Eigenschaften, z.B. sind sie alle zueinander diffeomorph, es gilt die Surjektivität der Periodenabbildung, und nach dem Theorem von Torelli kann man sie durch die Hodge-Struktur (also durch das transzendete Gitter und das Picard-Gitter) klassifizieren. Sie wurden in den letzten Jahren eingehend untersucht, z.B. wegen ihrer arithmetischen Eigenschaften und nicht zuletzt wegen ihrer Rolle in der Physik und insbesondere in der String-Theorie: Sie sind Calabi-Yau-Mannigfaltigkeiten der Dimension zwei und spielen eine wichtige Rolle in der Spiegel-Symmetrie. Mit diesen Flächen habe ich mich sehr intensiv in den letzten Jahren beschäftigt, und insbesondere habe ich mich mit den folgenden Themen befaßt:
K3-Flächen mit großer Picard-Zahl. In den Arbeiten [BaSa], [Sa4] und [Sa5] beschäftige ich mich mit Familien von K3-Flächen mit großer Picard-Zahl (das Maximum für eine K3-Fläche ist 20). Es ist schwierig, Beispiele von solchen Familien zu konstruieren und die Flächen

in der Familie zu identifizieren, die eine höhere Picard-Zahl haben. Damit verbunden ist das Problem, Büschel von K3-Flächen mit großer Picard-Zahl und minimaler Anzahl von singulären Fasern zu konstruieren. Einige Beispiele sind in Arbeiten von Beauville, Belcastro, Verrill-Yui, Narumiya-Shiga enthalten. In [BaSa] zusammen mit Wolf Barth beschreibe ich die Quotienten der Familien $\{X_\lambda\}_{\lambda\in\mathbb{P}_1}$ nach den Gruppen G_d (s. Arbeit [Sa1]): Diese sind Familien von K3-Flächen, bei denen die allgemeine Fläche Picard-Zahl 19 hat und es vier singuläre Fasern gibt, die Picard-Zahl 20 haben. Insgesamt enthält die Familie aber fünf singuläre Fasern. In dem Artikel berechnen wir das Picard-Gitter der Flächen explizit. In [Sa4] beschreibe ich weitere Familien von K3-Flächen mit großer Picard-Zahl und kleiner Anzahl von singulären Fasern. Hier betrachte ich spezielle Untergruppen G von G_d. Dann ist die G_d-invariante Familie $\{X_\lambda\}_{\lambda\in\mathbb{P}_1}$ auch G-invariant und unter einigen Bedingungen sind die Quotienten X_λ/G wieder K3-Flächen mit großer Picard-Zahl. Wenn außerdem G Normalteiler von G_d mit $[G : G_d] = 2, 3$ ist, kann man X_λ/G als 2- bzw. 3-zyklische Überlagerung von X_λ/G_d betrachten. Mit Hilfe dieser Überlagerung kann man das Picard-Gitter von X_λ/G in vielen Fällen genau identifizieren. Mit Hilfe der Gitter-Theorie und Ergebnissen über quadratische Formen kann man das transzendente Gitter der Flächen berechnen. Das wurde von Barth für die Flächen aus [BaSa] durchgeführt. In [Sa5] berechne ich es für die Flächen aus [Sa4]. Damit kann ich dann die K3-Flächen klassifizieren.
Ich beschäftige mich weiter mit diesen Flächen in der Arbeit in Vorbereitung [Sa6], in der ich projektive Modelle der Flächen untersuche.

Symplektische Automorphismen auf K3-Flächen. Im Rahmen meines DFG-Forschungsprojekts in Mailand *Die Geometrie einiger Familien von K3-Flächen und symplektische Automorphismen auf K3-Flächen* habe ich mich mit Automorphismen auf K3-Flächen beschäftigt, die die 2-holomorphe Form invariant lassen (*symplektische Automorphismen*). Solche Automorphismen der Ordnung zwei heißen *Nikulin-Involutionen*. In dem Artikel [GS], untersuche ich sie zusammen mit Bert van Geemen.
Nach einer Arbeit von Nikulin induzieren sie eine eindeutige Operation auf $H^2(X,\mathbb{Z})$. Wir studieren die Neron-Severi-Gruppe und das transzendente Gitter. Insbesondere zeigen wir, dass, wenn X eine Nikulin-Involution besitzt, die Picard-Zahl ≥ 9 ist und die Neron-Severi-Gruppe eine Kopie des Gitters $E_8(-2)$ enthält (das ist das Gitter E_8 mit der Bilinearform multipliziert mit -2). Im Fall $\rho = 9$ bestimmen wir mit Hilfe der Gitter-Theorie vollständig die Struktur der Neron-Severi-Gruppe in Abhängigkeit von der Polarisierung der K3-Fläche. Wir geben an und untersuchen viele konkrete Beispiele, die die allgemeinen Sätze beleuchten u. a. doppelte Überlagerungen der Ebene, Quartiken in $\mathbb{P}_3$, vollständige Durchschnitte und insbesondere K3-Flächen mit elliptischer Faserung.
In der Arbeit [GaSa1] beschäftige ich mich zusammen mit Alice Garbagnati (Universität Mailand) mit symplektischen Automorphismen der Ordnung $3, 5, 7$. Nach einer Arbeit von Nikulin sind diese zusammen mit den Automorphismen der Ordnung zwei alle möglichen Primordnungen für solche Automorphismen. Mit Hilfe von elliptischen Faserungen auf K3-Flächen und der Gittertheorie konnten wir die Wirkung auf $H^2(X,\mathbb{Z})$ vollständig beschreiben. Wie in dem Fall der Ordnung zwei (nach einem Ergebnis von Nikulin) ist diese Wirkung eindeutig, d.h. unabhängig von der Wahl der K3-Fläche. Ich beschäftige mich mit ähnlichen Problemen (auch im Fall von nicht-symplektischen Automorphismen) in den Arbeiten in Vorbereitung: [AS] zusammen mit Michela Artebani (Universität Mailand) und [GaSa3] zusammen mit Alice Garbagnati.

Zwei-teilbaren Mengen von acht disjunkten rationalen Kurven. In der Arbeit [GaSa2] zusammen mit Alice Garbagnati (Universität Mailand) untersuchen wir K3-Flächen mit einer *2-teilbaren Menge* von acht (-2)-rationalen Kurven (das heißt die Summe der

Kurven ist äquivalent zu zweimal einem Divisor in der Picard Gruppe) und mit einer 2-teilbaren Menge von gewöhnlichen Doppelpunkten (das heißt, die (-2)-rationalen Kurven in der minimalen Auflösung sind eine 2-teilbare Menge). Solche K3-Flächen sind minimale Auflösungen und Quotienten einer K3-Fläche nach einer Nikulin-Involution, ihre Untersuchung vervollständigt die Ergebnisse der Artikel [GS] im folgenden Sinn: Gegeben sei eine K3-Fläche mit Automorphismus. Es ist natürlich zu fragen, was für eine Fläche der Quotient ist und mit welchen Eigenschaften (z.B. Singularitäten). In [GaSa2] studieren wir K3-Flächen mit gerader Menge von rationalen Kurven und mit kleinstmöglicher Picard-Zahl, die neun ist. Es werden die Flächen klassifiziert und es wird deren Modulraum beschrieben. Insbesondere beschreiben wir viele projektive Modelle, mit denen wir die Untersuchung erweitern und fortsetzen, die von Barth angefangen worden ist.

5) Arbeit über die McKay-Korrespondenz in Dimension zwei und drei: In der Arbeit [BoSa1] zusammen mit Samuel Boissière (Universität Nizza) geben wir eine Beziehung zwischen der McKay-Korrespondenz in Dimension zwei und in Dimension drei. Sei G eine endliche Untergruppe der $SO(3,\mathbb{R})$ und sei $\tilde{G} \subset SU(2)$ die binäre Gruppe zur Gruppe G. Die Gruppe $\widetilde{G}$ operiert auf $\mathbb{C}^2$ und der Quotient ist eine ADE-Flächensingularität. Ihre Auflösung besteht aus glatten, rationalen (-2)-Kurven mit einem ADE-Dynkin Diagramm als dualem Graph. Die McKay-Korrespondenz assoziert die Ecken des Graphs mit den irreduziblen Darstellungen ($\neq 1$) von $\widetilde{G}$. Die Auflösung der Quotienten $\mathbb{C}^2/\widetilde{G}$ und $\mathbb{C}^3/G$ sind Hilbert-Nakamura-Schemata und die exzeptionellen Kurven beider Auflösungen über dem Ursprung haben sehr ähnliche Eigenschaften. Wir zeigen, dass es einen Morphismus zwischen diesen beiden Auflösungen gibt, der bestimmte Kurven in der exzeptionellen Faser kontrahiert. Für den Beweis benutzen wir die McKay-Korrespondenz zwischen Darstellungen und exzeptionelle Kurven in Dimension zwei und drei, sowie die Theorie von Hilbert-Nakamura-Schemata.

2. Introduction

In the papers which I collect for my Habilitation at the University of Mainz, I concentrate my attention to algebraic varieties with some special geometric properties (for example with many singularities or many lines) with particular attention to K3 surfaces, which occupy an important place in the classification of algebraic surfaces. An example of such a surface is given on the cover: this is a surface of equation

$$1 + x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2 + 1)^2 = 0, \quad a = -0,49.$$

in fact any quartic surface in $\mathbb{P}_3$ is an example of a K3 surface.
About varieties with special geometric property there are easy questions with difficult answers, for example which is the maximal number of lines a surface of degree d in $\mathbb{P}_3$ can have? Or which is the maximal number of nodes? The first question has an answer up to the degree four, in general there are bounds of Segre and Miyaoka. It is well known that a smooth cubic contains 27 lines; for surfaces of degree four this maximal number is also known: it is 64, 16 if we assume that the lines are skew; these are results of Segre and Nikulin. The problem is still open for degree $d \geq 5$. Classical examples are the Fermat surfaces $x^d + y^d + z^d + t^d = 0$, which contain $3d^2$ lines. Other examples are the surfaces of the kind $\varphi(x, y) - \psi(z, t) = 0$, where φ and ψ are homogeneous polynomials of degree d. There are more results given by Caporaso-Harris-Mazur in [CHM], who construct examples of surfaces with many lines in each degree and there are results of Rams in [Ram2] about skew lines. In the case of the question about the nodes there is an answer up to the degree six. Unfortunately there are no-standard methods to construct examples. A successful idea is to consider surfaces with many symmetries, this was used by Barth, Endraß and other people to construct examples of surfaces with many nodes. I used it to construct examples of surfaces with many nodes or with many lines (cf. [Sa2], [Sa3], [BoSa2]). Strictly connected to the problem of lines on algebraic varieties is the problem to determine when a variety is coverd by lines or more precisely when it is uniruled by lines. The answer to the problem in the case of surfaces is given independently by M. Reid in [Re] and by Xiao in [Xi]. They show that for $d < (4/3)(n-2)$ a surface $X \subset \mathbb{P}_n$ of degree d is uniruled by lines (except when $n = 9$ and $(X, \mathcal{O}_X(1)) = (\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2})$). For varieties of higher degree this bound was unknown, in [KNS] we give a bound for varieties of any degree and we show that this is optimal in the case of threefolds.
Another topic of my work are the K3 surfaces, these are smooth, compact complex surfaces which are simply connected and have trivial canonical bundle. The K3 surfaces are of particular interest because of their important properties, for example they are all diffeomerphic to eachother, the period map is surjective and due to a theorem of Torelli, they can be classified by their Hodge structure which involves the transcendental lattice and the Picard lattice. They have been very much studied in the last years for example for their arithmetic properties and for their importance in Physics and in particular in the String-Theory: they are Calabi-Yau manifold of dimension two and play an important role in the Mirror-Symmetry. I have worked very much with these surfaces in the last years in particular with the following topics: K3 surfaces with big Picard number and automorphisms on K3 surfaces. The maximal Picard number for a K3 surface is 20, however non-trivial families have at most Picard number 19. It is difficult to construct such families and to identify the surfaces with Picard number 20 (i.e. the *singular* K3 surfaces). Strict connected to this, there is the problem of constructing families of K3 surfaces with big Picard number and small number of singular (in the usual sense) K3 surfaces. Some examples are studied by Beauville, Belcastro, Verrill-Yui, Narumiya-Shiga.

During my DFG-researchproject *Die geometrie einiger Familien von K3-Flächen und symplektische Automorphismen auf K3-Flächen* in Milan by B. van Geemen, I studied symplectic automorphisms on K3 surfaces (i.e. those automorphisms leaving the holomorphic two form invariant). These were extensively studied in the recent past. In a famous paper from 1979 [N1], Nikulin described the finite automorphism groups of K3 surfaces, in particular he classified all abelian groups which act symplectically on a K3 surface, that is, they leave the holomorphic two form invariant. This classification was completed in 1988 by Mukai in [Muk]: he classified all isomorphism classes of finite groups acting symplectically on a K3 surface. The first case to study are the symplectic involutions (called *Nikulin involutions* by Morrison in [Mo]). These are also important for their relation with the Shioda-Inose structure introduced by Morrison in [Mo]. This structure relates K3 surfaces with large Picard number to abelian surfaces and was studied for example in [L], [NS], [vGT]. For my research also the results on elliptic fibrations are important. In fact, given a K3 surface with an elliptic fibration and a section, the group of sections of the fibrations is the Mordell-Weil group of the surface. A section of finite order defines a symplectic automorphism of the same order. The study of these automorphisms is often very useful for gaining an understanding of the general case. The literature on elliptic fibrations is vast, works of particular importance for my research are of Shioda, [Shio] and Shimada, [Shim]. The last paper classifies all fibres of type ADE in an elliptic fibration and also describes the torsion group of the Mordell-Weil group (that is the part generated by sections of finite order).

Nikulin showed that the action induced by these automorphisms on the second cohomology group with integer coefficients, $H^2(X, \mathbb{Z})$, is determined by its order and does not depend on the particular choice of the K3 surface X. In particular, we have a canonical decomposition into an invariant lattice and its perpendicular. In the case of Nikulin involutions, this decomposition was identified by Morrison in [Mo], he showed that the invariant part is isometric with $U \oplus U \oplus U \oplus E_8(-2)$ (where U is a copy of the unimodular even hyperbolic plane and $E_8(-2)$ is the lattice E_8 with the bilinear form multiplied by -2) and its perpendicular is $E_8(-2)$. The question about this decomposition is of course of interest for any other group in the classification of Nikulin. In [GaSa1] we give this decomposition in the case of prym order automorphisms.

Given an automorphism of a surface it is natural to ask about the properties of the quotient surface. In the case of symplectic automorphisms, the quotient has only ADE singularities and its minimal resolution is a K3 surface. The first case to study are the quotients by a Nikulin involutions. More in general it is interesting to study surfaces with an even set of eight rational (-2)-curves (that is, the sum of the curves is twice a divisor in the Picard group) or with even sets of nodes (that is, the (-2)-curves in the resolution are an even set). In [B2] Barth gives a description of some projective models of such surfaces. In [GaSa2] we continue this description.

It is also interesting to study non-symplectic automorphisms, examples of such finite order automorphisms are given in the papers [DGK], [K] and in the paper [A1], [A2], [A3], in the particular case of order four automorphisms. In this case the study of the K3 surfaces and of concrete examples is more complicated, for instance, there are no results on their classification.

3. Description of the scientific works

The works which I submit for my Habilitation are the papers:

(1) with Wolf Barth, *Polyhedral Groups and Pencils of K3-Surfaces with maximal Picard Number*, **Asian J. of Math.** Vol. 7, No. 4, pp. 519–538, 2003.

(2) *Symmetric surfaces with many Singularities*, **Comm. in Algebra** Vol. 32, No. 10, pp. 3745-3770, 2004.

(3) *A geometrical construction for the generators of some reflection group*, **Serdica Math. J.**, 31, pp. 229-242, 2005.

(4) with Andreas Knutsen, Carla Novelli, *On Varieties that are uniruled by lines*, **Compositio Math.** 142, pp. 889–906, 2006.

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(6) *Transcendental lattices of some K3 surfaces*, to appear in **Math. Nachr.**

(7) with Bert van Geemen, *Nikulin involutions on K3 surfaces*, to appear in **Math. Z.**

(8) with Samuel Boissière, *Contraction of excess fibres between the Mckay correspondence in dimensions two and three*, to apper in **Ann. Inst. Fourier**

(9) with Alice Garbagnati, *Symplectic automorphisms of prime order on K3 surfaces*, Preprint math.AG/0603742, submitted.

(10) with Samuel Boissière, *Counting lines on surfaces*, Preprint math.AG/0606100, submitted.

(11) with Alice Garbagnati, *Projective models of K3 surfaces with an even set*, Preprint math.AG/0611182, submitted.

these are the papers [BaSa], [Sa2], [Sa3], [KNS], [Sa4], [Sa5], [GS], [BoSa1], [GaSa1], [BoSa2], [GaSa2] from the reference list. First I give a quick overwiev of the contents and then I will explain more in details:
1) the papers [Sa2], [Sa3], [BoSa2] deal with surfaces with many nodes, lines and many symmetries,
2) the papers [BaSa], [Sa4], [Sa5], deal with some special families of K3 surfaces with Picard number 19,
3) the papers [GS],[GaSa1],[GaSa2] are about symplectic automorphisms of K3 surfaces,
4) the paper [KNS] give a criterion for varieties in any degree to be uniruled by lines,
5) finally the paper [BoSa1] is about a special case of the Mckay correspondence, and it is related to the works of 2).

1) Surfaces with many double points. In my PhD thesis I worked about the question: which is the maximal number of nodes a surface of degree d in $\mathbb{P}_3$ can have. I have found three new one dimensional families of surfaces $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ in $\mathbb{P}_3$, they have degree $d=6,8$ resp. 12 and have the symmetries of the so called bipolyhedral tetrahedralgroup ($=G_6$), octahedralgroup ($=G_8$) resp. icosahedralgroup ($=G_{12}$), this means that the polynomials which define the families are invariant for the operation of $G_d\subset SO(4,\mathbb{R})$, $d=6,8,12$. Each family contains exactly four surfaces with nodes. In particular the family of degree 12 contains a

surface with 600 nodes (these results are contained in my paper [Sa1]).
The groups G_6 and G_{12} are subgroups of the reflection groups $[3,4,3]$ and $[3,3,5]$, which are the symmetry groups of some special four dimensional polyhedra. By using the G_6- resp. $G_{12}-$invariant surfaces in $\mathbb{P}_3$ I give an easy geometric construction for the generators of the rings of the invariant polynomials, these have degree $2,6,8,12$ resp. $2,12,20,30$ (these were described before in a different way by Racah in [Rac]). In the paper [Sa2] I consider other subgroups of $SO(4,\mathbb{R})$ and I study the one dimensional families of invariant surfaces in $\mathbb{P}_3$. I restrict my study to the subgroups which contain the Heisenberg group. These together with the groups G_d give a complete list of subgroups of $SO(4,\mathbb{R})$ which contain the Heisenberg group and have a one dimensional family of invariant surfaces.
2) Surfaces with many (disjoint) lines. In the paper [BoSa2] together with Samuel Boissière (University of Nizza) we construct surfaces in $\mathbb{P}_3$ with many (disjoint) lines. First we describe the surfaces $\varphi(x,y)-\psi(z,t)=0$ completely and we give for any degree d all the possible numbers of lines. We study also surfaces with many symmetries and we give an example of a surface of degree eight with 352 lines. This result improves a preceding result of Caporaso-Harris-Mazur [CHM], who construct a surface with 256 lines. We give also some new examples of surfaces with many disjoint lines, which improve some previous result of Rams [Ram2].
3) Classification of uniruled varieties. In [KNS] together with Andreas Knutsen (University of Rom) and Carla Novelli (University of Genova) we show the following result: Let X be an irreducible variety of dimension $k\geq 3$, $\mathcal{H}$ a globally generated and big line bundle on X with $\mathcal{H}^k := d$, $n = dimH^0(X,\mathcal{H})-1$. If $d<2(n-k)-4$ and $(k,d,n)\neq(3,27,19)$ then X is uniruled by lines. In the case of threefolds this is an optimal bound, since for $d=2n-10$ there are examples of threefolds which are not uniruled by lines. Our result holds in particular for varieties in $\mathbb{P}_n$. In the case of surfaces a similar bound was given by M. Reid and Xiao. Until now the best result for smooth k-folds X in $\mathbb{P}_n$ was a result of Horowitz. He showed that for $d<(3/2)(n-k-1)$, then X is uniruled by lines. Our result improves the result of Horowitz and moreover it holds without any assumption on the singularities if X. To prove our result we use Mori-Theory and the Minimal-Model-program, in particular we use some previous results of Mella.
4) K3 surfaces with big Picard number. In the papers [BaSa], [Sa4], [Sa5] I work with families of K3 surfaces with Picard number 19. In [BaSa] together with Wolf Barth (University of Erlangen) I describe the quotients of the one dimensional families $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ by the groups G_d (cf. **1** above and the paper [Sa1]): these are families of K3 surfaces, in which the general K3 surface has Picard number 19 and there are exactly five singular fibers: one is a degeneration and four have nodes, the latter have Picard number 20. In the paper we describe completely the Picard lattice. In [Sa4] I describe more families of K3-surfaces with big Picard number and small number of singular fibers, I consider some special subgroup G of G_d, then clearly the G_d-invariant family $\{X^d_\lambda\}_{\lambda\in\mathbb{P}_1}$ is also G-invariant and under some assumption on the groups G, the quotients X^d_λ/G are again K3 surfaces with big Picard number. Moreover if G is a normal subgroup with $[G:G_d]=2,3$ then I describe X^d_λ/G as 2-cyclic, resp. 3-cyclic covering of X^d_λ/G_d. This description is very helpful to identify the Picard lattice of the covering surfaces. Then in [Sa5] with the help of lattice theory (cf. [N2]) and results on quadratic forms I describe the transcendental lattices of the surfaces described in [Sa4] and I classify them.
I still work on these families of surfaces, in fact I'm looking for projective models of them. This is the topic of the work in progress [Sa6].
5) Symplectic automorphisms on K3 surfaces. The symplectic automorphisms of order

two are called *Nikulin involutions*. In the article [GS] I study them together with Bert van Geemen (University of Milan).
By a paper of Nikulin they induce a unique (up to isometry) action on $H^2(X,\mathbb{Z})$, this means that the operation is independent from the choice of the K3 surface. We study the Picard lattice and the transcendental lattice, in particular we show that if a K3 surface X has a Nikulin involution then the Picard number is $\rho \geq 9$ and the Picard lattice contain a copy of the lattice $E_8(-2)$. In the case of $\rho = 9$ with the help of lattice theory we describe completely the structure of the Picard lattice. We discuss also many concrete examples, which explains the general results, in particular double covers of the plane, quartics in $\mathbb{P}_3$, complete intersections and in particular elliptic fibrations.
In the paper [GaSa1] together with Alice Garbagnati (University of Milan) I study symplectic automorphisms of order $3, 5, 7$. Together with the order two these are all possible prime orders for such automorphisms (at least in characteristic zero). By using lattice theory and elliptic fibrations we identify completely the action on $H^2(X,\mathbb{Z})$. In the case of the order three automorphism we show that the orthogonal complement to the invariant sublattice of $H^2(X,\mathbb{Z})$ is the well known rank twelve Coxeter-Todd lattice.
I still work on similar questions in the paper in preparation [GaSa3] with Alice Garbagnati and in the case of non-symplectic automorphisms of order three in the work in progress [AS] with Michela Artebani (University of Milan).
6) Even sets of eight disjoint rational curves. In the submitted preprint [GaSa2] together with Alice Garbagnati, we study K3 surfaces with an even set of eight disjoint (-2)-rational curves or with an even set of eight nodes. Such K3 surfaces are minimal resolution and quotient of a K3 surface by a Nikulin involution, their study complete the results of the paper [GS] in the following meaning: we consider a K3 surface with a Nikulin involution, then it is natural to ask what is the quotient surface and which properties has, for example which kind of singularities. In [GaSa2] we study K3 surfaces with an even set of rational curves and with the smallest possible Picard number, which is nine. We classify the surfaces and we describe their moduli space. In particular we describe many projective models, which continue and complete the study started by Barth in [B2] of such surfaces.
7) The Mckay correspondence in dimension two and three. In the paper [BoSa1] together with Samuel Boissière (University of Nizza) we give a relation between the Mckay correspondence in dimension two and in dimension three. Let G be a finite subgroup of $SO(3,\mathbb{R})$ and let $\tilde{G} \subset SU(2)$ be the binary group associated to the group G. The group $\tilde{G}$ operates on $\mathbb{C}^2$ and the quotient is an ADE-surface singularity. Its resolution consists of smooth (-2)-rational curves with an ADE-Dynkin diagram as dual graph. The Mckay correspondence associates to the vertices of the graph the irreducible representations ($\neq 1$) of $\tilde{G}$. The resolutions of the quotients $\mathbb{C}^2/\tilde{G}$ and $\mathbb{C}^3/G$ are Hilbert-Nakamura-Schemes and the exceptional curves of the resolutions on the origin have very similar properties. We show that there exists a morphism between these two resolutions, which contracts some curves in the exceptional fiber. For the proof we use the Mckay correspondence in dimension two and three, and also the theory of Hilbert-Nakamura-Schemes. The study of these resolutions is related to the study of the resolutions of the singularities of the four special K3 surfaces of the families X^d_λ/G_d, which are fibrations of the singular space $\mathbb{P}_3/G_d$ (cf. **3**).

4. Short description of the other scientific works

1. In the paper [Sa1] from my PhD thesis I work on the question: which is the maximal number of nodes a surface of degree d in $\mathbb{P}_3$ can have. For the degree $d \leq 6$ this problem

is solved by results of Cayley, Kummer, Beauville and Barth. For $d \geq 7$ the exact number is unknown. There are bound of Varchenko and Miyaoka. In this paper I find a surface of degree 12 with 600 nodes, which improves the previous lower bound of 576 nodes of Kreiß for a surface in this degree.
2. In the paper [ES] together with Philippe Ellia (University of Ferrara) I prove the Hartshorne conjecture for codimension two subvarieties in the case of 2-arithmetic Buchsbaum varieties. The exactly formulation of the Hartshorne conjecture for varieties of codimension two is the following: each smooth variety of codimension two in $\mathbb{P}_n$, $n \geq 7$ is complete intersection.

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