LINEAR SYSTEMS ON IHS

/ $\mathbb{C}$, variety = red. irred. scheme

Linear systems: X smooth, $D \in \mathrm{Div}(X)$

$|D| := \{ D' \in \mathrm{Div}(X) \mid D' \underset{\text{lin eq}}{\sim} D,\ D' \geq 0 \} \cong \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee)$

└ complete linear system assoc. to D

$H^0(X, \mathcal{O}_X(D)) = \langle s_0, \dots, s_N \rangle \qquad s_i : X \xrightarrow{\text{hol}} \mathbb{C}$

$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^N, \quad x \longmapsto (s_0(x) : \dots : s_N(x))$

DEF: • the base locus of D

$\mathrm{Bs}|D| := \{ x \in X \mid s_i(x) = 0 \quad \forall i = 0, \dots, N \}$

• D is basepoint free if $\mathrm{Bs}|D| = \emptyset$, i.e., $\varphi_{|D|}$ is a morphism.

IHS : Irreducible Holomorphic Symplectic

DEF: X is a proj IHS variety if it is a smooth variety st

- X is simply connected
- $H^0(X, \Omega_X^2) \cong \mathbb{C} \cdot \omega_X$, ω_X nowhere vanishing hol. 2-form

symplectic form

Rmk:
- $\dim X$ even
- $K_X \sim \mathcal{O}_X$
- $H^1(X, \mathcal{O}_X) = 0 \implies \mathrm{Pic}(X) \cong NS(X) \hookrightarrow H^2(X, \mathbb{Z})$

Néron-Severi group

Thm: (Beauville-Bogomolov)
X smooth, $c_1(X) = 0$. Then $\exists$ finite étale cover $\tilde{X} \to X$ st
$\tilde{X} = A \times \prod_i X_i \times \prod_j Y_j$

A abelian var
X_i IHS
Y_j Calabi-Yau

Ex: • K3 surfaces dim = 2

$S_2 \longrightarrow \mathbb{P}^2$ double cover of $\mathbb{P}^2$ ramified over a smooth sextic curve

$S_4 \subseteq \mathbb{P}^3$ smooth quartic surface of $\mathbb{P}^3$

• S K3 $\rightsquigarrow S^{[n]} = \mathrm{Hilb}^n(S)$

scheme parametrizing closed subschemes of length n, dim 0 on S

e.g. $n=2$, $S^{[2]}$ = Hilbert square of S

points on $S^{[2]}$:

- $p+q$ $p, q \in S$, $p \neq q$
- (p, t) $p \in S$, $t \in \mathbb{P}^1$ tangent direction through p

Thm: (Beauville-Bogomolov-Fujiki)
X IHS $\Rightarrow$ There exists a quadratic form on $H^2(X,\mathbb{Z})$:
$$q_X : H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z} \qquad \text{BBF-form}$$
and $(H^2(X,\mathbb{Z}), q_X)$ is a lattice.

Ex: if X K3 $\Rightarrow q_X \equiv$ intersection form

<u>Linear systems on IHS</u>

Thm: (Saint-Donat)
X K3 surface st $\exists D$ ample, $D^2=2$
$\Rightarrow \varphi_{|D|} : X \longrightarrow \mathbb{P}^2$ is the double cover of $\mathbb{P}^2$ over a smooth sextic curve.

$\rightsquigarrow Bs|D| = \emptyset$, $\varphi_{|D|}$ is described

<u>Now</u> : $X = S_{2t}^{[2]}$, S_{2t} <u>generic</u> K3 surface, i.e,
$\mathrm{Pic}(S_{2t}) = \mathbb{Z}h$, $h^2 = 2t$
$t \in \mathbb{Z}_{>0}$

$\mathrm{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$ ($2\delta = [E]$
$\begin{pmatrix} 2t & 0 \\ 0 & -2 \end{pmatrix}$ E exceptional divisor presentation Pietro)

<u>Suppose</u> $\exists$ D ample on X
$q_X(D) = 2$

e.g., $t = 2, 10, 13, 17, \ldots$

<u>Qst</u> : • $\mathrm{Bs}|D| = ?$
• Describe $\varphi_{|D|}$

Thm: [BCNS]

$X = S_{2t}^{[2]}$, S_{2t} generic K3, $t \geq 2$.

Then $\mathrm{Aut}(X) \neq \{id\}$ iff $\exists$ D ample, $q_X(D) = 2$.

In this case, $\mathrm{Aut}(X) = \langle \iota \rangle$, ι is an anti-symplectic involution, i.e., $\iota^* \omega_X = -\omega_X$.

Moreover, $\iota^* D \cong D \in \mathrm{Pic}(S_{2t}^{[2]})$

$\boxed{t=2}$ $S_4 \subseteq \mathbb{P}^3$ smooth quartic surface

Generic $S_4 \Rightarrow S_4$ does <u>not</u> contain any line

$D = h - \delta$ ample, $q_X(D) = 2$

$$\varphi_{|D|} \equiv f : S_4^{[2]} \longrightarrow \underbrace{G(1, \mathbb{P}^3)}_{\text{quadric}} \subseteq \mathbb{P}^5$$

$$p + q \longmapsto \text{line}(p, q)$$

S_4 does not contain lines $\Rightarrow \begin{cases} f \text{ finite} \\ \deg f = 6 \end{cases}$

$\text{line}(p,q) \cap S_4 = \{p, q, r, s\}$

$\varphi_{|D|}$ is a morphism $(\varphi_{|D|} \equiv f) \Rightarrow Bs|D| = \emptyset$

$$\iota : S_4^{[2]} \xrightarrow{\ \sim\ } S_4^{[2]} \qquad \text{(it is regular since } S_4 \text{ does not contain lines)}$$
$$p+q \longmapsto r+s$$

Rmk: $$\begin{array}{ccc} \mathrm{line}(p,q) & = & \mathrm{line}(r,s) \\ \| & & \| \\ \varphi_{|D|}(p+q) & = & \varphi_{|D|}(\iota(p+q)) \end{array}$$

$\Rightarrow \varphi_{|D|}$ factorises through $\pi : S_4^{[2]} \longrightarrow S_4^{[2]}/\langle \iota \rangle$

eq:
$$\begin{tikzcd} S_4^{[2]} \arrow[rr, "\varphi_{|D|}"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "/\!/"] & \mathbb{G}(1,\mathbb{P}^3) \subseteq \mathbb{P}^5 \\ & S_4^{[2]}/\langle \iota \rangle \arrow[ur, hook] & \end{tikzcd}$$

commutes!

Following an idea of O'Grady

Thm: $X = S_{2t}^{[2]}$, S_{2t} generic, $t \neq 2$, st $\exists\, D$ ample, $q_X(D) = 2$.

Then

- $$\begin{tikzcd} \varphi_{|D|}: S_{2t}^{[2]} \arrow[rr, dashed] \arrow[dr] & \arrow[d, phantom, "\parallel"] & Y \subseteq \mathbb{P}^5 \\ & X/\langle\iota\rangle \arrow[ur, dashed] & \end{tikzcd}$$ Commutative!

- $Bs|D| = \emptyset$

- $\varphi_{|D|}: S_{2t}^{[2]} \longrightarrow Y \subseteq \mathbb{P}^5$ is either st
  - □ $\deg \varphi_{|D|} = 2$, $\deg Y = 6$ or
  - □ $\deg \varphi_{|D|} = 4$, $\deg Y = 3$

Tools: • commutativity :

Beauville : $F := \mathrm{Fix}(\iota)$ is a .

Lagrangian surface

Compute $cl(F) \in H^4(X, \mathbb{Z})$

By contradiction: if the diagram is not commutative $\Rightarrow$ $cl(F) \notin H^4(X, \mathbb{Q})$

impossible

• $D_1, D_2 \in |D|$ distinct divisors $\Rightarrow D_1 \cap D_2$ red. irred surface

(not true for $t=2$!

$D = f^*\mathcal{O}_G(1)$, $\mathcal{O}_G(1) := \mathcal{O}_{\mathbb{P}^5}(1)|_{G(1,\mathbb{P}^3)}$

$D^2 = (f^*\mathcal{O}_G(1))^2 = f^*\sigma_{1,1} + f^*\sigma_2$. $f^*\sigma_{1,1}, f^*\sigma_2$ effective

where $H^4(G(1,\mathbb{P}^3), \mathbb{Z}) = \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2$)

Thank you!