Divisorial contractions to codimension three orbits

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I will present some recent results obtained in collaboration with Enrica Floris on the description of high dimensional equivariant divisorial contractions with nice geometric properties.

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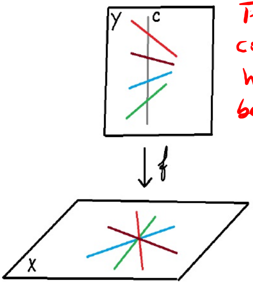
Overview

Introduction

At the beginning of today's story, there is...

Contraction theorem of Guido Castelnuovo (1893)



The curve C is contracted..
Which curves can be contracted?

From X, it is a blow-up
From Y, it is the contraction of a curve

Introduction

At the beginning of today's story, there is...

Contraction theorem of Guido Castelnuovo (1893)

Y : smooth projective complex surface.

$\cup$

C : smooth rational curve with $C^2 = -1$.

There is a smooth projective surface X and a morphism $f : Y \longrightarrow X$ contracting C to a point p, such that $f : Y \setminus C \longrightarrow X \setminus \{p\}$ is an isomorphism.

By adjunction, $\underbrace{2g(C)}_{=0} - 2 = C \cdot (C + K_Y)$ gives:

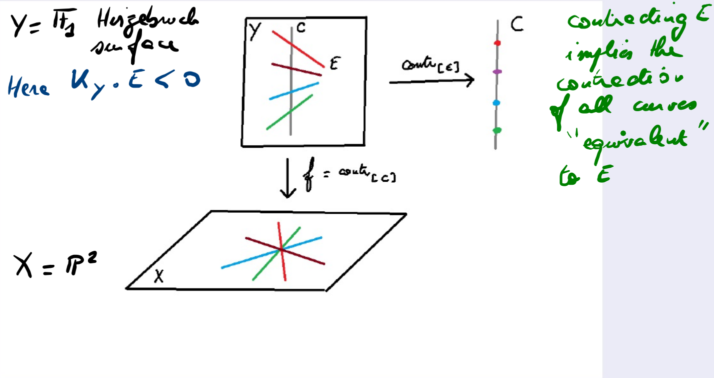
$K_Y \cdot C = -1 < 0$

This is the key to a generalization!

Introduction

...and nowadays we use:

Cone and contraction theorem of Kollar–Mori (1998)



Introduction

...and nowadays we use:

Cone and contraction theorem of Kollar–Mori (1998)

Y: normal, locally $\mathbb{Q}$-factorial projective variety with at worst terminal singularities.

→ normal: all local rings are integrally closed

→ locally $\mathbb{Q}$-factorial: every Weil divisor on Y has a Cartier multiple

→ terminal singularities: for every resolution of singularities $\mu: \tilde{Y} \to Y$, one has:

$$K_{\tilde{Y}} = \mu^* K_Y + \sum a_i E_i$$

prime exceptional divisors

with $a_i > 0 \quad \forall i$

Introduction

...and nowadays we use:

Cone and contraction theorem of Kollar–Mori (1998)

Let Y be a normal, locally $\mathbb{Q}$-factorial projective variety with at worst terminal singularities.

$N_1(Y) = \{1\text{-cycles}\}/\text{numerical equivalence}$

$\overset{\cup}{NE}(Y)$: the cone generated by irreducible curves.

If $R \subset \overline{NE(Y)}$ is a K_Y-negative extremal ray, there exists a morphism $Y \longrightarrow X$, called an extremal contraction, contracting exactly those curves whose class belongs to R. There are 3 possibilities for f:

i) $\dim X < \dim Y$: Mori fiber space

ii) $\dim X = \dim Y$ and $\mathrm{Exc}(f)$ has codimension 2 : small contraction

iii) $\dim X = \dim Y$ and $\mathrm{Exc}(f)$ has codimension 1 : divisorial contraction

Introduction

Let us focus on **divisorial contractions**.

Definition of a divisorial contraction

A morphism $f: Y \to X$ with connected fibers, between normal projective varieties Y and X is called a *divisorial contraction* if it satisfies all the following conditions:

1. Y is locally $\mathbb{Q}$-factorial with terminal singularities;
2. the morphism f is birational and its exceptional locus E is a prime divisor;
3. the canonical divisor K_Y is f-antiample.

f contracts K_Y - negative curves

Introduction

The challenging project

classify all divisorial contractions $f : Y \longrightarrow X$;

to X

from Y

two different approaches

Weighted blow-ups

Definition of a weighted blow-up.

X : d-dimensional variety, $Z \subset X$ smooth, Z smooth of codimension n.
$\omega = (\omega_1, \dots, \omega_n) \in (\mathbb{N}^*)^n$ weights.

Locally analytically on X, Z is given by the vanishing of n coordinate functions $x_1 = \dots = x_n = 0$.

The <u>weighted blow-up</u> of Z is defined as the closure of the graph of the morphism

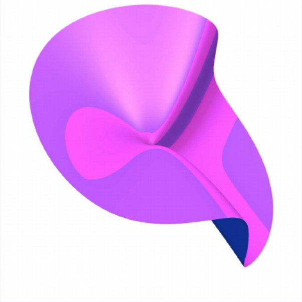
$$\begin{aligned} X \setminus Z &\longrightarrow \mathbb{P}(\omega) \\ (x_1, \dots, x_n) &\longmapsto (x_1 : \dots : x_n) \end{aligned}$$

where $\mathbb{P}(\omega) = \mathbb{C}^n \setminus \{0\} / \mathbb{C}^*$ with $\lambda \cdot (x_1, \dots, x_n) = (\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_n} x_n)$.

We denote by $Bl_Z^\omega X \longrightarrow X$ the weighted blow-up.

Weighted blow-ups

Example due to Miles Reid: resolve the E_8 surface singularity?



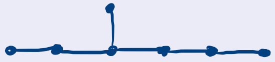
$$x^2 + y^3 + z^5 = 0$$

Weighted blow-ups

Example due to Miles Reid: resolve the E_8 surface singularity?

Classical successive blow-ups produce the singularities:

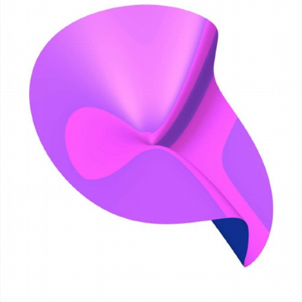
$$\begin{tikzcd}
 & & & A_1 & \\
E_8 \arrow[r] & E_7 \arrow[r] & D_6 \arrow[ur] \arrow[dr] & & A_1 \\
 & & & D_4 \arrow[ur] \arrow[r] \arrow[dr] & A_1 \\
 & & & & A_1
\end{tikzcd}$$



The way the exceptional curves organize is quite complicated and produces the E_8 configuration.

Weighted blow-ups

Example due to Miles Reid: resolve the E_8 surface singularity?



$$x^2 + y^3 + z^5 = 0$$

The equation is weighted homogeneous for the weight $\omega = (15, 10, 6)$, its degree is 30.

We consider the weighted blow-up $Bl_0^\omega X \subset \mathbb{C}^3 \times \mathbb{P}(\omega)$

Weighted blow-ups

Example due to Miles Reid: resolve the E_8 surface singularity?

$\mathbb{C}^3 \setminus \{0\} \longrightarrow \mathbb{P}(15,10,6) \cong \mathbb{P}^2$ ⚠ trick

$(x,y,z) \longmapsto (x:y:z) \longmapsto (x^2:y^3:z^5)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \rotatebox{90}{$=$}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u:v:w)$

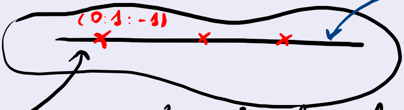
So $Bl_0^w X \subset \mathbb{C}^3 \times \mathbb{P}^2$ has equations:

$vx^2 = uy^3, \quad wx^2 = uy^5, \quad wy^3 = vz^5$

and $u+v+w = 0$.

singular line so $Bl_0^w X$ is nonnormal

$Bl_0^w X$:



3 singular points

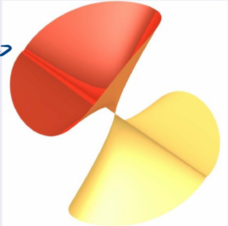
On the chart $v=1$, $x^2 = uy^3$ shows that x/y is not integral over the quotient field, but $(x/y)^2 = \dfrac{uy^3}{y^2} = uy$.

Weighted blow-ups

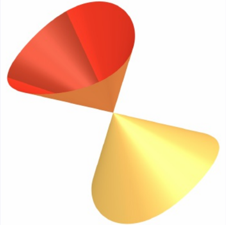
Example due to Miles Reid: resolve the E_8 surface singularity?

Locally around the point $(0 : 1 : -1)$, the situation looks like:

singular line



$\longleftarrow$



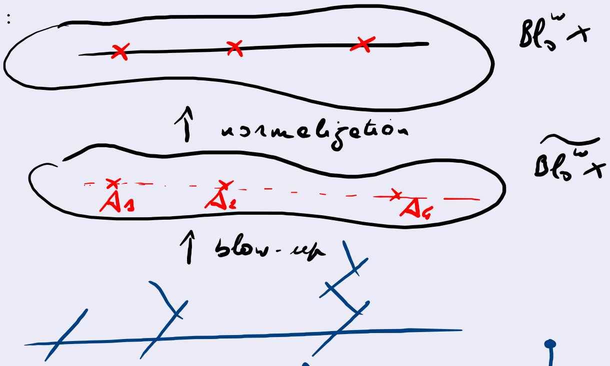
$$\frac{\mathbb{C}[x,y,u]}{uy^3 - x^2} \xleftarrow[t = x/y]{\text{normalization}} \frac{\mathbb{C}[x,y,u,t]}{uy^3 - x^2,\ ty - x} \simeq \frac{\mathbb{C}[x,y,t]}{uy - t^2}$$

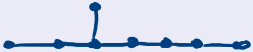
A_1-singularity

Weighted blow-ups

Example due to Miles Reid: resolve the E_8 surface singularity?

We observe :





we get very easily the E_8- configuration after a weighted blow-up.

Some classical results in dimension three

Consider a 3-dimensional divisorial contraction *to a point*:

$$f\colon Y \to X, \quad f(E) = p$$

where E is the exceptional divisor.

Y	p	f	
smooth	smooth	blow-up	Mori (1982)
smooth	A_1	blow-up	Mori (1982)
smooth	A_2	blow-up	Mori (1982)
smooth	$0 \in \mathbb{C}^3/\pm 1$	blow-up	Mori (1982)
-	A_1	blow-up	Corti (2000)
-	$0 \in \mathbb{C}^3/\frac{1}{r}(1, a, -a)$	weighted blow-up	Kawamata (1996)

no assumption

singular center
for the contraction

Some classical results in dimension three

Consider a 3-dimensional divisorial contraction *to a point*:

$$f\colon Y \to X, \quad f(E) = p$$

where E is the exceptional divisor.

" *While it seems that singularities on Y*
make it hard to tackle the problem,
the singularity of p may be useful
because it restricts the way
to take natural local description
of X at p. "
(Kawakita)

Some classical results in dimension three

Consider a 3-dimensional divisorial contraction *to a point*:

$$f\colon Y \to X, \quad f(E) = p$$

where E is the exceptional divisor. Assume that p is a smooth point of X.

Theorem of Mazayuki Kawakita (2001)

The divisorial contraction f is a weighted blow-up with weights $(n, m, 1)$ where n, m are characterized by the valuation on the local ring $\mathcal{O}_{Y,E}$.

$\mathcal{O}_{Y,E}$ is a discrete valuation ring

Fact : two divisorial contractions over X whose exceptional divisor define the same valuation are isomorphic : this is the starting idea of Kawakita's proof.

Generalization

Consider a d-dimensional divisorial contraction $f: Y \to X$ with $\dim Y = \dim X \geq 3$, which contracts its exceptional divisor to its center, which is assumed to be a smooth subvariety contained in the smooth locus of X. Assume that $\operatorname{codim} Z = d-3$.

e.g.: if $d=3$, Z is a smooth point (Kawakita's case)
if $d=4$, Z is a smooth curve

Bad news: if $d \geq 4$, in general f is NOT a weighted blow-up!
(we have counter-examples)

Generalization

Definition

Let G be a connected algebraic group. A divisorial contraction $f\colon Y \to X$ is called G-**equivariant** if it satisfies the following conditions:

1. X and Y are endowed with a regular action of G;
2. the contraction f is G-equivariant.

Theorem (S. B. and Enrica Floris - 2020

Every d-dimensional G-equivariant divisorial contraction to a G-simply connected G-orbit of dimension $d-3$ contained in the smooth locus is a G-equivariant weighted blow-up.

↑
means that $\underset{\circlearrowleft G}{\mathrm{Bl}_0^{w} X} \longrightarrow \underset{\circlearrowleft G}{X}$ G-equivariant

Generalization

Theorem (S. B. and Enrica Floris - 2020)

Every d-dimensional G-equivariant divisorial contraction to a G-simply connected G-orbit of dimension $d-3$ contained in the smooth locus is a G-equivariant weighted blow-up.

- G-orbit implies that Z is smooth and that all finite G-equivariant morphisms are étale
- G-simply connected means $\pi_1^G(Z) = \{1\}$ (equivariant fundamental group)

  this means that any connected, étale, G-equivariant morphism $Z' \to Z$ is an isomorphism.
- If G is a semi-simple linear group, then necessarily Z is a Fano variety and both assumptions are automatically satisfied.

Elements of proof

Kawakita's construction made G-equivariant, in dimension d

The tower construction

Stop when $Z_n = E_n$, i.e. when Z_n is a prime divisor

$$\begin{tikzcd}
W \arrow[r, "f'"] \arrow[ddd, twoheadrightarrow, "h'"'] & X_n \arrow[r, "b_n"] \arrow[d, "g_n"] \arrow[dd, bend right, "h_1"'] \arrow[ddd, bend right=60, "h"'] \arrow[dddl, dashed, "g"'] & \mathrm{Bl}_{Z_{n-1}} X_{n-1} \arrow[dl, "b'_n"] \\
 & X_{n-1} \arrow[r, "b_{n-1}"] \arrow[d, dotted] & \mathrm{Bl}_{Z_{n-2}} X_{n-2} \arrow[dl, dotted] \\
 & X_1 \arrow[r, "b_1"] \arrow[d, "g_1"] & \mathrm{Bl}_{Z_0} X_0 \arrow[dl, "b'_1"] \\
Y \arrow[r, "f"] & X = X_0 &
\end{tikzcd}$$

$E_n \cup Z_n$

$E_1 \cup Z_1^c$

E

$\check{Z} = Z_0$

Elements of proof

Why does the process stop?

Because the discrepancy of E_i in X increases at each step.

We have $K_Y \sim_{\mathbb{Q}} f^* K_X + a E$, $a = a(E, X)$ the discrepancy of E,

and $2 = a(E_1, X) < a(E_2, X) < \dots < a(E_n, X) = a(E, X) = a$

so there is at worst $a - 1$ steps.

What are the weights? $(n, m, 1)$

$n \leq a - 1$ is the number of floors of the tower.

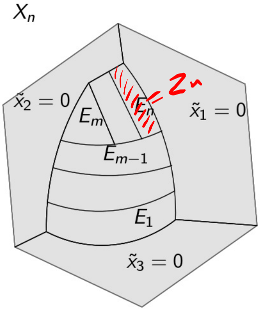
$$\dim Z = \dim Z_0 = \dim Z_1 = \dots = \dim Z_{m-1} < \dim Z_m \leq \dots \leq \underset{\substack{\shortparallel \\ d-1}}{\dim Z_n}$$

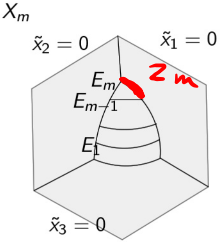
m = number of steps before the dimension of the center increases.

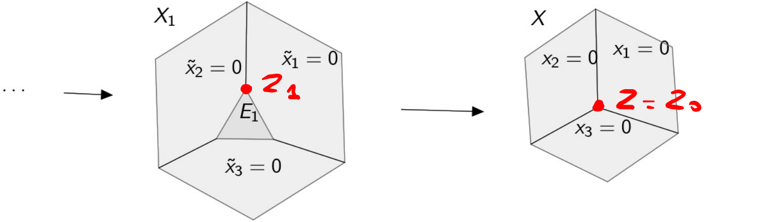
Then $Y \longrightarrow X$ is a G-equivariant blow-up of weights $(n, m, 1)$.

Elements of proof

The tower construction



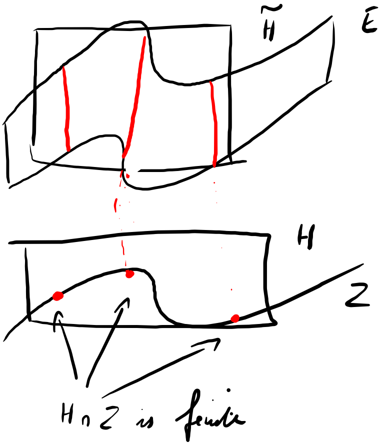




Elements of proof

Induction process

Let us explain it in the case $\dim Y = \dim X = 4$, Z is a curve.
Take a hyperplane section $H \subset X$ and $\tilde{H} = f^{-1}(H)$.



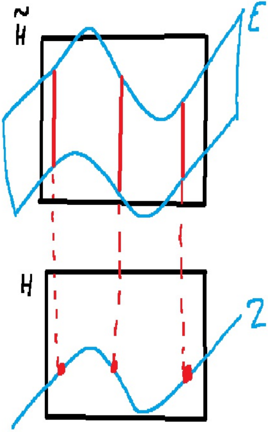
$$\begin{array}{ccc} f: Y & \longrightarrow & X \\ \cup & & \\ E & \longrightarrow & Z \end{array}$$

$f|_{\tilde{H}} : \tilde{H} \longrightarrow H$ is birational in dimension 3,

$\mathrm{Exc}(f|_{\tilde{H}}) = E \cap \tilde{H}$

$f|_{\tilde{H}}(E \cap \tilde{H}) = Z \cap H$ is finite

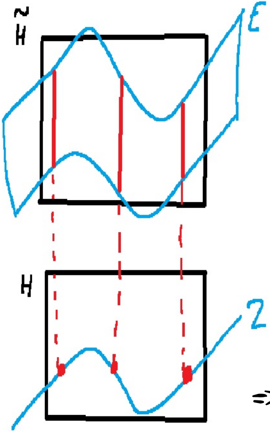
Elements of proof



Some technical points

- We prove that $f|_{\tilde{H}}$ is again a divisorial contraction (we run an ad hoc MMP)
- at each point Z, we have a weighted blow-up over it, by Kawakita's result. Since Z is a G-orbit, the weights do not change, so we can prove that the tower is isomorphic to a weighted blow-up of weights $(n, m, 1)$.

Elements of proof



Some technical points

In $f|_{\tilde H} : E \cap \tilde H \longrightarrow Z \cap H$, we need to ensure that the general fiber is irreducible. For this, we consider the Stein factorisation

$$\begin{tikzcd} \tilde E \arrow[r, "\nu"] \arrow[dr, "\nu'"'] & E \arrow[r, "f|_E"] & Z \\ & Z' \arrow[ur, "\eta"'] & \end{tikzcd}$$

ν : normalization
ν' : connected fibers
η : finite surjective

all G-equivariant.

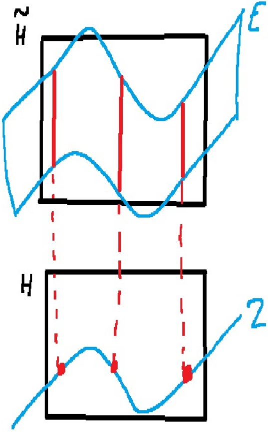
$\Rightarrow \eta$ is étale $\Longrightarrow \eta$ isomorphism

assumption on Z

$\Rightarrow f|_E \circ \nu$ has connected fibers.

$\tilde E$ is normal $\Rightarrow$ by Seidenberg theorem, the generic fiber of ν' is normal, hence irreducible since it is connected.

Elements of proof



Some technical points

$f|_{\tilde{H}} : \tilde{H} \longrightarrow H$

why is $\tilde{H}$ locally $\mathbb{Q}$-factorial ?

a hyperplane section of a locally $\mathbb{Q}$-factorial variety does not necessarily behave this property

We use a work of Ravindra - Srinivas (thanks to Janos Kollar):

the point is: if we choose $H \subset X$ general enough, then $\tilde{H}$ is locally $\mathbb{Q}$-factorial.

Thanks to the organizers for this event!

¡Gracias!