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K3 surfaces associated to a double EPW sextic

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$K3$ surfaces

Definition

A $K3$ surface is a complex, compact smooth surface

- *which is simply connected*
- *whose canonical bundle is trivial.*

$K3$ surfaces appear in the Enriques-Kodaira classification of complex algebraic surfaces as surfaces of Kodaira dimension 0, together with Enriques surfaces, complex tori and bielliptic surfaces.

Definition

A $\langle 2t \rangle$-polarized $K3$ surface is a pair (S, H) where S is a $K3$ surface and H is a primitive ample divisor with $H^2 = 2t$.

Some examples

Proposition

For every $t \geq 1$, the ample divisor H induces $\phi_{|H|} : S \to \mathbb{P}^{t+1}$. For $t \geq 2$ and (S, H) general, the morphism $\phi_{|H|}$ is a closed embedding.

If (S, H) is general, then

$t = 2$: $\phi_{|H|}(S) \subset \mathbb{P}^3$ is a quartic hypersurface. Every smooth quartic hypersurface in $\mathbb{P}^3$ is a $K3$ surface.

$t = 3$: $\phi_{|H|}(S) \subset \mathbb{P}^4$ is the complete intersection of a quadric and a cubic. Every such intersection is a $K3$ surface.

$t = 4$: $\phi_{|H|}(S) \subset \mathbb{P}^5$ is the complete intersection of three quadrics . Every such intersection is a $K3$ surface.

Remark

These are all the $K3$ surfaces that can be obtained as complete intersections inside a projective space!

The case $t = 5$

For $H^2 = 10$ we have $\phi_{|H|} : S \to \mathbb{P}^6$. Take H very ample and consider $S \subset \mathbb{P}^6$.

Theorem (Saint-Donat, 1974)

If S is cut out by quadrics, we call $V_6 = H^0(\mathbb{P}^6, \mathcal{I}_S(2))$ the 6-dimensional space of symmetric bilinear forms associated to quadrics containing S. Then there exists

- *a hyperplane $V_5 \subset V_6$*
- *an embedding $\mathbb{P}^6 \hookrightarrow \mathbb{P}(\bigwedge^2 V_5)$*

such that, via the embedding,

$$S = \mathbb{P}^6 \cap G(2, V_5) \cap Q(x),$$

where $Q(x)$ is the projective quadric hypersurface of $\mathbb{P}^6$ associated to any $x \in V_6 - V_5$.

Gushel-Mukai varieties

Definition

In this situation, we say that the embedded $K3$ surface $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ is a (smooth) ordinary Gushel-Mukai variety of dimension 2.

We say that $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ is *strongly smooth* if the Fano variety $M_S = \mathbb{P}^6 \cap G(2, V_5)$ is smooth.

Definition

We call $\mathrm{Aut}(S, \mathbb{P}^6)$ the group of automorphisms of $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ induced by restriction of linear automorphisms of $\mathbb{P}^6$.

Lemma (B.)

If S is strongly smooth, the group $\mathrm{Aut}(S, \mathbb{P}^6)$ is a finite subgroup of $PGL(2, \mathbb{C})$. In particular it appears in the following list

$$\mathbb{Z}/n\mathbb{Z} \text{ for } 1 \leq n \leq 66, \qquad D_n \text{ for } 2 \leq n \leq 66, \qquad A_4, \qquad \mathfrak{S}_4 \qquad A_5.$$

Hyperkähler manifolds

Definition

Let X be a complex smooth variety which is compact and Kähler. We say that X is a hyperkähler manifold if

- *it is simply connected,*
- $H^0(X, \Omega^2_X) = \mathbb{C} \cdot \varphi$ *i.e. there is, up to constant, one and only one holomorphic 2-form on X,*
- *the rank of φ is maximal on every point of X.*

Hyperkähler manifolds are a natural generalization of $K3$ surfaces.

Remark

In dimension 2, hyperkähler manifolds coincide with $K3$ surfaces.

A first example of hyperkähler 4-fold: Hilbert squares on $K3$ surfaces

Let S be a projective $K3$ surface. We call $S^{[2]}$ the Hilbert square on S i.e. the scheme parametrizing 0-dimensional subschemes $(Z, \mathcal{O}_Z)$ of S whose length is two.

- $S^{[2]}$ can be thought of as the blow up of $S^{(2)} =: S^2/\langle \mathfrak{S}_2 \rangle$ along the diagonal $\Delta = \{2p \mid p \in S\}$.
- outside the exceptional divisor, a point of $S^{[2]}$ is described by a non-ordered pair $p + q$ for $p, q \in S$, $p \neq q$.

Theorem (Fujiki, 1983)

The scheme $S^{[2]}$ is a hyperkähler 4-fold.

Let (S, H) be a polarized $K3$ surface.

- The ample divisor $H \in \mathrm{Pic}(S)$ induces a *non-ample* divisor $H_2 \in \mathrm{Pic}(S^{[2]})$.
- We call $E \in \mathrm{Pic}(S^{[2]})$ the class of the exceptional divisor of the blowing-up $S^{[2]} \to S^{(2)}$. ($E = 2\delta$ for a primitive class $\delta \in \mathrm{Pic}(S^{[2]})$).

The definition of EPW sextics

A volume form on $V_6 \cong \mathbb{C}^6 \rightsquigarrow$ a bilinear form $\omega : \left(\bigwedge^3 V_6\right) \times \left(\bigwedge^3 V_6\right) \to \mathbb{C}$.

- Let $A \subset \bigwedge^3 V_6$ a Lagrangian subspace i.e. $A \cong \mathbb{C}^{10}$ such that $\omega_{|_{A\times A}} = 0$.

Remark

We always suppose $\mathbb{P}(A) \cap G(3, V_6) = \emptyset$ i.e. A does not contain any vector in the form $v_1 \wedge v_2 \wedge v_3$ with $v_i \in V_6$, $i = 1, 2, 3$.

We define

$$Y_A = \{[v] \in \mathbb{P}(V_6) \text{ such that } A \cap (v \wedge \bigwedge^2 V_6) \neq 0\}$$

Proposition (O'Grady, 2006)

The variety $Y_A \subset \mathbb{P}(V_6)$ is a singular and normal sextic hypersurface. We call it EPW sextic.

$\mathrm{Sing}(Y_A)$ is an irreducible surface.

Dual EPW sextics

Let A be a Lagrangian subspace. The orthogonal complement $A^{\perp} \subset \left(\bigwedge^3 V_6\right)^{\vee}$ is again a Lagrangian subspace.

Definition

The EPW sextic $Y_{A^{\perp}} \subset \mathbb{P}(V_6^{\vee})$ is called dual EPW sextic.

The dual EPW sextic $Y_{A^{\perp}}$ is the projective dual of Y_A.

A second example of hyperkähler 4-fold: double EPW sextics

Theorem (O'Grady 2006)

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace and Y_A the corresponding EPW sextic. There exists a double cover $f_A : X_A \to Y_A$ ramified over the surface $\mathrm{Sing}(Y_A)$. The variety X_A is called double EPW sextic.

Theorem (O'Grady 2006)

The set of Lagrangian subspaces $A \subset \bigwedge^3 V_6$ such that X_A is smooth is open inside the space parametrizing Lagrangian subspaces inside $\bigwedge^3 V_6$.
In this case, the double cover X_A is a hyperkähler 4-fold (deformation equivalent to a Hilbert square on a $K3$ surface).

Double EPW sextics, with a natural polarization induced by the cover structure, are a locally complete family of polarized hyperkähler 4-folds.

The point of the situation

Let $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ be a *strongly smooth* ordinary Gushel-Mukai variety of dimension 2 (general $\langle 10 \rangle$-polarized $K3$ surface).

We consider the Hilbert square $S^{[2]}$, with the linear system $|H_2 - E|$, where

- H_2 is induced by the divisor $H \in \mathrm{Pic}(S)$ given by the embedding $S \hookrightarrow \mathbb{P}^6$,
- E is the class of the exceptional divisor of the blow-up $S^{[2]} \to S^{(2)}$.

Proposition (O'Grady, 2010)

$H^0(S^{[2]}, H_2 - E) \cong V_6$, the 6-dimensional space of symmetric bilinear forms associated to quadrics containing S.
The morphism $\phi_{|H_2 - E|}$ is everywhere defined if S does not contain lines.

So, under these conditions, we have a morphism $\phi_{|H_2 - E|} : S^{[2]} \to \mathbb{P}(V_6^\vee)$.

Once we fixed a volume form on V_6, there is a family of (dual) EPW sextics in the form $Y_{A^\perp} \subseteq \mathbb{P}(V_6^\vee)$ for $A \subset \bigwedge^3 V_6$ Lagrangian subspace.

The dual EPW sextic associated to a Gushel-Mukai variety of dimension 2

- We call $P_S \subset S^{[2]}$ the closure of the scheme parametrizing lines inside $M_S = \mathbb{P}^6 \cap G(2, V_5)$. We have $P_S \cong \mathbb{P}^2$ (Iskovskih, 1977).
- For any $C \subset S$ smooth conic, we have $C^{(2)} \subset S^{[2]}$ with $C^{(2)} \cong \mathbb{P}^2$.

Theorem (O'Grady, 2010)

1) There exists a Lagrangian subspace $A(S) \subset \bigwedge^3 V_6$ such that $\overline{\phi_{|H_2-E|}}(S^{[2]}) = Y_{A(S)^\perp} \subset \mathbb{P}(V_6^\vee)$.

2)Let $C_1, \ldots, C_k$ be the smooth conics contained in S. If S does not contain any line, then $\phi_{|H_2-E|}$ factorizes as

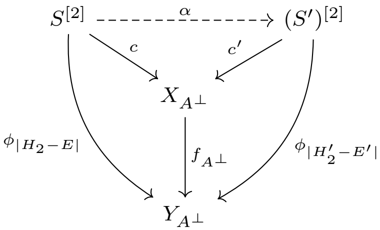
$$S^{[2]} \xrightarrow{c} X_{A(S)^\perp} \xrightarrow{f_{A(S)^\perp}} Y_{A(S)^\perp} \hookrightarrow \mathbb{P}(V_6^\vee),$$

where c is an isomorphism on the complement of $P_S \cup C_1^{(2)} \cup \ldots \cup C_N^{(2)}$.

3)The morphism c contracts each of $P_S, C_1^{(2)}, \ldots, C_k^{(2)}$ to a point, and $\mathrm{Sing}(X_{A(S)^\perp}) = \{c(P_S), c(C_1^{(2)}), \ldots, c(C_N^{(2)})\}$.

Gushel-Mukai varieties associated to the same EPW sextic

Two strongly smooth ordinary Gushel-Mukai varieties (S, H), (S', H') are associated to the same Lagrangian subspace, $A(S) = A(S') =: A$, if and only if there exists a birational map $\alpha : S^{[2]} \dashrightarrow (S')^{[2]}$ with $\alpha^*(H_2' - E') = H_2 - E$ i.e.



commutes, see O'Grady, 2010.

Corollary (B.)

For $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ generic between Gushel-Mukai varieties containing a conic, there is exactly one 2-dimensional Gushel-Mukai variety S' non isomorphic to S such that $A(S) = A(S')$.

Some questions

As S is strongly smooth, we have $\mathbb{P}(A(S)) \cap G(3, V_6) = \mathbb{P}(A(S)^\perp) \cap G(3, V_6^\vee) = \emptyset$ (Debarre, Kuznetsov, 2015).

Remark

$X_{A(S)^\perp}$ is never a hyperkähler manifold, since $c(P_S) \in \mathrm{Sing}(X_{A(S)^\perp})$. A priori, we do not know if the double EPW sextic $X_{A(S)}$ is.

First question

Can we give conditions on S to have $X_{A(S)}$ smooth, hence hyperkähler?

Second question

Can we get rid of the additional hypothesis "S does not contain lines"?

Some answers

Theorem (B.)

If $S = \mathbb{P}^6 \cap G(2, V_5) \cap Q$ *contains a line, then* $X_{A(S)}$ *is not a hyperkähler manifold. Moreover, if* S *contains neither a line nor an elliptic pencil of degree* 5*, then* $X_{A(S)}$ *is a hyperkähler manifold.*

A modular point of view

The conditions we found can be described as divisorial conditions on the moduli space $\mathcal{K}_{10}$ of $\langle 10\rangle$-polarized $K3$ surfaces.

We consider on $\mathcal{K}_{10}$ the divisor

$$\mathcal{D}_{x,y} = \{(S,H) \in \mathcal{K}_{10} \text{ such that there is a primitive sublattice}$$

$$\mathbb{Z}H + \mathbb{Z}D \subseteq \mathrm{Pic}(S) \text{ whose Gram matrix is } \begin{bmatrix} 10 & x \\ x & y \end{bmatrix}\}.$$

Corollary (B.)

Consider $(S,H) \in \mathcal{K}_{10}$. If $(S,H) \notin \mathcal{D}_{h,0}$ for $h = 1,\ldots,5$ and $(S,H) \notin \mathcal{D}_{1,-2}$, then $X_{A(S)}$ is a hyperkähler manifold.

Automorphisms

Definition

An automorphism σ on a hyperkähler manifold is said to be symplectic if it acts trivially on a 2-form on the manifold. Otherwise σ is non-symplectic.

Consider $(S, H) \in \mathcal{K}_{10}$ with associated Lagrangian subspace $A(S) \subset \bigwedge^3 V_6$.
We choose $(S, H) \in \mathcal{K}_{10}$ such that the double EPW sextic $X_{A(S)}$ is a hyperkähler manifold.

We are interested in the biregular automorphisms of $X_{A(S)}$.

Remark

There is an inclusion $\mathrm{Aut}(S, \mathbb{P}^6) \subset \mathrm{Aut}(Y_{A(S)})$, described by Debarre and Kuznetsov, 2015.

Lifting groups of automorphisms to double EPW sextics

We denote by $\mathrm{Aut}_{\iota_A}(X_A)$ the group of biregular automorphisms of X_A commuting with the covering involution ι_A. There is a short exact sequence

$$1 \to \{\mathrm{id}, \iota_A\} \to \mathrm{Aut}_{\iota_A}(X_A) \to \mathrm{Aut}(Y_A) \to 1,$$

see Debarre, Kuznetsov, 2015.

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with associated EPW sextic Y_A. Take A such that X_A is a hyperkähler manifold.

Proposition (Mongardi, 2013)

Consider $G \subseteq \mathrm{Aut}(Y_A)$. If G acts trivially on a section of K_{Y_A}, then G lifts to a symplectic action on X_A.

Proposition (B.)

Consider $G \subseteq \mathrm{Aut}(Y_A)$. If G has odd order, then G lifts to an action on X_A. Moreover, for every $\alpha \in G$ the lifting of α acts as a n-th rooth of the identity on a 2-form on X_A if and only if α acts as a n-th rooth on a section of K_{Y_A}.

Automorphisms of EPW sextics associated to $K3$ surfaces

Consider $(S, H) \in \mathcal{K}_{10}$ such that the double EPW sextic $X_{A(S)}$ is a hyperkähler manifold. We have $\phi_{|H|} : S \to \mathbb{P}^6$.

Definition

We call $\mathrm{Aut}_{\mathcal{C}}(S, \mathbb{P}^6)$ *the group of automorphisms of* S

- *induced by* $\mathbb{P}^6$*, i.e. contained in* $\mathrm{Aut}(S, \mathbb{P}^6)$*,*
- *that send every conic inside* S *in itself.*

$\mathrm{Aut}_{\mathcal{C}}(S, \mathbb{P}^6)$ appears in the following list

$$\mathbb{Z}/n\mathbb{Z} \text{ for } 1 \leq n \leq 66, \qquad D_n \text{ for } 2 \leq n \leq 66, \qquad A_4, \qquad \mathfrak{S}_4 \qquad A_5.$$

Theorem (B.)

There is an exact sequence

$$1 \to \mathrm{Aut}_{\mathcal{C}}(S, \mathbb{P}^6) \to \mathrm{Aut}(Y_{A(S)}) \to \mathfrak{S}_{N+1},$$

where N *is the number of conics on* S*.*

Symplecticity of automorphisms

Consider $(S, H) \in \mathcal{K}_{10}$ such that the double EPW sextic $X_{A(S)}$ is a hyperkähler manifold.

Proposition (B.)

For any $\sigma \in \mathrm{Aut}(S, \mathbb{P}^6)$, the induced action on $Y_{A(S)}$

- *lifts to a symplectic action on $X_{A(S)}$ if and only if σ is symplectic,*
- *lifts to an automorphism that acts as a n-th root of the identity if and only if σ do.*

Inducing automorphims

There is a lot of literature on automorphisms of $K3$ surfaces, for example actions of cyclic groups of prime order on $K3$ surfaces are classified, both

- symplectic (Nikulin, 1979 - van Geemen, Sarti, 2007 - Sarti, Garbagnati, 2007) and
- non-symplectic (Nikulin, 1979 - Artebani, Sarti, 2008 - Artebani, Sarti, Taki, 2011).

Corollary (B.)

Let $p = 2, 3, 5$. We can induce a symplectic action of $\mathbb{Z}/p\mathbb{Z}$ on a family of (hyperkähler) double EPW sextics associated to a family of $K3$ surfaces.

Thanks for your attention!