Cox ring of K3 surfaces of Picard number three

This is a joint work with M. Artebani and A. Laface

Claudia Correa Deisler

Universidad de Concepción

November 19, 2020

Definition 1

Let X be a normal projective algebraic variety, defined over $\mathbb{C}$, such that the Class group $\mathrm{Cl(X)}$ is finitely generated and free. The **Cox ring** of X is defined to be the ring

$$R(X) := \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D)).$$

Where we choose a subgroup K of $\mathrm{Div(X)}$ such that the class homomorphism $K \to \mathrm{Cl(X)}$, $D \mapsto [D]$, is an isomorphism.

The variety X is called **Mori dream space** when the Cox ring is finitely generated.

Definition 1

Let X be a normal projective algebraic variety, defined over $\mathbb{C}$, such that the Class group $\mathrm{Cl(X)}$ is finitely generated and free. The **Cox ring** of X is defined to be the ring

$$R(X) := \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D)).$$

Where we choose a subgroup K of $\mathrm{Div(X)}$ such that the class homomorphism $K \to \mathrm{Cl(X)}$, $D \mapsto [D]$, is an isomorphism.

The variety X is called **Mori dream space** when the Cox ring is finitely generated.

Remarks:

- We observe that $R(X)$ is a $\mathrm{Cl(X)}$-graded algebra over $\mathbb{C}$, i.e it has a direct sum decomposition into complex vector spaces $H^0(X, \mathcal{O}_X(D))$, where $[D] \in \mathrm{Cl(X)}$ such that if $f_1 \in H^0(X, \mathcal{O}_X(D_1))$, $f_2 \in H^0(X, \mathcal{O}_X(D_2))$ then $f_1 \cdot f_2 \in H^0(X, \mathcal{O}_X(D_1 + D_2))$.
- Given a homogeneous element $f \in H^0(X, \mathcal{O}_X(D))$ we define its *degree* to be $\deg(f) = [D]$.

Definition 2

A Weil divisor D of X is a formal finite sum $D = a_1 D_1 + \cdots + a_s D_s$, where $a_i \in \mathbb{Z}$ and D_i are irreducible closed hypersurfaces of X, is an effective divisor if all a_i are nonnegative.

Is called **Effective cone** the cone $\mathrm{Eff}(\mathrm{X}) \subseteq \mathrm{Cl}_{\mathbb{Q}}(\mathrm{X})$ generated by the classes of effective divisors.

The nef cone is the dual of the effective cone with respect to the intersection product.

Definition 2

A Weil divisor D of X is a formal finite sum $D = a_1 D_1 + \cdots + a_s D_s$, where $a_i \in \mathbb{Z}$ and D_i are irreducible closed hypersurfaces of X, is an effective divisor if all a_i are nonnegative.

Is called **Effective cone** the cone $\mathrm{Eff(X)} \subseteq \mathrm{Cl_{\mathbb{Q}}(X)}$ generated by the classes of effective divisors.

The nef cone is the dual of the effective cone with respect to the intersection product.

Remarks:

- If D is an effective divisor such that $[D]$ is an element of the Hilbert basis of $\mathrm{Eff}(X) \subseteq \mathrm{Cl(X)}_{\mathbb{Q}}$, then the Cox ring $R(X)$ has a generator in degree $[D]$.

  Because, the degrees of a set of homogeneous generators of $R(X)$ generate $\mathrm{Eff(X)}$, see [AHL10, Proposition 2.1]

In this context there are two main **problems**:

1. determine conditions on X such that $R(X)$ is finitely generated,
2. find an explicit presentation for $R(X)$.

Examples of Mori dream surfaces (explicit computations) are:

- toric surfaces (Cox 1995)
- Del Pezzo surfaces (Batyrev, Popov 2002)
- $\mathbb{C}^*$-surfaces (Hausen, Suss 2010)
- all smooth rational surfaces of Picard number at most six (Hausen, Keicher, Laface 2016)
- extremal rational elliptic surfaces (Artebani, Garbagnati, Laface 2016)
- some K3 surfaces:

  (Artebani, Hausen, Laface 2010; Ottem 2013)

Definition 3

A **K3 surface** is a projective and smooth complex surface X, with $H^1(X, \mathcal{O}_X) = \{0\}$ and trivial canonical class.

Remarks:

- For a K3 surface X,

$$\mathrm{Cl(X)} \cong \mathrm{Pic(X)} \cong \mathrm{NS(X)} \subset \mathrm{H^2(X, \mathbb{Z})}.$$

- In particular the Picard group is an even, non-degenerate lattice whose rank is at most 20 in $H^2(X, \mathbb{Z})$ ($1 \leq \rho(X) \leq 20$), which is known as the Picard lattice of X.
- The theory of linear systems on K3 surfaces has been developed in [SD74].
- `LSK3Lib.m`: library for linear systems on $K3$ surfaces, contains Magma [BCP97] programs created in [ACDL21].

Definition 3

A **K3 surface** is a projective and smooth complex surface X, with $H^1(X, \mathcal{O}_X) = \{0\}$ and trivial canonical class.

Remarks:

- For a K3 surface X,

$$\mathrm{Cl(X)} \cong \mathrm{Pic(X)} \cong \mathrm{NS(X)} \subset \mathrm{H^2(X, \mathbb{Z})}.$$

- In particular the Picard group is an even, non-degenerate lattice whose rank is at most 20 in $H^2(X, \mathbb{Z})$ ($1 \leq \rho(X) \leq 20$), which is known as the Picard lattice of X.
- The theory of linear systems on K3 surfaces has been developed in [SD74].
- `LSK3Lib.m`: library for linear systems on $K3$ surfaces, contains Magma [BCP97] programs created in [ACDL21].

Known explicit presentations of the cox ring of:

[AHL10] all generic K3 surfaces with a non-symplectic involution when $2 \leq \rho(X) \leq 5$. Moreover, they computed the Cox ring of K3 surfaces that are general double covers of del Pezzo surfaces.

[Ott13] several examples of K3 surfaces of Picard number 2.

The first characterization of Mori dream spaces for K3 surfaces is as follows

Theorem 1 ([AHL10], [McK10])

Let X be a complex projective K3 surface. Then the following statements are equivalent.

1. *X is a Mori dream surface.*
2. *The effective cone $\mathrm{Eff(X)} \subseteq \mathrm{Cl}_{\mathbb{Q}}(\mathrm{X})$ is polyhedral.*
3. *The automorphism group of X is finite.*

Moreover, if the Picard number is at least three, then X is a Mori dream surface if and only if X contains a finite non-zero number of smooth rational curves.
In this case, these curves are (-2)-curves and their classes generate the effective cone.

K3 surfaces with the properties in the previous Theorem have been classified in a series of papers by Piatetski-Shapiro and Shafarevich [PŠŠ71], Nikulin [Nik79, Nik84, Nik00] and Vinberg [Vin07].

For Picard number ≥ 3 there is a finite number of families with such property. (Explicit classification, see book [ADHL15, Theorem 5.1.5.3]).

Theorem 2 ([Nik84])

Let X be a complex projective K3 surface with Picard number $\rho(X) = 3$. Then X is a Mori dream surface if and only if $\mathrm{Cl(X)}$ is isometric to one of the following 26 lattices:

$$\begin{aligned}
S'_{4,1,2} &= \langle 2e_1 + e_3, e_2, 2e_3 \rangle, \\
S_{k,1,1} &= \langle ke_1, e_2, e_3 \rangle, \;\; k \in \{4, 5, 6, 7, 8, 10, 12\}, \\
S_{1,k,1} &= \langle e_1, ke_2, e_3 \rangle, \;\; k \in \{2, 3, 4, 5, 6, 9\}, \\
S_{1,1,k} &= \langle e_1, e_2, ke_3 \rangle, \;\; k \in \{1, 2, 3, 4, 6, 8\},
\end{aligned}$$

where the intersection matrix of e_1, e_2, e_3 is $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix}$*, and*

$$S_1 = (6) \oplus A_1^2, \qquad S_2 = (36) \oplus A_2, \qquad S_3 = (12) \oplus A_2,$$

$$S_4 = \begin{bmatrix} 6 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \qquad S_5 = (4) \oplus A_2, \qquad S_6 = \begin{bmatrix} 14 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}.$$

Theorem 2 ([Nik84])

Let X be a complex projective K3 surface with Picard number $\rho(X) = 3$. Then X is a Mori dream surface if and only if $\mathrm{Cl(X)}$ is isometric to one of the following 26 lattices:

$$\begin{aligned} S'_{4,1,2} &= \langle 2e_1 + e_3, e_2, 2e_3 \rangle, \\ S_{k,1,1} &= \langle ke_1, e_2, e_3 \rangle, \ k \in \{4,5,6,7,8,10,12\}, \\ S_{1,k,1} &= \langle e_1, ke_2, e_3 \rangle, \ k \in \{2,3,4,5,6,9\}, \\ S_{1,1,k} &= \langle e_1, e_2, ke_3 \rangle, \ k \in \{1,2,3,4,6,8\}, \end{aligned}$$

where the intersection matrix of e_1, e_2, e_3 is $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix}$, *and*

$$S_1 = (6) \oplus A_1^2, \qquad S_2 = (36) \oplus A_2, \qquad S_3 = (12) \oplus A_2,$$

$$S_4 = \begin{bmatrix} 6 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \qquad S_5 = (4) \oplus A_2, \qquad S_6 = \begin{bmatrix} 14 & 0 & -1 \\ 0 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}.$$

Problem: Find a minimal set of generators for Cox ring of families of K3 surface of Picard number 3.

Techniques: Koszul type sequences

Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$, such that $(\mathrm{div}(f_1) + E_1) \cap (\mathrm{div}(f_2) + E_2) = \emptyset$.
Then there is an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_X(-E_1 - E_2) \longrightarrow \mathcal{O}_X(-E_1) \oplus \mathcal{O}_X(-E_2) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

tensoring with $\mathcal{O}_X(D)$:

$$0 \longrightarrow \mathcal{O}_X(D - E_1 - E_2) \longrightarrow \mathcal{O}_X(D - E_1) \oplus \mathcal{O}_X(D - E_2) \longrightarrow \mathcal{O}_X(D) \longrightarrow 0$$

and taking the associated long exact sequence in cohomology

$$\cdots \to H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D) \to$$

$$\to H^1(X, D - E_1 - E_2) \to H^1(X, D - E_1) \oplus H^1(X, D - E_2) \to \dots$$

Techniques: Koszul type sequences

Proposition 1 ([Bea96])

Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$, such that $(\mathrm{div}(f_1) + E_1) \cap (\mathrm{div}(f_2) + E_2) = \emptyset$.
If $D \in \mathrm{Div}(X)$ is such that $h^1(X, D - E_1 - E_2) = 0$, then there is a surjective map

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D),$$
$$(g_1, g_2) \mapsto g_1 f_1 + g_2 f_2.$$

Techniques: Koszul type sequences

Proposition 1 ([Bea96])

Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$, such that $(\mathrm{div}(f_1) + E_1) \cap (\mathrm{div}(f_2) + E_2) = \emptyset$.
If $D \in \mathrm{Div}(X)$ is such that $h^1(X, D - E_1 - E_2) = 0$, then there is a surjective map

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D),$$
$$(g_1, g_2) \mapsto g_1 f_1 + g_2 f_2.$$

Observe that the surjectivity of the morphism in the proposition implies that $R(X)$ does not need a generator in degree $[D]$.

Techniques: Koszul type sequences

Considering the case of three disjoint divisors, we obtain the following result.

Proposition 2 ([ACDL21])

Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$, such that $\cap_{i=1}^3(\operatorname{div}(f_i) + E_i) = \emptyset$. If $D \in \operatorname{Div}(X)$ then the map

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D),\ (g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3,$$

is surjective if $h^1(X, D - E_i - E_j) = 0$ for all distinct $i, j \in \{1, 2, 3\}$ and $h^2(X, D - E_1 - E_2 - E_3) = 0$.

Techniques: Koszul type sequences

Considering the case of three disjoint divisors, we obtain the following result.

Proposition 2 ([ACDL21])

Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$, such that $\cap_{i=1}^3 (\mathrm{div}(f_i) + E_i) = \emptyset$. If $D \in \mathrm{Div}(X)$ then the map

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \ (g_1, g_2, g_3) \mapsto g_1 f_1 + g_2 f_2 + g_3 f_3,$$

is surjective if $h^1(X, D - E_i - E_j) = 0$ for all distinct $i, j \in \{1, 2, 3\}$ and $h^2(X, D - E_1 - E_2 - E_3) = 0$.

Observe that the surjectivity of the morphism in above proposition implies that $R(X)$ does not need a generator in degree $[D]$.

Cox rings of K3 surfaces

The following theorem gives the possible degrees of a minimal set of generators of the Cox ring of a (not necessarily Mori dream) K3 surface.

Theorem 3 ([ACDL21])

Let X be a projective K3 surface over $\mathbb{C}$. Then the degrees of a minimal set of generators of its Cox ring $R(X)$ are either:

(i) *classes of (-2)-curves,*

(ii) *classes of nef divisors which are sums of at most three elements of the Hilbert basis of the nef cone (allowing repetitions),*

(iii) *or classes of divisors of the form $2(F+F')$ where F, F' are smooth elliptic curves with $F \cdot F' = 2$.*

Cox rings of K3 surfaces

The following theorem gives the possible degrees of a minimal set of generators of the Cox ring of a (not necessarily Mori dream) K3 surface.

Theorem 3 ([ACDL21])

Let X be a projective K3 surface over $\mathbb{C}$. Then the degrees of a minimal set of generators of its Cox ring $R(X)$ are either:

(i) *classes of (-2)-curves,*

(ii) *classes of nef divisors which are sums of at most three elements of the Hilbert basis of the nef cone (allowing repetitions),*

(iii) *or classes of divisors of the form $2(F+F')$ where F, F' are smooth elliptic curves with $F \cdot F' = 2$.*

There are examples for the case (ii)3, in this way, we can say that the previous theorem is optimal.

Sketch of the proof of Theorem 3

- $R(X)$ has a generator in all the degrees of the (-2)-curves.
- If D is effective and not nef then there exists a (-2)-curve C such that $D \cdot C < 0$, this implies that $C \in Bs(|D|)$ and, the multiplication

  $$H^0(X, D - C) \to H^0(X, D)$$

  by a non-zero element of $H^0(X, C)$ is surjective.

  Thus $R(X)$ has no generators in degree $[D]$, and we can assume D to be nef.
- Assume that $D \sim \sum_{i=1}^{r} a_i N_i$, where $a_i \in \mathbb{Z}^+$ with $\sum_{i=1}^{r} a_i \geq 4$ and $[N_i] \in \mathrm{BNef}(X)$.

  By the hypothesis on D we can find three effective nef divisors N_1, N_2, N_3 such that $D - \sum_{i=1}^{3} N_i$ is nef and not linearly equivalent to zero.
- The divisors N_i can be chosen with $N_1 \cap N_2 \cap N_3 = \emptyset$, unless D is of the following types: $D \sim 4(2F + E)$ or $D \sim 3(2F + E) + F$, where F is a smooth elliptic curve and E is a (-2)-curve with $F \cdot E = 1$.

Sketch of the proof of Theorem 3

- Let $A_{ij} := D - N_i - N_j$, with distinct $i, j \in \{1, 2, 3\}$. The divisors A_{ij} are nef and by [KL07]
$$h^1(X, D - N_i - N_j) = 0$$
unless $A_{ij} \sim kF$, where F is a smooth elliptic curve and $k \geq 2$ is an integer.

- Since $D - N_1 - N_2 - N_3$ is an effective non zero divisor then
$$h^2(X, D - N_1 - N_2 - N_3) = h^0(X, N_1 + N_2 + N_3 - D) = 0.$$
Thus, by Proposition 2 with $E_i = N_i$ for $i = 1, 2, 3$ we conclude that $R(X)$ has no generators in degree $[D]$, unless $A_{ij} \sim kF$.

- If $A_{ij} \sim kF$, that is $D \sim N_i + N_j + kF$ with k and F as above.
We obtain that $R(X)$ is not generated in degree $[D]$ unless $D \sim 2(F + F')$ with $F \cdot F' = 2$.

K3 surfaces of Picard number three

From Theorem 2, we know that in the case of Picard number three there are 26 families of Mori dream K3 surfaces.

We described the extremal rays and the Hilbert basis of the effective cone and the nef cone for each of the 26 families [ACDL21, Proposition 3.1], by means of a Magma [BCP97] library `Find-2.m`.

Moreover, we obtained the following main result.

Theorem 4 ([ACDL21])

Let X be a Mori dream K3 surface of Picard number three. The degrees of a set of generators of the Cox ring $R(X)$ are given in the following Table. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.

Degrees of a set of generators of $R(X)$ for $\rho(X) = 3$

N°	Cl(X)	Degrees of generators of $R(X)$
1	S_1	BEff
2	S_2	E, BNef
3	S_3	E, BNef
4	S_4	E, BNef
5	S_5	$E, \mathrm{BNef}[i], i = 1, 2, 5$
6	S_6	$E, \mathrm{BNef}[i], i = 1-7, 10, 11^*, 12, 13, 15, 16, 18-20, 25, 27, 28, 30, 33,$ $34, 38, 40^*, 41, 42^*, 43^*$
7	$S'_{4,1,2}$	E, BNef
8	$S_{4,1,1}$	E, BNef
9	$S_{5,1,1}$	E, BNef
10	$S_{6,1,1}$	E, BNef
11	$S_{7,1,1}$	E, BNef
12	$S_{8,1,1}$	E, BNef
13	$S_{10,1,1}$	$E, \mathrm{BNef}, 3\,\mathrm{BNef}[2]$ $\mathrm{BNef}[2] + \mathrm{BNef}[i],\ i = 1^*, 5^*, 6^*, 8^*, 10^*, 13^*, 16^*, 17^*, 19^*, 20^*, 21^*, 24^*$
14	$S_{12,1,1}$	$E \cup \mathrm{BNef}$

Degrees of a set of generators of $R(X)$ for $\rho(X) = 3$

N°	Cl(X)	Degrees of generators of $R(X)$
15	$S_{1,2,1}$	E, BNef
16	$S_{1,3,1}$	E, BNef
17	$S_{1,4,1}$	$E, \mathrm{BNef}[i], i = 1-12, 13^*, 14, 15^*$
18	$S_{1,5,1}$	$E, \mathrm{BNef}[i], i = 1-17, 18^*, 19-26, 27^*$
19	$S_{1,6,1}$	E, BNef
20	$S_{1,9,1}$	$E, \mathrm{BNef}, \mathrm{BNef}[11] + \mathrm{BNef}[i], i = 20^*, 34^*, 43^*$ $\mathrm{BNef}[2] + \mathrm{BNef}[i], i = 11^*, 12^*, 13^*, 20^*, 29^*, 33^*$
21	$S_{1,1,1}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[3], \mathrm{BNef}[2] + \mathrm{BNef}[3]$
22	$S_{1,1,2}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[2], \mathrm{BNef}[1] + \mathrm{BNef}[2] + \mathrm{BNef}[3]$
23	$S_{1,1,3}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[2], \mathrm{BNef}[6]$
24	$S_{1,1,4}$	$E, \mathrm{BNef}[1], \mathrm{BNef}[3], \mathrm{BNef}[7]$
25	$S_{1,1,6}$	$E, \mathrm{BNef}[i],\ i = 1, 2, 3, 5, 6, 9, 13-17$
26	$S_{1,1,8}$	$E, \mathrm{BNef}[i],\ i = 1, 2, 3, 5, 7, 11, 15, 19, 20, 24, 29, 34, 35, 38, 39, 41$ $\mathrm{BNef}[2] + \mathrm{BNef}[i],\ i = 7^*, 11^*, 19^* \quad \mathrm{BNef}[19] + \mathrm{BNef}[i],\ i = 7^*, 11^*$

K3 surfaces of Picard number four

By [Vin07, Theorem 1] there are 14 families of K3 surfaces with Picard number four whose general member has finitely generated Cox ring.

In [ACDR20, Theorem 2.1] we described the set of extremal rays of the effective cone and its Hilbert basis, and the set of extremal rays of the nef cone and its Hilbert basis by means of the Magma library `Find-2.m`.

We also describe projective models for all Mori dream K3 surfaces of Picard number four and we geometrically identified the degrees of the generators of the Cox ring. These results are contained in [ACDR20].

Projective models for all families of Mori dream K3 surfaces have been recently given by Roulleau ([Rou20b], [Rou20a]).

Our main result is the following:

Theorem 5 ([ACDR20])

Let X be a Mori dream K3 surface of Picard number four such that $\mathrm{Cl}(\mathrm{X})$ is not isometric to V_{14}. The degrees of a set of generators of the Cox ring $R(X)$ are given in the following Table. All degrees in the Table are necessary to generate $R(X)$, except possibly for those marked with a star.

Degrees of a set of generators of $R(X)$ for $\rho(X) = 4$

N°	Cl(X)	Degrees of generators of $R(X)$
1	V_1	BEff
2	V_2	$E, \mathrm{BNef}[i], i = 1-5, 7, 11, 14, 15, 20, 23, 25, 29, 32-35$
3	V_3	$E, \mathrm{BNef}[i], i = 1, 2, 4, 7, 9$
4	V_4	$E, \mathrm{BNef}[3], \mathrm{BNef}[3] + \mathrm{BNef}[4]$
5	V_5	$E, \mathrm{BNef}[1] + \mathrm{BNef}[2]$
6	V_6	$E, \mathrm{BNef}[15]$
7	V_7	E
8	V_8	$E, \mathrm{BNef}[i], i = 1, 2, 3, 5$
9	V_9	$E, \mathrm{BNef}[i], i = 1-4, 6, 7$
10	V_{10}	$E, \mathrm{BNef}[i], i = 2-5$
11	V_{11}	$E, \mathrm{BNef}[i], i = 4, 7, 9, 11, 13, 15, 16, 18, 20, 21, 23, 24, 26, 27$ $\mathrm{BNef}[i]^*, i = 1-3, 5, 6, 8, 10, 12, 14, 17, 19, 22, 25$
12	V_{12}	$E, \mathrm{BNef}[30], \mathrm{BNef}[31]$
13	V_{13}	$E, \mathrm{BNef}[i], i = 1^*, 7, 9, 12, 14, 16, 17, 27, 28, 30-39$
14	V_{14}	Contains the degrees in the following Table

Degrees of a set of generators of $R(X)$ for $\rho(X) = 4$

N°	Degrees of generators
14	$(1,0,-1,-2),(0,-1,1,-1),(1,2,-1,0),(2,2,-3,-2),(0,0,1,0),$ $(0,0,-1,1),(1,1,0,-2),(2,3,-3,-1),(1,-1,0,-1),(1,0,-1,-1),$ $(1,0,0,0),(1,1,-1,-1),(1,1,-1,0),(2,-3,2,-1),(2,-2,1,-1),$ $(2,-1,-2,-3),(2,1,-3,-2),(2,1,0,-4),(2,2,-3,-1),(2,2,0,-3),$ $(2,3,-3,0),(2,3,-2,1),(3,0,-4,-4),(3,1,2,-7),(3,2,-2,-5),$ $(3,2,2,-6),(3,3,-5,-2),(3,3,-2,-4),(3,4,-5,-1),(3,4,-2,-3),$ $(3,5,-4,1),(4,1,0,-9),(4,4,-4,-5),(4,5,-4,-4),(4,5,0,-5),$ $(5,2,-2,-10),(5,3,-8,-5),(5,7,-2,-5),(5,8,-8,0),(6,-2,-7,-9),$ $(6,5,-10,-5),(6,9,-10,-1),(6,10,-7,3),(7,-3,-8,-11),(7,3,6,-16),$ $(7,5,-6,-11),(7,8,-12,-4),(7,9,-12,-3),(7,10,-6,-6),(7,12,-8,4),$ $(8,3,8,-19),(8,7,-8,-11),(8,11,-8,-7),(9,10,-10,-10),(9,11,-10,-9),$ $(11,3,-2,-24),(11,15,-2,-12),(13,3,-2,-29),(13,18,-2,-14),$ $(14,10,-23,-13),(14,22,-23,-1),(17,12,-28,-16),(17,27,-28,-1),$ $(18,22,-31,-9),(19,15,-18,-28),(19,27,-18,-16),(22,27,-38,-11),$ $(23,18,-22,-34),(23,27,-26,-24),(23,33,-22,-19),(28,33,-32,-29)$

References I

[ACDL21] Michela Artebani, Claudia Correa Deisler, and Antonio Laface, *Cox rings of K3 surfaces of Picard number three*, J. Algebra **565** (2021), 598–626. ↑7, 8, 15, 16, 17, 18, 21

[ACDR20] Michela Artebani, Claudia Correa Deisler, and Xavier Roulleau, *Mori dream K3 surfaces of Picard number four: projective models and Cox rings, preprint* (2020), available at `https://arxiv.org/abs/2011.00475`. ↑24

[ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. ↑9

[AHL10] Michela Artebani, Jürgen Hausen, and Antonio Laface, *On Cox rings of K3 surfaces*, Compos. Math. **146** (2010), no. 4, 964–998. ↑4, 5, 7, 8, 9

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). ↑7, 8, 21

References II

[Bea96] Arnaud Beauville, *Complex algebraic surfaces*, Second, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. ↑13, 14

[KL07] Andreas Leopold Knutsen and Angelo Felice Lopez, *A sharp vanishing theorem for line bundles on $K3$ or Enriques surfaces*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3495–3498. ↑20

[McK10] James McKernan, *Mori dream spaces*, Jpn. J. Math. **5** (2010), no. 1, 127–151. ↑9

[Nik00] Viacheslav V. Nikulin, *A remark on algebraic surfaces with polyhedral Mori cone*, Nagoya Math. J. **157** (2000), 73–92. ↑9

[Nik79] ———, *Quotient-groups of groups of automorphisms of hyperbolic forms of subgroups generated by 2-reflections*, Dokl. Akad. Nauk SSSR **248** (1979), no. 6, 1307–1309. ↑9

[Nik84] ———, *$K3$ surfaces with a finite group of automorphisms and a Picard group of rank three*, Trudy Mat. Inst. Steklov. **165** (1984), 119–142. Algebraic geometry and its applications. ↑9, 10, 11

References III

[Ott13] John Christian Ottem, *Cox rings of K3 surfaces with Picard number 2*, J. Pure Appl. Algebra **217** (2013), no. 4, 709–715. ↑7, 8

[PŠŠ71] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type* $\mathrm{K3}$, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572. ↑9

[Rou20a] Xavier Roulleau, *An atlas of K3 surfaces with finite automorphism group* (2020), available at https://arxiv.org/abs/2003.08985. ↑24

[Rou20b] ———, *On the geometry of K3 surfaces with finite automorphism group and no elliptic fibrations* (2020), available at https://arxiv.org/pdf/1909.01909.pdf. ↑24

[SD74] B. Saint-Donat, *Projective models of* $K-3$ *surfaces*, Amer. J. Math. **96** (1974), 602–639. ↑7, 8

[Vin07] È. B. Vinberg, *Classification of 2-reflective hyperbolic lattices of rank 4*, Tr. Mosk. Mat. Obs. **68** (2007), 44–76. ↑9, 24