On large prime actions on Riemann surfaces

Sebastián Reyes Carocca

Departamento de Matemática y Estadística
Universidad de La Frontera
Temuco, Chile
(joint work with Anita M. Rojas)

ECOS-ANID workshop in Algebraic Geometry

Partially supported by Fondecyt Grant 11180024, 1190991 and Redes Grant 170071

November 12, 2020

The moduli space

Let $\mathscr{M}_g$ denote the **moduli space** of isomorphism classes of compact Riemann surfaces of genus $g \geqslant 2$.

- $\mathscr{M}_g$ has a structure of complex analytic space,
- its dimension is $3g - 3$, and
- if $g \geqslant 4$ its singular locus is

$$\text{Sing}(\mathscr{M}_g) = \{[S] : S \text{ has non-trivial symmetries}\}$$

Equivalences

We assume $g \geqslant 2$.

Theorem.

There is an equivalence between isomorphism classes of:

- compact Riemann surfaces,
- (complex projective smooth) algebraic curves, and
- orbit spaces of the upper-half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by the action of a co-compact Fuchsian group Δ.

Fuchsian groups

We recall that a **Fuchsian group** is a discrete subgroup Δ of

$$\mathrm{Aut}(\mathbb{H}) \cong \mathbb{P}\mathrm{SL}(2, \mathbb{R})$$

and the co-compact ones are those such that $\mathbb{H}/\Delta$ is compact.

Definition. The **signature** of Δ is the tuple

$$\sigma(\Delta) = (h; m_1, \ldots, m_l)$$

where h is the genus of the quotient $\mathbb{H}/\Delta$ and $m_1, \ldots, m_l$ are the branch indices in the universal canonical projection

$$\mathbb{H} \to \mathbb{H}/\Delta.$$

Uniformization theorem

If Δ is a Fuchsian group of signature $(h; m_1, \ldots, m_l)$ then Δ has a canonical presentation with generators

$$\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h, \gamma_1, \ldots, \gamma_l$$

and relations

$$\gamma_1^{m_1} = \cdots = \gamma_l^{m_l} = \Pi_{i=1}^{h}[\alpha_i, \beta_i]\Pi_{i=1}^{l}\gamma_i = 1,$$

where the brackets stands for commutator.

Uniformization theorem Let S be a compact Riemann surface of genus g. There is a unique, up to conjugation, Fuchsian group Γ of signature $(g; -)$ such that

S is **isomorphic** to $\mathbb{H}/\Gamma$.

Automorphisms

Let S be a Riemann surface of genus $g \geqslant 2$.

Well-known facts.

- $\mathrm{Aut}(S)$ is finite group and $|\mathrm{Aut}(S)| \leq 84(g-1)$.
- Each finite group can be realized as a group of automorphisms of a compact Riemann surface of a suitable genus.
- If $G \leqslant \mathrm{Aut}(S)$ then the space of orbits S/G is endowed with a Riemann surface structure in such a way that

  $$\pi : S \to S/G$$

  is holomorphic

Riemann's existence theorem

Let $S \cong \mathbb{H}/\Gamma$ be a compact Riemann surface of genus $g \geqslant 2$.

Riemann's existence theorem

A finite group G acts on S **if and only if**:

1. There is a Fuchsian group Δ and a group epimorphism

$$\theta : \Delta \to G \text{ such that } \ker(\theta) = \Gamma.$$

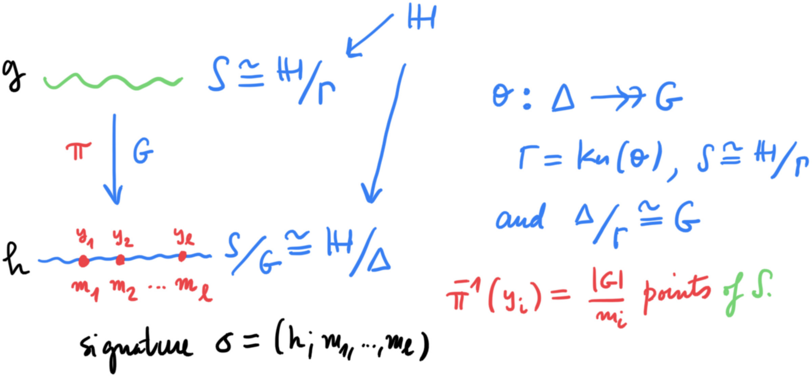
2. The Riemann-Hurwitz formula is satisfied

$$2(g-1) = |G|(2h - 2 + \Sigma_{j=1}^{l}(1 - \tfrac{1}{m_j}))$$

G acts on S with signature $\sigma(\Delta) = (h; m_1, \ldots, m_l)$.

Riemann's existence theorem

- $\mathbb{H} \to S \cong \mathbb{H}/\Gamma$ is smooth.
- the ramification arises in $S \to S/G$.
- m_i is the order of an isotropy group.



Bounds

Let N_g denote the maximal order of the full automorphism group among the compact Riemann surfaces of genus g.

Hurwitz's upper bound:

$$N_g \leqslant 84(g-1)$$

Accola-Maclachlan's lower bound:

$$8(g+1) \leqslant N_g$$

Accola-Maclachlan curve

Let X_8 be the compact Riemann surface of genus g defined by

$$y^2 = x^{2(g+1)} - 1$$

Then (Accola-Maclachlan)

1. The full automorphism group of X_8 has order $8(g+1)$.
2. Aut(X_8) acts on X_8 with signature $(0; 2, 4, 2(g+1))$.
3. there are infinitely many values of g for which there is no compact Riemann surfaces of genus g with more than $8(g+1)$ automorphisms.

Moreover, up to finitely many values of the genus, if $g \not\equiv 3 \bmod 4$ then X_8 is the **unique** compact Riemann surface of genus g with exactly $8(g+1)$ automorphisms (Kulkarni).

One-dimensional families

Let $g \geqslant 2$. Then (Costa-Izquierdo)

- there exists a closed equisymmetric one-dimensional family $\bar{\mathscr{C}_g}$ of Riemann surfaces of genus g with a group of automorphisms isomorphic to

$$\mathbf{D}_{g+1} \times C_2 \text{ acting with signature } (0; 2, 2, 2, g+1).$$

- $4(g+1)$ is the largest order of the full automorphism group of complex one-dimensional families of compact Riemann surfaces of genus g appearing **for all** g.
- $X_8 \in \bar{\mathscr{C}_g}$

Similar results for 3 and 4-dimensional families were recently obtained by Izquierdo-RC-Rojas.

Large prime actions

If a compact Riemann surface of genus $g \geqslant 2$ has an automorphism of prime order q such that $q > g$ then either

$$q = 2g + 1 \quad \text{or} \quad q = g + 1.$$

The former case corresponds to the so-called Lefschetz surfaces. This talk deals with the **latter case**.

Let $q \geqslant 5$ be a prime number. Consider the singular sublocus

$$\mathscr{M}_{q-1}^{q} \subset \mathrm{Sing}(\mathscr{M}_{q-1})$$

consisting of all those compact Riemann surfaces of genus $q - 1$ endowed with an automorphism of order q.

Goal

To classify and describe the surfaces lying in $\mathscr{M}_{q-1}^{q}$.

In other words, we shall consider all those compact Riemann surfaces of genus $g \geqslant 4$ with a group of automorphisms of order

$$\lambda(g+1) \text{ where } \lambda \geqslant 1 \text{ is an integer,}$$

under the assumption that $q := g+1$ is a prime number.

Remark: This locus has been considered by Arakelian-Speziali, Costa-Izquierdo, Hidalgo and Urzúa, among others.

Classification

Theorem 1. Let $q \geqslant 7$ be a prime number. If S is a compact Riemann surface of genus $g = q - 1$ endowed with a group of automorphisms of order λq for some integer $\lambda \geqslant 1$ then

$$\lambda \in \{1, 2, 3, 4, 8\}.$$

Assume $\lambda = 8$. Then S is isomorphic to the Accola-Maclachlan curve X_8.

Assume $\lambda = 4$.

- If $q \equiv 3 \bmod 4$ then S belongs to the closed family $\bar{\mathscr{C}}_g$.
- If $q \equiv 1 \bmod 4$ then S belongs to the closed family $\bar{\mathscr{C}}_g$ or S is isomorphic to the unique compact Riemann surface X_4 with full automorphism group isomorphic to

$$C_q \rtimes_4 C_4 \text{ acting with signature } (0; 4, 4, q).$$

Classification

Moreover, if $\mathscr{C}_g$ stands for the interior of $\bar{\mathscr{C}_g}$ then

$$\bar{\mathscr{C}_g} - \mathscr{C}_g = \{X_8\}.$$

Assume $\lambda = 3$. Then S is isomorphic to the unique Riemann surface X_3 with full automorphism group isomorphic to

$$C_q \times C_3 \text{ acting with signature } (0; 3, q, 3q).$$

Assume $\lambda = 2$. Then one of the following statements holds.

- S is isomorphic to one of the $\frac{q-3}{2}$ pairwise non-isomorphic compact Riemann surfaces $X_{2,k}$ for $k \in \{1, \dots, \frac{q-3}{2}\}$ with full automorphism group isomorphic to

$$C_q \times C_2 \text{ acting with signature } (0; q, 2q, 2q).$$

Classification

- S belongs to the closed family $\bar{\mathscr{K}_g}$ of compact Riemann surfaces with a group of automorphisms isomorphic to

$$\mathbf{D}_q \text{ acting with signature } (0; 2, 2, q, q).$$

Moreover, the closed family $\bar{\mathscr{K}_g}$ consists of at most

$$\begin{cases} \frac{q+3}{4} & \text{if } q \equiv 1 \bmod 4 \\ \frac{q+1}{4} & \text{if } q \equiv 3 \bmod 4 \end{cases}$$

equisymmetric strata; one of them being $\mathscr{C}_g$. Furthermore, if $\mathscr{K}_g$ stands for the interior of $\bar{\mathscr{K}_g}$ then the full automorphism group of $S \in \mathscr{K}_g - \mathscr{C}_g$ is isomorphic to $\mathbf{D}_q$ and

$$\bar{\mathscr{K}_g} - \mathscr{K}_g = \begin{cases} \{X_4, X_8\} & \text{if } q \equiv 1 \bmod 4 \\ \{X_8\} & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

Some remarks

- If $S \in \mathscr{M}_{q-1}^{q}$ then either Aut$(S) \cong C_q$ or S lies in one of the cases described in the theorem. These two possible situations were considered by Costa and Izquierdo where the focus was put on finding isolated equisymmetric strata of Sing$(\mathscr{M}_g)$.
- The case $q = 5$ is slightly different. As a matter of fact, if S has genus 4 and is endowed with a group of automorphisms of order 5λ for some $\lambda \geqslant 1$ then, in addition to the case Aut$(S) \cong C_5$ and the possibilities given in the theorem, λ can equal 12 and 24. In the last two cases, S is isomorphic to the classical Bring's curve.

Idea of the proof

The proof consists of several steps.

- The case $\lambda = 3$. There is only one possible signature $(0; 3, q, 3q)$ and this implies that G is cyclic. The action can only be extended **non-normally** to a group G' of order $12q$ acting with signature $(0; 2, 3, 3q)$ but this is not possible: the non-normality implies that $G' \cong C_q \times A_4$.
- Numerically, the cases $\lambda = 5, 6, 7$ are not possible.
- To prove that $\lambda \geqslant 8$ implies $\lambda = 8$. Most cases are numerically disregarded. Group theory considerations allow us to show that $\lambda = 8$ and $(0; 2, 4, 2q)$. This together with results of Kulkarni show that $S = X_8$.

Idea of the proof

- $C_q \times C_2^2$ acts with signature $(0; 2, 2q, 2q)$ and there is only one surface X, up to isomorphism. The action extends to a group of order $8q$. The uniqueness of X_8 implies that $X = X_8$.
- $C_q \rtimes_4 C_4$ acts with signature $(0; 4, 4, q)$ and there is only one surface X_4, up to isomorphism. The action **does not extend**. Thus, X_4 has exactly $4q$ automorphisms and is not isomorphic to X_8.
- etc... (13 propositions)

Algebraic description

Theorem 2. Let $q \geqslant 5$ be a prime number and let $g = q - 1$. Set $\omega_l = \exp(\frac{2\pi i}{l})$.

If S belongs to the closed family $\bar{\mathscr{C}_g}$ then S is isomorphic to

$$\mathscr{X}_t : y^2 = (x^q - 1)(x^q - t) \text{ for some } t \in \mathbb{C} - \{0, 1\}.$$

In addition, if $S \in \mathscr{C}_g$ then the full automorphism group of $S \cong \mathscr{X}_t$ is generated by

$$(x, y) \mapsto (\omega_q x, -y) \text{ and } (x, y) \mapsto (\tfrac{1}{x}, \tfrac{1}{x^q}\psi_t(x)y)$$

where $\psi_t(x) = \sqrt{\frac{tx^q - 1}{x^q - t}}$.

Algebraic description

Assume $q \equiv 1 \bmod 4$ and choose $\rho \in \{2, \dots, q-2\}$ such that $\rho^4 \equiv 1 \bmod q$. Then X_4 is isomorphic to

$$y^q = (x-1)(x-i)^{\rho}(x+1)^{q-1}(x+i)^{q-\rho}$$

where $i^2 = -1$. In the previous model the full automorphism group of X_4 is generated by

$$(x,y) \mapsto (x, \omega_q y) \text{ and } (x,y) \mapsto (ix, \varphi(x) y^{\rho})$$

where $\varphi(x) = \frac{-(x+i)^{e-\rho}}{(x-i)^{e-1}(x+1)^{\rho-1}}$ and $e = \frac{\rho^2+1}{q}$.

X_3 isomorphic to $y^3 = x^q - 1$ and, in this model, its full automorphism group is generated by $(x,y) \mapsto (\omega_q x, \omega_3 y)$.

Algebraic description

For each $k \in \{1, \ldots, \frac{q-3}{2}\}$ there exists $n_k \in \{1, \ldots, q-1\}$ different from $q-2$ such that $X_{2,k}$ is isomorphic to

$$y^q = x^{n_k}(x^2 - 1)$$

and, in this model, its full automorphism group is generated by

$$(x, y) \mapsto (x, \omega_q y) \text{ and } (x, y) \mapsto (-x, (-1)^{n_k} y).$$

If S belongs to the closed family $\bar{\mathscr{K}_g}$ then S is isomorphic to

$\mathscr{Z}_t : y^q = (x-1)(x+1)^{q-1}(x-t)(x+t)^{q-1}$ for some $t \in \mathbb{C} - \{0, \pm 1\}$

and, if $S \neq X_4$ and $S \notin \bar{\mathscr{C}_g}$ then the full automorphism group of $S \cong \mathscr{Z}_t$ is generated by

$$(x, y) \mapsto (x, \omega_q y) \text{ and } (x, y) \mapsto (-x, \phi_t(x) y^{-1})$$

where $\phi_t(x) = (x^2 - 1)(x^2 - t^2)$.

Hyperelliptic surfaces in $\mathscr{M}_{q-1}^{q}$

Arakelian and Speziali studied groups of automorphisms of large prime order of projective absolutely irreducible algebraic curves over algebraically closed fields of any characteristic.

They succeeded in proving that

$$S \in \mathscr{M}_{q-1}^{q} \text{ is non-hyperelliptic implies } \lambda \in \{1, 2, 3, 4\}.$$

We lengthen the implication above.

Proposition. The Riemann surfaces lying in $\mathscr{M}_{q-1}^{q}$ that are non-hyperelliptic are $X_{2,k}, X_3, X_4$, the surfaces which belong to $\mathscr{K}_g - \mathscr{C}_g$ and the ones for which $\text{Aut}(S) \cong C_q$.

Jacobian varieties

Let S be a compact Riemann surface of genus $g \geqslant 2$.

Well-known facts.

- JS is an irreducible principally polarised abelian variety of dimension g.
- up to isomorphism, the surface is determined by its Jacobian (Torelli's theorem).
- The action of $G \leqslant \mathrm{Aut}(S)$ induces an action of G on JS.
- The latter action induces the so-called group algebra decomposition of JS

$$JS \sim A_1 \times \cdots \times A_r \sim B_1^{n_1} \times \cdots \times B_r^{n_r}$$

The group algebra decomposition

- The group algebra decomposition only depends on the structure of G (Lange-Recillas, Carocca-Rodríguez)
- The dimension of the factors depends on the way G acts (Rojas)

Example: The group algebra decomposition of JX_3 is trivial whilst the one of $JX_{2,k}$ agrees with the classical decomposition

$$JX_{2,k} \sim J(X_{2,k}/H) \times \mathrm{Prym}(X_{2,k} \to X_{2,k}/H)$$

where $H \leqslant \mathrm{Aut}(X_{2,k})$ is isomorphic to C_2.

Jacobian varieties

Theorem 3. Let $q \geqslant 5$ be a prime number and let $g = q - 1$.

The Jacobian variety JX_8 decomposes, up to isogeny, as

$$JX_8 \sim JY_8^2$$

where Y_8 is quotient given by the action of $\langle z \rangle$ on X_8, where

$$\mathrm{Aut}(X_8) \cong \langle x, y, z : x^{2q} = y^2 = z^2 = 1, \\ [x, y] = [z, y] = 1, zxz = x^{-1}y \rangle.$$

The Jacobian variety JX_4 decomposes, up to isogeny, as

$$JX_4 \sim JY_4^4$$

where Y_4 is quotient given by the action of $\langle B \rangle$ on X_4, where

$$\mathrm{Aut}(X_4) \cong \langle A, B : A^q = B^4 = 1, BAB^{-1} = A^\rho \rangle$$

and ρ is a primitive fourth root of unity in $\mathbb{Z}_q$.

Jacobian varieties

The Jacobian variety JS of each $S \in \mathscr{K}_g$ decomposes, up to isogeny, as

$$JS \sim JX^2$$

where X is quotient given by the action of $\langle s \rangle$ on S, where

$$\mathrm{Aut}(S) \cong \begin{cases} \mathbf{D}_q & \text{if } S \in \mathscr{K}_g - \mathscr{C}_g \\ \mathbf{D}_q \times C_2 & \text{if } S \in \mathscr{C}_g \end{cases}$$

and $\mathbf{D}_q = \langle r, s : r^q = s^2 = (sr)^2 = 1 \rangle$.

Idea: to study the representations of the involved groups and to study the induced decomposition of the intermediate covers.

Field of moduli and fields of definition

Let Gal($\mathbb{C}/\mathbb{Q}$) denote the group of field automorphisms of $\mathbb{C}$.

The correspondence

$$\mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \times \mathscr{M}_g \to \mathscr{M}_g \text{ given by } (\sigma, [S]) \mapsto [S^\sigma]$$

where S^σ is the Galois σ-transformed of S, defines an action.

The **field of moduli** of S is the fixed field $\mathcal{M}(S)$ of the isotropy group of S under the aforementioned action, namely

$$\mathcal{M}(S) = \mathrm{fix}\{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) : S^\sigma \cong S\}.$$

Field of moduli and fields of definition

Known facts.

- Koizumi: the field of moduli of S agrees with the intersection of all its fields of definition and S can be defined over a finite degree extension of $\mathcal{M}(S)$.
- Weil: necessary and sufficient conditions under which S can be defined over its field of moduli were provided; these conditions are trivially satisfied if S has no non-trivial automorphisms.
- Wolfart: if S is quasiplatonic then S can be defined over its field of moduli.

Field of moduli and fields of definition

The general question of deciding whether or not the field of moduli is a field of definition is a challenging problem.

If the genus of $S/\mathrm{Aut}(S)$ is zero then (Debes-Emsalem)

- S can be defined over $\mathcal{M}(S)$ or
- S can be defined over a quadratic extension of $\mathcal{M}(S)$

Example. The surfaces $X_3, X_{2,k}$ and X_8 are defined over $\mathbb{Q}$ (Theorem 2) and therefore their fields of moduli are $\mathbb{Q}$.

Example. As X_4 is quasiplatonic, it can be defined over its field of moduli. Moreover, the uniqueness of X_4 implies that its field of moduli is $\mathbb{Q}$. In fact, X_4 is isomorphic to

$$y^q = x(x+1)^\rho(x-1)^{q-\rho}$$

Field of moduli and fields of definition

Proposition. Let $q \geqslant 5$ be prime and let $g = q - 1$. If S belongs to the family $\mathscr{K}_g$ and

$$S \cong \mathscr{Z}_t = \{(x,y) : y^q = (x-1)(x+1)^{q-1}(x-t)(x+t)^{q-1}\}$$

for $t \in \mathbb{C} - \{0, \pm 1\}$ then the field of moduli of S is $\mathbb{Q}(t)$.

Remark: S and JS can be defined over the same fields and that their fields of moduli agree (Sekiguchi, Milne). Thus:

Corollary. The Riemann surfaces of Theorem 1 and their Jacobian varieties can be defined over their fields of moduli.

The sublocus of $\mathcal{A}_g$ with G-action

Known facts.

- The moduli space $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g is isomorphic to the quotient

$$\pi : \mathscr{H}_g \to \mathcal{A}_g \cong \mathscr{H}_g/\mathsf{Sp}(2g, \mathbb{Z})$$

- If the isomorphism class of JS is represented by $Z_S \in \mathscr{H}_g$ then there is an isomorphism of groups

$$\mathsf{Aut}(JS) \cong \Sigma_S := \{R \in \mathsf{Sp}(2g, \mathbb{Z}) : R \cdot Z_S = Z_S\},$$

where Σ_S is well-defined up to conjugation in $\mathsf{Sp}(2g, \mathbb{Z})$.

The sublocus of $\mathcal{A}_g$ with G-action

- The subset of $\mathscr{H}_g$ given by

$$\mathscr{S}_S := \{Z \in \mathscr{H}_g : R \cdot Z = Z \text{ for all } R \in \Sigma_S\}$$

consists of those matrices representing abelian varieties admitting an action equivalent to the one of $\mathrm{Aut}(JS)$.

If $\bar{\mathscr{U}}_g$ is an equisymmetric family of surfaces of genus g and if S is any surface lying in the interior $\mathscr{U}_g$ of $\bar{\mathscr{U}}_g$ then

$$\{JX : X \in \mathscr{U}_g\} \subseteq \pi(\mathscr{S}_S).$$

In general, those loci of $\mathcal{A}_g$ **do not agree**.

The sublocus of $\mathcal{A}_g$ with G-action

Although a satisfactory description of the matrices in $\mathscr{S}_S$ seems to be a difficult problem, there is a simple representation theoretic way to compute the dimension N_S of $\mathscr{S}_S$.

Theorem. Let $q \geqslant 5$ be prime, let $g = q - 1$ and let $S \in \mathscr{K}_g$. Then

$$N_{X_8} = N_{X_3} = N_{X_{2,k}} = 0,\ N_{X_4} = \tfrac{q-1}{4} \ \text{ and } \ N_S = \tfrac{q-1}{2}.$$

According to results due to Streit (and later generalised by Frediani-Ghigi-Penegini), if N_S equals zero then the full automorphism group of S determines the period matrix for JS and JS admits complex multiplication.

The sublocus of $\mathcal{A}_g$ with G-action

As a direct consequence of the previous theorem we recover the following known result.

Corollary. The Jacobian varieties of $X_3, X_{2,k}$ and X_8 admit complex multiplication.

Final remark: The fact that $N_{X_8} = 0$ allows us to determine **explicitly** the period matrix of the Accola-Maclachlan curve X_8 of genus 4 (a theoretical method was given by Bujalance, Costa, Gamboa and Riera).

$$\begin{pmatrix} \frac{25}{22} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} - \frac{70}{11} i \sqrt{\frac{2}{5}\sqrt{5}+5} & \frac{1}{2}\sqrt{5}-\frac{3}{2} & -\frac{1}{2}\sqrt{5}+1 & -\frac{1}{2}\sqrt{5}+1 \\ \frac{1}{2}\sqrt{5}-\frac{3}{2} & -\frac{5}{22} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} + \frac{39}{22} i \sqrt{\frac{2}{5}\sqrt{5}+5} & \frac{5}{44} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} - \frac{39}{44} i \sqrt{\frac{2}{5}\sqrt{5}+5} & \frac{25}{44} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} - \frac{151}{44} i \sqrt{\frac{2}{5}\sqrt{5}+5} \\ -\frac{1}{2}\sqrt{5}+1 & \frac{5}{44} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} - \frac{39}{44} i \sqrt{\frac{2}{5}\sqrt{5}+5} & \frac{1}{2} i \sqrt{\frac{2}{5}\sqrt{5}+5} & -\frac{15}{22} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} + \frac{42}{11} i \sqrt{\frac{2}{5}\sqrt{5}+5} \\ -\frac{1}{2}\sqrt{5}+1 & \frac{25}{44} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} - \frac{151}{44} i \sqrt{\frac{2}{5}\sqrt{5}+5} & -\frac{15}{22} i \left(\frac{2}{5}\sqrt{5}+5\right)^{\frac{3}{2}} + \frac{42}{11} i \sqrt{\frac{2}{5}\sqrt{5}+5} & \frac{1}{2} i \sqrt{\frac{2}{5}\sqrt{5}+5} \end{pmatrix}$$

Thanks!