THE BERGLUND-HÜBSCH-CHIODO-RUAN MIRROR SYMMETRY FOR K3 SURFACES

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ABSTRACT. We prove that the mirror symmetry of Berglund-Hübsch-Chiodo-Ruan, applied to K3 surfaces with a non-symplectic involution, coincides with the lattice mirror symmetry.

1. Introduction

The most famous exemple of mirror symmetry between Calabi-Yau varieties was given in 1991 by the physicists Candelas, de la Ossa, Green and Parkes [7], where they describe the mirror family of a one parameter family of smooth quintic threefolds in $\mathbb{P}^4$. The mirror family is a desingularization of a quotient of the family by a finite group acting symplectically on it. Since then a big amount of work has been done, and mirror symmetry has found its expression in many (mathematical) ways e.g. through toric geometry or Landau-Ginzburg theory. In 1992, Berglund and Hübsch [3] described a special construction of mirror pairs of Calabi-Yau manifolds given as finite quotients of certain hypersurfaces in weighted projective spaces extending the construction of [7]. Later, Chiodo and Ruan in [8], using results of Krawitz in [18] proved that the transposition rule of Berglund-Hübsch provides pairs of Calabi-Yau manifolds whose Hodge diamonds have the symmetry required in mirror symmetry. In this paper we apply the transposition rule to certain K3 surfaces carrying a non-symplectic involution and we relate this to a mirror construction between families of lattice polarized K3 surfaces (we call it *lattice mirror symmetry*) due to Dolgachev and Nikulin [14, 24, 13], Voisin [28] and Borcea [4]. Our main Theorem 1.1 is that the transposition rule by Berglund and Hübsch, in this case, provides pairs of K3 surfaces which belong to lattice mirror families. The results of [13] and [28, Lemma 2.5 and §2.6] has in particular fundamental for our theorem (see subsection 4.2).

Let W denote a Delsarte type polynomial, i.e. a polynomial having as many monomials as variables (this will be called "potential" in the sequel, following the terminology of physicists). Assume that the matrix A of exponents of W is invertible (over $\mathbb{Q}$), that $\{W = 0\}$ has an isolated singularity at the origin and that it defines a well-formed hypersurface in some normalized weighted projective space. We denote by $\mathrm{Aut}(W)$ the group of diagonal symmetries of W, by $\mathrm{SL}(W)$ the group of diagonal symmetries of determinant one, and by J_W the monodromy group of the affine Milnor fibre associated to W. Let W^T denote the *transposed* potential defined by the matrix A^T. For any subgroup $G \subset \mathrm{Aut}(W)$, we denote by G^T

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the *transposed* group of automorphisms of the potential W^T, this can be defined as in [2, Section 4] as the kernel of the dual morphism between the dual groups $\mathrm{Aut}(W)^* \longrightarrow G^*$ (see Section 3 for many equivalent descriptions of G^T). The main result of this paper is the following.

Theorem 1.1. *Let W be a K3 surface defined by a non-degenerate and invertible potential of the form:*

$$x^2 = f(y, z, w)$$

in some weighted projective space. Let G_W be a subgroup of $\mathrm{Aut}(W)$ such that $J_W \subset G_W \subset \mathrm{SL}(W)$. Put $\widetilde{G_W} := G_W/J_W$ and $\widetilde{G_W^T} := G_W^T/J_{W^T}$. Then the Berglund-Hübsch-Chiodo-Ruan mirror orbifolds $[W/\widetilde{G_W}]$ and $[W^T/\widetilde{G_W^T}]$ belong to lattice mirror families.

The Berglund-Hübsch-Chiodo-Ruan (BHCR for short) mirror symmetry applies to Calabi-Yau varieties in weighted projective spaces which are not necessarily Gorenstein. As remarked by Chiodo and Ruan in [8, Section 1], this is the main difference with Batyrev mirror symmetry [1]. Most of our K3 surfaces are not contained in a Gorenstein weighted projective space.

The paper is organized as follows. In Section 2 we give some preliminary results about hypersurfaces in weighted projective spaces, potentials and the Berglund-Hübsch construction. In Section 3 we describe the group $\mathrm{Aut}(W)$ of diagonal automorphisms of a potential and we make a bridge between the different descriptions of the transposed group G^T of a subgroup G of $\mathrm{Aut}(W)$. Most of the results of the section are already contained in (or follow directly from results contained in) [18] and [19]. We give easier proofs using some simple (linear) algebra. Section 4 contains preliminary facts about non-symplectic involutions on K3 surfaces and introduces the lattice mirror construction. Section 5 deals with K3 surfaces defined by a potential as in the statement of Theorem 1.1: we study their singularities and we determine the basic invariants of the non-symplectic involution $x \mapsto -x$. In Section we give the proof of Theorem 1.1.

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2. The Berglund-Hübsch-Chiodo-Ruan construction

2.1. **Hypersurfaces in weighted projective spaces.** We start by recalling some basic facts about hypersurfaces in weighted projective spaces, see for example [16]. Let $x_1, \ldots, x_n$ be affine coordinates on $\mathbb{C}^n$, $n \geq 3$, and let $(w_1, \ldots, w_n)$ be a sequence of positive weights. The group $\mathbb{C}^*$ acts on $\mathbb{C}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n)$$

and the *weighted projective space* $\mathbb{P}(w_1, \ldots, w_n)$ is the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$. The weighted projective space is called *normalized* if

$$\gcd(w_1, \ldots, \widehat{w_i}, \ldots, w_n) = 1 \text{ for } i = 1, \ldots, n.$$

Weighted projective spaces are singular in general and the singularities arise only on the fundamental simplex Δ with vertices in the points $P_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, $i = 1, \ldots, n$. The vertices are singularities of type $1/w_i(w_1, \ldots, \widehat{w_i}, \ldots, w_n)$ and they are not necessarily isolated, since the higher dimensional toric strata of Δ can be singular too. For example, if $h_{i,j} := \gcd(w_i, w_j) > 1$, then the generic point of the edge P_iP_j is a singularity of type $1/h_{i,j}(w_1, \ldots, \widehat{w_i}, \ldots, \widehat{w_j}, \ldots, w_n)$. The weighted projective space $\mathbb{P}(w_1, \ldots, w_n)$ has Gorenstein singularities if and only if $w_j | \sum_{i=1}^n w_i$ for all j. This is also equivalent to say that the weighted projective space is Fano or finally, regarding $\mathbb{P}(w_1, \ldots, w_n)$ as toric variety, that its associated polytope is reflexive [11, Section 3.5].

A quasihomogeneous polynomial $W(x_1, \ldots, x_n)$ of total degree d defines a hypersurface in $\mathbb{P}(w_1, \ldots, w_n)$, which is also denoted by W in the sequel.

Assume that the weighted projective space is normalized. The hypersurface W is called

- *well-formed* if $\mathrm{codim}_W(W \cap \mathbb{P}_{sing}) \geq 2$ where $\mathbb{P}_{sing}$ denotes the singular locus of $\mathbb{P}(w_1, \ldots, w_n)$ (cf. [12, Definition 1], where the author calls it *general position*). This is equivalent to ask that
$$\gcd(w_1, \ldots, \widehat{w_i}, \ldots, \widehat{w_j}, \ldots, w_n) \text{ divides } d$$
for all $i, j = 1, \ldots, n$ (see e.g. [16, Definition 6.9, §6.10, Note 6.13]);
- *quasismooth* if the polynomial W is non-degenerate, i.e. its affine cone is smooth outside its vertex $(0, \ldots, 0)$;
- *Calabi-Yau* (*K3 surface* in the two-dimensional case) if it has canonical singularities (in particular W is Gorenstein), its canonical bundle is trivial and $H^i(W, \mathcal{O}_W) = 0$ for all $i = 1, \ldots, n-3$.

Observe that by [10, Lemma 1.12] a well-formed and quasismooth hypersurface W in $\mathbb{P}(w_1, \ldots, w_n)$ is Calabi-Yau if and only if $d = \sum_{i=1}^n w_i$. On the other hand by [12, Proposition 6] if $n \geq 5$ a quasismooth hypersurface in a normalized weighted projective space that satisfies the Calabi-Yau condition is automatically well-formed. Reid in [26] and Yonemura in [29] give a list of all possible families of K3 surfaces in weighted projective spaces. These are 95 in total and only 14 of the weighted projective spaces are Gorenstein. For each type Reid describes the singularities of the K3 surface. By [10] the 95 projective spaces have canonical singularities, and in fact one can determine 104 families of weights such that the weighted projective spaces have canonical singularities. However in 9 cases one can not obtain K3 surfaces with canonical singularities [10, Theorem 1.17].

Finally, we recall that the genus of a smooth curve C_d of total degree d in $\mathbb{P}(w_1, w_2, w_3)$ is given by the formula (see [16, Theorem 12.2]):

$$g(C_d) = \frac{1}{2}\left(\frac{d^2}{w_1w_2w_3} - d\sum_{i>j}\frac{\gcd(w_i, w_j)}{w_iw_j} + \sum_{i=1}^{3}\frac{\gcd(d, w_i)}{w_i} - 1\right). \tag{1}$$

2.2. **Invertible potentials.** We briefly recall the mirror construction of Berglund-Hübsch in [3] and Chiodo-Ruan in [8]. Consider a *potential*:

$$W(x_1, \ldots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}},$$

that is a polynomial in n variables containing n monomials. Since we have n monomials it is not a restriction to consider all the coefficients to be equal to 1, so that a potential is determined by the matrix $A := (a_{ij})_{i,j=1,\ldots,n}$. The potential is called *invertible* if the matrix A is invertible over $\mathbb{Q}$. In this case we denote by

$A^{-1} := (a^{ij})_{i,j=1,\dots,n}$ the inverse matrix and define the *charge* $q_i := \sum_{j=1}^n a^{ij}$ as the sum of the entries of the i-th row of A^{-1}. Clearly the charges q_i satisfy:

$$A \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Let d be the least common denominator of the charges and let $w_i := dq_i$. Then $\{W = 0\}$ defines a hypersurface W in $\mathbb{P}(w_1, \dots, w_n)$ of total degree d. We assume that the weighted projective space is normalized, that W is quasismooth and verifies the Calabi-Yau condition. Hence if $n \geq 5$ it is well-formed as observed above. If $n = 3$ or 4 one can check by hand, using [20, Theorem 1] (see also subsection 2.2 below), that every quasismooth invertible potential in a normalized weighted projective space satisfying the Calabi-Yau condition is well-formed.

By [20, Theorem 1] a potential W is invertible and non-degenerate (i.e. the corresponding hypersurface is quasismooth) if and only if it can be written as a sum of invertible potentials of *atomic types*:

$$\begin{array}{lcl} W_{Fermat} & := & x^a, \\ W_{loop} & := & x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, \\ W_{chain} & := & x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}. \end{array}$$

If W is a Fermat type potential (i.e. sum of W_{Fermat}) then W defines a hypersurface in a Gorenstein weighted projective space. In the other cases this is not true in general.

2.3. **The Berglund-Hübsch-Chiodo-Ruan construction.** Given an invertible and non-degenerate potential W as in the previous subsection, we consider the group of diagonal automorphisms:

$$\mathrm{Aut}(W) := \{\gamma = (\gamma_1, \dots, \gamma_n) \in (\mathbb{C}^*)^n \mid W(\gamma_1 x_1, \dots, \gamma_n x_n) = W(x_1, \dots, x_n)\}$$

and its subgroup

$$\mathrm{SL}(W) := \mathrm{Aut}(W) \cap \mathrm{SL}_n(\mathbb{C}).$$

To each column of A^{-1} we associate the diagonal matrix

$$\rho_j := \mathrm{diag}(\exp(2\pi i a^{1,j}), \dots, \exp(2\pi i a^{n,j})) \in \mathrm{Aut}(W)$$

and we define the matrix j_W to be the product

$$\rho_1 \cdots \rho_n = \mathrm{diag}(\exp(2\pi i q_1), \dots, \exp(2\pi i q_n)).$$

Observe that the group J_W generated by j_W is cyclic of order d and acts trivially on the hypersurface W, since it acts trivially on the weighted projective space $\mathbb{P}(w_1, \dots, w_n)$. In what follows we assume the hypersurface W to be Calabi-Yau. Then $\sum_i q_i = 1$, so that $J_W \subset \mathrm{SL}(W)$.

Let G_W be a group of diagonal automorphisms such that $J_W \subset G_W \subset \mathrm{SL}(W)$ and let $\widetilde{G_W} := G_W / J_W$. We will now construct a potential W^T and a group G_W^T of symmetries of W^T. The potential W^T is defined by transposing the matrix A:

$$W^T := W^T(x_1, \dots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ji}}.$$

Similarly as before, we denote by q_j^T the charges of W^T, which are the sums of the entries of the j-th columns of A^{-1}. Observe that $\sum_j q_j = \sum_j q_j^T = \sum_{i,j} a^{i,j} = 1$.

Since the potential W^T is non-degenerate by the classification in [20] and the charges satisfy $\sum_j q_j^T = 1$, it is easy to show that the equation $\{W^T = 0\}$ defines a variety in a normalized weighted projective space. By the discussion in subsection 2.1 this is also well-formed.

Remark 2.1. Without the condition $\sum_i q_i = 1$ it is not true that the equation $\{W^T = 0\}$ defines a variety in a normalized projective space. For example $W = x_1^5 x_2 + x_2^2 x_3 + x_3^3 x_4 + x_4^9$ defines a surface in $\mathbb{P}(7, 19, 16, 6)$ and W^T defines a surface in $\mathbb{P}(9, 18, 9, 4)$, which is clearly not normalized.

The group G_W^T is defined by Krawitz in [18] as:

$$G_W^T = \left\{ \prod_{j=1}^n (\rho_j^T)^{m_j} \mid \prod_{j=1}^n x_j^{m_j} \text{ is } G_W\text{-invariant} \right\},$$

where the definition of the automorphisms ρ_j^T of W^T is similar to the definition of ρ_j using the matrix A^T. Equivalent definitions for the group G_W^T will be given in the next section. The group satisfies $J_{W^T} \subset G_W^T \subset \mathrm{SL}_{W^T}$. Putting $\widetilde{G_W^T} := G_W^T / J_{W^T}$, we have the following result.

Theorem 2.2. [8, Theorem 2] *The Calabi-Yau orbifolds $[W/\widetilde{G_W}]$ and $[W^T/\widetilde{G_W^T}]$ form a mirror pair, i.e. we have*

$$H_{\mathrm{CR}}^{p,q}([W/\widetilde{G_W}], \mathbb{C}) \cong H_{\mathrm{CR}}^{n-2-p,q}([W^T/\widetilde{G_W^T}], \mathbb{C})$$

where $H_{\mathrm{CR}}(-, \mathbb{C})$ stands for the Chen-Ruan orbifold cohomology.

The previous result clearly gives no information in the case of K3 surfaces, since all K3 surfaces have the same Hodge diamond. However, it is a strong motivation for considering this as a good mirror correspondence even in the two-dimensional case.

We now show that the action of $\mathrm{SL}(W)$ is symplectic. This was surely already known by the experts but since we have not found an explicit proof in the literature, for convenience we give it again.

Proposition 1. *Let W be a non-degenerate potential defining a Calabi-Yau manifold in $\mathbb{P}(w_1, \ldots, w_n)$. Then the action of $\mathrm{SL}(W)$ on the volume form is trivial.*

Proof. We can write the volume form locally for $x_1 \neq 0$ and $\frac{\partial W}{\partial x_n} \neq 0$ as

$$\xi := \frac{dx_2 \wedge \ldots \wedge dx_{n-1}}{\frac{\partial W}{\partial x_n}}.$$

Let $g = (\exp(2\pi i \alpha_1), \ldots, \exp(2\pi i \alpha_n)) \in \mathrm{SL}(W)$. We can normalize g by multiplying by $\exp(2\pi i(-\alpha_1/w_1))$, so that we obtain $g = (1, \exp(2\pi i \beta_2), \ldots, \exp(2\pi i \beta_n))$ with $\beta_i = \alpha_i - (w_i/w_1)\alpha_1$. If we apply this transformation to W, it is multiplied by $\exp(2d\pi i(-\alpha_1/w_1))$. We have that

$$g \frac{\partial W}{\partial x_n} = \exp(2\pi i(-\beta_n - (\alpha_1/w_1)d))W$$

hence the form ξ is multiplied by $\exp(2\pi i \delta)$, with

$$\delta = \beta_1 + \ldots + \beta_n + \frac{\alpha_1}{w_1} d = \alpha_1 + \alpha_2 + \ldots + \alpha_n \in \mathbb{Z}. \quad \square$$

3. The group of diagonal automorphisms

3.1. **Description of** $\mathrm{Aut}(W)$**.** Let $W : \mathbb{C}^n \to \mathbb{C}$ be a non-degenerate, invertible potential and let $A = (a_{ij})_{i,j} \in \mathrm{GL}(n, \mathbb{Q})$ be the associated matrix. In this section we describe the group $\mathrm{Aut}(W)$ of diagonal automorphisms of W and its subgroups. Several results (Lemma 1, Proposition 2, Lemma 2, Proposition 3) were already proved by Krawitz in [18] or follow directly from his results. He was in fact the first to start a systematic study of Berglund-Hübsch construction and to generalize the construction to any subgroup $J_W \subset G_W \subset \mathrm{SL}(W)$. Proposition 2 is also an immediate consequence of [19]. We give here new easier proofs using simple (linear) algebra arguments.

Lemma 1. $\mathrm{Aut}(W)$ *is finite.*

Proof. Since $\mathrm{Aut}(W)$ is abelian, it is enough to prove that any of its elements has finite order. If $\gamma \in \mathrm{Aut}(W)$ then $\prod_{j=1}^n \gamma_j^{a_{ij}} = 1$ for any $i \in \{1, \ldots, n\}$, in particular $\prod_{j=1}^n |\gamma_j|^{a_{ij}} = 1$. Thus, taking the logarithm we obtain that

$$(\ln|\gamma_1|, \ldots, \ln|\gamma_n|) \in \ker(A) = \{0\},$$

which implies that $|\gamma_i| = 1$. Thus $\gamma_i = \exp(2\pi i a_i)$, $a_i \in \mathbb{R}$, and the previous condition on γ can be translated as $A \cdot a \in \mathbb{Z}^n$, where $a = (a_1, \ldots, a_n)$. Since A has integral entries, then $a \in \mathbb{Q}^n$, so that γ has finite order. □

After writing $\gamma = (\exp(2\pi i a_1), \ldots, \exp(2\pi i a_n))$ with $a_i \in \mathbb{Q}$, we can identify $\mathrm{Aut}(W)$ with

$$\{a = (a_1, \ldots, a_n) \in (\mathbb{Q}/\mathbb{Z})^n | A \cdot a \in \mathbb{Z}^n\} = A^{-1}\mathbb{Z}^n/\mathbb{Z}^n.$$

In particular, $|\mathrm{Aut}(W)| = |\det(A)|$. Similarly, we will identify $\mathrm{Aut}(W^T)$ with $(A^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. We observe that, since A and A^T have the same Smith normal form, then $\mathrm{Aut}(W) \cong \mathrm{Aut}(W^T)$.

By means of the previous description, since $\mathrm{Aut}(W)$ is generated by the columns of A^{-1}, we obtain the following result (see also [18, Lemma 1.6] and [19]).

Proposition 2.

1) $\mathrm{Aut}(W_{Fermat}) \cong \mathbb{Z}/a\mathbb{Z}$ *and a generator is* $\frac{1}{a}$,
2) $\mathrm{Aut}(W_{loop}) \cong \mathbb{Z}/(a_1 \cdots a_n + (-1)^{n+1})\mathbb{Z}$ *and a generator is* $(\varphi_1, \ldots, \varphi_n)$, *where*
$$\varphi_1 := \frac{(-1)^n}{\Gamma}, \quad \varphi_i := \frac{(-1)^{n+1-i} a_1 \cdots a_{i-1}}{\Gamma}, \ i \geq 2.$$
3) $\mathrm{Aut}(W_{chain}) \cong \mathbb{Z}/(a_1 \cdots a_n)\mathbb{Z}$ *and a generator is given by* $(\varphi_1, \ldots, \varphi_n)$, *where*
$$\varphi_i := \frac{(-1)^{n+i}}{a_i \cdots a_n}.$$

A subgroup G of $\mathrm{Aut}(W)$ is given by $C^{-1}\mathbb{Z}^n/\mathbb{Z}^n$, where $C \in \mathrm{M}(n, \mathbb{Z})$ is a matrix invertible over $\mathbb{Q}$ such that the columns of $C^{-1} \in \mathrm{M}(n, \mathbb{Q})$ are spanned by the columns of A^{-1}, i.e. $C^{-1} = A^{-1}B$ for some $B \in \mathrm{M}(n, \mathbb{Z})$.

Remark 3.1. Let $J_W = \langle q \rangle$, where $q = (q_1, \ldots, q_n)$ is the vector of charges and let $C_0 \in \mathrm{M}(n, \mathbb{Z})$ be such that $J_W = C_0^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. In this case the columns of C_0^{-1} are a basis of the lattice L generated by the canonical basis $e_1, \ldots, e_n$ and the vector q. Such a basis can be obtained as follows: let $w = (w_1, \ldots, w_n)$ be the vector of weights and let $M \in \mathrm{GL}(n,\mathbb{Z})$ such that $Mw = e_1$ (this is possible since w is primitive), then a basis of L is given by $q, M^{-1}e_2, \dots, M^{-1}e_n$:

$$C_0^{-1} = \begin{pmatrix} q & M^{-1}e_2 & \dots & M^{-1}e_n \end{pmatrix}$$

In what follows e will be the column vector with all entries equal to 1.

Lemma 2. *$J_W \subset G$ if and only if $B^{-1}e \in \mathbb{Z}^n$ and $G \subset \mathrm{SL}(W)$ if and only if $(C^T)^{-1}e \in \mathbb{Z}^n$.*

Proof. Recall that J_W is generated by $q = A^{-1}e$. Thus $J_W \subset G$ if and only if $CA^{-1}e = B^{-1}e \in \mathbb{Z}^n$. The group G is contained in $\mathrm{SL}(W)$ if and only if $\sum_{i=1}^n a_i = a \cdot e \in \mathbb{Z}$ for all $a = (a_1, \dots, a_n) \in G$. Equivalently

$$(C^{-1}u)^T e = u^T (C^T)^{-1} e \in \mathbb{Z}$$

for all $u \in \mathbb{Z}^n$, i.e. $(C^T)^{-1}e \in \mathbb{Z}^n$. □

3.2. **Description of G_W^T.** Given a subgroup $G = C^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ of $\mathrm{Aut}(W)$, where $C^{-1} = A^{-1}B \in \mathrm{M}(n,\mathbb{Z})$, we define the *transposed group* G^T in $\mathrm{Aut}(W^T)$ as:

$$G^T := (B^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n.$$

As a consequence of the previous description of the group G we have the following properties.

Proposition 3.
1) $|G| = |\det(C)|$ *and* $|G^T| = |\det(B)|$,
2) $(G^T)^T = G$,
3) $\{0\}^T = \mathrm{Aut}(W^T)$ *and* $\mathrm{Aut}(W)^T = \{0\}$,
4) $J_W^T = \mathrm{SL}(W^T)$,
5) *if* $G_1 \subset G_2$, *then* $G_2^T \subset G_1^T$ *and* $G_2/G_1 \cong G_1^T/G_2^T$.

Proof. We will prove statement 4), the remaining ones follow easily from the definition. Let C_0^{-1} be a matrix corresponding to J_W as in Remark 3.1 and let $B_0 = AC_0^{-1}$. By Lemma 2, J_W^T is contained in $\mathrm{SL}(W^T)$ if $B_0^{-1}e \in \mathbb{Z}^n$. Equivalently:

$$C_0 A^{-1} e = C_0 q \in \mathbb{Z}^n,$$

which clearly holds since $q = C_0^{-1}e_1$.

Conversely, let $a = (A^T)^{-1}v = (B_0^T)^{-1}(C_0^T)^{-1}v \in \mathrm{Aut}(W^T)$, $v \in \mathbb{Z}^n$, such that $a \cdot e \in \mathbb{Z}$. We show that $(C_0^T)^{-1}v \in \mathbb{Z}^n$. The condition $a \cdot e \in \mathbb{Z}$ is equivalent to:

$$a^T A q = a^T A C_0^{-1} e_1 = v^T C_0^{-1} e_1 \in \mathbb{Z},$$

i.e. $(C_0^T)^{-1}v \cdot e_1 \in \mathbb{Z}^n$. Since the columns of C_0^{-1}, except for the first one, have integral entries, this is enough to prove that $(C_0^T)^{-1}v \in \mathbb{Z}^n$. Thus $a \in J_W^T$. □

Remark 3.2. By Proposition 3 it follows that $J_W \subset G$ if and only if $G^T \subset \mathrm{SL}(W^T)$ and $J_{W^T} \subset G^T$ if and only if $G \subset \mathrm{SL}(W)$. Moreover, $\mathrm{SL}(W^T)/J_{W^T}$ is isomorphic to $\mathrm{SL}(W)/J_W$, so that $\mathrm{SL}(W^T) = J_{W^T}$ if and only if $J_W = \mathrm{SL}(W)$.

3.3. **The group** $\mathrm{SL}(W)$**.** We will now determine the order of the subgroup $\mathrm{SL}(W) = \mathrm{Aut}(W) \cap \mathrm{SL}_n(\mathbb{C})$.

Corollary 1. *The order of* $\mathrm{SL}(W)$ *is equal to* $|\det(A)|/d^T$*, where* d^T *is the least common denominator of the charges of* W^T*.*

Proof. By Proposition 3, $\mathrm{SL}(W^T) = J_W^T$ and $|\,\mathrm{SL}(W^T)| = |\det(B_0)|$ where $B_0 = AC_0^{-1}$ and C_0 is given in Remark 3.1. Observe that

$$MC_0^{-1} = \begin{pmatrix} e_1/d & e_2 & \dots & e_n \end{pmatrix}$$

since $Mq = M(w/d) = e_1/d$. Thus

$$|\,\mathrm{SL}(W^T)| = |\det(A)\det(C_0^{-1})| = |\det(A)\det(MC_0^{-1})| = \frac{|\det(A)|}{d}.$$

Changing W with W^T we get the statement. □

Proposition 4. *Let* $W : \mathbb{C}^4 \to \mathbb{C}$ *be a well-formed potential of the form*

$$W(x, y, z, w) = x^2 - f(y, z, w)$$

and let $A = (a_{ij})_{i,j=1,2,3}$ *be the matrix associated to* f*.*

- *If* f *is of chain type, then* $|\,\mathrm{SL}(W)| = 2\gcd(a_1a_2a_3, 1 - a_1 + a_1a_2)$*, where* $a_{11} = a_1$*,* $a_{12} = 1$*,* $a_{22} = a_2$*,* $a_{23} = 1$*,* $a_{33} = a_3$*.*
- *If* f *is of loop type, then* $|\,\mathrm{SL}(W)| = 2\gcd(1 + a_1a_2a_3, 1 - a_1 + a_1a_2)$*, where* $a_{11} = a_1$*,* $a_{12} = 1$*,* $a_{22} = a_2$*,* $a_{23} = 1$*,* $a_{31} = 1$*,* $a_{33} = a_3$*.*
- *If* f *is of Fermat type, then* $|\,\mathrm{SL}(W)| = \frac{2a_1a_2a_3}{\mathrm{lcm}(a_1,a_2,a_3)}$*, where* $a_{11} = a_1$*,* $a_{22} = a_2$*,* $a_{33} = a_3$*.*
- *If* f *is of chain+Fermat type, then* $|\,\mathrm{SL}(W)| = 2a_3\gcd(a_1a_2, a_1 - 1)$ *if* $\gcd(d^T, w_3^T) = 1$ *and* $|\,\mathrm{SL}(W)| = a_3\gcd(a_1a_2, a_1 - 1)$ *otherwise (i.e.* $\gcd(d^T, w_3^T) = 2$*), where* $a_{11} = a_1$*,* $a_{12} = 1$*,* $a_{22} = a_2$*,* $a_{33} = a_3$*.*
- *If* f *is of loop+Fermat type, then* $|\,\mathrm{SL}(W)| = 2a_3\gcd(a_1a_2 - 1, a_2 - 1)$ *if* $\gcd(d^T, w_3^T) = 1$ *and* $|\,\mathrm{SL}(W)| = a_3\gcd(a_1a_2 - 1, a_2 - 1)$ *otherwise (i.e.* $\gcd(d^T, w_3^T) = 2$*), where* $a_{11} = a_1$*,* $a_{12} = 1$*,* $a_{21} = 1$*,* $a_{22} = a_2$*,* $a_{33} = a_3$*.*

Proof. In all cases we will apply Corollary 1. We denote by A the matrix associated to the potential W, by $w^T = (w_1^T, w_2^T, w_3^T, w_4^T)$ the weight vector of W^T and by d^T the degree of the hypersurface W^T in $\mathbb{P}(w^T)$.

If f is of chain type, then $\det(A) = 2a_1a_2a_3$. From the linear system

$$A^T \cdot w^T = d^T e \tag{2}$$

we obtain that

$$\frac{2a_1a_2a_3}{d^T} w_4^T = 2(1 - a_1 + a_1a_2).$$

Let $m := \gcd(d^T, w_4^T)$. Observe that m divides w_2^T, w_3^T, w_4^T by (2). Since W is well-formed, this implies that $m = 1$, so that $|\,\mathrm{SL}(W)| = \frac{\det(A)}{d^T} = 2\gcd(a_1a_2a_3, 1 - a_1 + a_1a_2)$. The case when f is of loop type is similar to the previous one.

If f is of chain+Fermat type, then looking at the equation of the linear system (2) coming from the chain part we obtain that

$$\frac{a_1a_2}{d^T} w_3^T = a_1 - 1.$$

Let $m := \gcd(d^T, w_3^T)$. Observe that m divides w_2^T, w_3^T, $2w_1^T$ and $a_3w_4^T$. Thus, since W is well-formed, m is either 1 or 2. Moreover, again since W is well-formed, $m = 2$ implies a_3 is even. If $m = 1$, then $|\operatorname{SL}(W)| = 2a_3 \gcd(a_1a_2, a_1 - 1)$, otherwise $|\operatorname{SL}(W)| = a_3 \gcd(a_1a_2, a_1 - 1)$. The case when f is of loop+Fermat type is similar. □

Remark 3.3. The formulas given in Proposition 4 for the chain, loop and Fermat case can be easily generalized to the case of a higher dimensional well-formed potential of type $x^2 = f(x_1, \dots, x_n)$. Let $\Theta = 1 + \sum_{j=1}^{n-1}(-1)^j a_1 \cdots a_j$. In the chain case $|\operatorname{SL}(W)| = 2\gcd(a_1 \cdots a_n, \Theta)$, in the loop case $|\operatorname{SL}(W)| = 2\gcd((-1)^{n-1} + a_1 \cdots a_n, \Theta)$ and finally in the Fermat case $|\operatorname{SL}(W)| = \frac{2a_1\cdots a_n}{\operatorname{lcm}(a_1,\dots,a_n)}$.

3.4. **Relation with Borisov's description.** In this subsection we relate the definition of transposed group with the one given in [5]. Let $M_0^* = N_0 = \mathbb{Z}^n$ and $\xi : N_0 \to M_0^*,\ u \mapsto Av$. We have an exact sequence

$$0 \to N_0 \xrightarrow{\xi} M_0^* \xrightarrow{f} \operatorname{Aut}(W) \to 0,$$

where $f(e_i) = A^{-1}e_i$. Thus f induces an isomorphism $\operatorname{Aut}(W) \cong M_0^*/\xi(N_0) = \mathbb{Z}^n/A\mathbb{Z}^n$. The dual of ξ gives the exact sequence

$$0 \to M_0 \xrightarrow{\xi^*} N_0^* \xrightarrow{f^T} \operatorname{Aut}(W^T) \to 0,$$

where $f^T(e_i) = (A^{-1})^T e_i$, giving isomorphisms

$$\operatorname{Aut}(W^T) \cong \operatorname{Ext}^1(\operatorname{Aut}(W), \mathbb{Z}) \cong N_0^*/\xi^*(M_0) = \mathbb{Z}^n/A^T\mathbb{Z}^n.$$

Let G be a subgroup of $\operatorname{Aut}(W)$. Then there is a submodule $N = B\mathbb{Z}^n$ of M_0^* containing $\xi(N_0)$ such that $G \cong N/\xi(N_0) \cong B\mathbb{Z}^n/A\mathbb{Z}^n$. Observe that we can write $A = BC$, where B, C are integral matrices invertible over $\mathbb{Q}$. Consider the chain of inclusions

$$\xi(N_0) = A\mathbb{Z}^n \hookrightarrow N = B\mathbb{Z}^n \hookrightarrow \mathbb{Z}^n = M_0^*,$$

and its dual

$$M_0 = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}^n) \to N^* = \operatorname{Hom}(B\mathbb{Z}^n, \mathbb{Z}) \to \xi(N_0)^* = \operatorname{Hom}(A\mathbb{Z}^n, \mathbb{Z}).$$

We identify $\operatorname{Hom}(A\mathbb{Z}^n, \mathbb{Z})$ with $\mathbb{Z}^n$ via the homomorphism given by the dual of $\xi : N_0 \to \xi(N_0)$:

$$h : \operatorname{Hom}(A\mathbb{Z}^n, \mathbb{Z}) \to \mathbb{Z}^n,\ (A^T)^{-1}e_i \mapsto e_i.$$

Thus $h(N^*) = C^T\mathbb{Z}^n$ and $h(M_0) = A^T\mathbb{Z}^n$. According to Borisov's definition

$$G^T := N^*/M_0 \overset{h}{\cong} C^T\mathbb{Z}^n/A^T\mathbb{Z}^n \overset{f^T}{\cong} (B^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n,$$

which agrees with the definition given in the first section.

3.5. *G* **and** *G^T* **as orthogonal groups.** Consider the bilinear form $b : \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$ given by $b(u, v) = u^T Av$. Observe that this induces a bilinear form

$$\bar{b} : \operatorname{Aut}(W^T) \times \operatorname{Aut}(W) \to \mathbb{Q}/\mathbb{Z},$$

where we recall that $\operatorname{Aut}(W^T) = (A^T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ and $\operatorname{Aut}(W) = A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. In fact $\bar{b}$ is well defined since, if $u, v \in \mathbb{Z}^n$, then

$$b(u, A^{-1}v) = u^T AA^{-1}v = u^T v \in \mathbb{Z}, \quad b((A^T)^{-1}u, v) = u^T A^{-1}Av = u^T v \in \mathbb{Z}.$$

Let $G = C^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ be a subgroup of $\operatorname{Aut}(W)$. We show that G^T is the orthogonal of G with respect to $\bar{b}$.

Lemma 3.
$$G^T = \{x \in \mathrm{Aut}(W^T) : \bar{b}(x,y) = 0,\ \forall y \in G\}.$$

Proof. Let $u \in \mathbb{Z}^n$ and $x = (A^T)^{-1}u \in \mathrm{Aut}(W^T)$. We have that
$$\bar{b}(x, C^{-1}v) = u^T A^{-1} A C^{-1} v = u^T C^{-1} v = 0$$
for all $v \in \mathbb{Z}^n$, if and only if $(C^T)^{-1}u \in \mathbb{Z}^n$, i.e. $(A^T)^{-1}u \in (A^T)^{-1}C^T\mathbb{Z}^n = (B^T)^{-1}\mathbb{Z}^n$, which means that $x \in G^T$. □

This remark relates our definition of transposed group with the one given in [15].

3.6. **Relation with Berglund and Henningson's description.** Let $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ be the dual group of G, that is isomorphic to the set of irreducible representations of G endowed with the tensor product. Then the natural inclusion of G in $\mathrm{Aut}(W)$ gives a dual surjective morphism $\mathrm{Aut}(W)^* \longrightarrow G^*$. In [2, Section 4] Berglund and Henningson define G^T as the kernel of this morphism which in our paper, following the literature, we call *transposed* group, because it arises when applying the "transposition rule" for the duality between the potentials. Using the preceding description it is easy to show that $\mathrm{Aut}(W)/G \cong B^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ so taking the duals the kernel of the dual morphism is isomorphic to $(B^{-1})^T\mathbb{Z}^n/\mathbb{Z}^n$, which corresponds to our definition of G^T.

3.7. **Relation with Krawitz's description.** According to Krawitz's definition in [18] the transposed group is
$$G^T = \left\{ \prod_{j=1}^n (\rho_j^T)^{m_j} : \prod_{j=1}^n x_j^{m_j} \text{ is } G\text{-invariant} \right\}.$$
Observe that $\prod_{j=1}^n (\rho_j^T)^{m_j}$ corresponds, in $\mathrm{Aut}(W^T)$, to $\sum_j m_j (A^T)^{-1} e_j = (A^T)^{-1} m$, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Moreover, $\prod_{j=1}^n x_j^{m_j}$ is G-invariant if and only if $\sum_j m_j a_j \in \mathbb{Z}$ for all $a = (a_1, \ldots, a_n) \in G$, i.e.
$$\sum_j m_j a_j = (m^T A^{-1}) A a = \bar{b}((A^T)^{-1} m, a) = 0.$$
Thus G^T is the orthogonal complement of G with respect to $\bar{b}$, in agreement with Lemma 3.

4. The lattice mirror symmetry for K3 surfaces

4.1. **K3 surfaces with non-symplectic involutions.** We briefly recall the classification theorem for non-symplectic involutions on K3 surfaces given by Nikulin in [22, §4] and [25, §4]. Let X be a K3 surface and ι be non-symplectic involution of X. The local action of ι at a fixed point is of type:
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
so that the fixed locus X^ι is the disjoint union of smooth curves and there are no isolated fixed points. The invariant lattice:
$$H^2(X,\mathbb{Z})^+ := \{x \in H^2(X,\mathbb{Z}) \mid \iota^* x = x\}$$
is 2-elementary, i.e the discriminant group $(H^2(X,\mathbb{Z})^+)^\vee / H^2(X,\mathbb{Z})^+$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ for some non negative integer a. According to Rudakov-Shafarevich

in [27], the isometry class of such lattice is determined by the invariants r, a and δ, where $r = \operatorname{rk} H^2(X,\mathbb{Z})^+$ and $\delta \in \{0,1\}$ is 0 if and only if $x^2 \in \mathbb{Z}$ for any $x \in (H^2(X,\mathbb{Z})^+)^\vee$. Equivalently, by [22, §4], $\delta = 0$ if and only if the class of the fixed locus of ι is divisible by two in $H^2(X,\mathbb{Z})$.

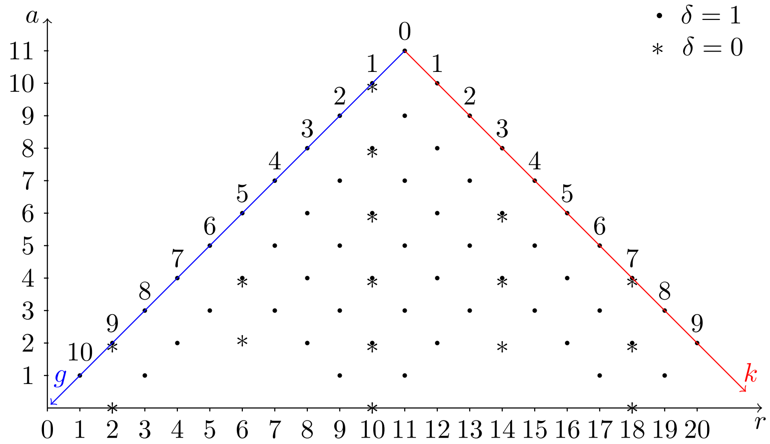

FIGURE 1. Nikulin classification

Theorem 4.1. [22, Theorem 4.2.2] *The fixed locus of a non-symplectic involution on a K3 surface is*

- *empty if $r = 10$, $a = 10$ and $\delta = 0$,*
- *the disjoint union of two elliptic curves if $r = 10$, $a = 8$ and $\delta = 0$,*
- *the disjoint union of a curve of genus g and k rational curves otherwise, where $g = (22 - r - a)/2$, $k = (r - a)/2$.*

Figure 1 shows all the values of the triple (r, a, δ) which are realized and the corresponding invariants (g, k) of the fixed locus.

We now assume that X carries a symplectic automorphism σ of prime order commuting with ι. The minimal resolution Z of $X/\langle\sigma\rangle$ is known to be a K3 surface and ι lifts to a non-symplectic involution j on Z. The following proposition relates the invariants of ι and j. We denote by $\delta(\iota)$ and $\delta(j)$ the δ-invariants of the invariant lattices of ι and j. We recall that the order of a symplectic automorphism of prime order p on a K3 surface is either $2, 3, 5$ or 7 by [23].

Proposition 5. *Let X be a K3 surface carrying a non-symplectic involution ι and a symplectic automorphism σ of prime order $p > 2$ commuting with ι. Then ι induces a non-symplectic involution j on the minimal resolution Z of $X/\langle\sigma\rangle$ such that $\delta(\iota) = \delta(j)$.*

Proof. Let $\pi : X \to X/\langle\sigma\rangle$ be the natural quotient map. Since σ is symplectic, the set F of its fixed points is finite and the quotient $X/\langle\sigma\rangle$ has singular points of type A_{p-1} at $S = \pi(F)$ (see [23]). We denote by $r : Z \to X/\langle\sigma\rangle$ the minimal resolution of $X/\langle\sigma\rangle$ and by $E_1^q, \dots, E_{p-1}^q$ the irreducible components of $r^{-1}(q)$, $q \in S$, with $E_i^q \cdot E_{i+1}^q = 1$, $i = 1, \dots, p-2$.

Since ι commutes with σ, it induces an involution on $X/\langle\sigma\rangle$ which lifts to a non-symplectic involution j on Z. Let X^ι be the fixed locus of ι and Z^j be the fixed locus of j. Since the orders of ι and σ are relatively prime, the fixed locus of the involution induced by ι on $X/\langle\sigma\rangle$ coincides with $\pi(X^\iota)$. Observe that in general this is not a Cartier divisor since it passes through the singular points of $X/\langle\sigma\rangle$. Taking the pull-back of $\pi(X^\iota)$ to Z we obtain the following $\mathbb{Q}$-divisor:

$$r^*(\pi(X^\iota)) = \tilde{X}^\iota + \frac{1}{p} \sum_{q\in S\cap\pi(X^\iota)} \sum_{i=1}^{p-1} iE_i^q, \tag{3}$$

where $\tilde{X}^\iota$ is the proper transform of $\pi(X^\iota)$ and it only intersects E_{p-1}^q for any $q \in S\cap\pi(X^\iota)$. Since j leaves invariant the exceptional divisors over $S\cap\pi(X^\iota)$ and, being non-symplectic, it only fixes smooth disjoint curves, we have

$$Z^j = \tilde{X}^\iota + \sum_{k=1}^{\frac{p-1}{2}} \sum_{q\in S\cap\pi(X^\iota)} E_{2k-1}^q. \tag{4}$$

Observe that the natural inclusions induce isomorphisms $\mathrm{Cl}(X) \cong \mathrm{Cl}(X-F)$ and $\mathrm{Cl}(X/\langle\sigma\rangle) \cong \mathrm{Cl}(X/\langle\sigma\rangle - S)$ since F and S have codimension two. Moreover $\pi_0 : X - F \to X/\langle\sigma\rangle - S$ is an unramified covering. Now assume that $\alpha := [X^\iota]$ is divisible by two in $H^2(X,\mathbb{Z})$ or, equivalently in $\mathrm{Cl}(X)$, and let $\beta := [\pi_0(X^\iota)]$. Then by projection formula $\pi_{0*}(\alpha) = \pi_{0*}\pi_0^*(\beta) = p\beta$ is also divisible by two. From equalities (3) and (4) we get:

$$r^*(p\beta) \equiv [Z^j] \pmod 2.$$

Thus $[Z^j]$ is divisible by two. Conversely, if $[Z^j]$ is divisible by two, the same is true for $r^*(p\beta)$ by the previous congruence. By projection formula β is divisible by two in $\mathrm{Cl}(X/\langle\sigma\rangle)$, thus the same is true for $\pi_0^*(\beta) = \alpha$. □

Remark 4.2. Equation 3 can be checked by means of a local computation at an A_{p-1} singularity, for example computing the pull-back of the invariant divisors of the toric variety associated to the fan with rays $(p, 1-p), (0,1)$ by means of MAGMA [6].

4.2. **The lattice mirror symmetry.** Let M be an even non-degenerate lattice of signature $(1, \rho-1)$, $1 \le \rho \le 19$.

Definition 4.3. An *M-polarized* K3 surface is a pair (X, j) where X is a K3 surface and $j : M \hookrightarrow \mathrm{Pic}(X)$ is a primitive lattice embedding.

Dolgachev in [13] constructs a (coarse) moduli space $\mathbf{K}_M$ parametrizing M-polarized K3 surfaces, which has dimension $20-\rho$. Assume now that

$$M^\perp \cap H^2(X,\mathbb{Z}) \cong U \oplus \bar{M},$$

where U is a copy of the hyperbolic plane. As described in [13] one can define the *mirror moduli space* of $\mathbf{K}_M$ as the moduli space $\mathbf{K}_{\bar{M}}$ of $\bar{M}$-polarized K3 surfaces: one can use the primitive embedding $\bar{M} \hookrightarrow M^\perp \subset H^2(X,\mathbb{Z})$ to get a primitive even non-degenerate sublattice of signature $(1, (20-\rho)-1)$ of the K3 lattice $U^3 \oplus E_8(-1)^2$. Observe that for generic K3 surfaces $X_M \in \mathbf{K}_M$ and $X_{\bar{M}} \in \mathbf{K}_{\bar{M}}$ we have

$$\dim \mathbf{K}_M = 20 - \rho = \mathrm{rk}\,\mathrm{Pic}(X_{\bar{M}}), \quad \dim \mathbf{K}_{\bar{M}} = \rho = \mathrm{rk}\,\mathrm{Pic}(X_M).$$

We now consider the special case when X is a K3 surface admitting a non-symplectic involution and $M = H^2(X,\mathbb{Z})^+$. We denote the anti-invariant lattice by

$$H^2(X,\mathbb{Z})^- := (H^2(X,\mathbb{Z})^+)^\perp \cap H^2(X,\mathbb{Z}).$$

Proposition 6. [28, Lemma 2.5, §2.3] *Assume that* $(r, a, \delta) \neq (14, 6, 0)$ *and* $g \geq 1$. *Then:*

- $H^2(X, \mathbb{Z})^- \cong U \oplus \bar{M}$;
- *the generic K3 surface* $X_{\bar{M}} \in \mathbf{K}_{\bar{M}}$ *has a non-symplectic involution;*
- *if* $X_M \in \mathbf{K}_M$ *has invariants* (r, a, δ) *then the invariants of* $X_{\bar{M}} \in \mathbf{K}_{\bar{M}}$ *are* $(20 - r, a, \delta)$.

Remark 4.4.

- In Figure 1 one can see the mirror couples making a reflection with respect to the axis through $r = 10$ and $1 \leq g \leq 10$ and deleting the axis with $g = 0$ and the point $(r, a, \delta) = (14, 6, 0)$.
- Since K3 surfaces with a non-symplectic involution are projective the invariant lattice contains an ample class. One can then consider instead of $\mathbf{K}_M$ the moduli space $\mathbf{K}_M^a$ of ample M-polarized K3 surfaces and do the same construction of mirror moduli spaces as above [13].

5. K3 surfaces in weighted projective spaces with non-symplectic involutions

In this section we will consider K3 surfaces obtained as desingularizations of hypersurfaces of the following type in some weighted projective space:

$$W(x, y, z, w) = x^2 - f(y, z, w) = 0. \tag{5}$$

Observe that any such surface carries the non-symplectic involution $\iota : x \mapsto -x$. We will describe their singularities and we will explain how to compute the triple of invariants (r, a, δ) of ι introduced in Section 4. We recall that $h_{ij} := \gcd(w_i, w_j)$.

Lemma 4. *Let* W *be a quasismooth, well-formed and Gorenstein hypersurface in* $\mathbb{P}(w_1, w_2, w_3, w_4)$ *defined by an invertible potential as in (5). Then the singular points of* W *are Du Val singularities of type* A_k *and can only occur at the vertices* P_2, P_3, P_4 *or along the edges* P_iP_j *with* $1 \leq i, j \leq 4$. *More precisely:*

a) *if* $w_i > 2$, $i = 2, 3, 4$, *then* $P_i \in W$ *if and only if* $w_i \not| d$ *and in this case it is a singular point of type* $A_{w_i - 1}$;
b) *if* $i, j > 1$, $w_i, w_j > 2$ *and* $h_{ij} > 1$, *then* W *intersects* $P_iP_j - \{P_i, P_j\}$ *at* $\lfloor \frac{dh_{ij}}{w_i w_j} \rfloor$ *singular points of type* $A_{h_{ij}-1}$;
c) *if* $i, j > 1$, $w_i = 2$ *and* $w_i | w_j$, *then* W *intersects* $P_iP_j - \{P_j\}$ *at* $\lfloor \frac{d}{w_j} \rfloor$ *singular points of type* A_1;
d) *if* $h_{1i} > 1$, *then* $P_i \notin W$ *and* W *intersects* P_1P_i *at two points if* $w_i | w_1$ *and at one point otherwise. In both cases the intersection points are singularities of type* $A_{h_{1i}-1}$.

Proof. Since W is quasismooth and well-formed, the singularities of W can only appear along the vertices P_i or the edges P_iP_j, where the singular points of the ambient projective space occur. Since W is Gorenstein and quasismooth, then it has only cyclic, canonical singularities [10], i.e. its singular points are Du Val of type A_k. More precisely, a vertex $P_i \in W$ is a singular point of type A_{w_i-1} and an intersection point of W with an edge P_iP_j (outside of the vertices) is a singularity of type $A_{h_{ij}-1}$ where $h_{ij} = \gcd(w_i, w_j)$.

We first observe that $P_1 = (1, 0, 0, 0) \notin W$. Moreover, if w_2 does not divide the degree d of W, then clearly $P_2 = (0, 1, 0, 0) \in W$. Now assume that $w_2 > 2$, w_2

divides d and $P_2 \in W$. Thus f is of one of the following types up to a change of coordinates:

$$y^a z + z^b w + w^c, \quad y^a z + z^b w + w^c y, \quad y^a z + z^b + w^c, \quad y^a z + z^b y + w^c.$$

In the first case, the linear system

$$\begin{cases} 2w_1 = d \\ aw_2 + w_3 = d \\ bw_3 + w_4 = d \\ cw_4 = d \end{cases}$$

implies that w_2 divides w_3, w_4, contradicting the fact that W is well-formed. Similarly, the second case does not occur. In the third case the analogous linear system gives that w_2 divides w_3 and $2w_1$. Since W is well-formed, this implies that $w_2 = 2$, giving a contradiction. The last case is similar. Thus, in case w_2 divides d and $w_2 > 2$, then $P_2 \notin W$. This proves a).

If $w_2 = 2$, then P_2 can be either on W or not, but in any case its singular type is the same of the generic point of the singular edges containing it. A similar discussion holds for P_3 and P_4.

Now assume that $w_2, w_3 > 2$ and $h_{23} = \gcd(w_2, w_3) > 1$. The number of intersection points of W with the edge $P_2P_3 \cong \mathbb{P}(w_2, w_3) \cong \mathbb{P}(v_2, v_3)$, where $v_i := \frac{w_i}{h_{23}}$, only depends on the weights. In fact, assume that $P_2, P_3 \in W$ and let $d' := d/h_{23}$. Then $\bar{f} := \frac{f(0,y,z,0)}{yz}$ is of the form

$$\bar{f} = y^{\frac{d'-v_2-v_3}{v_2}} + z^{\frac{d'-v_2-v_3}{v_3}}.$$

Thus we obtain that $\bar{f}/z^{\frac{d'-v_2-v_3}{v_3}} = (\frac{y^{v_3}}{z^{v_2}})^{\frac{d'-v_2-v_3}{v_2v_3}} + 1$, so that, since y^{v_3}/z^{v_2} is an affine coordinate, f has $\frac{d'-v_2-v_3}{v_2v_3}$ distinct points outside of the vertices. Observe that this number equals $\lfloor \frac{dh_{23}}{w_2w_3} \rfloor$. In fact

$$\frac{dh_{23}}{w_2w_3} - \frac{d'-v_2-v_3}{v_2v_3} = \frac{a_2+a_3}{a_2a_3},$$

where $a_i := w_i/h_{23}$, $i = 2, 3$. If $a_2 > 2$ and $a_3 > 2$, then the right hand side is clearly smaller than one. Otherwise, since a_2 and a_3 are relatively prime, one of them would be equal to one, for example $w_2 = h_{23}$. This contradicts the fact that w_2 does not divide d, since $P_2 \in W$, by the first point in the proof.

The case when either P_2 or P_3, or both, do not belong to W is similar (see also the proof of [16, Lemma I.6.3]). This proves b) and c).

Finally, assume that $h_{12} = \gcd(w_1, w_2) > 1$. It can be easily proved that $P_2 \notin W$ since W is well-formed. In this case the intersection of f with the edge P_1P_2 is given by the solutions of an equation of type $x^2 - y^a = 0$ in $\mathbb{P}(w_1, w_2) \cong \mathbb{P}(v_1, v_2)$, where $v_i := w_i/h_{12}$. Observe that w_2 divides $2w_1$ in this case. The previous equation has two solutions if w_2 divides w_1, since in this case $\frac{x}{y^{1/2}}$ is a coordinate. Otherwise, $w_2 = 2\gcd(w_1, w_2)$, and the equation has a unique solution since a coordinate function is $\frac{x^2}{y^a}$. This proves d). □

Example 5.1. Consider a quasismooth and invertible potential $W(x, y, z, w) = x^2 - f(y, z, w)$ defining a degree 18 hypersurface in $\mathbb{P}(9, 4, 3, 2)$. Observe that W has a singular point of type A_3 at P_2 since w_2 does not divide 18. On the other hand, $P_3 \notin W$ since $w_3 > 2$ and w_3 divides 18. The point P_4 can be either in

W or not, depending on f. The surface W intersects the edge P_1P_3 in 2 points, exchanged by ι, since w_3 divides w_1. These are singularities of type A_2. Finally, we consider the intersection of W with the edge $P_2P_4 \cong \mathbb{P}(2,1)$. Observe that in this case we obtain a degree 9 equation $f'(y,w)=0$. If $P_4 \in W$, then W intersects the edge in $\frac{9-1}{2}=4$ points outside the vertices. Otherwise, if $P_4 \notin W$, then W intersects the edge in $\frac{9-1-2}{2}=3$ points outside the vertices. In any case we have exactly one singular point of type A_3 and four singular points of type A_1 along the edge.

Let $W \subset \mathbb{P}(w_1,w_2,w_3,w_4)$ be defined by an equation of type (5) and let $\gamma : X \to W$ be its minimal resolution. We will denote by ι both the involution $x \mapsto -x$ on W and the involution induced by this on X. We will consider the following commutative diagram, where $\mathbb{P} := \mathbb{P}(w_2,w_3,w_4)$ and $\gamma_1 : \widetilde{\mathbb{P}} \to \mathbb{P}$ is its minimal resolution, π and $\tilde{\pi}$ are the quotients by ι and γ_2 is the blow up of $\widetilde{\mathbb{P}}$ at the singular points of the pull-back of the branch locus of π.

$$\begin{tikzcd}
X \arrow[rr, "\gamma"] \arrow[d, "\tilde{\pi}"'] & & W \arrow[d, "\pi"] \\
Y \arrow[r, "\gamma_2"] & \widetilde{\mathbb{P}} \arrow[r, "\gamma_1"] & \mathbb{P}
\end{tikzcd}$$

Lemma 5. *Assume that the fixed locus of ι on X is of the form*

$$C \cup E_1 \cup \cdots \cup E_k,$$

where $g(E_i)=0$ for any i. Then the invariant lattice $H^2(X,\mathbb{Z})^+$ is generated by $\tilde{\pi}^* \operatorname{Pic}(Y)$ and by the classes of $E_1,\dots,E_k$.

Proof. We first observe that $\tilde{\pi}^* \operatorname{Pic}(Y) \otimes \mathbb{Q} = H^2(X,\mathbb{Q})^+$. In fact, given $y \in \operatorname{Pic}(Y)$, we can assume that y represents an irreducible and effective divisor that we call again y. Then $\tilde{\pi}^*(y)=z$ if z is ι^*-invariant or $\tilde{\pi}^*(y) = z + \iota^*(z)$ if $\tilde{\pi}^{-1}(y)$ consists of two components on X (which are switched by ι). In any case we get a ι^*-invariant class. On the other hand, if $x \in H^2(X,\mathbb{Z})^+$ then $x = (\tilde{\pi}^*\tilde{\pi}_*(x))/2 \in \tilde{\pi}^* \operatorname{Pic}(Y) \otimes \mathbb{Q}$. In particular $r := \operatorname{rk} H^2(X,\mathbb{Z})^+ = \operatorname{rk} \operatorname{Pic}(Y)$.

Since ι is a non-symplectic involution, then $Y = X/\langle \iota \rangle$ is a smooth rational surface. In particular $\operatorname{Pic}(Y) = H^2(Y,\mathbb{Z})$ is a unimodular lattice, thus the determinant of the lattice $\tilde{\pi}^* \operatorname{Pic}(Y) = \operatorname{Pic}(Y)(2)$ equals $\pm 2^r$ and the index of $\tilde{\pi}^* \operatorname{Pic}(Y)$ in $H^2(X,\mathbb{Z})^+$ equals $2^{\frac{r-a}{2}} = 2^k$ by Theorem 4.1.

Observe that the curves $\tilde{\pi}(E_i)$ are (-4)-curves in Y and the sum of their classes is not equal to $-2K_Y$ (in fact $[\tilde{\pi}(C) + \sum_i \tilde{\pi}(E_i)] = -2K_Y$). It follows by [21, Lemma 2.2] that no subset of $\{E_1,\dots,E_k\}$ is an even set. This implies that the classes of the E_i's are not contained in $\tilde{\pi}^* \operatorname{Pic}(Y)$ and give independent elements in $H^2(X,\mathbb{Z})^+/\tilde{\pi}^* \operatorname{Pic}(Y) \cong \mathbb{Z}/2\mathbb{Z}^{\oplus k}$. Thus the index of the lattice generated by $\tilde{\pi}^* \operatorname{Pic}(Y)$ and by the classes of $E_1,\dots,E_k$ in the lattice $H^2(X,\mathbb{Z})^+$ is one. $\square$

In order to compute the triple (r,a,δ) for the lattice $H^2(X,\mathbb{Z})^+$ we follow these steps:

- we identify the irreducible components of the fixed locus of ι in W and the number of their intersection points: W^ι always contains the curve C defined by $x=0$ and possibly one more curve, defined by the vanishing of another coordinate;

- denoting by B the branch locus of π, we identify the singularities of $\mathbb{P}$ on B;
- we compute $r = \operatorname{rk}\operatorname{Pic}(Y)$ as the sum of the Picard number of $\widetilde{\mathbb{P}}$ with the number s of singular points of $\gamma_1^* B$;
- we recall that $a = 22 - r - 2g$ by Theorem 4.1, thus to obtain a it is enough to compute the genus of the curve $x = 0$ by means of the formula given in §2.1;
- in order to identify δ, we compute the invariant lattice of X as follows: we observe that $\tilde{\pi}^* \operatorname{Pic}(Y) = M(2) \oplus (-2)^s$, where $M = \operatorname{Pic}(\tilde{\mathbb{P}})$, and we add to this lattice the classes of the rational curves in the ramification locus of $\tilde{\pi}$ (their classes can be computed by looking at their intersection with the generators of $\tilde{\pi}^* \operatorname{Pic}(Y)$).

The invariant r can also be computed as follows: let $\operatorname{Exc}(\gamma)$ be the lattice generated by the exceptional divisors of γ. Then $r = 1 + \operatorname{rk}\operatorname{Exc}(\gamma)^\iota$, where $1 = \operatorname{rk} H^2(W, \mathbb{Z})^+ = \operatorname{rk}\operatorname{Cl}(\mathbb{P})$ and $\operatorname{rk}\operatorname{Exc}(\gamma)^\iota$ equals the number of ι-orbits in the exceptional locus of γ.

Remark 5.2. We observe that the triple (r, a, δ) only depends on the weight vector $w = (w_1, w_2, w_3, w_4)$. In fact, the configuration of the irreducible components of W^ι (i.e. their number and mutual intersections) only depends on w, and the same holds for the singularities of W by Lemma 4.

Example 5.3. We now compute the triple (r, a, δ) for the surface W in Example 5.1. The projective plane $\mathbb{P} = \mathbb{P}(4, 3, 2) \cong \mathbb{P}(2, 3, 1)$ has a singular point of type A_1 at $(1, 0, 0)$ and one of type A_2 at $(0, 1, 0)$. Its minimal resolution is a toric variety $\tilde{\mathbb{P}}$ whose fan has six rays:

$$r_1 = (-1, 1),\ r_2 = (0, -1),\ r_3 = (2, 1),\ r_4 = (1, 1),\ r_5 = (0, 1),\ r_6 = (1, 0),$$

where r_6 corresponds to the exceptional divisor over the A_1 singularity, r_4, r_5 to the two components of the exceptional divisor over the A_2 singularity and r_3 to the proper transform of the line through the two singular points of $\mathbb{P}$. A basis of $\operatorname{Pic}(\tilde{\mathbb{P}})$ is given by the classes v_1, v_2, v_3, v_4 of the last four rays. With respect to this basis, the classes of the six rays are given by the columns of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the Picard lattice of $\tilde{\mathbb{P}}$ has intersection matrix:

$$M := \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

The branch locus B of π is the union of the curves

$$B_1 : f(y, z, w) = 0 \qquad B_2 :\ z = 0.$$

Observe that B_1 and B_2 intersect at $(1, 0, 0)$ and at $4 = \lfloor \frac{d \cdot h_{24}}{w_2 w_4} \rfloor$ other points (see the second point in Lemma 4). The pull-back $\gamma_1^* B$ in $\widetilde{\mathbb{P}}$ has three irreducible components: the proper transforms $\tilde{B}_1, \tilde{B}_2$ and the exceptional divisor E over the singular point, with $\tilde{B}_1 \cdot \tilde{B}_2 = 4$ and $\tilde{B}_i \cdot E = 1,\ i = 1, 2$. The surface Y is the blow-up of $\widetilde{\mathbb{P}}$ at the six singular points of $\gamma_1^* B$, thus its Picard lattice has intersection matrix $M \oplus (-1)^6$. We still denote by $\tilde{B}_1, \tilde{B}_2$ the proper transforms of the curves in Y.

Let $v_5, \ldots, v_8$ be the classes of the exceptional divisors over the points in $\tilde{B}_1 \cap \tilde{B}_2$, v_9 the one over $E \cap \tilde{B}_1$ and v_{10} over $E \cap \tilde{B}_2$. We now compute $H^2(X, \mathbb{Z})^+$: this is obtained by adding to the lattice $\tilde{\pi}^* \operatorname{Pic}(Y) = M(2) \oplus (-2)^6$ the classes of the rational curves in the fixed locus, in this case $\tilde{\pi}^*([\tilde{B}_2])/2 = (3v_1 + 2v_2 + v_3 + v_4 - v_5 - v_6 - v_7 - v_8 - v_{10})/2$ and $\tilde{\pi}^*([E])/2 = (v_4 - v_9 - v_{10})/2$. Computing the discriminant group of the lattice by means of a computer algebra program, we see that $\delta = 1$. Thus $(r, a, \delta) = (10, 6, 1)$.

6. The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces

In this section we prove Theorem 1.1 by means of a classification of K3 surfaces defined by a non-degenerate invertible potential of the form $W(x, y, z, w) = x^2 - f(y, z, w)$ in some weighted projective space. The possible decompositions of the polynomial $f(y, z, w)$ as a sum of atomic types are the following, up to a permutation of the variables y, z, w:

i) chain: $W_c = x^2 - y^{a_1} z + z^{a_2} w + w^{a_3}$,

ii) loop: $W_l = x^2 - y^{a_1} z + z^{a_2} w + w^{a_3} y$,

iii) Fermat: $W_f = x^2 - y^{a_1} + z^{a_2} + w^{a_3}$,

iv) chain+Fermat: $W_{cf} = x^2 - y^{a_1} z + z^{a_2} + w^{a_3}$,

v) loop+Fermat: $W_{lf} = x^2 - y^{a_1} z + z^{a_2} y + w^{a_3}$.

Borcea in [4, Tables 1, 2, 3] and Yonemura in [29, Table 2.2] classified equations of K3 surfaces in weighted projective 3-spaces, but these are not always of Delsarte type. Thus our first aim is to identify which weights w admit a quasi-homogeneous equation of type $x^2 = f(y, z, w)$, where f is as in i), ii), iii), iv) or v), and then to write the possible equations for a given weight. The result of this classification is contained in the first two columns of Tables 1, 2, 3, 4, 5.

We briefly explain the notation in the tables. In the first column we number the K3 surface $W :\ x^2 - f(y, z, w) = 0$ following [4] and we put in parenthesis the number corresponding to the transposed K3 surface W^T. In the fourth column appear the Nikulin's invariants (r, a, δ) of the involution ι on the resolution X of W, computed as explained in Section 5. In the last two columns we compute the orders of the groups $\mathrm{SL}(W)$ (by means of Proposition 4) and J_W.

6.1. **Trivial $\widetilde{\mathrm{SL}(W)}$.** Observe that in every case, except for the ones marked with $*$, we have that the group $\mathrm{SL}(W)/J_W$ is trivial, so that W and W^T are BHCR-mirror of each other. As the tables show, for such pairs the invariants (r, a, δ) are lattice mirror (i.e. they are (r, a, δ) and $(20 - r, a, \delta)$), thus the theorem is proved in these cases.

Example 6.1. We consider the case of the weight vector $w = (5, 3, 1, 1)$. In order to determine which f can appear in this weight, we need to solve the linear system

$$Aw = 10e,$$

where e is the column vector with all entries equal to 1 and A is the matrix associated to one of the potentials $W_c, W_l, W_f, W_{cf}, W_{lf}$ (and the ones obtained from them by a coordinate change).

No. 3a and 28 in Table 5. If A is associated to the potential W_c, the only solution is $(a_1, a_2, a_3) = (3, 9, 10)$, which gives the surface No. 3a in Table 5. This has only one A_2 singularity and $g = 9$ so that $(r, a, \delta) = (3, 1, 1)$ (here δ is uniquely determined, see Figure 1). The surface W^T is No. 28 in Table 5. Its configuration of singular fibers is $A_1+A_3+A_4+A_8$, so that $(r, a, \delta) = (17, 1, 1)$. By Proposition 4 we find that $J_{W^T} = \mathrm{SL}(W^T)$, so W and W^T are BHCR-mirror and belong to lattice mirror families.

No. 3b and 5 in Table 5. If we consider the potential W_c with the variables y ad z exchanged, we find another solution with $(a_1, a_2, a_3) = (7, 3, 10)$. This gives case No. 3b in Table 5, which has again $(r, a, \delta) = (3, 1, 1)$. The surface W^T is No. 5 in Table 5 and has $3A_1+A_3$ singular points invariant for ι, so that $(r, a, \delta) = (7, 3, 1)$. Here $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/3\mathbb{Z}$, so that W and W^T are not BHCR-mirror.

No. 3 and 23 in Table 2. If A is of loop type then the only solution is $(a_1, a_2, a_3) = (3, 9, 7)$, which gives the surface No. 3 in Table 2. This surface has again $(r, a, \delta) = (3, 1, 1)$ (see Remark 5.2). The surface W^T is No. 23 in Table 2. Here again $J_W = \mathrm{SL}(W)$, so that W and W^T are BHCR-mirror and belong to lattice mirror families.

No. 3 and 18 in Table 4. In the chain+Fermat case we obtain as a unique solution $(a_1, a_2, a_3) = (3, 10, 10)$, which gives No. 3 in Table 4. The surface W^T is given by No. 18 in the same table, but in this case $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/2\mathbb{Z}$, so that W and W^T are not BHCR-mirror.

No. 3 in Table 3. We obtain the solution $(a_1, a_2, a_3) = (3, 7, 10)$ in the loop+Fermat case. Here $W = W^T$ and $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/4\mathbb{Z}$.

We will discuss the last cases in the next subsection.

6.2. **Non trivial $\widetilde{\mathrm{SL}(W)}$.** In this case the BHCR-mirror pairs are given by the minimal resolutions of $W/\tilde{G}$ and $W^T/\tilde{G}^T$, where $\tilde{G}^T = \mathrm{SL}(W^T)/J_{W^T}$ by Proposition 3. We recall that, by Proposition 1, the group $\tilde{G}$ acts symplectically on W and its minimal resolution X. Moreover, since it is finite and abelian it appears in the list of the 15 possible finite symplectic abelian groups given by Nikulin in [23]. Since the involution ι commutes with $\tilde{G}$ (which is generated by diagonal automorphisms), then ι clearly induces a non-symplectic involution j on $X/\tilde{G}$ and on its minimal resolution Y. We are thus interested in computing the triple (r, a, δ) for such involution on Y. We have a commutative diagram:

(6)
$$\begin{tikzcd} & X \arrow[r, "\gamma"] \arrow[d] & W \arrow[d, "q"] \\ Y \arrow[r, "\eta_2"] & X/\tilde{G} \arrow[r, "\eta_1"] & W/\tilde{G} \end{tikzcd}$$

where we still denote by $\tilde{G}$ its lifting to X, η_2 is the minimal resolution of $X/\tilde{G}$ and $\eta_2 \circ \eta_1$ the minimal resolution of $W/\tilde{G}$, whose singular locus is the image of the singular locus of W and of the points with non trivial stabilizer for G. The rank r of the invariant lattice $H^2(Y,\mathbb{Z})^j$ equals 1 plus the number of j-orbits of the exceptional locus in Y and the curve of maximal genus in $\mathrm{Fix}(j)$ is isomorphic to $q(C)$, thus its genus g can be computed by means of the Riemann-Hurwitz formula. Finally a can be computed by means of the formula in Theorem 4.1.

If $\tilde{G}$ is cyclic of prime order $p > 2$, then the invariant δ of (Y, j) equals the one of (X, ι) by Proposition 5. Otherwise, we need a deeper analysis to compute explicitely a basis of $H^2(Y,\mathbb{Z})^+$ as explained in Section 5.

Example 6.2. We now show that the surfaces No. 3b and No. 5 in Table 5 are lattice mirror.

No. 3b in Table 5. By Proposition 2 and Corollary 1 a generator for $\mathrm{SL}(W)$ and $\widetilde{G} \cong \mathbb{Z}/3\mathbb{Z}$ is $\tilde{g} := (1, 14/15, 7/15, 3/5)$, with respect to the coordinates x, z, y, w. A local analysis in the charts shows that the point $(0 : 1 : 0 : 0) \in W$ is an A_2 singularity fixed by $\tilde{g}$, hence it induces an A_8 singularity in the quotient $W/\widetilde{G}$. The remaining fixed points of $\tilde{g}$ are $(0 : 0 : 1 : 0)$, $(1 : 0 : 1 : 0)$ and $(-1 : 0 : 1 : 0)$, which give 3 singularities of type A_2 in the quotient $W/\widetilde{G}$, two of them interchanged by ι. Thus X contains 12 j-orbits of exceptional curves and $r = 13$. The automorphism $\tilde{g}$ clearly preserves the curve C and it fixes two points on it (corresponding to the A_8 singularity and to the first A_2 singularity). By Riemann-Hurwitz formula, its image $q(C)$ has genus 3. In conclusion the invariants of j are $(r, a, \delta) = (13, 3, 1)$ (here δ is uniquely determined, see Figure 1), thus Y and the surface No. 5. in Table 5 are BHCR-mirror and belong to lattice mirror families.

No. 5 in Table 5. We recall that W has one A_3 and $3A_1$ singularities fixed by ι. By Proposition 2 and Corollary 1 a generator for $\mathrm{SL}(W)/(J_W)$ is $\tilde{g} := (1, 20/21, 10/21, 4/7) \in \mathrm{SL}(W)$. A local analysis in the charts shows that the point $(0 : 1 : 0 : 0) \in W$ is an A_3 singularity fixed by $\tilde{g}$, which gives an A_{11} singularity in $W/\widetilde{G}$. Moreover, the points $(1 : 0 : 1 : 0), (0 : 0 : 0 : 1)$ are smooth points in W fixed by $\tilde{g}$, thus giving two A_2 singularities of the quotient. The 3 singularities of type A_1 are permuted by $\tilde{g}$ and give a point of type A_1 in the quotient. Thus Y contains $11 + 2 \cdot 2 + 1 = 16$ orbits of exceptional curves. Moreover, the genus of the curve of maximal genus is 2, so that the invariants of j are $(r, a, \delta) = (17, 1, 1)$, so Y and the surface No. 3b in Table 5 are BHCR-mirror and belong to lattice mirror families.

Example 6.3. We now show that the surfaces No. 3 and No. 18 in Table 4 belong to lattice mirror families.

No. 3 in Table 4. In this case we need a deeper analysis to determine the invariant δ. The involution σ induces the involution $\bar{\sigma}(y, z, w) = (y, z, -w)$ in $\mathbb{P} := \mathbb{P}(3, 1, 1)$ and the involution ι induces an involution $\bar{\iota}$ on $W/\langle\sigma\rangle$ and Y. We observe that we have the following commutative diagram. The map $Y \to W/\langle\sigma\rangle$ is the minimal resolution and $b \circ r : Z \to \mathbb{P}/\langle\bar{\sigma}\rangle$ is obtained composing the minimal resolution r of $\mathbb{P}/\langle\bar{\sigma}\rangle$ with the blow-up b of the singular locus of r^*B, where B is the branch locus of $\bar{\pi}$.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\pi"'] & W/\langle\sigma\rangle \arrow[d, "\bar{\pi}"] & Y \arrow[l] \arrow[d, "\bar{\pi}"] \\ \mathbb{P} \arrow[r] & \mathbb{P}/\langle\bar{\sigma}\rangle & Z \arrow[l] \end{tikzcd} \tag{7}$$

Observe that $\mathbb{P}/\langle\bar{\sigma}\rangle \cong \mathbb{P}(3,1,2)$ and B is the union of the curves B_1, B_2 defined by $f(y,z,w) = 0$ and $w = 0$, which intersect at three smooth points and at the singular point $Q_1 := (1:0:0)$. The projective plane $\mathbb{P}(3,1,2)$ has a singular point of type A_2 at Q_1 and a point of type A_1 at $Q_2 = (0:0:1)$, thus its resolution is a toric variety with Picard number 4. Moreover r^*B has 6 double points, thus the surface Z has Picard number 10 and its Picard lattice can be explicitely computed as in Example 5.3. The invariant lattice $H^2(Y,\mathbb{Z})^+$ has rank 10 and, by Lemma 5, it is the lattice obtained by adding to $\pi^*\operatorname{Pic}(Z)$ the classes of the two rational curves in $\operatorname{Fix}(\bar{\iota})$. An explicit computation, following the method explained in §5, gives that $\delta = 0$.

We discuss one more case in detail, since here the group acting on the surface W^T is not cyclic.

Example 6.4. No. 1 in Table 1. The equation $x^2 = y^6 + z^6 + w^6$ defines a smooth K3 surface W of degree $d = 6$ in $\mathbb{P}(3,1,1,1)$ and $(r,a,\delta) = (1,1,1)$. The group $\widetilde{G} = \mathrm{SL}(W)/J_W$ is of order 12. By Nikulin's classification [23] of finite abelian groups acting symplectically on a K3 surface we have that $\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The group J_W is generated by the element $(1/2, 1/6, 1/6, 1/6)$ and observe that the elements $(1/2, 1/2, 1, 1)$ and $(1, 1/6, 5/6, 1)$ generate $\widetilde{G}$. Denote by $(1,0)$ the generator of order 2 and by $(0,1)$ the generator of order 6. Again by [23] we know that we have the following configuration of fixed points:

$$\begin{tikzcd}[arrows=dash] H^2_{(0,3)}(6) \arrow[r] & H^6_{(0,1)}(2) \arrow[r] & H^3_{(0,2)}(0) \arrow[r] \arrow[d] & H^6_{(1,2)}(2) \arrow[r] & H^2_{(1,0)}(6) \\ & & H^6_{(1,1)}(2) \arrow[r] & H^2_{(1,3)}(6) & \end{tikzcd}$$

where we follow the notation of [23] denoting by $H^m_x(t)$ the cyclic group of order m with generator x, and t denotes the number of fixed points having H^m_x as stabilizer.

Looking at the diagram and by a local analysis one sees that it is enough to study the fixed points of the elements $(1,0)$, $(0,3)$ and $(1,3)$. The fixed points of $(1,0)$ are $(1:1:0:0)$, $(-1:1:0:0)$ and $(0:0:1:\xi^j)$, with $\xi = \exp(2\pi i/12)$ and $j = 1,3,5,7,9,11$. The first two points are interchanged by ι and have in fact stabilizer of order 6. The computation for the elements $(0,3)$ and $(1,3)$ is similar. We find that the quotient $W/\widetilde{G}$ has in total $3A_5$ and $3A_1$ singularities which gives $r = 19$. Finally, by an easy computation, one sees that the curve C contains 18 points with stabilizer group of order 2 hence by Riemann-Hurwitz formula the curve C_1 has genus 1. In this case $\delta = 1$ by Figure 1, thus the invariants for Y are $(r,a,\delta) = (19,1,1)$. This shows that the surfaces Y and W belong to lattice mirror families.

In Table 1 and 3 there are cases where $\mathrm{SL}(W)/J_W$ has non trivial proper subgroups $\widetilde{G} = G/J_W$. By making similar computations of the Nikulin invariants

(r, a, δ) for $W/\widetilde{G}$ and $W/\widetilde{G^T}$ one obtains that the corresponding minimal resolutions are mirror K3 surfaces, proving Theorem 1.1 also in these cases. We specify however one more case, in which the method for computing δ uses a fake weighted projective plane.

Definition 6.5. A fake weighted projective space is a $\mathbb{Q}$-factorial toric variety with Picard number one.

By [9, Proposition 4.7] (see also [17, Corollary 2.3]) every fake weighted projective space is a quotient of a weighted projective space by a finite group acting freely in codimension one.

Example 6.6. No. 30 in Table 1. Let $W = x^2 - y^4 - z^8 - w^8 = 0$ in $\mathbb{P}(4, 2, 1, 1)$. The surface has two A_1 singular points at $(1 : 1 : 0 : 0), (-1 : 1 : 0 : 0)$ which are exchanged by ι and ι fixes a curve C of genus 9, so that $(r, a) = (2, 2)$ and by Nikulin's table $\delta = 0$. Moreover $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and it is generated by $(1/2, 1/2, 1, 1), (1, 1/4, 3/4, 1)$, which for simplicity we will call $(1, 0)$ and $(0, 1)$ respectively. By the results of the previous sections one can easily compute the values for (r, a) for the surface W and for its quotients by subgroups of $\widetilde{G}$. To compute the invariant δ of the quotient $W/\widetilde{G}$, one has a similar diagram as diagram (7), just replace $\langle\bar{\sigma}\rangle$ by the induced group on $\mathbb{P} := \mathbb{P}(2, 1, 1)$ generated by $\langle(1/2, 1, 1), (1/4, 3/4, 1)\rangle$. The quotient of $\mathbb{P}$ by this group is a fake weighted projective plane, with fan of its minimal resolution defined by 8 rays (computation with MAGMA [6]):

$$(-1, 0), \quad (1, -2), \quad (1, 2), \quad (0, -1), \quad (0, 1), \quad (1, 1), \quad (1, 0), \quad (1, -1)$$

One thus proceeds as described in the previous sections to compute δ. The results are resumed in Table 7.

7. Tables

No.	(w_1,w_2,w_3,w_4)	$f(y,z,w)$	(r,a,δ)	$\lvert \mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$	$\mathrm{SL}(W)/J_W$
$*1$	$(3,1,1,1)$	$y^6+z^6+w^6$	$(1,1,1)$	72	6	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$
$*2$	$(5,2,2,1)$	$y^5+z^5+w^{10}$	$(6,4,0)$	50	10	$\mathbb{Z}/5\mathbb{Z}$
$*8$	$(9,6,2,1)$	$y^3+z^9+w^{18}$	$(6,2,0)$	54	18	$\mathbb{Z}/3\mathbb{Z}$
18	$(15,10,3,2)$	$y^3+z^{10}+w^{15}$	$(10,4,0)$	30	30	1
26	$(21,14,6,1)$	$y^3+z^7+w^{42}$	$(10,0,0)$	42	42	1
$*30$	$(4,2,1,1)$	$y^4+z^8+w^8$	$(2,2,0)$	64	8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$
$*34$	$(10,5,4,1)$	$y^4+z^5+w^{20}$	$(6,4,0)$	40	20	$\mathbb{Z}/2\mathbb{Z}$
$*41$	$(6,3,2,1)$	$y^4+z^6+w^{12}$	$(4,4,1)$	48	12	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
$*42$	$(6,4,1,1)$	$y^3+z^{12}+w^{12}$	$(2,0,0)$	72	12	$\mathbb{Z}/6\mathbb{Z}$
$*45$	$(12,8,3,1)$	$y^3+z^8+w^{24}$	$(6,2,0)$	48	24	$\mathbb{Z}/2\mathbb{Z}$

TABLE 1. The Fermat mirror cases

No.	(w_1,w_2,w_3,w_4)	$f(y,z,w)$	(r,a,δ)	$\lvert \mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$
$*(1)1$	$(3,1,1,1)$	$y^5z+z^5w+w^5y$	$(1,1,1)$	6	42
$(23)3$	$(5,3,1,1)$	$y^3z+z^9w+w^7y$	$(3,1,1)$	10	10
$(13)11$	$(11,7,3,1)$	$y^3w+w^{19}z+z^5y$	$(9,1,1)$	22	22
$(11)13$	$(13,7,5,1)$	$y^3z+z^5w+w^{19}y$	$(11,1,1)$	26	26
$(3)23$	$(19,11,5,3)$	$y^3z+z^7w+w^9y$	$(17,1,1)$	38	38

TABLE 2. The loop mirror cases

No.	(w_1,w_2,w_3,w_4)	$f(y,z,w)$	(r,a,δ)	$\lvert \mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$	$\mathrm{SL}(W)/J_W$
$*1$	$(3,1,1,1)$	$y^5z+z^5y+w^6$	$(1,1,1)$	48	6	$\mathbb{Z}/8\mathbb{Z}$
$*2$	$(5,2,2,1)$	$y^4z+z^4y+w^{10}$	$(6,4,0)$	30	10	$\mathbb{Z}/3\mathbb{Z}$
$*3$	$(5,3,1,1)$	$y^3z+z^7y+w^{10}$	$(3,1,1)$	40	10	$\mathbb{Z}/4\mathbb{Z}$
$*5$	$(7,4,2,1)$	$y^3z+z^5y+w^{14}$	$(7,3,1)$	28	14	$\mathbb{Z}/2\mathbb{Z}$
6	$(9,4,3,2)$	$y^4w+w^7y+z^6$	$(10,6,1)$	18	18	1
10	$(11,6,4,1)$	$y^3z+z^4y+w^{22}$	$(10,2,1)$	22	22	1
$*30$	$(4,2,1,1)$	$z^7w+w^7z+y^4$	$(2,2,0)$	48	8	$\mathbb{Z}/6\mathbb{Z}$
$*31$	$(8,4,3,1)$	$z^5w+w^{13}z+y^4$	$(6,4,0)$	32	16	$\mathbb{Z}/2\mathbb{Z}$
$*32$	$(8,5,2,1)$	$y^3w+w^{11}y+z^8$	$(6,2,0)$	32	16	$\mathbb{Z}/2\mathbb{Z}$
36	$(14,9,4,1)$	$y^3w+w^{19}y+z^7$	$(10,0,0)$	28	28	1
$*42$	$(6,4,1,1)$	$z^{11}w+w^{11}z+y^3$	$(2,0,0)$	60	12	$\mathbb{Z}/5\mathbb{Z}$
47	$(18,12,5,1)$	$z^7w+w^{31}z+y^3$	$(10,0,0)$	36	36	1

TABLE 3. The loop+Fermat mirror cases

No.	(w_1, w_2, w_3, w_4)	$f(y, z, w)$	(r, a, δ)	$\lvert \mathrm{SL}(W) \rvert$	$\lvert J_W \rvert$
$*(15)1$	$(3,1,1,1)$	$y^5z + z^6 + w^6$	$(1,1,1)$	12	6
$*(33b)2a$	$(5,2,2,1)$	$y^4z + z^5 + w^{10}$	$(6,4,0)$	20	10
$(39)2b$	$(5,2,2,1)$	$w^8y + y^5 + z^5$	$(6,4,0)$	10	10
$*(18)3$	$(5,3,1,1)$	$y^3z + z^{10} + w^{10}$	$(3,1,1)$	20	10
$(35a)4$	$(7,3,2,2)$	$y^4z + z^7 + w^7$	$(10,6,0)$	14	14
$(24)5$	$(7,4,2,1)$	$y^3z + z^7 + w^{14}$	$(7,3,1)$	14	14
$*(41a)6$	$(9,4,3,2)$	$y^4w + w^9 + z^6$	$(10,6,0)$	36	18
$*(8a)7$	$(9,5,3,1)$	$y^3z + z^6 + w^{18}$	$(7,3,1)$	36	18
$*(7)8a$	$(9,6,2,1)$	$z^6y + y^3 + w^{18}$	$(6,2,0)$	36	18
$(46)8b$	$(9,6,2,1)$	$w^{12}y + y^3 + z^9$	$(6,2,0)$	18	18
$(48)8c$	$(9,6,2,1)$	$w^{16}z + z^9 + y^3$	$(6,2,0)$	18	18
$*(1)15$	$(15,6,5,4)$	$w^6y + y^5 + z^6$	$(12,6,1)$	60	30
$(34a)16$	$(15,7,6,2)$	$y^4w + w^{15} + z^5$	$(14,4,0)$	30	30
$(19)17$	$(15,8,6,1)$	$y^3z + z^5 + w^{30}$	$(11,1,1)$	30	30
$*(3)18$	$(15,10,3,2)$	$w^{10}y + y^3 + z^{10}$	$(10,4,0)$	60	30
$(17)19$	$(15,10,4,1)$	$z^5y + y^3 + w^{30}$	$(9,1,1)$	30	30
$(5)24$	$(21,14,4,3)$	$z^7y + y^3 + w^{14}$	$(13,3,1)$	42	42
$(45b)25$	$(21,14,5,2)$	$z^8w + y^3 + w^{21}$	$(14,2,0)$	42	42
$(36)26a$	$(21,14,6,1)$	$w^{28}y + y^3 + z^7$	$(10,0,0)$	42	42
$(47)26b$	$(21,14,6,1)$	$w^{36}z + y^3 + z^7$	$(10,0,0)$	42	42
$(42b)29$	$(33,22,6,5)$	$w^{12}z + z^{11} + y^3$	$(18,0,0)$	66	66
$*(35b)30a$	$(4,2,1,1)$	$z^7w + w^8 + y^4$	$(2,2,0)$	16	8
$*(43)30b$	$(4,2,1,1)$	$z^6y + y^4 + w^8$	$(2,2,0)$	16	8
$*(31a)31a$	$(8,4,3,1)$	$z^4y + y^4 + w^{16}$	$(6,4,0)$	32	16
$*(34b)31b$	$(8,4,3,1)$	$z^5w + w^{16} + y^4$	$(6,4,0)$	32	16
$*(45a)32$	$(8,5,2,1)$	$y^3w + w^{16} + z^8$	$(6,2,0)$	32	16
$*(41b)33a$	$(10,5,3,2)$	$z^6w + w^{10} + y^4$	$(8,6,1)$	40	20
$*(2a)33b$	$(10,5,3,2)$	$z^5y + y^4 + w^{10}$	$(8,6,1)$	40	20
$(16)34a$	$(10,5,4,1)$	$w^{15}y + y^4 + z^5$	$(6,4,0)$	20	20
$*(31b)34b$	$(10,5,4,1)$	$w^{16}z + z^5 + y^4$	$(6,4,0)$	40	20
$(4)35a$	$(14,7,4,3)$	$w^7y + y^4 + z^7$	$(10,6,0)$	28	28
$*(30a)35b$	$(14,7,4,3)$	$w^8z + z^7 + y^4$	$(10,6,0)$	56	28
$(26a)36$	$(14,9,4,1)$	$y^3w + w^{28} + z^7$	$(10,0,0)$	28	28
$(2b)39$	$(20,8,7,5)$	$z^5w + y^5 + w^8$	$(14,4,0)$	40	40
$*(6)41a$	$(6,3,2,1)$	$w^9y + y^4 + z^6$	$(4,4,1)$	24	12
$*(33a)41b$	$(6,3,2,1)$	$w^{10}z + z^6 + y^4$	$(4,4,1)$	24	12
$*(44)42a$	$(6,4,1,1)$	$z^8y + y^3 + w^{12}$	$(2,0,0)$	24	12
$(29)42b$	$(6,4,1,1)$	$z^{11}w + w^{12} + y^3$	$(2,0,0)$	12	12
$*(30b)43$	$(12,5,4,3)$	$y^4z + z^6 + w^8$	$(10,6,1)$	48	24
$*(42a)44$	$(12,7,3,2)$	$y^3z + z^8 + w^{12}$	$(10,4,0)$	48	24
$*(32)45a$	$(12,8,3,1)$	$w^{16}y + y^3 + z^8$	$(6,2,0)$	48	24
$(21)45b$	$(12,8,3,1)$	$w^{21}z + z^8 + y^3$	$(6,2,0)$	24	24
$(8b)46$	$(18,11,4,3)$	$y^3w + w^{12} + z^9$	$(14,2,0)$	36	36
$(26b)47$	$(18,12,5,1)$	$z^7w + w^{36} + y^3$	$(10,0,0)$	36	36
$(8c)48$	$(24,16,5,3)$	$z^9w + w^{16} + y^3$	$(14,2,0)$	48	48

TABLE 4. The chain+Fermat mirror cases

No.	(w_1, w_2, w_3, w_4)	$f(y,z,w)$	(r,a,δ)	$\lvert\mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$
(27)1	$(3,1,1,1)$	$y^5z+z^5w+w^6$	$(1,1,1)$	6	6
(37)2	$(5,2,2,1)$	$w^8z+z^4y+y^5$	$(6,4,0)$	10	10
(28)3a	$(5,3,1,1)$	$y^3z+z^9w+w^{10}$	$(3,1,1)$	10	10
$*$(5)3b	$(5,3,1,1)$	$z^7y+y^3w+w^{10}$	$(3,1,1)$	30	10
$*$(30)4	$(7,3,2,2)$	$y^4z+z^6w+w^7$	$(10,6,0)$	42	14
$*$(3b)5	$(7,4,2,1)$	$z^7+zy^3+yw^{10}$	$(7,3,1)$	42	14
(14)7	$(9,5,3,1)$	$w^{13}y+y^3z+z^6$	$(7,3,1)$	18	18
(38)8	$(9,6,2,1)$	$w^{16}z+z^6y+y^3$	$(6,2,0)$	18	18
(17)11	$(11,7,3,1)$	$z^5y+y^3w+w^{22}$	$(9,1,1)$	22	22
(31a)12	$(13,6,5,2)$	$z^4y+y^4w+w^{13}$	$(14,4,0)$	26	26
(19)13	$(13,7,5,1)$	$y^3z+z^5w+w^{26}$	$(11,1,1)$	26	26
(7)14	$(13,8,3,2)$	$w^{13}+wy^3+yz^6$	$(13,3,1)$	26	26
(31b)16	$(15,7,6,2)$	$y^4w+w^{12}z+z^5$	$(14,4,0)$	30	30
(11)17	$(15,8,6,1)$	$z^5+zy^3+yw^{22}$	$(11,1,1)$	30	30
(13)19	$(15,10,4,1)$	$y^3+yz^5+zw^{26}$	$(9,1,1)$	30	30
(32)25	$(21,14,5,2)$	$z^8w+w^{14}y+y^3$	$(14,2,0)$	42	42
(1)27	$(25,10,8,7)$	$y^5+yz^5+zw^6$	$(19,1,1)$	50	50
(3a)28	$(27,18,4,5)$	$y^3+yz^9+zw^{10}$	$(17,1,1)$	54	54
$*$(4)30	$(4,2,1,1)$	$y^4+yz^6+zw^7$	$(2,2,0)$	24	8
(12)31a	$(8,4,3,1)$	$y^4+yz^4+zw^{13}$	$(6,4,0)$	16	16
(16)31b	$(8,4,3,1)$	$y^4+yw^{12}+wz^5$	$(6,4,0)$	16	16
(25)32	$(8,5,2,1)$	$z^8+zw^{14}+wy^3$	$(6,2,0)$	16	16
(47)36	$(14,9,4,1)$	$y^3w+w^{24}z+z^7$	$(10,0,0)$	28	28
(2)37	$(16,5,7,4)$	$w^8+wz^4+zy^5$	$(14,4,0)$	32	32
(8)38	$(16,9,5,2)$	$w^{16}+wz^6+zy^3$	$(14,2,0)$	32	32
(42)40	$(22,13,5,4)$	$y^3z+z^8w+w^{11}$	$(18,0,0)$	44	44
(40)42	$(6,4,1,1)$	$y^3+yz^8+zw^{11}$	$(2,0,0)$	12	12
(36)47	$(18,12,5,1)$	$y^3+yw^{24}+wz^7$	$(10,0,0)$	36	36

TABLE 5. The chain mirror cases

$\widetilde{G}$	generators	(r,a,δ)	$\widetilde{G^T}$	generators	(r,a,δ)
$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1)$	$(8,6,1)$	$\mathbb{Z}/6\mathbb{Z}$	$(1,1/3,1/6,1/2)$	$(12,6,1)$
$\mathbb{Z}/2\mathbb{Z}$	$(1,1/2,1/2,1)$	$(8,6,1)$	$\mathbb{Z}/6\mathbb{Z}$	$(1/2,1/3,1,1/6)$	$(12,6,1)$
$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1,1/2,1)$	$(8,6,1)$	$\mathbb{Z}/6\mathbb{Z}$	$(1/2,1/3,1/6,1)$	$(12,6,1)$
$\mathbb{Z}/3\mathbb{Z}$	$(1,1/3,2/3,1)$	$(7,7,1)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1)$, $(1,1/2,1/2,1)$	$(13,7,1)$

TABLE 6. The subgroups in the Fermat case No. 1

$\widetilde{G}$	generators	(r,a,δ)	$\widetilde{G^T}$	generators	(r,a,δ)
0	0	$(2,2,0)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$(1/2,1/2,1,1),(1,1/4,3/4,1)$	$(18,2,0)$
$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1)$	$(10,6,0)$	$\mathbb{Z}/4\mathbb{Z}$	$(1/2,3/4,3/4,1)$	$(10,6,0)$
$\mathbb{Z}/4\mathbb{Z}$	$(1,1/4,3/4,1)$	$(10,6,1)$	$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1,1/2,1)$	$(10,6,1)$
$\mathbb{Z}/2\mathbb{Z}$	$(1,1/2,1/2,1)$	$(6,6,1)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1),(1,1/2,1/2,1)$	$(14,6,1)$

TABLE 7. The subgroups in the Fermat case No. 30

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