

ORDER EIGHT NON-SYMPLECTIC AUTOMORPHISMS ON ELLIPTIC K3 SURFACES

DIMA AL TABBAA AND ALESSANDRA SARTI

ABSTRACT. In this paper we classify complex $K3$ surfaces with non-symplectic automorphism of order 8 that leaves invariant a smooth elliptic curve. We show that the rank of the Picard group is either 10, 14 or 18 and the fixed locus is the disjoint union of elliptic curves, rational curves and points, whose number does not exceed 1, 2, respectively 14. We give examples corresponding to several types of fixed locus in the classification.

INTRODUCTION

The study of automorphisms of $K3$ surfaces started with the pioneering work of Nikulin [11] and several people contributed then to the development of the theory, see [17] for a short survey. In the paper we investigate purely non-symplectic automorphisms of order 8, i.e. automorphisms that multiply the non-degenerate holomorphic two form by a primitive 8th root of unity.

The study of non-symplectic automorphisms of prime order was completed by several authors: Nikulin in [12], Artebani, Sarti and Taki in [3, 5, 14]. The study of non-symplectic automorphisms of non-prime order turns out to be more complicated. Indeed, in this situation the "generic" case does not imply that the action of the automorphism is trivial on the Picard group [8, Section 11]. In the paper [15], Taki completely describes the case when the order of the automorphism is a prime power and the action is trivial on the Picard group. If we consider non-symplectic, non-trivial automorphisms of order 2^t , then by results of Nikulin we have $1 \leq t \leq 5$, and by a recent paper by Taki [16] there is only one $K3$ surface that admits a non-symplectic automorphism of order 32. Some further results in this direction are contained in a paper by Schütt in the case of automorphisms of a 2-power order [13] and in a paper by Artebani and Sarti in the case of order 4 [4]. Recently in [2] the two authors and Taki completed the study for purely non-symplectic automorphisms of order 16.

This paper mainly deals with purely non-symplectic automorphisms σ of order eight on elliptic $K3$ surfaces under the assumption that their fourth power σ^4 is the identity on the Picard lattice. This corresponds to the situation for the generic $K3$ surface in the moduli space of $K3$ surfaces with non-symplectic automorphism of order 8 and fixed action on the second cohomology with integer coefficients, see [8, Section 10]. The fixed locus $\text{Fix}(\sigma)$ of such an automorphism σ is the disjoint union of smooth curves and points. In the paper we give a complete classification in the case that $\text{Fix}(\sigma^4)$ contains an elliptic curve. More precisely let X be a $K3$ surface, ω_X a generator of $H^{2,0}(X)$, $\sigma \in \text{Aut}(X)$ such that $\sigma^*\omega_X = \zeta_8\omega_X$, where ζ_8

2010 *Mathematics Subject Classification.* Primary 14J28; Secondary 14J50, 14J10.
Key words and phrases. non-symplectic automorphisms, $K3$ surfaces.

denotes a primitive 8th root of unity. We denote by r, l, m and m_1 the rank of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1, i$ and ζ_8 respectively. We also denote by k the number of smooth rational curves fixed by σ , by N the number of isolated fixed points in $\text{Fix}(\sigma)$. We prove the following result:

Theorem 0.1. *Let σ be a purely non-symplectic order eight automorphism acting on a K3 surface X such that σ^4 acts identically on $\text{Pic}(X)$. Assume that $\text{Fix}(\sigma^4)$ contains a curve C of genus $g(C) = 1$, then we have the following possibilities for the fixed locus*

- If $\text{Fix}(\sigma)$ contains the elliptic curve C , then

$$(k, N, \text{rk Pic}(X)) = (0, 2, 10), (0, 4, 14).$$

- If $\text{Fix}(\sigma^2)$ contains the elliptic curve C and there is no elliptic curve fixed by σ , then

$$(k, N, \text{rk Pic}(X)) = (0, 2, 10), (0, 6, 10), (0, 4, 14), (1, 10, 14).$$

- If σ^i for $i = 1, 2$ does not fix an elliptic curve, then

$$(k, N, \text{rk Pic}(X)) = (0, 2, 10), (0, 4, 14), (0, 2, 14), (0, 6, 14), \\ (1, 8, 14), (0, 2, 18), (0, 6, 18), (2, 14, 18).$$

In Table 5 we list all the possibilities in detail and we describe the action of σ on the fibers of the elliptic fibration. We have in total 16 possibilities and we give examples for all the cases except for some special cases when the action of σ is a translation on the invariant elliptic curve or it acts as a rotation on some reducible fibers of the fibration.

The results of the paper are partially contained in the PhD thesis [1] of the first author under the supervision of the second author.

Acknowledgements: This work was partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. Part of the work was done by the second author during the Polish Algebraic Geometry Mini-Semester, 18 april-18 june 2016. She warmly thanks the organizers for the stimulating working atmosphere.

1. BASIC FACTS

Let X be a K3 surface and $\sigma \in \text{Aut}(X)$. We assume that $\sigma^*\omega_X = \zeta_8\omega_X$, where $\zeta_8 = e^{\frac{2\pi i}{8}}$ is a primitive 8th root of the unity. Such a σ is called *purely non-symplectic*, for simplicity we just call it *non-symplectic*, always meaning that the action is by a primitive 8th root of unity.

We denote by $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$ and m_1 for $j = 1, 2, 4$ the rank of the eigenspace of $(\sigma^j)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1, i$ and ζ_8 respectively (clearly $m_{\sigma^4} = 0$). For simplicity we just write r, l, m for $j = 1$. We recall the invariant lattice:

$$S(\sigma^j) = \{x \in H^2(X, \mathbb{Z}) \mid (\sigma^j)^*(x) = x\},$$

and its orthogonal complement:

$$T(\sigma^j) = S(\sigma^j)^\perp \cap H^2(X, \mathbb{Z}).$$

Since the automorphism acts purely non-symplectically, X is projective, see [11, Theorem 3.1]) so that if we denote $\text{rk } S(\sigma^j) = r_{\sigma^j}$, we have that $r_{\sigma^j} > 0$ for all $j = 1, 2, 4$ (one can always find an invariant ample class). On the other hand, one

can easily show that $S(\sigma^j) \subseteq \text{Pic}(X)$ for $j = 1, 2, 4$ so that the transcendental lattice satisfies $T_X \subseteq T(\sigma^j)$ for $j = 1, 2, 4$. For simplicity we write $T(\sigma) := T(\sigma^1)$. Recall that the action of σ on T_X is by primitive 8th roots of the unity, see [11, Theorem 3.1 (c)].

Remark 1.1. *It is a straightforward computation to show that the invariants $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$ and m_1 with $j = 1, 2, 4$ satisfy the following relations:*

$$\begin{aligned} r_{\sigma^2} &= r + l; & r_{\sigma^4} &= r + l + 2m; \\ l_{\sigma^2} &= 2m; & l_{\sigma^4} &= 4m_1; \\ 2m_{\sigma^2} &= 4m_1. \end{aligned}$$

We remark that the invariants l_{σ^2} and m_{σ^2} are even numbers.

The moduli space of $K3$ surfaces carrying a non-symplectic automorphism of order 8 with a given action on the $K3$ lattice is known to be the quotient of a complex ball of dimension $m_1 - 1$, see [8, §11],

$$\mathbb{B} = \{[w] \in \mathbb{P}(V) : (w, \bar{w}) > 0\},$$

where V is the ζ_8 -eigenspace of σ^* in $T(\sigma^4) \otimes \mathbb{C}$. This implies that the Picard group of a $K3$ surface, corresponding to the generic point, equals $S(\sigma^4)$ (see [8, Theorem 11.2]).

By [11, Theorem 3.1] the eigenvalues of σ on T_X are primitive 8th roots of unity so $\text{rk}(T_X) = 4m_1$. Since $0 < \text{rk}(T_X) \leq 21$ we have that $m_1 \leq 5$.

2. THE FIXED LOCUS

We denote by $\text{Fix}(\sigma^j)$, $j = 1, 2, 4$ the fixed locus of the automorphism σ^j such that

$$\text{Fix}(\sigma^j) = \{x \in X \mid \sigma^j(x) = x\}.$$

Clearly $\text{Fix}(\sigma) \subseteq \text{Fix}(\sigma^2) \subseteq \text{Fix}(\sigma^4)$. To describe the fixed locus of order 8 non-symplectic automorphisms we start recalling the following result about non-symplectic involutions (see [12, Theorem 4.2.2]).

Theorem 2.1. *Let τ be a non-symplectic involution on a $K3$ surface X . The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and k smooth rational curves.*

Moreover, its fixed lattice $S(\tau) \subset \text{Pic}(X)$ is a 2-elementary lattice with determinant 2^a such that:

- $S(\tau) \cong U(2) \oplus E_8(2)$ iff the fixed locus of τ is empty;
- $S(\tau) \cong U \oplus E_8(2)$ iff τ fixes two elliptic curves;
- $2g = 22 - \text{rk}S(\tau) - a$ and $2k = \text{rk}S(\tau) - a$ otherwise.

The action of σ at a point in $\text{Fix}(\sigma)$ can be locally diagonalized as follows (up to permutation of the coordinates, but this does not play any role in the classification):

$$A_{1,0} = \begin{pmatrix} \zeta_8 & 0 \\ 0 & 1 \end{pmatrix}, A_{2,7} = \begin{pmatrix} i & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, A_{3,6} = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & -i \end{pmatrix}, A_{4,5} = \begin{pmatrix} -1 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}.$$

In the first case the point belongs to a smooth fixed curve, while in the other cases it is an isolated fixed point. We say that an isolated point $x \in \text{Fix}(\sigma)$ is of *type* (t, s) if the local action at x is given by $A_{t,s}$. We denote by $n_{t,s}$ the number of isolated fixed points by σ with matrix $A_{t,s}$. We further denote by $N_{\sigma^j}, k_{\sigma^j}$, $j = 1, 2, 4$ the number of isolated points and smooth rational curves in $\text{Fix}(\sigma^j)$. We observe that

$N_{\sigma^4} = 0$ since σ^4 only fixes curves (or is empty) as explained in Theorem 2.1. For simplicity we write $N := N_\sigma$ and $k := k_\sigma$. The fixed locus of an automorphism σ is then

$$\text{Fix}(\sigma) = C \cup R_1 \cup \dots \cup R_k \cup \{p_1, \dots, p_N\}$$

where C is a smooth curve of genus $g \geq 0$, R_i are smooth disjoint rational curves and p_i are isolated points. We will see that the fixed locus can never contain two elliptic curves or be empty.

Proposition 2.2. *Let σ be a purely non-symplectic automorphism of order 8 acting on a K3 surface X . Then $\text{Fix}(\sigma)$ is never empty nor it can be the union of two smooth elliptic curves. It is the disjoint union of smooth curves and $N \geq 2$ isolated points. Moreover, the following relations hold:*

$$n_{2,7} + n_{3,6} = 2 + 4\alpha, \quad n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha, \quad N = 2 + r - l - 2\alpha.$$

Here we denote $\alpha = \sum_{K \subset \text{Fix}(\sigma)} (1 - g(K))$.

Proof. Let σ be a purely non-symplectic automorphism of order 8, then $\sigma^*(\omega_X) = \zeta_8 \omega_X$ where $\zeta_8 = e^{\frac{2\pi i}{8}}$. The Lefschetz number of σ is :

$$L(\sigma) = \sum_{j=0}^2 (-1)^j (\text{tr}(\sigma^* | H^j(X, \mathcal{O}_X))) = 1 + \zeta_8^7,$$

By [6, Theorem 4.6] we have also

$$L(\sigma) = \sum_{t,s} \frac{n_{t,s}}{\det(I - \sigma^* | T_x)} + \frac{1 + \zeta_8}{(1 - \zeta_8)^2} \sum_{C \subset \text{Fix}(\sigma)} (1 - g(C))$$

where T_x denotes the tangent space at an isolated fixed point x . Comparing the two expressions for $L(\sigma)$ and since $L(\sigma) \neq 0$ the fixed locus is never empty nor it can be the union of two elliptic curves. Then using the expression for the local action of σ at a fixed point we get the equations:

$$(1) \quad \begin{cases} n_{2,7} + n_{3,6} &= 2 + 4\alpha. \\ n_{4,5} + n_{2,7} - n_{3,6} &= 2 + 2\alpha. \end{cases}$$

Since $\alpha \geq 0$ we observe that σ fixes at least two isolated points. We consider now the topological Lefschetz fixed point formula for σ . Recall that we write $r = r_\sigma$ and $l = l_\sigma$. We have

$$\begin{aligned} N + \sum_{K \subset \text{Fix}(\sigma)} (2 - 2g(K)) &= \chi(\text{Fix}(\sigma)) = \sum_{h=0}^4 (-1)^h \text{tr}(\sigma^* | H^h(X, \mathbb{R})) \\ &= 2 + \text{tr}(\sigma^* | H^2(X, \mathbb{R})). \end{aligned}$$

This gives $N + 2\alpha = \chi(\text{Fix}(\sigma)) = r - l + 2$, so that $r - l = N + 2\alpha - 2$ and we have the third relation in the statement. \square

Remark 2.3. *The isolated fixed points, by a non-symplectic order eight automorphism σ of type (2, 7) and (3, 6) are also isolated fixed points in $\text{Fix}(\sigma^2)$. The points of type (4, 5) in $\text{Fix}(\sigma)$ are contained in a smooth fixed curve by σ^2 . In fact the action of σ^2 at such points is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_8^2 \end{pmatrix}$ which implies that these points belong to a smooth curve in $\text{Fix}(\sigma^2)$.*

From now on we denote by $n_t = n_{t,s}$ the number of isolated fixed points by σ of type (t, s) , where $t = 2, 3, 4$ and $t + s = 9$. Recall the useful lemma.

Lemma 2.4. [7, Lemma 8.1], [4] *Let $\tau = \sum_i R_i$ be a tree of smooth rational curves on a K3 surface X such that each R_i is invariant under the action of a purely non-symplectic automorphism σ of order q . Then, the points of intersection of the rational curves R_i are fixed by σ and the action at one fixed point determines the action on the whole tree.*

Remark 2.5. *In the case of an automorphism of order 8, with the assumptions of Lemma 2.4, the local actions at the intersection points of the curves R_i appear in the following order (we give only the exponents of ζ_8 in the matrix of the local action):*

$$\dots, (0, 1), (7, 2), (6, 3), (5, 4), (4, 5), (3, 6), (2, 7), (1, 0), \dots$$

Assuming that $\tau = R$ consists only of one rational curve, which is not pointwise fixed and do not intersect a fixed curve of higher genus, one get immediately that σ has either one fixed point of type $(2, 7)$ and another one of type $(3, 6)$, two fixed points of type $(4, 5)$ or one fixed point of type $(4, 5)$ and one of type $(3, 6)$.

Proposition 2.6. *Let σ be a non-symplectic automorphism of order 8 on a K3 surface X . Assume that $\text{Pic}(X) = S(\sigma^4)$ and $C \subset \text{Fix}(\sigma)$ with $g(C) = 1$. Then the following relations hold:*

$$\begin{aligned} N_{\sigma^2} &= 2k_{\sigma^2} + 4, \\ 4k_{\sigma^2} &= r_{\sigma^2} - l_{\sigma^2} - 2 = 2(10 - l_{\sigma^2} - m_{\sigma^2}). \end{aligned}$$

Proof. Observe that the fixed locus of σ^2 is the disjoint union of k_{σ^2} smooth rational curves, a smooth elliptic curve and N_{σ^2} isolated points. Moreover we have that an isolated fixed point by σ^2 is given by the local action $\begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}$. We obtain the relations in the statement by applying holomorphic and topological Lefschetz's formulas. \square

3. THE CLASSIFICATION

Lemma 3.1. *If a K3 surface X carries a σ -invariant elliptic fibration, such that σ^4 fixes an irreducible smooth fiber C of this fibration, then σ acts with order 8 on the basis of the fibration and fixes two points on it.*

Proof. Let $\pi_C : X \rightarrow \mathbb{P}^1$ be a σ -invariant elliptic fibration having C as a smooth fiber and such that C is fixed by the involution σ^4 . Observe that σ^2 (respectively σ^4) is not the identity on the basis of π_C , since otherwise it would act as the identity on the tangent space at a point of C , contradicting the fact that σ^2 (respectively σ^4) is purely non-symplectic. Hence σ acts as an order eight automorphism on \mathbb{P}^1 and has two fixed points on it corresponding to C and another fiber C' . \square

Remark 3.2. *Let C be an elliptic curve fixed by the involution σ^4 . Then C is also invariant by σ^i for $i = 1, 2$. And σ either fixes C or it acts on C as a translation with no-fixed points or σ preserves C and fixes isolated points on it. In this last case we have two possibilities:*

- a) The automorphism σ acts on C as an automorphism of order four with two isolated fixed points, that must be of type (2,7) and/or (3,6). In fact σ can not have a fixed point of type (4,5) on C otherwise this point would be contained on a fixed curve for σ^2 , that would be also fixed by σ^4 , but σ^4 already fixes C and fixed curves do not intersect.
- b) The automorphism σ acts on C as an involution with four isolated fixed points. These must be clearly of type (4,5).

Theorem 3.3. *Let σ be a purely non-symplectic automorphism of order 8 on a K3 surface X , such that the involution σ^4 fixes a smooth elliptic curve. Then the number $\#C$ of fixed smooth elliptic curves by σ^4 is at most 2. Moreover, the action of σ is described in the Table 5.*

Proof. Observe first that $\text{rk Pic}(X) \in \{18, 14, 10, 6, 2\}$ since $\varphi(8) | \text{rk } T_X$ by [11]. The involution σ^4 fixes an elliptic curve by Theorem 2.1 (see also [5, Figure 1]) so that we get $\text{rk Pic}(X) \in \{10, 14, 18\}$. On the other hand, by Theorem 2.1 again we have that σ^4 fixes at most two elliptic curves, moreover we know that for the number $\#C$ of smooth fixed elliptic curves by σ^4 and the number k_{σ^4} of rational fixed curves holds:

$$(2) \quad (\text{rk Pic}(X), \#C, k_{\sigma^4}) = (10, 2, 0), (14, 1, 4), (18, 1, 8).$$

Let C denote a σ^4 -fixed smooth elliptic curve. Since σ preserves C there is a σ -invariant elliptic fibration

$$\pi_C : X \longrightarrow \mathbb{P}^1$$

with generic fiber C . Observe that by Lemma 3.1 the automorphism σ has order eight on the basis \mathbb{P}^1 and it fixes two points corresponding to the fiber C and a fiber C' . The fiber C' can be smooth elliptic and so $k_{\sigma} = 0$ or it can be reducible and then it contains all rational curves fixed by σ^4 . In the case $(\text{rk Pic}(X), \#C, k_{\sigma^4}) = (14, 1, 4)$, the curve C' is a reducible fiber containing four smooth rational curves fixed by σ^4 . Since the non-symplectic involution σ^4 does not fix isolated points and the fixed curves by σ^4 do not intersect, one can easily see that the reducible fiber C' is either of Kodaira type IV^* or I_8 . By the same argument we get that C' is of type I_{16} in the case $(\text{rk Pic}(X), \#C, k_{\sigma^4}) = (18, 1, 8)$ (see also [4, Theorem 3.1, Theorem 8.4]).

The case $\text{rk Pic}(X) = 10$. Since $k_{\sigma^4} = 0$ we have also $k = 0$, thus $n_2 + n_3 = 2$ by Proposition 2.2. Observe that σ acts as an automorphism of order four on one of the two σ^4 -fixed elliptic curves C, C' , in fact σ^2 has order four and can not fix two elliptic curves, see [4, Proposition 1]. On the other hand either σ acts on the other σ^4 -fixed elliptic curve as a translation (of order 2 or 4, recall that the fixed locus of σ^4 consists of two smooth elliptic curves) or it is the identity, in this case $n_4 = 0$ so that $(n_2, n_3, n_4) = (2, 0, 0)$ or it acts as an involution when $(n_2, n_3, n_4) = (0, 2, 4)$ (see Proposition 2.2 and Remark 3.2). Since $\text{rk Pic}(X) = 10$ we have that $\text{rk } T_X = 12$ so that $m_1 = 3$. This gives $m_{\sigma^2} = 6$ by Remark 1.1. Since $k_{\sigma^2} = 0$ using Proposition 2.6 we get that $10 - l_{\sigma^2} - m_{\sigma^2} = 0$, so we have $l_{\sigma^2} = 2m = 4$, $m = 2$ and $r_{\sigma^2} = 6$ by Proposition 2.6 again. By Proposition 2.2 and Remark 1.1 one can find easily that $(r, l, m) = (3, 3, 2)$, respectively $(5, 1, 2)$. This gives the **first four cases** in Table 5.

The case $\text{rk Pic}(X) = 14$. As we have already seen, the fiber C' is either of type IV^* or I_8 . We study here these two possibilities separately.

- C' of type IV^* : Since the fiber IV^* is preserved by σ^i ; $i = 1, 2, 4$, the automorphism σ either preserves each component of IV^* or it exchanges the two branches of IV^* . In these two cases σ^2 preserves each component of C' and fixes the central component of multiplicity 3 since it has at least three fixed points by σ^2 . Then we have that $k \in \{0, 1\}$ by Remark 2.5. If $k = 1$ then σ preserves each component of C' , hence it fixes six points on C' three of each type (2, 7) and (3, 6) by Remark 2.5, thus $(n_2, n_3, n_4) \geq (3, 3, 0)$ (we write $(x_1, x_2, x_3) \geq (y_1, y_2, y_3)$ if and only if $x_i \geq y_i$ for all $i = 1, 2, 3$). By Proposition 2.2 we get $(n_2, n_3, n_4) = (3, 3, 4)$, this implies that σ acts as an involution on C and fixes four points on it of type (4, 5). This is **case 11** in Table 5.

If $k = 0$ the automorphism σ exchanges the two branches of C' . So the fiber C' contains two fixed points of type (4, 5) on the central component of multiplicity 3, which is then fixed by σ^2 , and one point of type (2, 7) and another one of type (3, 6), hence we get $(n_2, n_3, n_4) \geq (1, 1, 2)$. By Proposition 2.2 we have that $(n_2, n_3, n_4) = (1, 1, 2)$ which means that σ acts on C either as the identity or as a translation of order two or four (since σ^4 fixes C). These are the **cases 5, 6, 7** in the Table 5.

Finally, since $k_{\sigma^2} = 1$ and by doing the same computation as in the previous case one gets immediately that $(r, l, m) = (10, 0, 2)$ and $(6, 4, 2)$ respectively for $k = 1$ and $k = 0$.

- C' of type I_8 : First observe that all components of C' are preserved by σ^4 since $\text{Pic}(X) = S(\sigma^4)$ and a component which is not fixed intersects two fixed ones. The automorphism σ either preserves each component of I_8 or acts on it as a reflection (i.e. σ preserves two components on I_8 and exchanges the remaining 6 components two by two) or it acts as a rotation, of order two or four. Applying Remark 2.5 one gets that in the first case σ fixes one component of C' (i.e. $k = 1$) and 2 isolated points of each type (2, 7), (3, 6) and (4, 5), so we get $(n_2, n_3, n_4) \geq (2, 2, 2)$. On the other hand, we have that $(n_2, n_3, n_4) = (4, 2, 2)$ by Proposition 2.2, which implies that σ acts on C as an automorphism of order four and fixes two points on it of type (2, 7) (see Remark 3.2). This is the **case 12** in the Table 5.

If σ acts on C' as a reflection, then $k = 0$ and σ fixes four points of type (4, 5) on the two invariant components of C' . Observe that these two components are fixed by σ^2 . Hence we get $(n_2, n_3, n_4) \geq (0, 0, 4)$. On the other hand, by Proposition 2.2 we have that $(n_2, n_3, n_4) = (0, 2, 4)$, thus σ acts as an automorphism of order four on C and fixes two points of type (3, 6) (see Remark 3.2). This is the **case 10** in the Table 5.

Using the relation $4k_{\sigma^2} = r_{\sigma^2} - l_{\sigma^2} - 2 = 2(10 - l_{\sigma^2} - m_{\sigma^2})$ of Proposition 2.6 since $k_{\sigma^2} = 2$ and $m_{\sigma^2} = 4$ we get in these two cases that $(r_{\sigma^2}, l_{\sigma^2}) = (12, 2)$. In fact in these two cases σ^2 preserves each component of I_8 and fixes two components in C' . We get then easily that (r, l, m) is equal respectively to $(10, 2, 1)$ and $(8, 4, 1)$ by Proposition 2.2.

Finally, if σ acts as a rotation of order two or of order four on C' , then in these both cases $k = 0$ and σ does not fix any point on C' . By Proposition 2.2 we have that $(n_2, n_3, n_4) = (2, 0, 0)$ and so σ also acts as an automorphism of order four on C and fixes two points of type (2, 7) (see Remark 3.2). Moreover we compute as above the invariants r, l, m in

the case that $k_{\sigma^2} = 2$ respectively 0 which correspond to the action of the automorphism on I_8 as a rotation of order two or of order four respectively. We get easily that (r, l, m) equals $(6, 6, 1)$ respectively $(4, 4, 3)$. These are the **cases 8 and 9** in the Table 5.

The case $\text{rk Pic}(X) = 18$. In this case the fiber C' is of type I_{16} as remarked at the beginning of the proof. As we have seen previously, there are four possibilities for the action of σ on I_{16} that are: either σ preserves each component of I_{16} or it acts as a reflection (here σ preserves two components of I_{16} and exchanges the remaining 14 components two by two) or it acts as a rotation of order two, respectively four. (These are the **cases 16, 15, 13 and 14** in the Table 5. If σ preserves each component of C' , then by applying Remark 2.5 we get that σ fixes two components of C' (i.e. $k = 2$) and 12 isolated points four of each type $(2, 7)$, $(3, 6)$ and $(4, 5)$. On the other hand, if σ acts as a reflection, then $k = 0$ and σ fixes four isolated points of type $(4, 5)$ on the two preserved components of C' . Finally, σ does not fix any point on C' if it acts as a rotation.

Observe that in these three cases σ acts as an automorphism of order four on the smooth σ^4 -fixed fiber C by Proposition 2.2. Using the same argument as before we get the values of the invariants r, l, m that appeared in the last four cases of Table 5. \square

4. EXAMPLES

In this section we give examples corresponding to several cases in the classification of the non-symplectic automorphisms of order eight on elliptic $K3$ surfaces.

Example 4.1. Consider the elliptic fibration $\pi_C : X \rightarrow \mathbb{P}^1$ with a Weierstrass equation:

$$y^2 = x^3 + a(t)x + b(t).$$

where $a(t) = at^8 + b$ and $b(t) = ct^8 + d$ with $a, b, c, d \in \mathbb{C}$. The fibration π_C admits the order eight automorphism:

$$\sigma(x, y, t) = (x, y, \zeta_8 t).$$

The fibers preserved by σ are over 0 and ∞ and the action of σ at infinity is:

$$(x/t^4, y/t^6, 1/t) \mapsto (-x/t^4, iy/t^6, \zeta_8^7/t)$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4a(t)^3 + 27b(t)^2 = h_1 t^{24} + h_2 t^{16} + h_3 t^8 + O(t^4),$$

where

$$h_1 = 4a^3, \quad h_2 = 12a^2b + 27c^2, \quad h_3 = 12ab^2 + 54cd.$$

Observe that $\Delta(t)$ has 24 simple zeros for a generic choice of the coefficients. By studying the zeros of $\Delta(t)$ and looking in the classification of singular fibers of elliptic fibrations on surfaces (see e.g. [10, section 3]) one obtains the following: for a generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$, both fibers over $t = 0$ and $t = \infty$ are smooth elliptic curves, moreover the automorphism σ fixes pointwisely the fiber over 0 and acts as an *order 4 automorphism* on the fiber over ∞ , hence it has two fixed points on this fiber. We show below that generically the rank of $\text{Pic}(X)$ is 10. This gives an example for the **first case in Table 5**. On the other hand, if $h_1 = 0$ so that $a = 0$ the fibration acquires a fiber of type IV^* at ∞ by a generic choice of the

parameters. The fiber over 0 is a smooth elliptic curve fixed pointwisely by σ . This gives an example for the **case 5 in Table 5**, this follows by Theorem 3.2 and by the fact that this case is the only one for which σ contains an elliptic curve in the fixed locus and the fibration has a fiber of type IV^* .

By using standard transformations on the parameters in the Weierstrass form we get that the number of moduli is 2. In fact both the polynomials $a(t), b(t)$ depend on 2 parameters, but we can apply the transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$; $\lambda \in \mathbb{C}^*$, to cancel one of the 4 parameters. Moreover the automorphisms of \mathbb{P}^1 commuting with $t \mapsto \zeta_8 t$ are of the form $t \mapsto \mu t$; $\mu \in \mathbb{C}^*$, so we can cancel a second parameter. This shows that the family depends on 2 parameters, so that generically $\text{rk } T_X = 12$ (recall that $\text{rk } T_X = 4m_1$ and $m_1 - 1$ equals to the number of moduli) and $\text{rk Pic}(X) = 10$.

Example 4.2. As in the previous example 4.1, consider again the elliptic fibration $\pi_C : X \rightarrow \mathbb{P}^1$ in Weierstrass form given by :

$$y^2 = x^3 + a(t)x + b(t).$$

where $a(t) = at^8 + b$ and $b(t) = ct^8 + d$ with $a, b, c, d \in \mathbb{C}$. This elliptic fibration carries the non-symplectic automorphism σ of order eight:

$$(x, y, t) \mapsto (x, -y, \zeta_8 t).$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is

$$(x/t^4, y/t^6, 1/t) \mapsto (-x/t^4, -iy/t^6, \zeta_8^7/t).$$

The discriminant polynomial of π_C is:

$$\Delta(t) := 4a(t)^3 + 27b(t)^2 = h_1 t^{24} + h_2 t^{16} + O(t^8),$$

where

$$h_1 = 4a^3 \text{ and } h_2 = 12a^2b + 27c^2.$$

We have seen in example 4.1 that for a generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 over the zeros of $\Delta(t)$ (see [10, section 3]), moreover σ acts as an *involution* on the fiber over 0 and it acts as an *order 4 automorphism* on the fiber over ∞ (both fibers are smooth). So we have an example for the **case 4 in Table 5**. If $h_1 = 0$, that implies $a = 0$, the fibration acquires a fiber of type IV^* at ∞ by a generic choice of the parameters. This is the **case 11 in Table 5** (this is the only case where the fibration has a fiber of type IV^* and the action on the smooth fiber is an involution).

Example 4.3. Consider the elliptic fibration $\pi_C : X \rightarrow \mathbb{P}^1$ in Weierstrass form given by

$$y^2 = x^3 + a(t)x + b(t),$$

where $a(t) = at^8 + b$ and $b(t) = ct^4 + dt^{12}$; $a, b, c, d \in \mathbb{C}$. Observe that it carries the order eight automorphism

$$\sigma : (x, y, t) \mapsto (-x, iy, \zeta_8^7 t).$$

For generic choice of the coefficients the fiber over $t = 0$ is smooth. The automorphism σ^4 is an involution fixing the smooth elliptic curve over $t = 0$. On the other hand, σ acts on the elliptic curve over $t = 0$ as an *order 4 automorphism* with 2 isolated fixed points. Moreover it acts as the identity on the fiber over $t = \infty$, in fact the action is

$$(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, y/t^6, \zeta_8/t).$$

The discriminant of π_C is :

$$\Delta(t) = h_1 t^{24} + h_2 t^{16} + h_3 t^8 + 4b^3,$$

where

$$h_1 = 4a^3 + 27d^2, \quad h_2 = 12a^2b + 54cd, \quad h_3 = 12ab^2 + 27c^2.$$

Observe that $\Delta(t)$ has 24 simple zeros for a generic choice of the coefficients. By studying the zeros of $\Delta(t)$ and looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g [10, section 3]) one obtains the following cases: for the generic choice of the coefficients of $a(t)$ and $b(t)$ the fibration has 24 fibers of type I_1 . We get again an example for **case 4 in Table 5**. If $h_1 = 0$ the fibration has a fiber of type I_8 at infinity. If $h_1 = h_2 = 0$ we get a fiber I_{16} . By [9, §3] a holomorphic two form is given by $\omega_X = (dt \wedge dx)/2y$ and so the action of σ on it is $\sigma^*(\omega_X) = \frac{-\zeta_8^7}{i} \omega_X = \zeta_8 \omega_X$. We determine now exactly the type of the local action of σ at the two fixed points on the elliptic curve C . We look at the elliptic fibration locally around the fiber over $t = 0$. The equation in $\mathbb{P}^2 \times \mathbb{C}$ is given by:

$$F(x, y, z, t) := zy^2 - (x^3 + (at^8 + b)z^2x + (ct^4 + dt^{12})z^3) = 0.$$

Where $(x : y : z)$ are the homogeneous coordinates of \mathbb{P}^2 . The fiber at $t = 0$ has equation

$$f := \{zy^2 - x^3 - bz^2x = 0\}$$

and for $b \in \mathbb{C}$ generic is smooth since the partial derivatives $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ are not simultaneously zero. The two fixed points by σ are $p := (0 : 1 : 0)$ and $p' := (0 : 0 : 1)$. The point $p' := (0 : 0 : 1)$ is contained in the chart $z = 1$ and it belongs to the open set $F_x := \partial F/\partial x \neq 0$, in fact $F_x(p') := \frac{\partial F(x, y, 1, 0)}{\partial x} \neq 0$. The one-form for the elliptic curve in this open subset is:

$$dy/F_x(p') = dy/(-3x^2 - b).$$

The action of σ here is a multiplication by i : ($\sigma^*(dy/F_x(p')) = i(dy/F_x(p'))$), so that the action on the holomorphic 2-form $dt \wedge (dy/(-3x^2(at^8 + b)))$ is the multiplication by ζ_8 as expected. In particular the local action at p' is of type (7,2). Similarly we can do the computation on the open subset in the chart $y = 1$ which contains the fixed point p , and we can find again the same action (7,2). Observe that since σ acts as the identity on the fiber over $t = \infty$, it preserves the curves of the singular fibers (of type I_8 or I_{16}). This gives an example for **the cases 12 and 16 in Table 5**.

On other hand, the fibration π_C admits also the automorphism $\tau(x, y, t) = (-x, -iy, \zeta_8^3 t)$. This automorphism acts also by multiplication by ζ_8 on the holomorphic 2-form ω_X , thus τ is not a power of σ (i.e it is a new automorphism). Moreover the square of τ preserves each components of the fiber at $t = \infty$ in fact the action at infinity is:

$$(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, -y/t^6, \zeta_8^5/t).$$

By a similar computation as above one sees that the local action at the fixed points on the fiber C is of type (3,6), so we have an example for **the cases 10 and 15 in Table 5** respectively.

Example 4.4. (*Translation*).

We give here an example for the **cases 2, 8 and 13 in Table 5**. Observe that

in these three cases the non-symplectic automorphism of order 8 on X acts as a translation of order two on the σ -invariant fiber C' .

It is well known that an elliptic $K3$ surface with a 2-torsion section can be written with equation:

$$y^2 = x(x^2 + a(t)x + b(t)),$$

where $a(t)$ and $b(t)$ are polynomials of degree 4 and 8 respectively. Or equivalently one can write the equation in Weierstrass form:

$$y^2 = x^3 + A(t)x + B(t),$$

where

$$A(t) = 9b(t) - 3a(t)^2 \text{ and } B(t) = 3a(t)^2 - 9a(t)b(t).$$

Now the map:

$$(4) \quad \tau : (x, y, t) \mapsto (y^2/x^2 - a(t) - x, (y/x).\tau(x), t),$$

with $\tau(x) := (y^2/x^2) - a(t) - x$ is an automorphism on X that acts as a translation of order two on the generic fiber of π .

Consider now the non-symplectic automorphism of order eight on X :

$$\sigma : (x, y, t) \mapsto (-x, iy, \zeta_8^7 t).$$

This automorphism preserves the jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ defined as follows:

$$y^2 = x(x^2 + a(t)x + b(t)),$$

where $a(t) = \alpha t^4, b(t) = \beta t^8 + \gamma$; $\alpha, \beta, \gamma \in \mathbb{C}$. Or equivalently the fibration can be written as:

$$y^2 = x^3 + A(t)x + B(t),$$

such that $A(t) = (9\beta - 3\alpha^2)t^8 + 9\gamma$ and $B(t) = (2\alpha^3 - 9\alpha\beta)t^{12} - 9\alpha\gamma t^4$. The discriminant of π is:

$$\Delta(t) = K(\beta t^8 + \gamma)^2[(\alpha^2 - 4\beta)t^8 - 4\gamma]; \quad K \in \mathbb{C} \text{ is a constant.}$$

Consider now the translation τ :

$$\tau : (x, y, t) \mapsto (y^2/x^2 - \alpha t^4 - x, (y/x).\tau(x), t).$$

As we have seen, τ is an automorphism of X and it acts as a translation of order two on the generic fiber of π . Moreover, one can get easily that $\tau \circ \sigma = \sigma \circ \tau$, thus the $K3$ surface X has the order eight non-symplectic automorphism $\sigma' := \sigma \circ \tau$. Observe that the automorphisms σ and σ' act with order eight on \mathbb{P}^1 , they preserve the two fibers over $t = 0$ and $t = \infty$ and act as an automorphism of order four on the smooth fiber over $t = 0$ given by $f := zy^2 - x^3 - 9\gamma xz^2$, for $\gamma \in \mathbb{C}$ generic. Moreover, σ acts as the identity on the fiber over $t = \infty$, while σ' acts on it as an order two translation (observe that the action of σ at infinity is $(x/t^4, y/t^6, 1/t) \mapsto (x/t^4, y/t^6, \zeta_8 1/t)$ and $\sigma'^2 = (\sigma \circ \tau)^2 = id$).

Studying the zeros of the discriminant $\Delta(t)$, and looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g [10, §3]) we get the following:

- For generic α, β, γ the fibration π has 8 fibers of type I_2 and 8 fibers of type I_1 over the zeros of $\Delta(t)$, σ' acts as an order four automorphism on the fiber over 0 and it acts as an order two translation on the fiber at ∞ (both fibers are smooth). This gives an example for the **case 2 in Table 5**.

- If $\alpha^2 - 4\beta = 0$ with $(\beta \neq 0)$, then $\Delta(t) = K(\beta t^8 + \gamma)^2(-4\beta)$; $K \in \mathbb{C}^*$. So that the fibration acquires a fiber of type I_8 over ∞ (in fact $\Delta(t)$ has a zero of order 8 over $t = \infty$ and $A(t), B(t)$ are non zero). The automorphism σ' acts as a rotation of order two on the fiber I_8 (see [10]). This corresponds to the **case 8 in Table 5**.
- If $\beta = 0$, $(\alpha \neq 0)$, the discriminant $\Delta(t) = K\gamma^2(\alpha^2 t^2 - 4\gamma)$ vanishes at $t = \infty$ with order 16, and $A(t), B(t)$ are nonzero at ∞ . Thus we get a fiber of type I_{16} , on which σ acts as a rotation of order two. We are in the **case 13 of Table 5**.

Example 4.5. *Quadruple Quartics.*

Take the fourfold cover of \mathbb{P}^2 :

$$t^4 = x_0(l_3(x_1, x_2) + x_0^2 l_1(x_1, x_2))$$

where $l_3(x_1, x_2)$ is homogeneous of degree three and $l_1(x_1, x_2)$ is homogeneous of degree 1. This is invariant for the action of the order 8 non symplectic automorphism:

$$(t, x_0, x_1, x_2) \mapsto (\zeta_8 t, -x_0, x_1, x_2)$$

it fixes the inverse image of the curve $\{x_0 = 0\}$ which is rational and 4 points on the curve $C : \{l_3(x_1, x_2) + x_0^2 l_1(x_1, x_2) = 0\}$ which is in fact elliptic. This gives another example for the **case 11 of Table 5**.

5. THE TABLE OF THE σ -INVARIANT ELLIPTIC FIBRATIONS

We give the table for the classification of the non-symplectic automorphisms of order 8 on an elliptic K3 surface. The cases for which we have an example are denoted with \checkmark . We list also in this table the invariants r, l, m and m_1 of σ which denote the rank of the eigenspaces of $(\sigma)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1, i$ and ζ_8 respectively. Recall that by $\#C$ we denote the number of σ -invariant smooth elliptic curves.

Example	r	l	m	k_{σ^2}	$\#C$	$\text{rk Pic}(X)$	k_{σ^4}	N	(n_2, n_3, n_4)	k	action of σ on fixed elliptic curves and singular fibers
1. \checkmark	3	3	2	0	2	10	0	2	(2,0,0)	0	(identity, order four)
2. \checkmark	3	3	2	0				2	(2,0,0)	0	(translation of order two, order four)
3. $-$	3	3	2	0				2	(2,0,0)	0	(translation of order four, order four)
4. \checkmark	5	1	2	0				6	(0,2,4)	0	(involution, order four)
5. \checkmark	6	4	2	1	1	14	4	4	(1,1,2)	0	(identity, reflection of IV^*)
6. $-$	6	4	2	1				4	(1,1,2)	0	(translation of order two, reflection of IV^*)
7. $-$	6	4	2	1				4	(1,1,2)	0	(translation of order four, reflection of IV^*)
8. \checkmark	6	6	1	2				2	(2,0,0)	0	(order four, rotation of order 2 on I_8)
9. $-$	4	4	3	0				2	(2,0,0)	0	(order four, rotation of order 4 on I_8)
10. \checkmark	8	4	1	2				6	(0,2,4)	0	(order four, reflection on I_8)
11. \checkmark	10	0	2	1				10	(3,3,4)	1	(involution, preserves each curve of IV^*)
12. \checkmark	10	2	1	2				8	(4,2,2)	1	(order four, preserves each curve of I_8)
13. \checkmark	9	9	0	4	1	18	8	2	(2,0,0)	0	(order four, rotation of order 2 on I_{16})
14. $-$	5	5	4	0				2	(2,0,0)	0	(order four, rotation of order 4 on I_{16})
15. \checkmark	11	7	0	4				6	(0,2,4)	0	(order four, reflection on I_{16})
16. \checkmark	17	1	0	4				14	(6,4,4)	2	(order four, preserves each curve of I_{16})

REFERENCES

1. D. Al Tabbaa, *On some automorphisms of K3 surfaces*, PhD thesis University of Poitiers, December 2015.
2. D. Al Tabbaa, A. Sarti, and S. Taki, *Classification of order sixteen non-symplectic automorphisms on K3 surfaces*, To appear in J. Korean Math. Soc.
3. M. Artebani and A. Sarti, *Non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Ann. **342** (2008), no. 4, 903–921.
4. ———, *Symmetries of order four on K3 surfaces*, J. Math. Soc. Japan **67** (2015), no. 2, 1–31.
5. M. Artebani, A. Sarti, and S. Taki, *K3 surfaces with non-symplectic automorphisms of prime order.*, Math. Z. **268** (2011), no. 1-2, 507–533, with an appendix by Shigeyuki Kondō.
6. M. F. Atiyah and I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. (2) **87** (1968), 546–604.
7. Jimmy Dillies, *On some order 6 non-symplectic automorphisms of elliptic K3 surfaces*, Albanian J. Math. **6** (2012), no. 2, 103–114. MR 3009163
8. I.V. Dolgachev and S. Kondō, *Moduli of K3 surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100.
9. S. Kondō, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98.
10. R. Miranda, *The basic theory of elliptic surfaces. Notes of lectures.*, Dottorato di Ricerca di Matematica, Università di Pisa, Dipartimento di Matematica. Pisa: ETS Editrice, 1989 (English).
11. V.V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Uspehi Mat. Nauk **31** (1976), no. 2(188), 223–224.
12. ———, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications*, J. Soviet. Math. **22** (1983), 1401–1475.
13. M. Schütt, *K3 surfaces with non-symplectic automorphisms of 2-power order*, J. Algebra **323** (2010), no. 1, 206–223.
14. S. Taki, *Classification of non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Nachr. **284** (2011), 124–135.
15. ———, *Classification of non-symplectic automorphisms on K3 surfaces which act trivially on the Néron-Severi lattice*, J. Algebra **358** (2012), 16–26.
16. ———, *On Oguiso’s K3 surface*, J. Pure Appl. Algebra **218** (2014), no. 3, 391–394.
17. D-Q. Zhang, *Automorphisms of K3 surfaces*, Proceedings of the International Conference on Complex Geometry and Related Fields (Providence, RI), AMS/IP Stud. Adv. Math., vol. 39, Amer. Math. Soc., 2007, pp. 379–392.

LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UMR CNRS 7348, UNIVERSITÉ DE POITIERS,
TÉLÉPORT 2, BOULEVARD MARIE ET PIERRE CURIE, 86962 FUTUROSCOPE CHASSENEUIL,
FRANCE

E-mail address: `Dima.Al.Tabbaa@math.univ-poitiers.fr`

LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS, UMR CNRS 7348, UNIVERSITÉ DE POITIERS,
TÉLÉPORT 2, BOULEVARD MARIE ET PIERRE CURIE, 86962 FUTUROSCOPE CHASSENEUIL,
FRANCE

E-mail address: `sarti@math.univ-poitiers.fr`

URL: `http://www-math.sp2mi.univ-poitiers.fr/~sarti/`