CLASSIFICATION OF ORDER SIXTEEN NON-SYMPLECTIC AUTOMORPHISMS ON K3 SURFACES

DIMA AL TABBAA, ALESSANDRA SARTI, AND SHINGO TAKI

Abstract. In the paper we classify complex K3 surfaces with non-symplectic automorphism of order 16 in full generality. We show that the fixed locus contains only rational curves and points and we completely classify the seven possible configurations. If the Picard group has rank 6, there are two possibilities and if its rank is 14, there are five possibilities. In particular if the action of the automorphism is trivial on the Picard group, then we show that its rank is six.

Introduction

Automorphisms of complex K3 surfaces were widely studied in the last years, in particular also for the recent relation with the Bloch conjecture, see e.g. [10], [9]. Here we study (purely) non-symplectic automorphisms of order d, i.e. automorphisms that multiply the non-degenerate holomorphic two form by a primitive dth root of unity. The study of non-symplectic automorphisms of prime order was completed by Nikulin in [17] in the case of involutions, and more recently by Artebani, Sarti and Taki in several papers [2, 4, 21] for the other prime orders. The study of non-symplectic automorphisms of nonprime order turns out to be more complicated, indeed in this situation the "generic" case does not imply that the action of the automorphism is trivial on the Picard group [8, Section 11]. In the paper [22] Taki completely describes the case when the automorphism is a prime power and the action is trivial on the Picard group. If we consider non-symplectic automorphisms that are of order 2^t, then by results of Nikulin we have $0 \leq t \leq 5$, and by a recent paper by Taki [23] there is only one K3 surface that admits an order 32 non-symplectic automorphism. Some further results in this direction are contained in a paper by Schütt [19] in the case of automorphisms of a 2-power order and in a paper by Artebani and Sarti [3], in the case of the order 4. In this last paper the hypothesis of trivial action on the Picard group is left out. Here we consider the case of the order 16 in full generality, which together with the order 8 remained quite unexplored.

Since Euler's totient function value of 16 must divide the rank of the transcendental lattice (see [15, Theorem 3.1]) the rank of the Picard group can only be equal to 6 or 14. More precisely let X be a K3 surface, ω_X a generator of $H^{2,0}(X)$,

Date: July 13, 2016.

2010 *Mathematics Subject Classification.* Primary 14J28; Secondary 14J50, 14J10.

Key words and phrases. non-symplectic automorphisms, K3 surfaces.

The third-named author was partially supported by Research Institute for Science and Technology of Tokyo Denki University Grant Number Q14K-06/Japan and Grant-in-Aid for Young Scientists (B) 15K17520 from JSPS.

σ an order 16 automorphism such that $\sigma^*\omega_X = \zeta_{16}\omega_X$, where ζ_{16} denotes a primitive 16th root of unity. We assume everywhere that σ^8 acts as the identity on $\mathrm{Pic}(X)$ that corresponds to the generic case in the moduli space (observe that if $\mathrm{rk}\,\mathrm{Pic}(X) = 6$ the condition is automatically satisfied, see Remark 5.3 and Proposition 5.4). We first show that if the fixed locus of σ contains a curve then its genus is zero (Proposition 2.9). We show also that the fixed locus of σ^4 contains always at least a curve of genus 0 or 1 (Proposition 2.10).
In the case that $\mathrm{rk}\,\mathrm{Pic}(X) = 6$ and so σ^8 acts trivially on $\mathrm{Pic}(X)$ we have the following number of isolated fixed points N and fixed rational curves k for σ (Theorem 4.1):

$$(\mathrm{Pic}(X), N, k) = (U \oplus D_4, 6, 1), \text{ or } (U(2) \oplus D_4, 4, 0).$$

In the first case the action of σ is trivial on $\mathrm{Pic}(X)$ but not in the second case. If σ^8 acts trivially on $\mathrm{Pic}(X)$, $\mathrm{rk}\,\mathrm{Pic}(X) = 14$ and σ^4 fixes an elliptic curve C, then σ leaves C invariant (but C is not point wise fixed by σ by Proposition 2.9) and the induced σ-invariant elliptic fibration has a reducible fiber of type IV^* (see Theorem 6.1). The number of isolated σ-fixed points and σ-fixed rational curves are as follows: $(N, k) = (8, 1)$ or $(6, 0)$. In the first case σ preserves each component of the fiber of type IV^* and in the second case it acts as a reflection on it. In any case the action of σ is nontrivial on $\mathrm{Pic}(X)$ (see Theorem 3.1). Finally if σ^8 acts trivially on $\mathrm{Pic}(X)$, $\mathrm{rk}\,\mathrm{Pic}(X) = 14$, $\mathrm{Fix}(\sigma^4)$ contains at least a curve and its genus is at most zero, we have three cases with $(\mathrm{Pic}(X), N, k)$ equal to:

$$(U \oplus D_4 \oplus E_8, 12, 1),\ (U(2) \oplus D_4 \oplus E_8, 4, 0) \text{ or } (U(2) \oplus D_4 \oplus E_8, 10, 1).$$

In these three cases the action of σ is not trivial on $\mathrm{Pic}(X)$, (Theorem 5.1). This in particular shows that there does not exist a K3 surface X with Picard number 14 with an automorphism of order 16 acting non-symplectically on it and trivially on $\mathrm{Pic}(X)$. This corrects a small mistake in the paper [22, Main Theorem (3)], where the author claims that such a K3 surface exists.

We construct the K3 surfaces in the Examples 3.2, 4.2, 5.2 (some of the examples are described in [22] and [7]). For the proofs of the Theorems 3.1, 4.1, 5.1, we use Lefschetz formulas, results on non-symplectic involutions, results on non-symplectic order four automorphisms, that are contained in [3], [22]. We recall these results in Appendix 6 for convenience of the reader. We use also and prove some results on non-symplectic automorphisms of order eight.

The results of this paper are partially contained in the PhD thesis of the first author under the supervision of the second author. The results on order eight non-symplectic automorphisms as well as a classification of K3 surfaces with non-symplectic automorphism of order eight is contained in the PhD thesis of Al Tabbaa, [1] too.

Acknowledgments: We warmly thank Michela Artebani, Samuel Boissière and Alice Garbagnati for several interesting discussions. We are also grateful to the anonymous referee for carefully reading of the paper and for the many useful comments that improved the first version of the paper.

1. Basic facts

Let X be a K3 surface and σ a *non-symplectic* automorphism of order 16 acting on it, this means that the action of σ^* on the vector space $H^{2,0}(X) = \mathbb{C}\omega_X$ of holomorphic two-forms is not trivial. More precisely we assume that $\sigma^*\omega_X = \zeta_{16}\omega_X$, where ζ_{16} is a primitive 16th root of unity. This action is called sometimes in the literature *purely non-symplectic*, in this paper for simplicity we omit "purely". We denote by $\zeta := \zeta_{16}$, $\xi := \zeta_{16}^2$ a primitive 8th root of unity, $i := \zeta_{16}^4$ a primitive 4th root of unity.

We denote furthermore by $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$, $m^1_{\sigma^j}$, $m^2_{\sigma^j}$, $j = 1, 2, 4, 8$ the rank of the eigenspaces of $(\sigma^j)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues $1, -1$, i, ξ and ζ (we follow the notation of [3, Section 1]). For simplicity for $j = 1$ we just write r_σ, $l_\sigma, \ldots$ or even $r, l, \ldots$. The following easy relations hold:

$$\begin{array}{ll} r_{\sigma^2} = r_\sigma + l_\sigma, & l_{\sigma^2} = 2m_\sigma, \quad m_{\sigma^2} = 2m^1_\sigma, \quad m^1_{\sigma^2} = 2m^2_\sigma, \\ r_{\sigma^4} = r_\sigma + l_\sigma + 2m_\sigma, & l_{\sigma^4} = 4m^1_\sigma, \quad m_{\sigma^4} = 4m^2_\sigma, \\ r_{\sigma^8} = r_\sigma + l_\sigma + 2m_\sigma + 4m^1_\sigma, & l_{\sigma^8} = 8m^2_\sigma, \\ r_\sigma + l_\sigma + 2m_\sigma + 4m^1_\sigma + 8m^2_\sigma = 22. & \end{array} \tag{1}$$

Let

$$S(\sigma^j) = \{x \in H^2(X, \mathbb{Z}) \,|\, (\sigma^j)^*(x) = x\},$$

$$T(\sigma^j) = S(\sigma^j)^\perp \cap H^2(X, \mathbb{Z}),$$

clearly $\operatorname{rk} S(\sigma^j) = r_{\sigma^j}$. An easy computation shows that $S(\sigma^j) \subset \operatorname{Pic}(X)$, $j = 1, 2, 4, 8$ so that the transcendental lattice $T_X = (\operatorname{Pic}(X))^\perp \cap H^2(X, \mathbb{Z})$ satisfies $T_X \subset T(\sigma^j)$.

The moduli space of K3 surfaces carrying a non-symplectic automorphism of order 16 with a given action on the K3 lattice is known to be a complex ball quotient of dimension $m^2_\sigma - 1$, see [8, §11]. The complex ball is given by:

$$B = \{[w] \in \mathbb{P}(V) : (w, \bar{w}) > 0\},$$

where V is the ζ-eigenspace of σ^* in $T(\sigma^8) \otimes \mathbb{C}$. This implies that the Picard group of a K3 surface corresponding to the generic point of such space equals $S(\sigma^8)$ (see [8, Theorem 11.2]).

By [15, Theorem 3.1] the eigenvalues of the action of σ on T_X are primitive 16th roots of unity so $\operatorname{rk}(T_X) = 8m^2_\sigma$. Since $0 < \operatorname{rk}(T_X) \leq 21$ we have in fact only two possibilities which are $m^2_\sigma = 1$ or 2 so that $\operatorname{rk} \operatorname{Pic}(X) = 14$ respectively 6. As remarked above in the generic case this is also the rank of $S(\sigma^8) = \operatorname{Pic}(X)$ and we have by orthogonality $T_X = T(\sigma^8)$. Observe moreover that $r_\sigma > 0$ since there is always an ample σ-invariant class on X (see [15, Theorem 3.1], [3, Lemma 1]).

2. The fixed locus

We start by recalling the following result about non-symplectic involutions (see [17, Theorem 4.2.2] and also [14, §4]).

Theorem 2.1. *Let τ be a non-symplectic involution on a K3 surface X. The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and k smooth rational curves.*

Moreover, its fixed lattice $S(\tau) \subset \operatorname{Pic}(X)$ is a 2-elementary lattice with determinant 2^a such that:

- *$S(\tau) \cong U(2) \oplus E_8(2)$ iff the fixed locus of τ is empty;*
- *$S(\tau) \cong U \oplus E_8(2)$ iff τ fixes two elliptic curves;*
- *$2g = 22 - \operatorname{rk} S(\tau) - a$ and $2k = \operatorname{rk} S(\tau) - a$ otherwise.*

Since $S(\tau)$ is 2-elementary its discriminant group $A_{S(\tau)} = S(\tau)^\vee/S(\tau) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$. We introduce the invariant $\delta_{S(\tau)}$ of $S(\tau)$ by putting $\delta_{S(\tau)} = 0$ if $x^2 \in \mathbb{Z}$ for any $x \in A_{S(\tau)}$ and $\delta_{S(\tau)} = 1$ otherwise. By [16, Theorem 3.6.2] and [18, §1] $S(\tau)$ is uniquely determined by the invariant $\delta_{S(\tau)}$, rank, signature and the invariant a. The situation is resumed in Figure 1 from [14, §4].

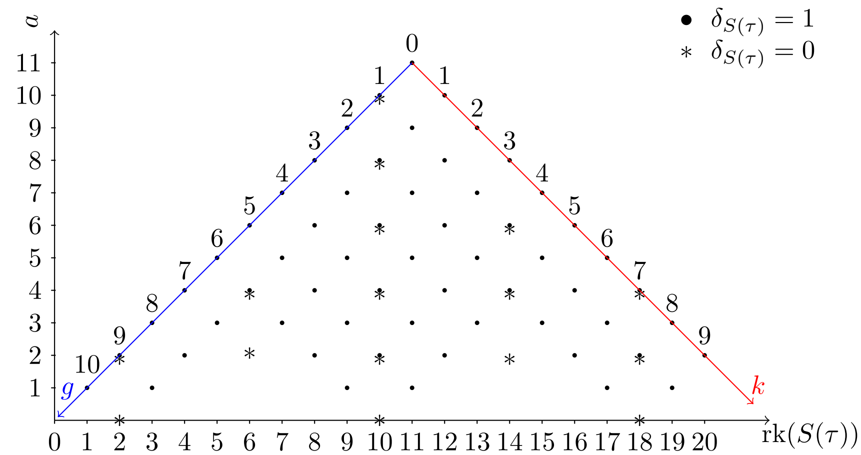

FIGURE 1. Order 2

At a fixed point for σ^j the action can be linearized (see e.g. [15, §5]) and is given by a matrix

$$A^j_{t,s} = \begin{pmatrix} \zeta^t_{(16/j)} & 0 \\ 0 & \zeta^s_{(16/j)} \end{pmatrix}$$

with $t + s = 1 \mod (16/j)$, $0 \leq t < s < 16/j$. This means that the fixed locus of σ^j is the disjoint union of smooth curves and isolated points (see [17, Section 4, §2] and [15, §5]). In the sequel of the paper when we consider curves in the fixed locus of some σ^j we always mean smooth curves. By Hodge index theorem $\mathrm{Fix}(\sigma^j)$ contains at most only one curve of genus $g > 1$. We denote by k_{σ^j} the number of fixed rational curves, by N_{σ^j} the number of isolated fixed points in $\mathrm{Fix}(\sigma^j)$. Moreover by $n^{\sigma^j}_{t,s}$ we denote the number of isolated fixed points of type (t, s) by σ^j. In several cases when it is clear which automorphism we are considering we just write k, N, $n_{t,s}$, and so on.

Lemma 2.2. *Let σ be a non-symplectic automorphism of order 16 acting on a K3 surface X and let A be the number of pairs of rational curves interchanged by σ^4 and fixed by σ^8, then $A \in 4\mathbb{Z}$.*

Proof. A curve as in the statement has stabilizer group in $\langle\sigma\rangle$ of order 2. Hence its σ-orbit has length 8, so we get that A is a multiple of 4. □

We formulate now Proposition 2.3 that we need to prove Proposition 2.7. We show then in Proposition 2.9 that the case $g(C) = 1$ is not possible.

Proposition 2.3. *Let σ be a non-symplectic automorphism of order 16 acting on a K3 surface X, with $\mathrm{Pic}(X) = S(\sigma^8)$. If $C \subset \mathrm{Fix}(\sigma)$ then $g(C) = 0, 1$, and we can not have two curves of genus one in the fixed locus.*

Proof. If $C \subset \mathrm{Fix}(\sigma)$ then this is also fixed by σ^4 which is non-symplectic of order 4. If $g(C) \geq 2$ by the relations (1) we have that l_{σ^4} and m_{σ^4} are multiples of 4, checking in Theorem 6.2 the only possible case is $(m_{\sigma^4}, r_{\sigma^4}, l_{\sigma^4}) = (4, 6, 8)$ and $N_{\sigma^4} = 2$, $k_{\sigma^4} = 0$, $g(C) = 2$. By the classification of Nikulin (see [14, §4]) the involution σ^8 fixes five rational curves other than the curve of genus 2. Since $k_{\sigma^4} = 0$, four of the rational curves are interchanged two by two by σ^4, one rational curve is preserved and contains the two fixed points. In this case $A = 2$ contradicting Lemma 2.2. If $g(C) = 1$ and there exists another genus one curve $C' \subset \mathrm{Fix}(\sigma)$, then by Theorem 2.1 we have $\mathrm{rk}\, S(\sigma^8) = 10$ but this is not possible, since the rank can be only equal to 6 or 14 as explained in Section 1. □

Remark 2.4. More in general by the same reason as in Proposition 2.3 if $\mathrm{Fix}(\sigma^8)$ contains an elliptic curve then this is the only one. We exclude also the case of $\mathrm{Fix}(\sigma^8) = \emptyset$ (here again is $\mathrm{rk}\, S(\sigma^8) = 10$ and this is not possible). The fact that $\mathrm{Fix}(\sigma^j) \neq \emptyset$, $j = 1, 2, 4$ follows immediately from the holomorphic Lefschetz formula, indeed the Lefschetz number is not zero (see Proposition 2.7, Proposition 2.11 and [3, Proposition 1]).

Recall the following useful Lemma and Remark, see e.g. [3, Lemma 4]:

Lemma 2.5. *Let $T = \sum_h R_h$ be a tree of smooth rational curves on a K3 surface X such that each R_h is invariant under the action of a non-symplectic automorphism η of order j. Then, the points of intersection of the rational curves R_h are fixed by η and the action at one fixed point determines the action on the whole tree.*

Remark 2.6. In the case of an automorphism of order 16, with the assumption of Lemma 2.5, the local actions at the intersection points of the curves R_h appear in the following order (we give only the exponents of ζ in the matrix of the local action):

$$\ldots, (0,1), (15,2), (14,3), (13,4), (12,5), (11,6), (10,7), (9,8),$$
$$(8,9), (7,10), (6,11), (5,12), (4,13), (3,14), (2,15), (1,0), \ldots$$

This remark will be particularly useful when we study elliptic fibrations on X.

Proposition 2.7. *Let σ be a non-symplectic automorphism of order* 16 *acting on a K3 surface X, $\mathrm{Pic}(X) = S(\sigma^8)$. Then the fixed locus is non-empty and*

$$\mathrm{Fix}(\sigma) = C \cup E_1 \cup \cdots \cup E_k \cup \{p_1, \cdots, p_N\}$$

or

$$\mathrm{Fix}(\sigma) = E_1 \cup \cdots \cup E_k \cup \{p_1, \cdots, p_N\},$$

where C is a curve of genus $g = 1$, the E_i's are rational fixed curves, $k = k_\sigma$ and the p_i's are isolated fixed points, $N = N_\sigma$. Moreover N is even, $4 \leq N \leq 16$ and the following relations hold :

(I) $$N = n_{3,14} + n_{4,13} + n_{5,12} + n_{6,11} + 2n_{7,10} + 2k + 1.$$

(II) $$N = 2n_{3,14} + 2n_{5,12} + 2n_{7,10} + 2k.$$

(III) $$N = 2 + r_\sigma - l_\sigma - 2k.$$

Proof. By Proposition 2.3 we know that the fixed locus may contain at most one curve of genus one. We use first the topological Lefschetz fixed point formula for σ. We write $r = r_\sigma$ and $l = l_\sigma$. We have

$$N + \sum_{\mathcal{K} \subset \mathrm{Fix}(\sigma)} (2 - 2g(\mathcal{K})) = \chi(\mathrm{Fix}(\sigma)) = \sum_{h=0}^{4} (-1)^h \operatorname{tr}(\sigma^* | H^h(X, \mathbb{R})) = 2 + \operatorname{tr}(\sigma^* | H^2(X, \mathbb{R})).$$

This gives $N + 2k = \chi(\mathrm{Fix}(\sigma)) = r - l + 2$ so that $r - l = N + 2k - 2$ (this gives (III)). Since $\operatorname{rk} S(\sigma) = 14$ or 6 in any case we have $N \leq 16$. We use now holomorphic Lefschetz formula (see [5, Theorem 4.6]). The Lefschetz number is

$$L(\sigma) = \sum_{h=0}^{2} (-1)^h \operatorname{tr}(\sigma^* | H^h(X, \mathcal{O}_X)) = 1 + \zeta_{16}^{-1} = 1 + \zeta_{16}^{15}, \tag{2}$$

on the other hand

$$L(\sigma) = \sum_{t,s} \frac{n_{t,s}}{\det(I - \sigma^* | T_x)} + \frac{1 + \zeta}{(1 - \zeta)^2} \sum_{\mathcal{K} \subset \mathrm{Fix}(\sigma)} (1 - g(\mathcal{K})) \tag{3}$$

where T_x denotes the tangent space at an isolated fixed point x. The action of σ on T_x is given by a matrix $A^1_{t,s}$ that we have introduced at the beginning of this section. Since the fixed locus contains at most one curve of genus one, this gives zero contribute for the Lefschetz number so we have

$$\sum_{\mathcal{K} \subset \mathrm{Fix}(\sigma)} (1 - g(\mathcal{K})) = k.$$

We can then expand equation (3) and collect the coefficients of the powers of ζ_{16}. Comparing with equation (2), we get the equations (one can do the computation by hand or using a computer algebra system as MAPLE):

$$n_{2,15} - n_{7,10} + n_{8,9} = 1 + 2k. \tag{4}$$

$$n_{2,15} - n_{3,14} + n_{4,13} - n_{5,12} + n_{6,11} - n_{7,10} + n_{8,9} = 2k. \tag{5}$$

$$n_{4,13} + n_{5,12} - 2n_{6,11} + 2n_{7,10} - n_{8,9} = 2k. \tag{6}$$

$$2n_{3,14} - 2n_{4,13} + 2n_{6,11} - n_{8,9} = 2k. \tag{7}$$

Combining (4) and (5) we get

$$n_{3,14} - n_{4,13} + n_{5,12} - n_{6,11} = 1. \tag{8}$$

From (4) and (5) and the fact that $N = \sum n_{t,s}$ we obtain the relations (I) and (II) in the statement respectively. By (I) we get that $N \geq 1$ and by (II) we find that N is an even number, thus $N \geq 2$. If $N = 2$ then by (I) we obtain $k = n_{7,10} = 0$ and either $n_{3,14}$ or $n_{5,12}$ is equal to 1 by relations (I) and (II) , thus $n_{4,13} = n_{6,11} = 0$ by (I) and either $n_{2,15}$ or $n_{8,9}$ is equal to one by (4). By (7) we obtain $n_{8,9} = 2n_{3,14}$ so $n_{8,9} = n_{3,14} = 0$. By using (6) we obtain $n_{5,12} = 0$ which is impossible. So $N \geq 4$. □

Remark 2.8. 1) As a direct consequence of formulas in Proposition 2.7, and also of formulas (4),. . .,(8), we find (one can compute by hand or use e.g. MAPLE to find the solutions):
- if $N = 4$ we have only the possibility with $(n_{3,14}, n_{7,10}, n_{8,9}, k) = (1, 1, 2, 0)$ (the other $n_{t,s}$ are zero) so that $r - l = 2$.

  - The case $(N,k)=(8,0)$ is not possible.
  - If $(N,k)=(6,0)$ then $(n_{5,12},n_{6,11},n_{7,10},n_{8,9})=(2,1,1,2)$, the other $n_{t,s}$ are zero and $r-l=4$.
  - If $(N,k)=(6,1)$ then $(n_{2,15},n_{3,14},n_{7,10})=(4,1,1)$, the other $n_{t,s}$ are zero and $r-l=6$.

2) The fixed points for σ with local action $(2,15)$, $(7,10)$, $(3,14)$, $(6,11)$ are isolated fixed points for σ^4, whence the points of type $(8,9)$, $(4,13)$ and $(5,12)$ are contained in a fixed curve for σ^4. The points of type $(8,9)$ are contained in a fixed curve for σ^2.

Proposition 2.9. *Let σ be a non-symplectic automorphism of order 16 acting on a K3 surface X, $\mathrm{Pic}(X)=S(\sigma^8)$. If $C\subset\mathrm{Fix}(\sigma)$ then C is rational.*

Proof. By Theorem 6.1 if $g(C)=1$ and since by formulas (1) we have $l_{\sigma^4},m_{\sigma^4}\in 4\mathbb{Z}$, we get $(m_{\sigma^4},r_{\sigma^4},l_{\sigma^4})=(4,10,4)$ and the fixed locus of σ^4 contains 1 rational fixed curve and 6 isolated fixed points (here $A=0$). Since $C\subset\mathrm{Fix}(\sigma)$ we have that also σ preserves the elliptic fibration determined by C. The automorphism σ^4 acts with order four on the base of the fibration by Theorem 6.1 so σ acts with order 16 on it and fixes two points. One point corresponds to the smooth elliptic curve C the other point to the fiber of type IV^* (as explained in Theorem 6.1). The component of multiplicity 3 in the fiber of type IV^* is clearly σ-invariant. If it is fixed by σ then each other component is preserved, so that $k=1$ and $N=6$. More precisely by Remark 2.6 we have $n_{2,15}=n_{3,14}=3$ which contradicts Remark 2.8. If the component of multiplicity 3 is σ-invariant then it contains 2 isolated fixed points. Two branches of the fiber are exchanged and we have $N=4$. By Remark 2.8 we have $n_{8,9}=2$, $n_{7,10}=1$, $n_{3,14}=1$ but this is not possible by using the Remark 2.6. □

Proposition 2.10. *Let σ be a non-symplectic automorphism of order 16 acting on a K3 surface X, $\mathrm{Pic}(X)=S(\sigma^8)$. The fixed locus $\mathrm{Fix}(\sigma^4)$ contains at least one fixed curve C of genus 0 or 1 (and no curves of higher genus).*

Proof. If $\mathrm{Fix}(\sigma^4)$ contains only isolated fixed points then by Remark 2.8 we have $n_{4,13}=n_{5,12}=n_{8,9}=k=0$. By equation (7) we obtain $n_{3,14}+n_{6,11}=0$ so they are both equal to 0. We get a contradiction to equation (8). Finally if $g(C)>1$ we have $(m_{\sigma^4},r_{\sigma^4},l_{\sigma^4})=(4,6,8)$ by Theorem 6.2. So by the same argument as in Proposition 2.3 this case is not possible, since $A=2$. □

Proposition 2.11. *Let σ be a non-symplectic automorphism of order 16 on a K3 surface X, $\mathrm{Pic}(X)=S(\sigma^8)$ and $C\subset\mathrm{Fix}(\sigma^2)$. Then $g(C)\leq 1$ and the following relations for the number of fixed points and curves by σ^2 hold:*

$$\begin{array}{lcl} n_{2,7}+n_{3,6} & = & 2+4k_{\sigma^2},\\ n_{4,5}+n_{2,7}-n_{3,6} & = & 2+2k_{\sigma^2},\\ N_{\sigma^2} & = & 2+r_{\sigma^2}-l_{\sigma^2}-2k_{\sigma^2}, \end{array}$$

where $n_{t,s}$ denote the number of fixed points of type (t,s) for the action of σ^2.

Proof. Observe that by Proposition 2.10 we have $g(C)\leq 1$ moreover an isolated fixed point for σ^2 is given by the local action $\begin{pmatrix}\xi^t & 0\\ 0 & \xi^s\end{pmatrix}$, $t+s=1 \mod (8)$, $0\leq t<s<8$. We obtain the relations in the statement by applying holomorphic and topological Lefschetz's formulas. □

Remark 2.12. By Lemma 2.5, and with the same notation there, the local action of σ^2 at the intersection points of the curves R_h appear in the following order:

$$\ldots,(0,1),(7,2),(6,3),(5,4),(4,5),(3,6),(2,7),(1,0),\ldots$$

moreover the σ-fixed points of type $(5,12)$ and $(4,13)$ give σ^2-fixed points of type $(4,5)$, the σ-fixed points of type $(2,15)$ and $(7,10)$ give σ^2-fixed points of type $(2,7)$ (up to the order). The σ-fixed points of type $(3,14)$ and $(6,11)$ give σ^2-fixed points of type $(3,6)$ (up to the order).

3. Elliptic Fibrations

Theorem 3.1. *Let σ be a non-symplectic automorphism of order 16 on a* K3 *surface X and assume that* $\mathrm{Pic}(X)=S(\sigma^8)$*, let $C\subset \mathrm{Fix}(\sigma^4)$. If $g(C)=1$ then σ acts as an automorphism of order four on C and we have the following cases*

m^2_σ	m^1_σ	m_σ	l_σ	r_σ	N_σ	k_σ	*type of* C'
1	*1*	*0*	*1*	*9*	*8*	*1*	IV*
1	*1*	*0*	*3*	*7*	*6*	*0*	IV*

Here C' denotes the invariant reducible fiber in the fibration determined by C. In particular in this case $\mathrm{rk}\,\mathrm{Pic}(X)=14$.

Proof. If $g(C)=1$ we are in the case $(m_{\sigma^4},r_{\sigma^4},l_{\sigma^4})=(4,10,4)$ by Theorem 6.1 and equations (1). In particular there is only one elliptic curve in the fixed locus of σ^4 so the curve C must be σ-invariant and the elliptic fibration induced by C is preserved. By Theorem 6.1 the automorphism σ^4 has order 4 on the base of the fibration, so that σ has order 16 on it. It fixes two points corresponding to the elliptic curve C and a singular fiber C' of type IV^*. The latter corresponds to the other fixed point for the action of σ on the base $\mathbb{P}^1$. By Proposition 2.9 the curve C can not be fixed by σ, hence σ has order 2 or 4 on it or it acts as a translation. By basic results on automorphisms on elliptic curves, in the first two cases σ fixes four, respectively two points on C. There are two possible actions on C' that we explain below.

First case: the singular fiber of type IV^* contains a fixed rational curve, which is necessarily the component of multiplicity 3. Then by using the Lemma 2.5 and the formulas in Proposition 2.7 we find $k=1$, $N=8$ with $n_{2,15}=n_{3,14}=3$ and $n_{4,13}=2$ the other $n_{t,s}$ are zero. In particular σ must have two fixed points on C this means that it acts as an automorphism of order four on C.

Second case: the singular fiber of type IV^* has a reflection of order 2. Then the curve of multiplicity 3 is preserved and contains two isolated fixed points with action $(8,9)$. In fact this curve must be fixed by σ^2 otherwise it would contain too many isolated fixed points for the action of σ^2. Combining Remark 2.8 and Proposition 2.7 we find $(N,k)=(6,0)$, with $n_{8,9}=2=n_{5,12}$, $n_{7,10}=1=n_{6,11}$, the other $n_{t,s}$ are zero. We observe that also in this case σ must have two fixed points on C, this means that it acts as an automorphism of order four on C.

Using the fact that $(m_{\sigma^4},r_{\sigma^4},l_{\sigma^4})=(4,10,4)$ we get immediately that in both cases $m^2_\sigma=m^1_\sigma=1$. Moreover we have that $r_\sigma+l_\sigma+2m_\sigma=10$ and in the first case we have $r-l=8$, in the second case $r-l=4$. In both cases we have $N_{\sigma^2}=10$ and $k_{\sigma^2}=1$ so using Proposition 2.11 we obtain the values of r,l,m given in the table.

□

Example 3.2. Consider the elliptic fibration in Weierstrass form given by :

$$y^2 = x^3 + ax + bt^8$$

where $a, b \in \mathbb{C}$ and the automorphism $\sigma(x, y, t) = (-x, iy, \zeta_{16}^{13}t)$ (recall that $i = \zeta_{16}^4$). By making the coordinate transformation that replace x by $\lambda^4 x$ and y by $\lambda^6 y$ for a suitable $\lambda \in \mathbb{C}$ we can assume that $a = 1$. Moreover since $b \neq 0$ we can apply a coordinate tranformation to t and so assume that $b = 1$ too. Our equation becomes:

$$y^2 = x^3 + x + t^8.$$

The fibers preserved by σ are over $0, \infty$ and the action at infinity is (see [11, §3]):

$$(x/t^4, y/t^6, 1/t) \longmapsto (-ix/t^4, \zeta_{16}^6 y/t^6, \zeta_{16}^3 1/t).$$

The discriminant of the fibration is

$$\Delta(t) = 4 + 27t^{16}.$$

We have that $t = \infty$ is an order eight zero of $\Delta(t)$, and $\Delta(t)$ has 16 simple zeros. Looking in the classification of singular fibers of elliptic fibrations on surfaces (e.g. [13, Section 3]) we see that the fiber over $t = \infty$ is of type IV* and the fibration has 16 fibers of type I_1. In particular the fiber over $t = 0$ is smooth. By [11, §3] a holomorphic two form is given by $(dt \wedge dx)/2y$ and so the action of σ on it is a multiplication by ζ_{16}. In fact we can understand the local action of the automorphism σ at the fixed points on C. If we look at the elliptic fibration locally around the fiber over $t = 0$ the equation in $\mathbb{P}^2 \times \mathbb{C}$ is given by:

$$G(x, y, z, t) := zy^2 - (x^3 + z^2x + z^3t^8) = 0$$

where $(x : y : z)$ are the homogeneous coordinates of $\mathbb{P}^2$ and the two fixed points for the automorphism σ on the fiber $t = 0$ are $p_0 := (0 : 1 : 0)$ and $p_1 := (0 : 0 : 1)$. In the chart $z = 1$ and on the open subset $\partial G(x, y, 1, 0)/\partial x \neq 0$ that contains the fixed point $p_1 = (0 : 0 : 1)$, a one form for the elliptic curve over $t = 0$ is:

$$dy/(\partial G(x, y, 1, 0)/\partial x) = dy/(-3x^2 - 1).$$

Here the action of σ is a multiplication by i so that the action on the holomorphic two form

$$dt \wedge (dy/(-3x^2 - 1))$$

is a multiplication by ζ_{16} as expected, and we see that the local action is of type $(4, 13)$. Doing a similar computation in an open subset of the chart $y = 1$ that contains the fixed point p_0 we find again the same local action. So we are in the first case of Theorem 3.1 with $N = 8$. On the other hand the fibration admits also the automorphism $\gamma(x, y, t) = (-x, -iy, \zeta_{16}^5 t)$. This acts also by multiplication by ζ_{16} on the holomorphic two form. Since σ acts also by multiplication by ζ_{16} on the holomorphic two form, then γ can not be a power of σ. In this case a similar discussion as above shows that the local action at the fixed points on the fiber C is of type $(5, 12)$, so we are in the second case of the Theorem 3.1.

Proposition 3.3. *Let σ be a non-symplectic automorphism of order 16 on a* K3 *surface X such that* $\mathrm{Pic}(X) = S(\sigma^8) \cong U \oplus L$ *where L is isomorphic to a direct sum of root lattices of types A_1, D_{4+n}, E_7 or E_8 and σ^8 fixes a curve of genus $g > 1$. Then X carries a jacobian elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$ whose fibers are σ^8-invariant and it has reducible fibers described by L and a unique section* $E \subset \mathrm{Fix}(\sigma^8)$. *Moreover, if $g > 4$ then π is $\sigma-$invariant .*

Proof. Since $\mathrm{Pic(X)} = \mathrm{S}(\sigma^8) \cong U \oplus L$ the first half of the statement follows from [11, Lemma 2.1, 2.2]. On other hand, since σ^8 fixes a curve C of genus $g > 1$, then C intersects each fiber of π in at least two points. This implies that σ^8 preserves each fiber of π and acts on it as an involution with four fixed points. By [20, Theorem 6.3] we have that the Mordell-Weil group of π is $MW(\pi) \cong \mathrm{Pic}(X)/T$ where T denote the subgroup of $\mathrm{Pic}(X)$ generated by the zero section and fiber components. Since L is a root lattice and $\mathrm{Pic}(X) \cong U \oplus L$ we have that $MW(\pi)$ is trivial, hence π has a unique section E. Since σ^8 preserves each fiber of π and E is invariant, we have that E is fixed by σ^8. This implies that C intersects each fiber in three points and one fixed point for the action of σ^8 is contained in the section E.

Now we will prove that π is $\sigma-$invariant if $g > 4$. Let f be the class of a fiber of π. The automorphism σ preserves the curve C, and we have that $CE = 0$ (the fixed curves for σ^8 can not intersect). Assume that $f \neq \sigma^*(f)$ then they intersect in at least 2 points. Indeed if $f \cdot \sigma^*(f) = 1$ then this is a fixed point of σ on f and so either C is fixed by σ which is not possible by Proposition 2.3, or E is fixed by σ. This is not possible too, since otherwise the action of σ on the base of the fibration would be the identity and so $f = \sigma^*(f)$, a contradiction. Now applying [3, Lemma 5] we find that:

$$2g - 2 = C^2 \leq \frac{2(C \cdot f)^2}{f \cdot \sigma^*(f) + 1} \leq \frac{2 \cdot 9}{3} = 6$$

This implies $g = g(C) \leq 4$. □

4. The rank six case

Theorem 4.1. *Let σ be an automorphism of order 16 acting non-symplectically on a* K3 *surface X and assume that* $\mathrm{Pic(X)} = \mathrm{S}(\sigma^8)$ *has rank* 6. *Then σ fixes at most one rational curve.*

The corresponding invariants of σ are given in the Table below. In any case $n_{4,13} = n_{5,12} = n_{6,11} = 0$ *and we have* $(n_{2,15}, n_{3,14}, n_{7,10}, n_{8,9}) = (4, 1, 1, 0)$ *in the first case and* $(n_{2,15}, n_{3,14}, n_{7,10}, n_{8,9}) = (0, 1, 1, 2)$ *in the second case.*

m_σ^2	m_σ^1	m_σ	l_σ	r_σ	N_σ	k_σ	N'	$g(C)$	Pic(X)
2	0	0	0	6	6	1	4	7	$U \oplus D_4$
2	0	0	2	4	4	0	2	6	$U(2) \oplus D_4$

Here C denotes the σ^8-fixed curve of genus > 1 and N' denotes the number of fixed points that are contained in C.

Proof. By the classification theorem for non-symplectic involutions on K3 surfaces given by Nikulin in [17, §4] we have that $(g(C), k_{\sigma^8})$ is either equal to $(5, 0), (6, 1)$ or $(7, 2)$. Observe that the case $g(C) = 5$ is not possible. Indeed in this case since $k_{\sigma^8} = 0$ then $k_{\sigma^4} = 0$ too and since C is not fixed by σ^4 by Proposition 2.10, we get a contradiction with Proposition 2.10 again. Observe that we have $m_\sigma^2 = 2$ so that $m_{\sigma^4} = 8$ by formulas (1). This means that the automorphism σ^4 can not have $l_{\sigma^4} > 0$ by Theorem 6.5. This implies that $l_{\sigma^4} = 0$ and by Theorem 6.4 or [22, Main Theorem 1] we have two possible cases that we recall below, both have $m_\sigma^1 = 0$.

<u>The case $(g(C), k_{\sigma^8}) = (6, 1)$.</u> The automorphism σ^4 of order 4 fixes one rational curve and six points on C by Theorem 6.4, [22, Proposition 4.3]. By Riemann-Hurwitz formula applied to the automorphism σ on C we find that either σ exchanges two fixed points and permutes the other four or σ fixes two points and the other four are exchanged two by two. The first case is not possible since then

$N = 2$ and by Proposition 2.7 we know that $N \geq 4$. So we are in the second case. Since again $N \geq 4$ then the rational curve is invariant but not fixed and so $N = 4$ and by Remark 2.8 we have $(n_{3,14}, n_{7,10}, n_{8,9}) = (1, 1, 2)$ the others $n_{t,s}$ are zero. We have moreover that $k_{\sigma^2} = 1$ and $N_{\sigma^2} = 6$ so combining the Lefschetz formulas we have $r + l + 2m = 6$, $4 = 2 + r - l$, $6 = 2 + r + l - 2m - 2$. That gives $m = 0$ and $r = 4$, $l = 2$. This is the second case in the table.

The case $(g(C), k_{\sigma^8}) = (7, 2)$. The automorphism σ^4 of order 4 fixes one rational curve, four points on C and two points on the other rational curve, see Theorem 6.4, [22, Proposition 4.3]. By Riemann-Hurwitz formula applied to the automorphism σ on C we find that either σ exchanges two by two the four points or it fixes each of the four points. In the first case since $N \geq 4$ we have that the two rational curves are invariant and they contain 2 fixed points each, so that $N = 4$ by Remark 2.8. Then $(n_{3,14}, n_{7,10}, n_{8,9}) = (1, 1, 2)$. Now by using Remark 2.6 this case is not possible. In fact clearly the two points of type $(8, 9)$ are contained on the same rational curve that is then fixed by σ^8, then on a σ-invariant (not pointwise fixed) rational curve we can not have a point of type $(3, 14)$ and of type $(7, 10)$.

So the action of σ on C fixes the four points. Observe that then the number of fixed points for σ^2 satisfies $n_{2,7} + n_{3,6} \geq 4$ so that $k_{\sigma^2} = 1$ by Proposition 2.11 (recall that $k_{\sigma^2} \leq 1$ since $k_{\sigma^4} = 1$). This again gives $n_{2,7} + n_{3,6} = 6$ and so $n_{4,5} = 0$ and $n_{2,7} = 5$, $n_{3,6} = 1$. So either $(N, k) = (8, 0)$ or $(N, k) = (6, 1)$. Observe that the case $(N, k) = (8, 0)$ is not possible for σ by Remark 2.8 and so we have $(N, k) = (6, 1)$. Again by Remark 2.8 we have $(n_{2,15}, n_{3,14}, n_{7,10}) = (4, 1, 1)$. In this case we have $r + l + 2m = 6$, $r - l = 6$, $r + l - 2m = 6$. We find $m = 0$, $r = 6$, $l = 0$ and $m^1_\sigma = 0$. So σ acts trivially on $\mathrm{Pic}(X)$ and this is the first case in the table. □

Example 4.2. 1) The case $g(C) = 7$, $(r_\sigma, l_\sigma) = (6, 0)$, $\mathrm{Pic}(X) = U \oplus D_4$.

Consider as in [19, Section 3.4] the elliptic fibration:

$$y^2 = x^3 + t^2x + (bt^3 + t^{11})$$

with $b \in \mathbb{C}$ and with the automorphism $\sigma(x, y, t) = (\zeta_{16}^2 x, \zeta_{16}^3 y, \zeta_{16}^2 t)$ (we write here the fibration in a slightly different way as given in [19]). On $t = 0$ the fibration has a fiber of type I_0^* and on $t = \infty$ the fibration has a fiber of type II. The action on the holomorphic two form $(dx \wedge dt)/2y$ is a multiplication by ζ_{16}. This is a one dimensional family and for generic b the action is trivial on $\mathrm{Pic}(X)$. So we are in the first case of Theorem 4.1. Observe that the fiber of type I_0^* contains the four fixed points with local action of type $(2, 15)$ and the invariant elliptic cuspidal curve over $t = \infty$ contains the fixed point with local action $(14, 3)$ (which is also contained on the section of the fibration) and the point of type $(7, 10)$. In particular observe that the curve C of genus 7 meets the fiber of type II at the singular point with multiplicity 3.

Observe that if $b = 0$ we get the elliptic fibration with the order 32 automorphism

$$\sigma_{32}(x, y, t) = (\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^2 t)$$

as described e.g. in [23]. The automorphism σ is the square of the automorphism σ_{32}^{25}.

2) The case $g(C) = 6$, $(r_\sigma, l_\sigma) = (4, 2)$, $\mathrm{Pic}(X) = U(2) \oplus D_4$.

The surfaces of this kind are described in the paper [12] and they are double covers of $\mathbb{P}^2$ ramified on a reducible sextic which is the product of a smooth quintic and a line. We consider the special family with equation in $\mathbb{P}(3,1,1,1)$:

$$z^2 = x_0(\alpha_0 x_0^4 x_2 + \beta_0 x_1^5 + \beta_1 x_1^3 x_2^2 + \beta_2 x_1 x_2^4).$$

Observe that the quintic curve is smooth and the K3 surface has five A_1 singularities over the points of intersection of the quintic curve and the line. The K3 surface carries the order 16 non-symplectic automorphism

$$\sigma(z : x_0 : x_1 : x_2) \mapsto (\zeta_{16}^3 z : x_0 : \zeta_8^7 x_1 : \zeta_8^3 x_2).$$

This acts by multiplication by ζ_{16} on the holomorphic two form:

$$(dx \wedge dy)/\sqrt{f}$$

where $f(x,y) = 0$ is the equation of the ramification sextic in the local coordinates x and y. An easy computation shows that the automorphism fixes the points:

$$(0:1:0:0), \qquad (0:0:1:0), \qquad (0:0:0:1).$$

Observe that the point $(0:0:0:1)$ is in fact one of the five A_1 singularities on the K3 surface. If we resolve it we find a fixed point on the strict transform of C which is the quintic curve on $\mathbb{P}^2$ (that have genus six) and one fixed point on the strict transform of L which denotes the curve $\{x_0 = 0\}$. The other two fixed points are contained respectively in C and L (and their respective strict transforms). Observe that the automorphism σ exchanges two by two the other points of intersection of C with L.

5. The rank fourteen case

Theorem 5.1. *Let σ be an automorphism of order 16 acting non symplectically on a K3 surface X and assume that $S(\sigma^8) = \mathrm{Pic}(X)$ has rank 14. Then the surface X is one of the surfaces described in Theorem 3.1 with a σ^4-fixed elliptic curve or it has:*

m_σ^2	m_σ^1	m_σ	l_σ	r_σ	N_σ	k_σ	N'	$g(C)$	Pic(X)
1	0	0	1	13	12	1	2	3	$U \oplus D_4 \oplus E_8$
1	0	1	1	11	10	1	2	2	$U(2) \oplus D_4 \oplus E_8$
1	0	1	5	7	4	0	2	2	$U(2) \oplus D_4 \oplus E_8$

Here C denotes the σ^8-fixed curve of genus > 1 and N' denotes the number of fixed points that are contained in C.

Proof. By the results of [17, §4] the possible values for the genus g of the σ^8-fixed curve C and the number k_{σ^8} of σ^8-fixed rational curves (different from C) are

$$(g, k_{\sigma^8}) = (0,3),\ (1,4),\ (2,5),\ (3,6)$$

The case $g(C) = 0$. The automorphism σ^4 satisfies the assumptions of Theorem 6.3 so that we have $(r_{\sigma^4}, l_{\sigma^4}, m_{\sigma^4}) = (10,4,4)$. Since $N_{\sigma^4} = 6$ and $k_{\sigma^8} = 3$, we have $N_\sigma \in \{4,6,8\}$ by Proposition 2.7. Moreover since $k_{\sigma^4} = 1$ then $k_{\sigma^2} \in \{0,1\}$ and also $k_\sigma \in \{0,1\}$.

Assume first $k_{\sigma^2} = 0$. The automorphism σ^4 fixes one rational curve and two points on each of the other three rational curves, this implies that the action of σ preserves each of the four rational curves. In fact if σ would permute the four curves or it would exchange them two by two, in any case one would find at least two fixed curves for σ^4, which is not possible. This implies that also σ^2 preserves the four curves. Now since σ^4 fixes exactly one curve we get $n_{4,5} = 2$, $n_{2,7} = 3 = n_{3,6}$ by Remark 2.12. This contradicts Proposition 2.11.

If $k_{\sigma^2} = 1$ then $n_{4,5} = 0$ and $n_{2,7} = 3 = n_{3,6}$. This again contradicts Proposition 2.11.

The case $g(C) = 1$. We can assume $C \not\subset \mathrm{Fix}(\sigma^4)$ otherwise we have discussed this case already in Theorem 3.1. Since C is fixed by σ^8 then C is also σ-invariant. Hence σ acts as (a composition of an automorphism and) a translation on the elliptic curve C (otherwise C would admits an automorphism of 2-power order bigger than 4, which is not possible). So that C does not contain fixed points for σ. By [3, Theorem 8.4] we get that $N_\sigma = 4, 6, 8$ and $k_\sigma = 1$ or 0. Studying the action of σ^2 on the four rational curves fixed by σ^8 and using the same argument as before, one shows easily that this case is not possible.

The case $g(C) = 2$. By Proposition 2.10 we have $k_{\sigma^4} \geq 1$ so that σ^4 fixes at least a rational curve. Moreover by formulas (1) we have $r_{\sigma^4} + l_{\sigma^4} = 14$ and $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$. Observe that $m_{\sigma^4} = 4m_\sigma^2 = 4$. By Theorem 6.5 if $l_{\sigma^4} > 0$ then we have $l_{\sigma^4} + m_{\sigma^4} = 4$ or 8. The first case is not possible. If $l_{\sigma^4} + m_{\sigma^4} = 8$, then $l_{\sigma^4} = 4$ (recall that $m_{\sigma^4} = 4$ and by Theorem 6.5 we have $k_{\sigma^4} = 1$). By Lemma 2.2 the automorphism σ^4 can not exchange two by two the remaining 4 rational curves neither it can permute them cyclically (this would not match with any action of σ on this set of four rational curves), so that the four rational curves are each σ^4-invariant. This gives $N_{\sigma^4} \geq 8$. By [3, Proposition 1] we have $N_{\sigma^4} = 6$ which contradicts the previous inequality. Hence $l_{\sigma^4} = 0$ and so σ^4 acts trivially on $\mathrm{Pic}(X)$. By Theorem 6.4 we have $(m_{\sigma^4}, r_{\sigma^4}, n_1, n_2, k_{\sigma^4}) = (4, 14, 4, 6, 3)$ where $N_{\sigma^4} = n_1 + n_2$ and n_2 is the number of fixed points on C. So we have 4 points contained in the two rational curves that are σ^4-invariant but not fixed. We call these curves R_1 and R_2. We study now the action of σ and σ^2 on the 5 rational curves, fixed by σ^8, and on C.

The automorphism σ^2. We have $k_{\sigma^2} \leq 3$ and at least one of the five curves is preserved or fixed. By using Remark 2.12 we have: $n_{4,5} \in 2\mathbb{Z}$ (points of this type can occur only on the rational curves) and $n_{2,7} + n_{3,6} \leq 10$. This follows from the fact that points of this type are contained in C or in the rational curves that are not fixed by σ^4. Since the number of σ^4-fixed points on C is $n_2 = 6$ and $k_{\sigma^4} = 3$ we get the previous inequality. By using now Proposition 2.11 we obtain that $k_{\sigma^2} \leq 2$. If $k_{\sigma^2} = 0$, since σ^4 fixes only 3 of the five σ^8-fixed curves, we obtain that σ exchanges two of these three curves and preserves the third curve or σ preserves all the three rational curves. In any case σ^2 preserves all the three σ^4-fixed rational curves and so the only possibility is that in fact preserves all the five σ^8-fixed rational curves. In particular $n_{4,5} = 6$ and $n_{2,7} \geq 2$ $n_{3,6} \geq 2$. This contradicts Proposition 2.11. We explain now the cases $k_{\sigma^2} = 1$ and $k_{\sigma^2} = 2$ below:

i) $k_{\sigma^2} = 2$. By Proposition 2.11 we get $n_{2,7} + n_{3,6} = 10$. This means that the curve C must contain six fixed points for σ^2 and the other four fixed points are contained in the two σ^4-invariant curves R_1 and R_2 (recall that σ^4 fixes the other three rational curves that then can not contain fixed points of this type). In particular we have $n_{2,7} \geq 2$ and $n_{3,6} \geq 2$, and $n_{4,5} = 2$. Since by Proposition 2.11 we have $n_{4,5} = 2n_{3,6} - 4$ we get $n_{3,6} = 3$, $n_{2,7} = 7$, $N_{\sigma^2} = 12$.

ii) $k_{\sigma^2} = 1$. By Proposition 2.11 we have $n_{2,7} + n_{3,6} = 6$. Observe that for the same reason as above the remaining rational curves can not be exchanged two by two. So these are invariant. This gives $n_{2,7} \geq 2$, $n_{3,6} \geq 2$ and $n_{4,5} = 4$. Using Proposition 2.11 we obtain that $n_{2,7} = n_{3,6} = 3$. And two fixed points are contained in C. The other points on C fixed by σ^4 form a σ-orbit of length four.

The automorphism σ. First observe that by using Riemann-Hurwitz formula on C we have two possibilities: the six points are exchanged two by two and so fixed by σ^2 (this is case i)) or C contains 2 fixed points and the other four points are permuted by σ in one orbit (this is case ii)).

i) In this case σ exchanges two by two the points on C. We have $n_{5,12} = n_{4,13} = 1$ since these two points correspond to the two fixed points with local action $(4, 5)$ for σ^2 and are contained in a rational curve (see Remark 2.6). Assume that R_1 and R_2 are not exchanged. We have $n_{2,15} + n_{7,10} + n_{3,14} + n_{6,11} = 4$ and $n_{2,15} = n_{3,14}$, $n_{7,10} = n_{6,11}$. But this contradicts equation (8) in Proposition 2.7. If R_1 and R_2 are exchanged we have $n_{3,14} = n_{6,11} = 0$, $n_{2,15} = n_{7,10} = 0$ and $n_{5,12} = n_{4,13} = 1$. But this contradicts the equality $n_{3,14} - n_{6,11} = 1$ in Proposition 2.7.

ii) In this case C contains two fixed points for σ. We have $n_{8,9} = 2w$, with $w = 0, 1$. Moreover by Remark 2.6 we have $n_{5,12} = n_{4,13} = 2$ or $n_{5,12} = n_{4,13} = 0$. If $n_{8,9} = 2$ so that $k_\sigma = 0$ an easy computation using the equations of Proposition 2.7 shows that the first case with $n_{5,12} = n_{4,13} = 2$ is not possible. If $n_{5,12} = n_{4,13} = 0$ again using Proposition 2.7 we find that $n_{3,14} = n_{7,10} = 1$ the other $n_{t,s}$ are zero. One computes $(r_\sigma, l_\sigma, m_\sigma) = (7, 5, 1)$ and we have $\mathrm{Pic}(X) = U(2) \oplus D_4 \oplus E_8$. Observe that in this case the remaining σ^8-fixed rational curves are exchanged two by two by σ. If $n_{8,9} = 0$ so that $k_\sigma = 1$ again one computes using Proposition 2.7 that :

$$(N, k, n_{8,9}, n_{2,15}, n_{3,14}, n_{4,13}, n_{5,12}, n_{6,11}, n_{7,10}) = (10, 1, 0, 3, 2, 2, 2, 1, 0)$$

and $(r_\sigma, l_\sigma, m_\sigma) = (11, 1, 1)$. Moreover we have $\mathrm{Pic}(X) = U(2) \oplus D_4 \oplus E_8$.

The case $g(C) = 3$. By Proposition 2.10 we have $k_{\sigma^4} \geq 1$ so that σ^4 fixes at least a rational curve. We have moreover by formulas (1) that $r_{\sigma^4} + l_{\sigma^4} = 14$ and $l_{\sigma^4}, m_{\sigma^4} \in 4\mathbb{Z}$ and observe that $m_{\sigma^4} = 4m_\sigma^2 = 4$. By Theorem 6.5 if $l_{\sigma^4} > 0$ then we have $l_{\sigma^4} + m_{\sigma^4} = 4$ or 8. The first case is not possible, if $l_{\sigma^4} + m_{\sigma^4} = 8$ then $l_{\sigma^4} = 4$ and by Theorem 6.5 we have $k_{\sigma^4} = 1$. Observe that σ preserves or permutes some of the five rational curves not fixed by σ^4 so that in any case $N_{\sigma^4} \geq 10$. By [3, Proposition 1] we have $N_{\sigma^4} = 6$, which is not possible. Hence $l_{\sigma^4} = 0$ and so σ^4 acts trivially on $\mathrm{Pic}(X)$. By Theorem 6.4 we have $(m_{\sigma^4}, r_{\sigma^4}, n_1, n_2, k_{\sigma^4}) = (4, 14, 6, 4, 3)$ where $N_{\sigma^4} = n_1 + n_2$ and n_2 is the number of fixed points on C. We have hence 6 points contained in the three rational curves that are σ^4-invariant but not fixed. We call these curves T_i, $i = 1, 2, 3$. We study now the action of σ and σ^2 on the 6 rational curves fixed by σ^8 and on C.

The automorphism σ^2. We have $k_{\sigma^2} \leq 3$. If σ would act as some permutation on the four curves, then σ^4 would fix more than three curves which is not possible since $k_{\sigma^4} = 3$. So each curve is preserved by σ and so by σ^2. Moreover we have $n_{4,5} \in 2\mathbb{Z}$, and these are at most 6, in fact points of this type can occur only on the rational curves, and $n_{2,7} + n_{3,6} \leq 10$ (since $n_2 = 4$ we have at most 4 σ^2-fixed points on C and points of this type are not contained in rational curves that are fixed for σ^4, but can be contained in the three rational curves that are only σ^4-invariant). Again by using Proposition 2.11 we find that $k_{\sigma^2} \leq 2$. If $k_{\sigma^2} = 0$ then $n_{2,7} + n_{3,6} = 2$ but since all the rational curves are preserved $n_{4,5} = 6$ and we get a contradiction using Proposition 2.11. We explain now below the cases $k_{\sigma^2} = 1$ and $k_{\sigma^2} = 2$:

i) $k_{\sigma^2} = 2$. Here we get $n_{2,7} + n_{3,6} = 10$ this means that the curve C must contain four fixed points for σ^2 and the other six points are contained in the three σ^4-invariant curves T_1, T_2 and T_3. In particular we have $n_{2,7} \geq 3$ and $n_{3,6} \geq 3$,

$n_{4,5}=2$, see Remark 2.12. Moreover $n_{4,5}=2n_{3,6}-4$ so we get $n_{3,6}=3$, $n_{2,7}=7$, $N_{\sigma^2}=12$ (by Proposition 2.11).

ii) $k_{\sigma^2}=1$: Here we get $n_{2,7}+n_{3,6}=6$ by Proposition 2.11. Observe that for the same reason as above the remaining rational curves can not be exchanged two by two. So these are invariant. This gives $n_{2,7}\geq 3$, $n_{3,6}\geq 3$ and $n_{4,5}=4$. We get using Proposition 2.11 that $n_{2,7}=n_{3,6}=3$, and so the four points on C fixed by σ^4 form a σ-orbit of length four.

The automorphism σ. By using Riemann-Hurwitz formula there are two possible actions on C: the automorphism σ exchanges 2 points and fixes the other two (this is case i) below) or the four points form a σ-orbit (this is case ii) below).

i) We have $n_{8,9}=2w$ and since $k_{\sigma^2}=2$ we have $0\leq w\leq 2$. Moreover $n_{5,12}=n_{4,13}=1$ (since these two points correspond to the two fixed points with local action $(4,5)$ for σ^2, see Remark 2.12). If $w=0$ and $k=0$, so that the two σ^2-fixed curves are exchanged by σ, then using Proposition 2.7 one sees that this case is not possible. If $w=0$ and $k=2$ using Proposition 2.7 we get $N=14$ which is impossible by looking at the geometry (indeed in this case we have $N\leq 12$).

If $w=1$, then $k=1$ and we find $N=12$ with

$$(N,k,n_{8,9},n_{2,15},n_{3,14},n_{4,13},n_{5,12},n_{6,11},n_{7,10})=(12,1,2,3,2,1,1,1,2).$$

This is the case in the statement.

If $w=2$ and $k=0$ this is not possible by using the equations in Proposition 2.7.

ii) We have $n_{8,9}=2w$ and since $k_{\sigma^2}=1$ we have $w=0,1$. If $w=0$ then $k=1$ and $n_{5,12}=n_{4,13}=2$ or $n_{5,12}=n_{4,13}=0$. If $n_{5,12}=n_{4,13}=2$ we obtain $n_{6,11}=1$ and $n_{7,10}=0$ which is impossible since the fixed points by σ are contained in the rational curves that are fixed by σ^8 (see Remark 2.6). If $n_{5,12}=n_{4,13}=0$ then two of the σ^4-fixed curves are exchanged. By using Proposition 2.7 we get $n_{7,10}=1$, $n_{2,15}=4$, $n_{3,14}=1$ (the other $n_{t,s}$ are zero), but this is not possible since the isolated points fixed by σ are contained in rational curves (see Remark 2.6).

If $w=1$ then $k=0$ then again $n_{5,12}=n_{4,13}=2$ or $n_{5,12}=n_{4,13}=0$. By using Proposition 2.7 we see that the first case is not possible. If $n_{5,12}=n_{4,13}=0$ then two of the σ^4-fixed curves are exchanged. By Proposition 2.7 we find $N=4$. This is not possible. Indeed if the curves T_i are preserved then $N=6$, if two of them are exchanged we get $N=2$. In any case we get a contradiction.

□

Example 5.2. 1) **The case** $g(C)=3$ (see [22]). Consider the elliptic fibration:

$$y^2=x^3+t^2x+t^7$$

This carries the order 16 automorphism $\sigma(x,y,t)=(\zeta_{16}^2x,\zeta_{16}^{11}y,\zeta_{16}^{10}t)$. The discriminant is $t^6(4+27t^8)$ so over $t=0$ the fibration has a fiber of type I_0^* and over $t=\infty$ the fibration has a fiber of type II^*. The automorphism σ preserves the fiber of type II^* and fixes the component of multiplicity 6. The genus 3 curve cuts the fiber of type II^* in the external component of multiplicity 3 and cuts the I_0^* fiber in three components of multiplicity one. The automorphism σ exchanges two curves in the fiber of type I_0^* (this corresponds to $l_\sigma=1$) and so the two intersection points of this fiber with C. It leaves invariant the component of multiplicity two and contains two fixed point on it. Using Remark 2.6 it is easy to find the local action at the 12 fixed points. In this case we have $\mathrm{Pic}(X)=U\oplus D_4\oplus E_8$.

2) **The case** $g(C)=2$ **and** $k_\sigma=0$. We consider the K3 surface double cover of $\mathbb{P}^2$ ramified on a special reducible sextic as in Example 4.2, 2). We consider the quintic with a special equation, more precisely we assume that the reducible sextic $(L=\{x_0=0\})\cup C$ has the equation:

$$x_0(x_0^4x_2+x_1^5-2x_1^3x_2^2+x_2^4x_1)=0,$$

and recall that the automorphism is:

$$\sigma:(z:x_0:x_1:x_2)\mapsto(\zeta_{16}^3z:x_0:\zeta_8^7x_1:\zeta_8^3x_2).$$

The line $L=\{x_0=0\}$ meets the quintic in the point $(0:0:1)$ and two further points $(0:1:1)$ and $(0:-1:1)$, that are in fact exchanged by the automorphism σ. By studying the partial derivatives of the equation of C one sees that these two last points are singular. These are in fact A_3 singularities. We explain the computations in detail for the point $(0:1:1)$. In the chart $x_2=1$ the equation of C becomes:

$$x_0^4+x_1^5-2x_1^3+x_1=0$$

We translate the point $(0,1)$ to the origin and we get an equation in new local coordinates (here $x_0=y$):

$$x^2(x^3+5x^2+8x+4)+y^4=0$$

So we have a double point at $(0,0)$ and by making a coordinates transformation as in [6, Ch. II, section 8] we obtain the local equation:

$$x^2+y^4=0$$

which is an A_3 singularity. Now as explained again in [6, Ch. II, section 8] or also in [12, Lemma 3.15] this gives a D_6 singularity of the reducible ramification sextic. The same happens at the point $(0:-1:1)$ since the two points are exchanged by σ. This means that the K3 surface defined by

$$z^2=x_0(x_0^4x_2+x_1^5-2x_1^3x_2^2+x_2^4x_1)$$

has two D_6 singularities and one A_1 singularity (coming from the intersection point $(0:0:1)$). Let X be the minimal desingularization of the double cover. The rank of the Picard group is at least 14 but since the automorphism of order 16 acts non-symplectically on it, the rank is exactly 14. The non-symplectic involution fixes a curve of genus 2 (the line $x_2=0$) and five rational curves: one is the strict transform of the A_1 singularity and the other four curves are contained in the two D_6 singularities. By looking in Figure 1 in Section 2 one finds that $S(\sigma^8)$ has rank 14. Since $S(\sigma^8)$ is primitively embedded in $\mathrm{Pic}(X)$ we get that $\mathrm{Pic}(X)=S(\sigma^8)$. By using Theorem 5.1 we conclude that $\mathrm{Pic}(X)=U(2)\oplus D_4\oplus E_8$. Observe that the (-2)-curve coming from the resolution of the A_1 singularity can not be fixed, because it intersects C and L on X (we call again in this way the strict transforms) that are σ^8-fixed. Moreover since the two D_6 singularities are exchanged we have $k=0$. Observe that the induced automorphism on $\mathbb{P}^2$ fixes also the point $(0:1:0)\in L$ and the point $(1:0:0)\in C$ which together with the two intersection points with L and C of the exceptional (-2)-curve on the A_1 singularity gives $N=4$.

3) **The case** $g(C)=2$ **and** $k_\sigma=1$ (see [7]). We consider the elliptic fibration in Weierstrass form with the non-symplectic automorphism of order 16 :

$$y^2=x^3+t^3(t^4-1)x,\qquad \sigma:(x,y,t)\mapsto(\zeta_{16}^6x,\zeta_{16}^9y,\zeta_{16}^4t)$$

This fibration has five fibers III (one over $t = \infty$) and one fiber III^* over $t = 0$. An easy computation using the local action at the fixed points shows that we have $k_\sigma = 1$ and 10 isolated fixed points.

Remark 5.3. If $\operatorname{rk} S(\sigma^8) = 14$ then the automorphism σ acts on $S(\sigma^8)^\perp \otimes \mathbb{C}$ by the eight primitive roots of unity ζ_{16}^i, $i = 1, 3, \ldots, 15$. In particular each eigenspace is one-dimensional, so by applying the construction for the moduli space of K3 surfaces with non-symplectic automorphisms as described in [8, §11], we see that in fact this is zero dimensional. In particular in these cases $\operatorname{Pic}(X) = S(\sigma^8)$, in fact if $S(\sigma^8)$ would be strictly contained in $\operatorname{Pic}(X)$ then σ would admit the eigenvalues ζ_{16}^i, $i = 1, 3, \ldots, 15$ on $\operatorname{Pic}(X) \otimes \mathbb{C}$ and this would imply $\operatorname{rk}\operatorname{Pic}(X) > 21$ which is impossible for a K3 surface. This is the case in Theorem 3.1 and in Theorem 5.1. If $\operatorname{rk} S(\sigma^8) = 6$ using the same construction as above one finds that the dimension of the moduli space is one, and in this case we could have K3 surfaces in the moduli spaces with $S(\sigma^8)$ strictly contained in $\operatorname{Pic}(X)$ so that we must have $\operatorname{rk}\operatorname{Pic}(X) = 14$. We show in the Proposition 5.4 below that the fixed locus of σ remains however of the same type as on the generic surface in the moduli space.

Proposition 5.4. *Let X be a K3 surface and let σ be an automorphism of order 16 acting purely non–symplectically on X. If $\operatorname{rk}\operatorname{Pic}(X) = 6$ then $\operatorname{Pic}(X) = S(\sigma^8)$ and the fixed locus of σ is described in the Theorem 4.1. If $\operatorname{rk}\operatorname{Pic}(X) = 14$ then*

- i) *if $\operatorname{Pic}(X) = S(\sigma^8)$ then the fixed locus of σ is described in the Theorems 3.1 and 5.1.*
- ii) *If $S(\sigma^8) \subset \operatorname{Pic}(X)$ but $S(\sigma^8) \neq \operatorname{Pic}(X)$ then X is a special member in the families of K3 surfaces whose generic element is described in Theorem 4.1 and the fixed locus of the automorphism on X remains of the same type as on the generic surface in the moduli space.*

Proof. Recall that the primitive 16-th root of the unity all have the same multiplicity for the action of σ on $H^2(X, \mathbb{Z})$. Since these are 8 and in the first case $\operatorname{rk}\operatorname{Pic}(X) = 6$, σ can not have such an eigenvalue on $\operatorname{Pic}(X) \otimes \mathbb{C}$ so that σ^8 acts as the identity on $\operatorname{Pic}(X)$. If rank of $\operatorname{Pic}(X)$ is 14, then σ may act on $\operatorname{Pic}(X) \otimes \mathbb{C}$ with primitive 16-roots of the unity. If it is the case then they multiplicity is 1 and $S(\sigma^8)$ is strictly contained in $\operatorname{Pic}(X)$. By the construction of the moduli space such a K3 surface belongs to one of the two families whose generic element is described in Theorem 4.1. We want to show that the fixed locus of σ on X is of the same type as the fixed locus on the generic surface in the family. We follow several steps. First remark that the fixed locus of σ^8 does not change, since it is the same for all the K3 surfaces that are $S(\sigma^8)$-polarized (it depends only on the properties of the lattice $S(\sigma^8)$ as we recall in Nikulin's Theorem 2.1), so that only the two following cases are possible (we keep the notations as in Theorem 4.1):

- a) $(g(C), k_{\sigma^8}) = (6, 1)$,
- b) $(g(C), k_{\sigma^8}) = (7, 2)$.

Recall that the topological Lefschetz theorem gives that (see proof of Proposition 2.7):

$$N + \sum_{\mathcal{K} \subset \operatorname{Fix}(\sigma)} (2 - 2g(\mathcal{K})) = 2 + \operatorname{tr}(\sigma^* | H^2(X, \mathbb{R})) = 2 + r - l.$$

Case a). Here we have:

$$N + \sum_{\mathcal{K} \subset \mathrm{Fix}(\sigma)} (2 - 2g(\mathcal{K})) = 4.$$

If the curve C is fixed by σ then the previous formula becomes:

$$N - 10 + 2\delta = 4$$

where $\delta = 0$ if the rational curve is σ-invariant not fixed, and $\delta = 1$ if the rational curve is fixed. In the first case $N = 2$ and in the second case $N = 0$. Replacing the values of δ and N in the equality one gets a contradiction. Hence C can not be fixed by σ so that the genus of a curve in $\mathrm{Fix}(\sigma)$ does not exceed 0. We can then use the relations in Proposition 2.7 in fact the hypothesis there on $\mathrm{Pic}(X) = S(\sigma^8)$ is used only in order to apply Proposition 2.3. So we have $N \geq 4$ and the only possibility is then to have 2 isolated fixed points on C and 2 isolated fixed points on the rational curve, so that $N' = 2$ and $k_\sigma = 0$, as described in the second case in the table of Theorem 4.1.
Case b). Here we have:

$$N + \sum_{\mathcal{K} \subset \mathrm{Fix}(\sigma)} (2 - 2g(\mathcal{K})) = 8.$$

As in the previous case by a similar argument one shows that C can not be fixed by σ, so that the genus of a curve in $\mathrm{Fix}(\sigma)$ does not exceed 0. As remarked before we can use the relations in Proposition 2.7. We can write

$$8 = N' + N_R + 2\delta,$$

where N' is the number of σ-fixed points on C, N_R is the number of σ-fixed points on the two rational curves, $\delta \in \{0, 1, 2\}$ denotes the number of rational curves that are σ-fixed.
If $\delta = 0$ then either $N_R = 0$ (the two curves are exchanged) and $N' = 8$ or each rational curve is preserved and contains 2 fixed points so that $N_R = 4$ and $N' = 4$. Both cases are excluded by Remark 2.8 since $(N, k) = (8, 0)$ is not possible.
If $\delta = 1$ then $N_R = 2$ since the other rational curve must be preserved and this gives $N' = 4$. This corresponds to the description of the fixed locus in the first case in the table of Theorem 4.1.
If $\delta = 2$ then $N_R = 0$ and $N' = 4 = N$ but this is again not possible by Remark 2.8 since in this case we must have $k = 0$.

□

6. Appendix: order four non–symplectic automorphisms

In this Appendix we recall some results of [3] that we frequently use in the paper. We denote (only in this section) by σ an automorphism of order four acting purely non–symplectically on a K3 surface X. We denote by m, r, l the multiplicities of the eigenvalues $i, 1, -1$ of σ on the complexified K3 lattice; by n, k the number of isolated σ-fixed points and σ-fixed rational curves, by a the number of rational curves that are exchanged two by two by σ and fixed by σ^2; and finally by C the curve fixed by σ^2 of genus $g \geq 1$.

Theorem 6.1. *Let σ be a purely non-symplectic order four automorphism on a K3 surface X with $\mathrm{Pic}(X) = S(\sigma^2)$ and $\pi_C : X \to \mathbb{P}^1$ be an elliptic fibration with a* *smooth fiber $C \subset \mathrm{Fix}(\sigma)$. Then σ preserves π_C and acts on its base as an order four automorphism with two fixed points corresponding to the fiber C and a fiber C' which is either smooth, of Kodaira type I_{4M} or IV^*. The corresponding invariants of σ are given in Table 1 and all the cases do exist.*

m	r	l	n	k	a	*type of* C'
6	6	4	4	0	0	I_0
5	7	5	4	0	0	I_4
4	10	4	6	1	0	IV^*
4	8	6	4	0	1	I_8 *or* IV^*
3	9	7	4	0	2	I_{12}
2	10	8	4	0	3	I_{16}

Table 1. The case $g = 1$

Theorem 6.2. *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it such that $\mathrm{Pic}(X) = S(\sigma^2)$. If $\mathrm{Fix}(\sigma)$ contains a curve of genus $g > 1$ then the invariants associated to σ are as in Table 2. All cases in the Table do exist.*

m	r	l	n	k	a	g
7	1	7	0	0	0	3
6	4	6	2	0	0	2
	2	8	0	0	1	3
5	5	7	2	0	1	2
4	6	8	2	0	2	2

Table 2. The case $g > 1$

Theorem 6.3. *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it with $\mathrm{Pic}(X) = S(\sigma^2)$. If $\mathrm{Fix}(\sigma)$ contains a smooth rational curve and all curves fixed by σ^2 are rational, then the invariants associated to σ are as in Table 3. All cases in the table do exist.*

Theorem 6.4. *Let σ be a purely non-symplectic automorphism on a K3 surface X of order four, such that $S(\sigma^2) = S(\sigma) = \mathrm{Pic}(X)$. Then the invariants of the fixed locus of σ and the lattices $S(\sigma^2)$ and $T(\sigma^2)$ (up to isomorphism) appear in Table 4. Moreover, all cases in the table do exist. We denote here by n_2 the number of σ-fixed points on C and by n_1 the σ-fixed points outside C, so that $n_1 + n_2 = n$).*

Theorem 6.5. *Let σ be a purely non-symplectic automorphism of order four on a K3 surface X such that $S(\sigma^2) = \mathrm{Pic}(X)$. Assume that the fixed locus of σ contains*

m	r	l	n	k	a
4	10	4	6	1	0
3	13	3	8	2	0
	11	5	6	1	1
2	16	2	10	3	0
	14	4	8	2	1
	12	6	6	1	2
1	19	1	12	4	0
	13	7	6	1	3

TABLE 3. The case $g = 0$

m	r	n_1	n_2	k	g	$S(\sigma^2)$	$T(\sigma^2)$
10	2	2	2	0	10	U	$U \oplus U \oplus E_8^{\oplus 2}$
	2	0	4	0	9	$U(2)$	$U \oplus U(2) \oplus E_8^{\oplus 2}$
8	6	2	4	1	7	$U \oplus D_4$	$U \oplus U \oplus E_8 \oplus D_4$
	6	0	6	1	6	$U(2) \oplus D_4$	$U \oplus U(2) \oplus E_8 \oplus D_4$
6	10	6	2	2	6	$U \oplus E_8$	$U \oplus U \oplus E_8$
	10	4	4	2	5	$U(2) \oplus E_8$	$U \oplus U(2) \oplus E_8$
	10	2	6	2	4	$U \oplus D_4^{\oplus 2}$	$U \oplus U \oplus D_4^{\oplus 2}$
	10	0	8	2	3	$U(2) \oplus D_4^{\oplus 2}$	$U \oplus U(2) \oplus D_4^{\oplus 2}$
4	14	6	4	3	3	$U \oplus E_8 \oplus D_4$	$U \oplus U \oplus D_4$
	14	4	6	3	2	$U(2) \oplus E_8 \oplus D_4$	$U \oplus U(2) \oplus D_4$
2	18	10	2	4	2	$U \oplus E_8^{\oplus 2}$	$U \oplus U$
	18	8	4	4	1	$U(2) \oplus E_8^{\oplus 2}$	$U \oplus U(2)$

TABLE 4. The case $l = 0$

n isolated fixed points and $k > 0$ rational curves, that the curve C fixed by σ^2 has $g = g(C) > 1$ and that $l > 0$. Then $g \leq m$ and we are in one of the following cases:

$m+l$	k	$g \leq$	$a \leq$
4	3	3	2
6	2	5	3
8	1	7	4

REFERENCES

1. D. Al Tabbaa, *On some automorphisms of K3 surfaces*, PhD thesis University of Poitiers, December 2015.
2. M. Artebani and A. Sarti, *Non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Ann. **342** (2008), no. 4, 903–921.
3. ______, *Symmetries of order four on K3 surfaces*, J. Math. Soc. Japan **67** (2015), no. 2, 1–31.
4. M. Artebani, A. Sarti, and S. Taki, *K3 surfaces with non-symplectic automorphisms of prime order.*, Math. Z. **268** (2011), no. 1-2, 507–533, with an appendix by Shigeyuki Kondō.
5. M. F. Atiyah and I. M. Singer, *The index of elliptic operators. III*, Ann. of Math. (2) **87** (1968), 546–604.

6. W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 4, Springer-Verlag, Berlin, 1984. MR 749574 (86c:32026)
7. J. Dillies, *Example of an order 16 non symplectic action on a K3 surface*, J. of Algebra **458** (2016).
8. I.V. Dolgachev and S. Kondō, *Moduli of $K3$ surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100.
9. L. Fu, *On the action of symplectic automorphisms on the CH_0-groups of some Hyperkähler Fourfolds*, preprint, arXiv:1302.6531v1.
10. D. Huybrechts, *Symplectic automorphisms of $K3$ surfaces of arbitrary order*, Math. Res. Lett. **19** (2012), 947–951.
11. S. Kondō, *Automorphisms of algebraic $K3$ surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98.
12. R. Laza, *Deformations of singularities and variation of GIT quotients*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 2109–2161.
13. R. Miranda, *The basic theory of elliptic surfaces. Notes of lectures.*, Dottorato di Ricerca di Matematica, Università di Pisa, Dipartimento di Matematica. Pisa: ETS Editrice, vi, 106 pages , 1989 (English).
14. V. V. Nikulin, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 654–671. MR 934268 (89d:11032)
15. V.V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type $K3$*, Uspehi Mat. Nauk **31** (1976), no. 2(188), 223–224.
16. ______, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1980), 103–167.
17. ______, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications*, J. Soviet. Math. **22** (1983), 1401–1475.
18. A. N. Rudakov and I. R. Shafarevich, *Surfaces of type $K3$ over fields of finite characteristic*, Current problems in mathematics, Vol. 18, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981, pp. 115–207. MR 633161 (83c:14027)
19. M. Schütt, *$K3$ surfaces with non-symplectic automorphisms of 2-power order*, J. Algebra **323** (2010), no. 1, 206–223.
20. M. Schütt and T. Shioda, *Elliptic surfaces*, Algebraic geometry in East Asia—Seoul 2008, Adv. Stud. Pure Math., vol. 60, Math. Soc. Japan, Tokyo, 2010, pp. 51–160.
21. S. Taki, *Classification of non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Nachr. **284** (2011), 124–135.
22. ______, *Classification of non-symplectic automorphisms on $K3$ surfaces which act trivially on the Néron-Severi lattice*, J. Algebra **358** (2012), 16–26.
23. ______, *On Oguiso's $K3$ surface*, J. Pure Appl. Algebra **218** (2014), no. 3, 391–394.

Laboratoire de Mathématiques et Applications, UMR CNRS 7348, Université de Poitiers, Téléport 2, Boulevard Marie et Pierre Curie, 86962 FUTUROSCOPE CHASSENEUIL, France

E-mail address: Dima.Al.Tabbaa@math.univ-poitiers.fr

Laboratoire de Mathématiques et Applications, UMR CNRS 7348, Université de Poitiers, Téléport 2, Boulevard Marie et Pierre Curie, 86962 FUTUROSCOPE CHASSENEUIL, France

E-mail address: sarti@math.univ-poitiers.fr
URL: http://www-math.sp2mi.univ-poitiers.fr/~sarti/

Department of Mathematics, Tokai University, 4-1-1, Kitakaname, Hiratsuka, Kanagawa, 259-1292, JAPAN

E-mail address: taki@tsc.u-tokai.ac.jp
URL: http://sm.u-tokai.ac.jp/~taki/