THE BERGLUND-HÜBSCH-CHIODO-RUAN MIRROR SYMMETRY FOR K3 SURFACES

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Abstract. We prove that the mirror symmetry of Berglund-Hübsch-Chiodo-Ruan, applied to K3 surfaces with a non-symplectic involution, coincides with the mirror symmetry described by Dolgachev and Voisin.

Contents

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1. Introduction

Berglund-Hübsch in [2] described a very concrete construction of mirror pairs of Calabi-Yau manifolds given as hypersurfaces in some weighted projective spaces. This construction has been used by Chiodo-Ruan in [5] to prove that the transposition rule of Berglund-Hübsch provides pairs of Calabi-Yau manifolds whose Hodge diamonds have the symmetry required in mirror symmetry. If one applies this construction to K3 surfaces given by Delsarte type equations in weighted projective spaces, this gives a priori no information, since all K3 surfaces have the same Hodge diamond. In this paper, we show that the transposition rule, in the case of K3 surfaces admitting a non-symplectic involution, provides pairs of K3 surfaces that belong to the mirror families constructed by Dolgachev in [9] and Voisin in [19].

Let W denote a Delsarte type polynomial having as many monomials as variables. Assume that the matrix of exponents of W is invertible, that $\{W = 0\}$ has an isolated singularity at the origin and that it defines a well-formed hypersurface in some normalized weighted projective space. We denote by $\mathrm{Aut}(W)$ the group of diagonal symmetries of W, by $\mathrm{SL}(W)$ the group of diagonal symmetries of determinant one, and by J_W the monodromy group of the affine Milnor fibre associated to W. For any group $G \subset \mathrm{Aut}(W)$, we denote by G^T the "transposed" group of automorphisms of the "transposed" potential W^T (see section 3 for more detail). The main result of this paper is the following:

Theorem 1.1. *Let W be a K3 surface admitting an equation in some weighted projective space given by a non-degenerate and invertible potential of the form:*

$$x^2 = f(y, z, w).$$

Let G_W be a group of diagonal symmetries of W such that $J_W \subset G_W \subset \mathrm{SL}(W)$. Put $\widetilde{G_W} := G_W/J_W$ and $\widetilde{G_W^T} := G_W^T/J_{W^T}$. Then the Berglund-Hübsch-Chiodo-Ruan mirror orbifolds $[W/\widetilde{G_W}]$ and $[W^T/\widetilde{G_W}^T]$ belong to the mirror families of Dolgachev and Voisin.

The theorem will be proved in several steps in the sections 5.1.1, 5.2.1, 5.3.1, 5.4.

The mirror symmetry we consider in this paper applies for Calabi-Yau varieties in weighted projective spaces which are not necessarily Gorenstein. This is the main difference with the mirror symmetry of Batyrev in [1]: the construction of Batyrev applies to reflexive polyhedra which produce hypersurfaces in weighted projective spaces with Gorenstein singularities. Most of our K3 surfaces are not contained in a Gorenstein weighted projective space. This is in fact a big difference between the Berglund-Hübsch-Chiodo-Ruan (BHCR for short) mirror symmetry and the Batyrev mirror symmetry, as remarked by Chiodo and Ruan in [5, Section 1]. However if the ambient space is Gorenstein the BHCR-mirror construction is essentially the same as the Batyrev-mirror construction.

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2. Mirror symmetry for lattice polarized K3 surfaces

2.1. **K3 surfaces with non-symplectic involution.** We briefly recall the classification theorem for non-symplectic involutions on K3 surfaces given by Nikulin in [14, §4] and [16, §4]. Let X be a K3 surface. The local action of a non-symplectic involution ι at a fixed point is of type:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that the fixed locus X^ι is the disjoint union of smooth curves and there are no isolated fixed points. The invariant lattice:

$$H^2(X,\mathbb{Z})^+ := \{x \in H^2(X,\mathbb{Z}) \mid \iota x = x\}$$

is 2-elementary, i.e the discriminant group $(H^2(X,\mathbb{Z})^+)^\vee/H^2(X,\mathbb{Z})^+ \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ for some integer a. According to Rudakov-Shafarevich in [18], its isometry class is determined by the invariants r, a and δ, where $r = \operatorname{rk} H^2(X,\mathbb{Z})^+$ and $\delta \in \{0,1\}$ and is 0 if and only if for any $x \in (H^2(X,\mathbb{Z})^+)^\vee$, $x^2 \in \mathbb{Z}$.

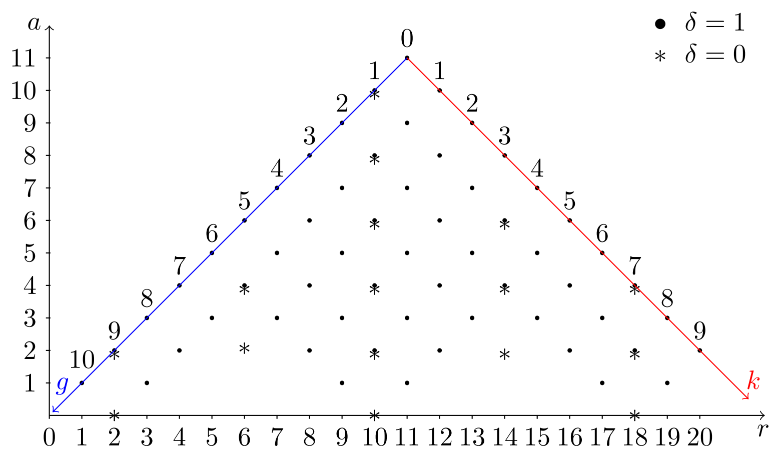

FIGURE 1. Nikulin classification

Theorem 2.1. [14, Theorem 4.2.2] *The fixed locus of a non-symplectic involution on a K3 surface is*

- *empty if* $r = 10$, $a = 10$ *and* $\delta = 0$,
- *the disjoint union of two elliptic curves if* $r = 10$, $a = 8$ *and* $\delta = 0$,
- *the disjoint union of a curve of genus* g *and* k *rational curves otherwise, where* $g = (22 - r - a)/2$, $k = (r - a)/2$.

Figure 1 shows all the values of the triple (r, a, δ) which are realized and the corresponding invariants (g, k) of the fixed locus.

2.2. **The Dolgachev-Voisin mirror symmetry.** Let M be an even non-degenerate lattice of signature $(1, \rho - 1)$, $1 \leq \rho \leq 19$.

Definition 2.2. An *M-polarized* K3 surface is a pair (X, j) where X is a K3 surface and $j : M \hookrightarrow \operatorname{Pic}(X)$ is a primitive lattice embedding.

Dolgachev in [9] constructs a (coarse) moduli space $\mathbf{K}_M$ parametrizing M-polarized K3 surfaces, which has dimension $20 - \rho$. Assume now that

$$M^\perp \cap H^2(X,\mathbb{Z}) = U \oplus \bar{M}$$

where U is a copy of the hyperbolic plane. As described in [9] one can define the *mirror moduli space* of $\mathbf{K}_M$ as the moduli space $\mathbf{K}_{\bar{M}}$ of $\bar{M}$-polarized K3 surfaces: one can use the primitive embedding $\bar{M} \hookrightarrow M^\perp \subset H^2(X,\mathbb{Z})$ to get a primitive even non-degenerate sublattice of signature $(1,(20-\rho)-1)$ of the K3 lattice $U^3 \oplus E_8(-1)^2$. Observe that for generic K3-surfaces $X_M \in \mathbf{K}_M$ and $X_{\bar{M}} \in \mathbf{K}_{\bar{M}}$ we have

$$\begin{array}{ll} \dim \mathbf{K}_M = 20-\rho, & \operatorname{rk}\operatorname{Pic}(X_M) = \rho, \\ \dim \mathbf{K}_{\bar{M}} = \rho, & \operatorname{rk}\operatorname{Pic}(X_{\bar{M}}) = 20-\rho. \end{array}$$

We consider now the special case where X is a K3 surface admitting a non-symplectic involution and $M = H^2(X,\mathbb{Z})^+$. Denote the anti-invariant lattice $H^2(X,\mathbb{Z})^- := (H^2(X,\mathbb{Z})^+)^\perp \cap H^2(X,\mathbb{Z})$.

Proposition 1. [19, Lemma 2.5, §2.6] *Let $(r,a,\delta) \neq (14,6,0)$ and $g \geq 1$. Then:*

- $H^2(X,\mathbb{Z})^- = U \oplus \bar{M}$,
- *the moduli spaces $\mathbf{K}_M$ and $\mathbf{K}_{\bar{M}}$ are mirror of each other,*
- *if $X_M \in \mathbf{K}_M$ has invariants (r,a,δ) then the invariants of $X_{\bar{M}} \in \mathbf{K}_{\bar{M}}$ are $(20-r,a,\delta)$, in particular $X_{\bar{M}}$ has also a non-symplectic involution.*

Remark 2.3.

- In the Figure 1 one can see the mirror couples making a reflection with respect to the axis through $r = 10$ and $1 \leq g \leq 10$ and deleting the axis with $g = 0$ and the point $(r,a,\delta) = (14,6,0)$.
- Since K3 surfaces with a non-symplectic involution are projective the invariant lattice contains an ample class. One can then consider instead of $\mathbf{K}_M$ the moduli space $\mathbf{K}^a_M$ of ample M-polarized K3 surfaces and do the same construction of mirror moduli spaces as above.

3. The Berglund-Hübsch-Chiodo-Ruan construction

3.1. **Hypersurfaces in weighted projective spaces.** Let $x_1,\ldots,x_n$ be affine coordinates on $\mathbb{C}^n$, $n \geq 3$, and let $(w_1,\ldots,w_n)$ be a sequence of positive weights. The group $\mathbb{C}^*$ acts by

$$\lambda(x_1,\ldots,x_n) = (\lambda^{w_1}x_1,\ldots,\lambda^{w_n}x_n)$$

and the *weighted projective space* $\mathbb{P}(w_1,\ldots,w_n)$ is the quotient $(\mathbb{C}^n\setminus\{0\})/\mathbb{C}^*$. The weighted projective space is called *normalized* if

$$\gcd(w_1,\ldots,\widehat{w_i},\ldots,w_n) = 1 \text{ for all } i.$$

Weighted projective spaces are singular in general and the singularities arise only on the fundamental simplex Δ with vertices the $P_i := (0,\ldots,0,1,0,\ldots,0)$. The vertices are singularities of type $1/w_i(w_1,\ldots,\widehat{w_i},\ldots,w_n)$, they are not necessarily isolated, since the higher dimensional toric strata of Δ can be singular too. For example we can have singularities along the edges P_iP_j, in this case each generic point of the edges is a singularity of type $1/h_{i,j}(w_1,\ldots,\widehat{w_i},\ldots,\widehat{w_j},\ldots,w_n)$ where $h_{i,j} := \gcd(w_i,w_j)$. The weighted projective space $\mathbb{P}(w_1,\ldots,w_n)$ has Gorenstein singularities if and only if $w_j | \sum_{i=1}^n w_i$ for all j, this is also equivalent to say that the weighted projective space is Fano, or finally, regarding $\mathbb{P}(w_1,\ldots,w_n)$ as toric variety, that its associated polytope is reflexive [7, Section 3.5].

A hypersurface W in $\mathbb{P}(w_1,\ldots,w_n)$ is defined by a quasihomogeneous polynomial $f(x_1,\ldots,x_n)$. Let d be its total degree, W is said *well formed* if the following conditions are satisfied for all i,j:

- $\gcd(w_1,\ldots,\widehat{w_i},\ldots,\widehat{w_j},\ldots,w_n)$ divides d,
- $\gcd(w_1,\ldots,\widehat{w_i},\ldots,w_n)=1$

and it is called *quasismooth* if it is non-degenerate, i.e. its affine cone is smooth outside its vertex $(0,\ldots,0)$. Finally W is said to be *Calabi-Yau* if it has canonical singularities (in particular W is Gorenstein), its canonical bundle is trivial and $H^i(W,\mathcal{O}_W)=0$ for all $i=1,\ldots,n-2$. Observe that by [6, Lemma 1.12] a well formed and quasismooth hypersurface W in $\mathbb{P}(w_1,\ldots,w_n)$ is Calabi-Yau if and only if $d=\sum_{i=1}^n w_i$.

The genus of a smooth curve C_d of total degree d in $\mathbb{P}(w_1,w_2,w_3)$ is given by the formula:

$$(1)\qquad g(C_d)=\frac{1}{2}\left(\frac{d^2}{w_1w_2w_3}-d\sum_{i>j}\frac{\gcd(w_i,w_j)}{w_iw_j}+\sum_{i=1}^{3}\frac{\gcd(d,w_i)}{w_i}-1\right).$$

Reid in [17] and Yonemura in [20] give a list of all possible families of K3 surfaces in weighted projective spaces. These are 95 in total and only 14 of the weighted projective spaces are Gorenstein. For each type Reid describes the singularities on the K3 surface. By [6] the 95 projective spaces have canonical singularities, and in fact one can determine 104 families of weights such that the weighted projective spaces have canonical singularities. However in 9 cases one can not obtain K3 surfaces with canonical singularities [6, Theorem 1.17].

3.2. **Invertible potentials.** We recall briefly the mirror construction of Berglund-Hübsch in [2] and Chiodo-Ruan in [5]. Consider the *potential*:

$$W:=W(x_1,\ldots,x_n)=\sum_{i=1}^{n}\prod_{j=1}^{n}x_j^{a_{ij}}.$$

This is a quasi-homogeneous polynomial in n variables, containing n monomials. Since we have n monomials it is not a restriction to consider all the coefficients equal to 1. To this polynomial we associate the matrix $A:=(a_{i,j})_{i,j=1,\ldots,n}$ and the potential is called *invertible* if the matrix A is invertible over $\mathbb{Q}$. Denote by $A^{-1}:=(a^{i,j})_{i,j=1,\ldots,n}$ the inverse matrix and define the *charge*, $q_i:=\sum_{j=1}^n a^{i,j}$, as the sum of the entries of the i-th row of A^{-1}. Clearly the charges q_i satisfy:

$$A\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

Let d be the least common denominator of the charges and let $w_i:=dq_i$. We assume that $\{W=0\}$ defines a well formed and quasismooth hypersurface in $\mathbb{P}(w_1,\ldots,w_n)$, which has total degree d. In the sequel we denote this hypersurface again by W. In particular this means that the potential is non-degenerate. By [8, Proposition 6] if the weighted projective space is normalized and the hypersurface is quasismooth of dimension ≥ 3 then it is well formed. In the case of K3 surfaces this is also true by checking in the Reid's 95 list.

We define the group of diagonal automorphisms by

$$\mathrm{Aut}(W):=\{\gamma=(\gamma_1,\ldots,\gamma_n)\in(\mathbb{C}^*)^n\,|\,W(\gamma_1x_1,\ldots,\gamma_nx_n)=W(x_1,\ldots,x_n)\}.$$

Each column of A^{-1} can be used to define the diagonal matrix

$$\rho_j:=\mathrm{diag}(\exp(2\pi i a^{1,j}),\ldots,\exp(2\pi i a^{n,j}))\in\mathrm{Aut}(W).$$

The matrix j_W is defined as the product

$$\rho_1 \cdots \rho_n = \mathrm{diag}(\exp(2\pi i q_1), \ldots, \exp(2\pi i q_n))$$

and acts trivially on the hypersurface W, since it acts trivially on the weighted projective space. The group J_W generated by j_W is cyclic of order d. Denote by

$$\mathrm{SL}(W) := \mathrm{Aut}(W) \cap \mathrm{SL}_n(\mathbb{C})$$

and assume that W is Calabi-Yau. Then $\sum_i q_i = 1$ and so $J_W \subset \mathrm{SL}(W)$.

Let G_W be a group of automorphisms such that $J_W \subset G_W \subset \mathrm{SL}(W)$ and $\widetilde{G_W} := G_W/J_W$. We associate to W in a natural way a potential W^T and a group G_W^T. The potential W^T is defined by transposing the matrix A:

$$W^T := W^T(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ji}}.$$

Similarly we denote by q_j^T the charges of W^T. Since the matrix of W^T is A^T, the charge q_j^T is the sum of the entries of the column ρ_j of A^{-1}. We have the relation

$$(A^{-1})^T A\,\mathbf{q} = \mathbf{q^T}$$

where $\mathbf{q}$ and $\mathbf{q^T}$ denote the $n \times 1$-matrices whose entries are the charges q_j, respectively q_j^T. Observe that $\sum_j q_j = \sum_j q_j^T = \sum_{i,j} a^{i,j}$.

The group G_W^T is defined by Krawitz in [11] as:

$$G_W^T = \left\{ \prod_{j=1}^{n} (\rho_j^T)^{m_j} \mid \prod_{j=1}^{n} x_j^{m_j} \text{ is } G_W\text{-invariant} \right\}$$

where the definition of the automorphisms ρ_j^T of W^T is similar as the definition of ρ_j using the matrix A^T. By [13, Theorem 1] a potential W given by a $n \times n$ matrix is nondegenerate and invertible if and only if it can be written as a sum of invertible potentials of *atomic types*:

$$\begin{array}{lll} W_{fermat} & := & x^a, \\ W_{loop} & := & x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, \\ W_{chain} & := & x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}. \end{array}$$

Recall that if W is a fermat type polynomial (i.e. sum of W_{fermat}) then $\{W = 0\}$ defines a variety in some weighted projective space and the weighted projective space is Gorenstein. In the other cases this is not true in general.

The potential W^T is quasismooth by the classification of [13] and the charges satisfy $q_1^T + \ldots + q_n^T = 1$: this implies easily that the equation $\{W^T = 0\}$ defines a variety in a normalized weighted projective space. By [8, Proposition 6] if $n \geq 5$ the hypersurface is well formed and so the potential W^T defines a Calabi-Yau variety. If $n = 3, 4$ it is also true by a quick case-by-case analysis.

Remark 3.1. Without the condition $\sum_i q_i = 1$ it is not true that the equation $\{W^T = 0\}$ defines a variety in a normalized projective space. For example $W = x_1^5x_2 + x_2^2x_3 + x_3^3x_4 + x_4^9$ defines a surface in $\mathbb{P}(7, 19, 16, 6)$ and W^T defines a surface in $\mathbb{P}(9, 18, 9, 4)$, which is clearly not normalized.

Proposition 2. *Let W be a nondegenerate potential defining a Calabi-Yau manifold in $\mathbb{P}(w_1, \ldots, w_n)$. Then the action of $\mathrm{SL}(W)$ on the volume form is trivial.*

Proof. We can write the volume form locally for $x_1 \neq 0$ and $\frac{\partial W}{\partial x_n} \neq 0$ as

$$\xi := \frac{dx_2 \wedge \ldots \wedge dx_{n-1}}{\frac{\partial W}{\partial x_n}}.$$

Let $g = (\exp(2\pi i\alpha_1), \ldots, \exp(2\pi i\alpha_n)) \in \mathrm{SL}(W)$. We can multiply this element by $\exp(2\pi i(-\alpha_1/w_1))$ to normalize it as $g = (1, \exp(2\pi i\beta_2), \ldots, \exp(2\pi i\beta_n))$ with $\beta_i = \alpha_i - (w_i/w_1)\alpha_1$. If we apply this transformation to W, this is multiplied by $\exp(2d\pi i(-\alpha_1/w_1))$. We have that

$$g\frac{\partial W}{\partial x_n} = \exp(2\pi i(-\beta_n - (\alpha_1/w_1)d))W.$$

Hence the form ξ is multiplied by $\exp(2\pi i\delta)$, with

$$\delta = \beta_1 + \ldots + \beta_n + \frac{\alpha_1}{w_1}d = \alpha_1 + \alpha_2 + \ldots + \alpha_n \in \mathbb{Z}.$$

■

The groups satisfy $J_{W^T} \subset G_W^T \subset \mathrm{SL}_{W^T}$. Putting $\widetilde{G_W^T} := G_W^T/J_{W^T}$, we have:

Theorem 3.2. [5, Theorem 2] *The Calabi-Yau orbifolds* $[W/\widetilde{G_W}]$ *and* $[W^T/\widetilde{G_W^T}]$ *form a mirror pair, i.e. we have*

$$H^{p,q}_{\mathrm{CR}}([W/\widetilde{G_W}], \mathbb{C}) \cong H^{n-2-p,q}_{\mathrm{CR}}([W^T/\widetilde{G_W^T}], \mathbb{C})$$

where $H_{\mathrm{CR}}(-, \mathbb{C})$ *stands for Chen-Ruan orbifold cohomology.*

4. The group of diagonal automorphisms

4.1. **Lattice theoretical description of** $\mathrm{Aut}(W)$**.** In this section we recall the description of Borisov [4, Proposition 2.3.1]. It is easy to see that $\mathrm{Aut}(W)$ is a finite abelian group. Writing $(\gamma_1, \ldots, \gamma_n) = (\exp(2i\pi a_1), \ldots, \exp(2i\pi a_n))$ we have an isomorphism

$$\mathrm{Aut}(W) \cong \left\{ (a_1, \ldots, a_n) \in (\mathbb{Q}/\mathbb{Z})^n \mid A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z}^n \right\} =: \mathrm{Aut}(W)_+$$

We identify ρ_i with the i-th column $(a^{1,i}, \ldots, a^{n,i})$ of A^{-1}. Since $A\rho_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, where the 1 is at the place i, we get $\rho_i \in \mathrm{Aut}(W)_+$. In fact the ρ_i generate $\mathrm{Aut}(W)_+$. We want to describe $\mathrm{Aut}(W)_+$ in a more precise way.

Let M_0 and N_0 be two free $\mathbb{Z}$-modules of rank n with basis respectively $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. We define a non-degenerate bilinear form:

$$\langle -, - \rangle : M_0 \times N_0 \longrightarrow \mathbb{Z}$$

by $\langle u_i, v_j \rangle := a_{i,j}$. This induces an immersion $\xi \colon N_0 \longrightarrow M_0^* := \mathrm{Hom}_{\mathbb{Z}}(M_0, \mathbb{Z})$ defined by $\xi(v)(u) = \langle u, v \rangle$. Let $u_1^*, \ldots, u_n^*$ be the dual basis of M_0^* defined by $u_i^*(u_j) = \delta_{i,j}$. Then for $v_j \in N_0$ we have $\xi(v_j)(u_i) = \langle u_i, v_j \rangle = a_{i,j}$ hence:

$$\xi(v_j) = \sum_{i=1}^{n} a_{i,j} u_i^*.$$

In conclusion, identifying v_j with its image $\xi(v_j)$ we identify $N_0 \subset M_0^*$ as a $\mathbb{Z}$-submodule of finite index equal to $\det(A)$. We have that $v_1, \dots, v_n$ form a basis of $M_0^* \otimes \mathbb{Q}$. We can write it in compact form as

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A^T \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}$$

hence the columns of A^{-1} are the coordinates of the $u_i^* \in M_0^* \otimes \mathbb{Q}$ in the basis v_j. We define a surjective map: $M_0^* \longrightarrow \mathrm{Aut}(W)_+,\ u_i^* \mapsto \rho_i^T$. For an element $\lambda_1 u_1^* + \ldots + \lambda_n u_n^* \in M_0^*$ (here $\lambda_i \in \mathbb{Z}$) its image is $(\lambda_1, \dots, \lambda_n)(A^{-1})^T$ so it is in the kernel of this map if and only if there exist $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ such that

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Hence an element is in the kernel if and only if

$$\lambda_1 u_1^* + \ldots + \lambda_n u_n^* = \sum_{j=1}^n a_{1,j}\alpha_j u_1^* + \ldots + \sum_{j=1}^n a_{n,j}\alpha_j u_n^* = \sum_{j=1}^n \alpha_j v_j$$

and so the kernel is exactly N_0. In conclusion $M_0^*/N_0 \cong \mathrm{Aut}(W)_+$ and in particular $\mathrm{Aut}(W)$ is of order $\det(A)$. The inverse morphism is given by:

$$(a_1, \dots, a_n) \in \mathrm{Aut}(W)_+ \mapsto a_1 v_1 + \ldots + a_n v_n + N_0 \in M_0^*/N_0$$
$$(\gamma_1, \dots, \gamma_n) \in \mathrm{Aut}(W) \mapsto \sum_{j=1}^n \frac{1}{2i\pi} \log(\gamma_i) v_i + N_0 \in M_0^*/N_0.$$

In the same way as before we define a surjective map: $\zeta : M_0 \longrightarrow N_0^* := \mathrm{Hom}_{\mathbb{Z}}(N_0, \mathbb{Z})$ by $\zeta(u)(v) = \langle u, v \rangle$, and in this case $u_i = \sum_{j=1}^n a_{i,j} v_j^*$. The coordinates of $v_j^* \in N_0^* \otimes \mathbb{Q}$ in the basis $u_0, \dots, u_n$ are the entries of the j-th line of the matrix A^{-1}. Finally observe that since M_0^*/N_0 is a torsion group we have $\zeta = \xi^*$.

Consider now the potential W^T associated to A^T. It is clear that $\mathrm{Aut}(W^T)_+ \cong N_0^*/M_0$ of order $\det(A^T) = \det(A)$. Looking at the Smith normal form of the matrix A we have (see also [11, Lemma 1.6]):

Corollary 1.

1) $|\mathrm{Aut}(W)| = |\mathrm{Aut}(W^T)| = \det(A)$,
2) $\mathrm{Aut}(W) \cong \mathrm{Aut}(W^T)$. *More precisely:*

$$\begin{array}{lcl} \mathrm{Aut}(W_{fermat}) & \cong & \mathbb{Z}/a\mathbb{Z} \\ \mathrm{Aut}(W_{loop}) & \cong & \mathbb{Z}/(a_1 \cdots a_n - (-1)^n)\mathbb{Z} \\ \mathrm{Aut}(W_{chain}) & \cong & \mathbb{Z}/(a_1 \cdots a_n)\mathbb{Z} \end{array}$$

Remark 4.1. In fact there is a natural coupling $\mathrm{Aut}(W) \times \mathrm{Aut}(W^T) \longrightarrow \mathbb{C}^*$ showing that $\mathrm{Aut}(W^T)$ is isomorphic to the dual group $\mathrm{Hom}(\mathrm{Aut}(W), \mathbb{C}^*)$ of $\mathrm{Aut}(W)$, see [10, Proposition 2] for more details.

4.2. **Description of G^T_W.** Identifying $\mathrm{Aut}(W)$ with M_0^*/N_0, the element j_W corresponds to $q_1v_1+\ldots+q_nv_n+N_0 \in M_0^*/N_0$. Denote $\deg^* := q_1v_1+\ldots+q_nv_n \in M_0^*$, this element is characterized by the relations $\deg^*(u_i) = 1$ for all i. The element $j_{W^T} = \mathrm{diag}(\exp(2i\pi q_1^T),\ldots,\exp(2i\pi q_n^T)) \in \mathrm{Aut}(W^T)$ is identified with $q_1^Tu_1+\ldots+q_n^Tu_n+M_0 \in N_0^*/M_0$. Denote $\deg := q_1^Tu_1+\ldots+q_n^Tu_n \in N_0^*$, this element is characterized by $\deg(v_i) = 1$ for all i.

Let now $G := G_W$ be a subgroup of $\mathrm{Aut}(W)$. It corresponds to a submodule $N_0 \subset N \subset M_0^*$, that we can describe as

$$N = \{(a_1v_1+\ldots+a_nv_n) \in M_0^* \,|\, (a_1,\ldots,a_n) \in G_+\}$$

where as before G_+ is the subgroup of $\mathrm{Aut}(W)_+$ which corresponds to $G \subset \mathrm{Aut}(W)$, that is $G_+ \cong N/N_0$. In particular $J_W \subset G$ if and only if $\deg^* \in N$. Denote $M := N^* = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, we have the inclusions $M_0 \subset M \subset N_0^*$. In particular $G \subset \mathrm{SL}(W) = \mathrm{Aut}(W) \cap \mathrm{SL}_n(\mathbb{C})$ if and only if each element $(a_1,\ldots,a_n) \in G_+$ satisfies $\sum_{i=1}^n a_i \in \mathbb{Z}$. But $\sum_{i=1}^n a_i = \deg(a_1v_1+\ldots+a_nv_n)$, hence we get the equivalence:

$$G \subset \mathrm{SL}(W) \Leftrightarrow \forall v \in N,\ \deg(v) \in \mathbb{Z} \Leftrightarrow \deg \in N^* = M.$$

The $\mathbb{Z}$-module M corresponds to a subgroup of $\mathrm{Aut}(W^T)$ which is precisely $G^T := G^T_W$, that is $(G^T)_+ \cong M/M_0$. This can be seen as follows. The elements of $M = N^*$ are by definition of the form $\beta_1v_1^*+\ldots+\beta_nv_n^* \in N_0^*$ with $(\beta_1v_1^*+\ldots+\beta_nv_n^*)(u) \in \mathbb{Z}$ for all $u \in N$ (here $\beta_i \in \mathbb{Z}$). Writing $u = \sum_{i=1}^n a_iv_i$ with $(a_1,\ldots,a_n) \in G_+$ the condition reads: $\beta_1a_1+\ldots+\beta_na_n \in \mathbb{Z}$, where the coefficients a_i are rational. This shows that $(G^T)_+ \cong M/M_0$.

Writing $u = \alpha_1u_1^*+\ldots+\alpha_nu_n^* \in M_0^*$ with $\alpha_i \in \mathbb{Z}$ we can give a more explicit description. Noting that

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

we get:

$$M = \left\{ \sum_{i=1}^n \beta_i v_i^* \in N_0^* \,|\, (\beta_1,\ldots,\beta_n)A^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{Z},\ \forall \sum_{i=1}^n \alpha_i u_i^* \in N \right\}.$$

Using the identification of v_i^* with the vector $\bar\rho_i^T \in \mathrm{Aut}(W^T)_+$ (the $\bar\rho_i^T$ are the columns of $(A^{-1})^T$) and the identification of u_i^* with $\rho_i^T \in \mathrm{Aut}(W)_+$ this gives

$$(G^T)_+ = \left\{ \sum_{i=1}^n \beta_i \bar\rho_i^T \,|\, (\beta_1,\ldots,\beta_n)A^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{Z},\ \forall \sum_{i=1}^n \alpha_i \rho_i \in G_+ \right\}.$$

Hence to give the couple (G, G^T) it is equivalent to give the couple $(N, M = N^*)$. Since $(N^*)^* = N$ one has $(G^T)^T = G$. If we take $G = \{1\}$ this corresponds to the submodule $N = N_0$, so $\{1\}^T = G^T$ corresponds to N_0^* which gives $G^T = \mathrm{Aut}(W^T)$. Similarly $\mathrm{Aut}(W)^T = \{1\}$.

Remark 4.2. The non-degenerate bilinear form $\langle -,- \rangle$ on $M_0 \times N_0$ induces a non-degenerate $\mathbb{Q}/\mathbb{Z}$-bilinear form on the product $\mathrm{Aut}(W)_+ \times \mathrm{Aut}(W^T)_+$, and so one can identify $(G^T)_+$ with the orthogonal module of G_+.

Proposition 3. *Consider two groups* $G_1 \subset G_2 \subset \mathrm{Aut}(W)$. *Then* $G_2^T \subset G_1^T \subset \mathrm{Aut}(W^T)$ *and we have a group isomorphism* $G_2/G_1 \cong G_1^T/G_2^T$.

Proof. The groups G_i can be identified with N_i/N_0, where $N_0 \subset N_1 \subset N_2 \subset M_0^*$, so G_2/G_1 corresponds to N_2/N_1. Applying the contravariant functor $\mathrm{Hom}(-,\mathbb{Z})$ to the exact sequence:

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_2/N_1 \longrightarrow 0$$

we get

$$\mathrm{Hom}(N_2/N_1,\mathbb{Z}) = \{0\} \longrightarrow N_2^* \longrightarrow N_1^* \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(N_2/N_1,\mathbb{Z}) \longrightarrow 0.$$

Since $\mathrm{Ext}^1_{\mathbb{Z}}(N_2/N_1,\mathbb{Z}) = N_2/N_1$, we have $N_1^*/N_2^* \cong N_2/N_1$ and we are done. ■

We now study $(J_W^T)_+$. The group J_W^T can be identified with

$$\left\{ \sum_{i=1}^n \beta_i v_i^* \in N_0^*/M_0 \ \mid \sum_{i=1}^n \beta_i q_i \in \mathbb{Z} \right\}.$$

On the other hand, writing $\beta_1 v_1^* + \ldots + \beta_n v_n^* = b_1 u_1 + \ldots + b_n u_n$ with $(b_1,\ldots,b_n) \in \mathrm{Aut}(W^T)_+$ we get

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = A^T \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

This gives:

$$\sum_{i=1}^n \beta_i q_i = (b_1 \ldots b_n) A \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = (b_1 \ldots b_n) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^n b_i \in \mathbb{Z}$$

hence

$$J_W^T = \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{Aut}(W^T) = \mathrm{SL}(W^T)$$
$$(J_{W^T})^T = \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{Aut}(W) = \mathrm{SL}(W).$$

Particularly interesting for us will be the case when $\mathrm{SL}(W^T) = J_{W^T}$. By proposition 3 it follows

Corollary 2. *We have* $\mathrm{SL}(W^T)/J_{W^T} \cong \mathrm{SL}(W)/J_W$ *and so* $\mathrm{SL}(W^T) = J_{W^T}$ *if and only if* $J_W = \mathrm{SL}(W)$.

Remark 4.3. If $n = 4$ and W is a K3 surface by proposition 2 the group $\mathrm{SL}(W)/J_W$ acts symplectically. Since $\mathrm{SL}(W)/J_W$ is finite and abelian it appears in the list of the 15 possible finite abelian groups (including the identity group) given by Nikulin in [15].

5. The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces

In this section we prove theorem 1.1 by a specific analysis of the possible decomposition of the polynomial $f(y,z,w)$ as a sum of atomic types. The possibilities are: chain, loop, fermat, chain+fermat and loop+fermat.

5.1. **The chain case.** Consider a potential of the form:

$$W_{chain} := x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$$

where

$$A = \begin{pmatrix} a_1 & 1 & & & \\ & a_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & 1 \\ & & & 0 & a_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{a_1} & \frac{-1}{a_1a_2} & & \cdots & \frac{(-1)^{n+1}}{a_1\cdots a_n} \\ 0 & \frac{1}{a_2} & \frac{-1}{a_2a_3} & \cdots & \frac{(-1)^{n+2}}{a_2\cdots a_n} \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{a_n} \end{pmatrix}$$

One can easily show that (cf. [11, Lemma 1.6]):

$$\mathrm{Aut}(W_{chain})_+ \cong \{\beta_1 u_1^* + \ldots + \beta_n u_n^* \,|\, \beta_i \in \mathbb{Z},\, 0 \leq \beta_i < a_i\}.$$

As shown in the corollary 1 this is also isomorphic to $\mathbb{Z}/(a_1\cdots a_n)\mathbb{Z}$ and as in [12, Section 4] a generator is given by $(\varphi_1, \ldots, \varphi_n)$ where

$$\varphi_i := \frac{(-1)^{n-i+1}}{a_i \cdots a_n}.$$

Proposition 4. *The order of* $\mathrm{SL}(W_{chain})$ *is*

$$|\,\mathrm{SL}(W_{chain})| = \gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1\cdots a_j\right),\ a_1\cdots a_n\right)$$

hence $\mathrm{SL}(W_{chain}) = J_{W_{chain}}$ *if and only if the total degree of* W_{chain} *is equal to* $\gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1\cdots a_j\right),\ a_1\cdots a_n\right)$.

Proof. An element $(\alpha\varphi_1, \ldots, \alpha\varphi_n) \in \mathrm{Aut}(W_{chain})_+$ with $\alpha \in \mathbb{Z}_{\geq 0}$ is contained in $\mathrm{SL}(W_{chain})_+$ if

$$\alpha \sum_{i=1}^{n} \varphi_i = \alpha\left(\frac{(-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1\cdots a_j}{a_1\cdots a_n}\right) \in \mathbb{Z}.$$

Let $\ell := \gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1\cdots a_j\right), a_1\cdots a_n\right)$ and denote

$$(-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1\cdots a_j = \gamma\ell \text{ and } a_1\cdots a_n = \beta\ell.$$

Then we must have $\alpha = \beta\lambda$ with $1 \leq \lambda \leq \ell$ and the assertion holds. ■
With the same notation as above we have:

Corollary 3. *A generator of* $\mathrm{SL}(W_{chain})_+$ *is* $(\beta\varphi_1, \ldots, \beta\varphi_n)$.

Remark 5.1. Particularly interesting for us is the case $n = 3$. In this case the formula reads $|\,\mathrm{SL}(W_{chain})| = \gcd(-1 + a_1 - a_1a_2,\ a_1a_2a_3)$.

We generalize now proposition 4 in the following form:

Proposition 5. *Let* W *denote a potential of the form:*

$$x^2 - (x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}).$$

We have $\mathrm{Aut}(W) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/a_1\cdots a_n\mathbb{Z}$*, and*

(1) *If* 2 *does not divide* $a_1 \cdots a_n$*, i.e.* $\mathrm{Aut}(W)$ *is cyclic of order* $2a_1 \cdots a_n$*, then:*

$$|\,\mathrm{SL}(W)| = \gcd\left(a_1 \cdots a_n + 2\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j\right),\ 2a_1 \cdots a_n\right).$$

(2) *Otherwise let* 2^k*,* $k \geq 1$*, be the greatest power of* 2 *dividing* $a_1 \cdots a_n$*, then* $\mathrm{Aut}(W)$ *is not cyclic and*

(2a) $|\,\mathrm{SL}(W)| = \gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j\right),\ a_1 \cdots a_n\right)$ *if* 2^k *divides* $(-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j$.

(2b) $|\,\mathrm{SL}(W)| = 2\gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j\right),\ a_1 \cdots a_n\right)$ *otherwise.*

Proof. **(1)** The proof in the case that $\mathrm{Aut}(W)$ is cyclic is the same as in the case of a potential of chain type, where this time a generator for $\mathrm{Aut}(W)_+$ is $(1/2, \varphi_1^{(n)}, \cdots, \varphi_n^{(n)})$.
(2) Observe that one gets elements of $\mathrm{SL}(W)$ by taking the elements in $\mathrm{SL}(W_{chain})$ with determinant 1 or -1 and multiplying them by 1 or -1, hence by proposition 4 one gets $|\,\mathrm{SL}(W)| \geq \gcd\left(\left((-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j\right),\ a_1 \cdots a_n\right)$. We show now **(2a)** and **(2b)**. First we assume that we can write:

$$a_1 \cdots a_n = 2^k\delta\beta \quad \text{and} \quad (-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j = 2^k\delta\gamma$$

and 2 does not divide δ and β. We want to determine $\alpha \in \mathbb{Z}$ such that:

$$\alpha\left(\frac{(-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j}{a_1 \cdots a_n}\right) = \alpha\left(\frac{\gamma}{\beta}\right)$$

is an integer or is an integer divided by 2. Since 2 does not divide β the second condition is not possible, so $|\,\mathrm{SL}(W)| = 2^k\delta$, this proves **(2a)**.

If we assume that:

$$a_1 \cdots a_n = 2^k\delta\beta \quad \text{and} \quad (-1)^n + \sum_{j=1}^{n-1}(-1)^{n-j}a_1 \cdots a_j = \delta\gamma$$

with 2 not dividing γ, we must determine an integer α such that $\frac{\alpha\gamma}{2^k\beta}$ is an integer or an integer divided by 2. Taking $\alpha = 2^k\beta s$ with $1 \leq s \leq \delta$ we get an element of $\mathrm{Aut}(W_{chain})$ with determinant 1 and taking $\alpha = 2^{k-1}\beta s$ with $s = 2t - 1$ and $1 \leq t \leq \delta$ we get an element of determinant -1. Hence we get **(2b)**. ∎

Remark 5.2. It is easy to see that if $n = 3$ the condition **(2a)** of the proposition 5 is not possible.

5.1.1. *BHCR-mirror construction for K3 surfaces of chain type.* We consider a potential W of the form:

$$x^2 - (y^{a_1}z + z^{a_2}w + w^{a_3})$$

and we assume that it satisfies the Calabi-Yau condition $\sum_{i=1}^4 q_i = 1$. Then $\{W = 0\} \subset \mathbb{P}(w_1, w_2, w_3, w_4)$ defines a K3 surface of total degree $d = \sum w_i$ with a non-symplectic involution $\iota\colon x \mapsto -x$. Applying the results of §4.1 the charges are

$$q_1 = \frac{1}{2},\ q_2 = \frac{a_2 a_3 - a_3 + 1}{a_1 a_2 a_3},\ q_3 = \frac{a_3 - 1}{a_2 a_3},\ q_4 = \frac{1}{a_3}.$$

The transposed potential W^T is

$$x^2 - (y^{a_1} + yz^{a_2} + zw^{a_3})$$

and defines a K3 surface in a weighted projective space with charges

$$q_1^T = \frac{1}{2},\ q_2^T = \frac{1}{a_1},\ q_3^T = \frac{a_1 - 1}{a_1 a_2},\ q_4^T = \frac{a_1 a_2 - a_1 + 1}{a_1 a_2 a_3}.$$

Borcea in [3, Tables 1, 2, 3] and Yonemura in [20, Table 2.2] write equations of K3 surfaces in weighted projective 3-spaces, which are not always of Delsarte type. Our first aim is to write down equations of the form $x^2 = W_{chain}$ when possible. The list of possible weights are given in [3, 17, 20] and for given weights (w_1, w_2, w_3, w_4) we get the relations

$$2w_1 = d,\ a_1 = \frac{d - w_3}{w_2},\ a_2 = \frac{d - w_4}{w_3},\ a_3 = \frac{d}{w_4}, \tag{2}$$

and the possible equations are listed in the table 1. As usual the weights are in non increasing order, so to get a solution of (2) one has to check all permutations of the coordinates y, z, w. We explain briefly the notation of the table 1. In the first column we write the number of the K3 surface following [3] and we put in parenthesis the number of the transposed K3 surface. In the fourth column we write the Nikulin's invariants (r, a) of the non-symplectic involution on $W = x^2 - f(y, z, w)$. In the fifth column we compute the order of the group $\mathrm{SL}(W)$ by using proposition 5. Except in the four cases denoted by $*$ the table proves theorem 1.1 in the chain case: the BHCR-mirror symmetry coincides with the mirror symmetry between the families described by Dolgachev and Voisin (observe that here the only possibility is $\widetilde{G_W} = \mathrm{id}$). The proof of theorem in the four special cases is given in §5.1.2.

To compute r one has to study the action of the non-symplectic involution ι on the singularities of the K3 surface. As explained in the §3.1 these arise on the edges or on the vertices of the fundamental simplex (the singularities are also listed in [20]). The pull-back of the hyperplane section gives a contribute of one to r, the other invariant classes are obtained by taking the ι-orbits of the exceptional curves in the resolution of the singularities. We compute $a = 22 - r - 2g$ by using the formula (1).

We compute explicitely two cases, the other cases are similar.

No. 1: $(w_1, w_2, w_3, w_4) = (3, 1, 1, 1)$. Here $d = 6$ and the only possibility is $(a_1, a_2, a_3) = (5, 5, 6)$, which gives the equation $x^2 = y^5 z + z^5 w + w^6$. The surface is smooth and so the only invariant class is the pull-back of the hyperplane section. The genus of the curve $\{y^5 z + z^5 w + w^6 = 0\}$ in $\mathbb{P}^2$ is 10, so $(r, a) = (1, 1)$. The potential W^T defines a K3 surface $x^2 = y^5 + yz^5 + zw^6$ in the weighted projective space with weights $(25, 10, 8, 7)$, which has $1A_1$, $1A_4$, $1A_6$ and $1A_7$ singularities so the invariants are $(r, a) = (19, 1)$ (this is No. 27 in the list). Using proposition 5 we find that $|\,\mathrm{SL}(W^T)| = |J_{W^T}| = 50$ and so the claim is proved.

No. 3a, No. 3b: $(w_1, w_2, w_3, w_4) = (5, 3, 1, 1)$. We have $d = 10$, the only solution with weights $(5, 3, 1, 1)$ in this order is $(a_1, a_2, a_3) = (3, 9, 10)$, which is the

$No.$	(w_1, w_2, w_3, w_4)	$f(y, z, w)$	(r, a)	$\|\mathrm{SL}(W)\|$	$\|J_W\|$
(1)27	$(25, 10, 8, 7)$	$y^5 + yz^5 + zw^6$	$(19, 1)$	50	50
(2)37	$(16, 7, 5, 4)$	$w^8 + wz^4 + zy^5$	$(14, 4)$	32	32
$(3a)$28	$(27, 18, 4, 5)$	$y^3 + yz^9 + zw^{10}$	$(17, 1)$	54	54
$*(3b)$5	$(7, 4, 2, 1)$	$z^7 + zy^3 + yw^{10}$	$(7, 3)$	*42	14
*(4)30	$(4, 2, 1, 1)$	$y^4 + yz^6 + zw^7$	$(2, 2)$	*24	8
$*(5)3b$	$(5, 3, 1, 1)$	$z^7y + y^3w + w^{10}$	$(3, 1)$	*30	10
(7)14	$(13, 8, 3, 2)$	$w^{13} + wy^3 + yz^6$	$(13, 3)$	26	26
(8)38	$(16, 9, 5, 2)$	$w^{16} + wz^6 + zy^3$	$(14, 2)$	32	32
(11)17	$(15, 8, 6, 1)$	$z^5 + zy^3 + yw^{22}$	$(11, 1)$	30	30
$(12)31a$	$(8, 4, 3, 1)$	$z^4 + zy^4 + yw^{13}$	$(6, 4)$	16	16
(13)19	$(15, 10, 4, 1)$	$y^3 + yz^5 + zw^{26}$	$(9, 1)$	30	30
(14)7	$(9, 5, 3, 1)$	$w^{13}y + y^3z + z^6$	$(7, 3)$	18	18
$(16)31b$	$(8, 4, 3, 1)$	$y^4 + yw^{12} + wz^5$	$(6, 4)$	16	16
(17)11	$(11, 7, 3, 1)$	$z^5y + y^3w + w^{22}$	$(9, 1)$	22	22
(19)13	$(13, 7, 5, 1)$	$y^3z + z^5w + w^{26}$	$(11, 1)$	26	26
(25)32	$(8, 5, 2, 1)$	$z^8 + zw^{14} + wy^3$	$(6, 2)$	16	16
(27)1	$(3, 1, 1, 1)$	$y^5z + z^5w + w^6$	$(1, 1)$	6	6
$(28)3a$	$(5, 3, 1, 1)$	$y^3z + z^9w + w^{10}$	$(3, 1)$	10	10
*(30)4	$(7, 3, 2, 2)$	$y^4z + z^6w + w^7$	$(10, 6)$	*42	14
$(31a)$12	$(13, 6, 5, 2)$	$z^4y + y^4w + w^{13}$	$(14, 4)$	26	26
$(31b)$16	$(15, 7, 6, 2)$	$y^4w + w^{12}z + z^5$	$(14, 4)$	30	30
(32)25	$(21, 14, 5, 2)$	$z^8w + w^{14}y + y^3$	$(14, 2)$	42	42
(36)47	$(18, 12, 5, 1)$	$y^3 + yw^{24} + wz^7$	$(10, 0)$	36	36
(37)2	$(5, 2, 2, 1)$	$w^8z + z^4y + y^5$	$(6, 4)$	10	10
(38)8	$(9, 6, 2, 1)$	$w^{16}z + z^6y + y^3$	$(6, 2)$	18	18
(40)42	$(6, 4, 1, 1)$	$y^3 + yz^8 + zw^{11}$	$(2, 0)$	12	12
(42)40	$(22, 13, 5, 4)$	$y^3z + z^8w + w^{11}$	$(18, 0)$	44	44
(47)36	$(14, 9, 4, 1)$	$y^3w + w^{24}z + z^7$	$(10, 0)$	28	28

Table 1. The chain mirror cases

surface $x^2 = y^3z + z^9w + w^{10}$. This has one A_2 singularity so $(r, a) = (3, 1)$ and W^T is the surface $x^2 = y^3 + yz^9 + zw^{10}$ in $\mathbb{P}(27, 18, 4, 5)$, which has $1A_1$, $1A_3$, $1A_4$ and $1A_8$ singularity, so $(r, a) = (17, 1)$ (this is the No. 28 in the list). Using proposition 5 we find $J_{W^T} = \mathrm{SL}(W^T)$ and the claim is proved.

If we permute the weights we have a solution only for $(5, 1, 3, 1)$, in this case $(a_1, a_2, a_3) = (7, 3, 10)$ and writing back in the coordinates in the space with weights $(5, 3, 1, 1)$ we get $x^2 = z^7y + y^3w + w^{10}$ which has $(r, a) = (3, 1)$. The surface defined by W^T is $x^2 = z^7 + zy^3 + yw^{10}$ in the weighted projective space with weights $(7, 4, 2, 1)$. This has $3A_1$ and $1A_3$ singularities so $(r, a) = (7, 3)$ (this is No. 5 in the list). This is not the mirror of W in the sense of Dolgachev and Voisin. In fact the order of J_{W^T} is 14 but the order of $\mathrm{SL}(W^T)$ is 42. We have to study the minimal resolution of the quotient of W^T by J_W^T/J_{W^T}. We will discuss this special case in detail in the next section.

5.1.2. *The four special cases.* We study more in detail the cases where the surface W is such that the group $\mathrm{SL}(W)/J_W$ is not trivial. We denote by $\gamma\colon \widetilde{W} \longrightarrow W$ the minimal resolution. Let $\widetilde{G} := \mathrm{SL}(W)/J_W$, which in all these cases is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Denote by γ_1 the blow-up of $\widetilde{W}$ at the fixed points of $\widetilde{G}$, which are isolated since $\widetilde{G}$ acts symplectically on W. We have a commutative diagram:

$$
\begin{array}{ccccccc}
\widehat{W} & \xrightarrow{\gamma_1} & \widetilde{W} & \xrightarrow{\gamma} & W & \xrightarrow{2:1} & \mathbb{P}(w_2, w_3, w_4) \\
\downarrow & & \downarrow & & \downarrow & & \\
W_1 := \widehat{W}/\widetilde{G} & \longrightarrow & \widetilde{W}/\widetilde{G} & \longrightarrow & W/\widetilde{G} & &
\end{array}
\tag{3}
$$

Here we denote by $\widetilde{G}$ the group acting on W, its pull-back to $\widetilde{W}$ and its pull-back to $\widehat{W}$. Moreover we denote by C the branching curve of the double cover $W \longrightarrow \mathbb{P}(w_2, w_3, w_4)$ and by $C_1 \subset W_1$ its pull-back to W_1.

No. 3b. The surface W has equation $x^2 = z^7y + y^3w + w^{10}$ and as previously seen $\widetilde{W}$ has $(r, a) = (3, 1)$. We study now the surface $\widehat{W}$. By corollary 3 a generator for $\mathrm{SL}(W_{chain})_+$ is $(14/15, 7/15, 3/5)$ hence we take $\tilde{g} := (1, 14/15, 7/15, 3/5)$ as generator of $\widetilde{G}_+ \cong \mathbb{Z}/3\mathbb{Z}$ with action on the coordinates in the order x, z, y, w. We must study the fixed locus of the group $\widetilde{G}$ on W to describe the singularities and so to get the values of the invariants (r, a) for the K3 surface W_1 (clearly since we always consider diagonal automorphisms, the surface W_1 inherits the non-symplectic involution ι). A local analysis in the charts shows: the point $(0 : 1 : 0 : 0) \in W$ is an A_2 singularity which is fixed by $\tilde{g}$ hence induces on the quotient $W/\widetilde{G}$ an A_8 singularity. The point $(0 : 0 : 1 : 0)$ is an A_2 singularity, and finally $(1 : 0 : 1 : 0)$ and $(-1 : 0 : 1 : 0)$ are $2A_2$ singularities which are interchanged by ι. On W_1 we find in total 13 invariant curves for the action of ι, hence $r = 13$. Observe that the automorphism $\tilde{g}$ preserves the curve C and it fixes two points on it (corresponding to the A_8 singularity and the first of the A_2 singularities). Using the Hurwitz formula we find that the genus of the curve C_1 is $g = 3$. In conclusion we get $(r, a) = (13, 3)$ so the surface W_1 is the mirror surface of the surface with equation $x^2 = w^{10}y + y^3z + z^7$ which is the No. 5. in the list.

No. 5. The surface W has equation $x^2 = w^{10}y + y^3z + z^7$ in $\mathbb{P}(7, 4, 2, 1)$. We have $d = 14$ and W has $1A_3$ and $3A_1$ singularities, which are all fixed by ι and so $\widetilde{W}$ has $(r, a) = (7, 3)$. We study the surface $\widehat{W}$. We have $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/3\mathbb{Z}$, so we take a generator of this cyclic group and we study the fixed locus. A generator for $\mathrm{Aut}(W_{chain})_+$ is $(-1/210, 1/21, -1/7)$ and by corollary 3 a generator for $\mathrm{SL}(W_{chain})_+$ is then $(-1/21, 10/21, -10/7) = (20/21, 10/21, 4/7)$. Consider now the element $\tilde{g} := (1, 20/21, 10/21, 4/7) \in \mathrm{SL}(W)_+$. If we consider its cube we get an element of $(J_W)_+$ and so we get a generator of $\mathrm{SL}(W)_+/(J_W)_+$. A local analysis in the charts shows: the point $(0 : 1 : 0 : 0) \in W$ is an A_3 singularity which in the quotient $W/\widetilde{G}$ is an A_{11} singularity. The point $(1 : 0 : 1 : 0) \in W$ is fixed by $\tilde{g}$ hence we have an A_2 singularity on $W/\widetilde{G}$. Finally the point $(0 : 0 : 0 : 1)$ is an A_2 singularity of the quotient. The 3 curves on $\widetilde{W}$ in the graphs of the A_1 singularities are permuted by $\tilde{g}$ and so they give only a contribute of 1 for r of W_1. We find in conclusion the invariants $(r, a) = (17, 1)$, so the surface W_1 is the mirror surface of the surface with equation $x^2 = z^7y + y^3w + w^{10}$ which is the No. 3b in the list.

No. 4. The surface W has equation $x^2 = y^4z + z^6w + w^7$ in $\mathbb{P}(7,3,2,2)$. We have $d = 14$ and W has $7A_1$ singularities and $1A_2$ singularity, so for $\widetilde{W}$ we have $(r,a) = (10,6)$. A generator of $\mathrm{SL}(W_{chain})_+$ is $(20/21, 4/21, 6/7)$ giving the generator $\tilde{g} = (1, 20/21, 4/21, 6/7)$ for $\mathrm{SL}(W)_+/(J_W)_+$. A local analysis in the charts shows: the point $(0:1:0:0) \in W$ is an A_2 singularity which is fixed by $\tilde{g}$ and it is an A_8 singularity in the quotient. The point $(0:0:1:0) \in W$ which is an A_1 singularity is fixed by $\tilde{g}$ and is an A_5 singularity in the quotient. Finally the point $(1:0:0:1)$ gives an A_2 singularity. The remaining $6A_1$ singularities on W are permuted in two length 3 orbits on W and so we have $2A_1$ singularities. We find $(r,a) = (18,2)$ so the surface W_1 is the mirror surface of the surface with equation $x^2 = w^7z + z^6y + y^4$ which is the No. 30 in the list.

No. 30. The surface W has equation $x^2 = w^7z + z^6y + y^4$ in $\mathbb{P}(4,2,1,1)$. We have $d = 8$ and W has $2A_1$ singularities which are interchanged by ι and so for $\widetilde{W}$ we have $(r,a) = (2,2)$. A generator for $\widetilde{G}_+$ is $\tilde{g} = (1, 11/12, 7/12, 1/2)$, which acts on the coordinates x, w, z, y in this order. We study its fixed points on W. A local analysis in the charts shows: the points $(1:1:0:0)$ and $(-1:1:0:0)$ are two A_1 singularities on W (interchanged by ι) which are fixed by $\tilde{g}$ so in the quotient we get $2A_5$ singularities. The points $(0:0:1:0)$ and $(0:0:0:1)$ are two A_2 singularities in the quotient. We find $(r,a) = (10,6)$ so the surface W_1 is the mirror surface of the surface with equation $x^2 = y^4z + z^6w + w^7$ which is the No. 4 in the list.

5.2. **The loop case.** Consider a potential of the form:

$$W_{loop} := x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1.$$

As in [12, Section 3] a generator of $\mathrm{Aut}(W_{loop})_+$ is given by $(\psi_1, \ldots, \psi_n)$ with

$$\psi_1 = \frac{(-1)^n}{\Gamma}, \quad \psi_j = \frac{(-1)^{n-j+1}a_1 \cdots a_{j-1}}{\Gamma}, \ j \geq 2$$

where $\Gamma := a_1a_2\cdots a_n - (-1)^n$, in the particular case of $n = 3$ we get the generator $\frac{1}{\Gamma}(-1, a_1, -a_1a_2)$, with $\Gamma = a_1a_2a_3 + 1$.

Proposition 6. *The order of* $\mathrm{SL}(W_{loop})$ *is*

$$|\,\mathrm{SL}(W_{loop})| = \gcd\left(\left((-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}\right),\ \Gamma\right)$$

hence $\mathrm{SL}(W_{loop}) = J_{W_{loop}}$ *if and only if the total degree of* W_{loop} *is equal to* $\gcd\left(\left((-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}\right),\ \Gamma\right)$.

Proof. The proof is similar as the proof of proposition 4. ■

Corollary 4. *A generator of* $\mathrm{SL}(W_{loop})$ *is* $(\beta\psi_1, \ldots, \beta\psi_n)$ *with* $\beta = \Gamma/|\,\mathrm{SL}(W_{loop})|$.

We generalize now proposition 6 in the following form:

Proposition 7. *Let* W *denote a potential of the form:*

$$x^2 - (x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1).$$

We have $\mathrm{Aut}(W) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\Gamma\mathbb{Z}$.

(1) *If* 2 *does not divide* Γ, *i.e.* $\mathrm{Aut}(W)$ *is cyclic of order* 2Γ, *then:*

$$|\,\mathrm{SL}(W)| = \gcd\left(\Gamma + 2\left((-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}\right),\ 2\Gamma\right).$$

(2) *Otherwise let* 2^k, $k \geq 1$, *be the greatest power of* 2 *dividing* Γ *then* $\mathrm{Aut}(W)$ *is not cyclic and*

(2a) $|\,\mathrm{SL}(W)| = \gcd\left(\left((-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}\right),\ \Gamma\right)$ *if* 2^k *divides* $(-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}$.

(2b) $|\,\mathrm{SL}(W)| = 2\gcd\left(\left((-1)^n + \sum_{j=2}^{n}(-1)^{n-j+1}a_1\cdots a_{j-1}\right),\ \Gamma\right)$ *otherwise.*

Proof. The proof is similar as the proof of proposition 5. ■

Remark 5.3. It is easy to see that if $n = 3$ the condition **(2a)** of the proposition 7 is not possible.

5.2.1. *BHCR-mirror construction for K3 surfaces of loop type.* We consider a potential W in the form

$$x^2 - (y^{a_1}z + z^{a_2}w + w^{a_3}y).$$

We assume that it defines a K3 surface (i.e. the sum of the charges is one) in a weighted projective space $\mathbb{P}(w_1, w_2, w_3, w_4)$ and denote by d its total degree. The possible weights for weighted projective spaces in which we can define K3 surfaces in the form W are listed by [3, 17, 20], and the exponents a_i must satisfy:

$$2w_1 = d,\ a_1 = \frac{d - w_3}{w_2},\ a_2 = \frac{d - w_4}{w_3},\ a_3 = \frac{d - w_1}{w_4}.$$

We collect the possible equations in table 2, where we use the same notation as in table 1. Except in the first case the table proves theorem 1.1 in the loop case. The proof of theorem in the special case is given in §5.2.2.

No.	(w_1, w_2, w_3, w_4)	$f(y, z, w)$	(r, a)	$\lvert\mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$
$*(1)1$	$(3, 1, 1, 1)$	$y^5z + z^5w + w^5y$	$(1, 1)$	6	42
$(23)3$	$(5, 3, 1, 1)$	$y^3z + z^9w + w^7y$	$(3, 1)$	10	10
$(13)11$	$(11, 7, 3, 1)$	$y^3w + w^{19}z + z^5y$	$(9, 1)$	22	22
$(11)13$	$(13, 7, 5, 1)$	$y^3z + z^5w + w^{19}y$	$(11, 1)$	26	26
$(3)23$	$(19, 11, 5, 3)$	$y^3z + z^7w + w^9y$	$(17, 1)$	38	38

TABLE 2. The loop mirror cases

5.2.2. *The special case.* **No. 1**. In this case $W = W^T$ and $\widetilde{G} = \mathrm{SL}(W)/J_W = \mathbb{Z}/7\mathbb{Z}$ which as shown in proposition 2 acts symplectically on W. We use the same notation as in diagram (3). The equation of the surface W is $x^2 = y^5z + z^5w + w^5y$ which is a smooth surface of degree 6 in $\mathbb{P}(3, 1, 1, 1)$, the genus of C is 10 so $(r, a) = (1, 1)$. By the corollary 4 a generator for $\mathrm{SL}(W_{loop})_+$ in this case is $(20/21, 5/21, 17/21)$ and a generator for $\mathrm{SL}(W)_+$ is $\tilde{g} := (1, 20/21, 5/21, 17/21)$. By a local analysis in the charts we find that the points $(0:1:0:0)$, $(0:0:1:0)$

and $(0:0:0:1)$ are fixed by $\tilde{g}$ and are contained on C. So on $W/\widetilde{G}$ we find $3A_6$ singularities and a fixed curve of genus 1, so the invariants for W_1 are $(r,a)=(19,1)$ and W_1 is the mirror of W.

5.3. **The fermat case.** We consider a potential

$$W_{f_n} := x_1^{a_1} + \ldots + x_n^{a_n}.$$

Recall that $\mathrm{Aut}(W_{f_n}) \cong \mathbb{Z}/a_1\mathbb{Z} \times \ldots \times \mathbb{Z}/a_n\mathbb{Z}$. We study now $\mathrm{SL}(W_{f_n})$. If $n=1$ clearly there are no diagonal symmetries of determinant one except the trivial one. If $n=2$ it is easy to see that the order is $\gcd(a_1,a_2)$, in general we have:

Proposition 8. *The order of* $\mathrm{SL}(W_{f_n})$ *is*

$$\begin{aligned} |\mathrm{SL}(W_{f_n})| &= \gcd(a_1\cdots a_{n-1},\ldots,a_1\cdots\widehat{a_i}\cdots a_n,\ldots,a_2\cdots a_n) \\ &= \prod_{k=2}^{n}\left(\prod_{\substack{I\subset\{1,\ldots,n\}\\ |I|=k}} \gcd\left(\{a_i\}_{i\in I}\right)\right)^{(-1)^k} \end{aligned}$$

Proof. An element of $\mathrm{Aut}(W_{f_n})$ has the form:

$$(\exp(2\pi i\alpha_1/a_1),\ldots,\exp(2\pi i\alpha_n/a_n))$$

where $\alpha_i \in \mathbb{Z}/a_i\mathbb{Z}$. It is in $\mathrm{SL}(W_{f_n})$ if the exponents satisfy the condition

$$\alpha_1(a_2\cdots a_n)+\ldots+\alpha_i(a_1\cdots\widehat{a_i}\cdots a_n)+\ldots+\alpha_n(a_1\cdots a_{n-1}) \equiv 0 \mod a_1\cdots a_n.$$

Let $\theta := \gcd(a_1\cdots a_{n-1},\ldots,a_1\cdots\widehat{a_i}\cdots a_n,\ldots,a_2\cdots a_n)$, then the equation is equivalent to:

$$\frac{\alpha_1(a_2\cdots a_n)+\ldots+\alpha_i(a_1\cdots\widehat{a_i}\cdots a_n)+\ldots+\alpha_n(a_1\cdots a_{n-1})}{\theta} \equiv 0 \mod \frac{a_1\cdots a_n}{\theta},$$

which has exactly θ solutions. The last equality follows from an easy combinatorial exercise. ■

Remark 5.4. For $n=3$ one has

$$|\mathrm{SL}(W_{f_3})| = \gcd(a_1a_2,a_1a_3,a_2a_3) = \frac{\gcd(a_1,a_2)\cdot\gcd(a_1,a_3)\cdot\gcd(a_2,a_3)}{\gcd(a_1,a_2,a_3)}.$$

Corollary 5. *Let* W *denote a potential of the form:*

$$x^2-(x_1^{a_1}+\ldots+x_n^{a_n}).$$

(1) *If* 2 *divides* $a_1\cdots a_n$ *then* $|\mathrm{SL}(W)| = 2|\mathrm{SL}(W_{f_n})|$,
(2) *otherwise* $|\mathrm{SL}(W)| = |\mathrm{SL}(W_{f_n})|$

Proof. It follows easily from proposition 8. ■

Remark 5.5. If the potential $x^2 = x^{a_1}+\ldots+x^{a_n}$ defines a hypersurface in a normalized weighted projective space, then the case (**1**) of corollary 5 is not possible.

5.3.1. *BHCR-mirror construction for K3 surfaces of fermat type.* Consider a potential W in the form

$$x^2 - (y^{a_1} + z^{a_2} + w^{a_3})$$

that defines a K3 surface. The possible weights are described by [3, 17, 20]. Clearly we have that $W = W^T$. To determine the possible exponents a_1, a_2, a_3 the relation with the weights is the following

$$2w_1 = d, \quad a_1 w_1 = d, \quad a_2 w_2 = d, \quad a_3 w_3 = d$$

and the possible equations are listed in the table 3. Using the results of corollary 5 we compute the order of $\mathrm{SL}(W)$. There are only two cases where the BHCR-mirror of W is W itself. This proves theorem 1.1 except for exactly 8 cases, where the group $\mathrm{SL}(W)/J_W$ is not trivial. We discuss some of them in §5.3.2. In the table 3 in the last column we describe explicitly $\mathrm{SL}(W)/J_W$ since it is not straightforward and we use Nikulin's classification [15] of finite abelian groups acting symplectically on a K3 surface.

No.	(w_1, w_2, w_3, w_4)	$f(y,z,w)$	(r,a)	$\lvert\mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$	$\mathrm{SL}(W)/J_W$
$*1$	$(3,1,1,1)$	$y^6+z^6+w^6$	$(1,1)$	72	6	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$
$*2$	$(5,2,2,1)$	$y^5+z^5+w^{10}$	$(6,4)$	50	10	$\mathbb{Z}/5\mathbb{Z}$
$*8$	$(9,6,2,1)$	$y^3+z^9+w^{18}$	$(6,2)$	54	18	$\mathbb{Z}/3\mathbb{Z}$
18	$(15,10,3,2)$	$y^3+z^{10}+w^{15}$	$(10,4)$	30	30	1
26	$(21,14,6,1)$	$y^3+z^7+w^{42}$	$(10,0)$	42	42	1
$*30$	$(4,2,1,1)$	$y^4+z^8+w^8$	$(2,2)$	64	8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$
$*34$	$(10,5,4,1)$	$y^4+z^5+w^{20}$	$(6,4)$	40	20	$\mathbb{Z}/2\mathbb{Z}$
$*41$	$(6,3,2,1)$	$y^4+z^6+w^{12}$	$(6,2)$	48	12	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
$*42$	$(6,4,1,1)$	$y^3+z^{12}+w^{12}$	$(2,0)$	72	12	$\mathbb{Z}/6\mathbb{Z}$
$*45$	$(12,8,3,1)$	$y^3+z^8+w^{24}$	$(6,2)$	48	24	$\mathbb{Z}/2\mathbb{Z}$

TABLE 3. The fermat mirror cases

5.3.2. *The special cases.* The study is essentially the same as in the chain case or in the loop case and we use again diagram (3). We discuss however some cases in detail, since here the group acting on the surface W^T is not always cyclic.

No. 1: The equation $x^2 = y^6 + z^6 + w^6$ defines a smooth K3 surface W of degree $d = 6$ in $\mathbb{P}(3,1,1,1)$ and $(r,a) = (1,1)$. The group $\widetilde{G} = \mathrm{SL}(W)/J_W$ is of order 12. By Nikulin's classification [15] of finite abelian groups acting symplectically on a K3 surface, $\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The group $(J_W)_+$ is generated by the element $(1/2, 1/6, 1/6, 1/6)$ and observe that the elements $(1/2, 1/2, 1, 1)$ and $(1, 1/6, 5/6, 1)$ generate $\widetilde{G}_+$. Denote by $(1,0)$ the generator of order 2 and by $(0,1)$ the generator of order 6. Again by [15] we know that we have the following number of fixed points:

$$\begin{array}{ccccccccc}
H^2_{(0,3)}(6) & — & H^6_{(0,1)}(2) & — & H^3_{(0,2)}(0) & — & H^6_{(1,2)}(2) & — & H^2_{(1,0)}(6) \\
 & & & & | & & & & \\
 & & & & H^6_{(1,1)}(2) & — & H^2_{(1,3)}(6) & &
\end{array}$$

where we follows the notation of [15] and we denote by $H_x^m(t)$ the cyclic group of order m with generator x, and the t in parentheses denotes the number of fixed points having H_x^m as stabilizer.

Looking at the diagram and by a local analysis one sees that it is enough to study the fixed points of the elements $(1,0)$, $(0,3)$ and $(1,3)$. The fixed points of $(1,0)$ are $(1:1:0:0)$, $(-1:1:0:0)$ and $(0:0:1:\xi^j)$, with $\xi=\exp(2\pi i/12)$ and $j=1,3,5,7,9,11$. The first two points are interchanged by ι and have in fact stabilizer of order 6. The computation for the elements $(0,3)$ and $(1,3)$ is similar. We find that the quotient $W/\widetilde{G}$ has in total $3A_5$ and $3A_1$ singularities which gives $r=19$. Finally by an easy computation one sees that the curve C contains 18 points with stabilizer group of order 2 hence by Hurwitz formula the curve C_1 has genus 1 so the invariants for W_1 are $(r,a)=(19,1)$. This shows that the surface W_1 is the mirror of the surface W.

No. 41. The K3 surface W has equation: $x^2=y^4+z^6+w^{12}$ of total degree 12 in $\mathbb{P}(6,3,2,1)$. The surface W has two A_1 singularities at the points $(1:0:1:0)$ and $(-1:0:1:0)$ which are interchanged by ι and two A_2 singularities at the points $(1:1:0:0)$ and $(-1:1:0:0)$ which are exchanged by ι too. We find $r=4$ and the genus of the fixed curve C is 7 hence $(r,a)=(4,4)$ (observe that this is different from the value given by Borcea in [3, Table 3]).
The group $\widetilde{G}$ has order 4 and by Nikulin's classification [15] it is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Here $(J_W)_+$ is generated by $(1/2,1/4,1/6,1/12)$. The two elements $(1,0):=(1,1/2,1/2,1)$ and $(0,1):=(1/2,1,1,1/2)$ are different and of order 2 in $\mathrm{SL}(W)_+/(J_W)_+$ hence $\mathrm{SL}(W)/J_W\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.
Fixed points of $(1,0)$: after blowing up the singularities of W, the element fixes 6 points on the two A_2-graphs of curves and the points $(1:0:0:\xi)$, with $\xi:=\exp(2\pi i/12)$ and $(1:0:0:1)$ which are interchanged by ι.
Fixed points of $(0,1)$: there are eight fixed points $(0:1:\xi:0)$, $(0:1:\xi^3:0)$, $(0:0:\xi^j:1)$, $j=1,3,5,7,9,11$.
Fixed points of $(1,1)$: after blowing up the singularities of W, the element fixes four points on the two A_1-graphs of curves and the 4 points $(0:\zeta^j:0:1)$, with $\zeta=\exp(2\pi i/8)$, $j=1,3,5,7$.
By a careful look at the action of $\widetilde{G}$ on $\widetilde{W}$ and $\widehat{W}$ we find that W_1 has the following configuration of curves: $(1A_1+1A_5)+4A_1+(1A_3+2A_1)$ so $r=16$. The curve C contains 12 points with stabilizer of order 2 hence the genus of C_1 is 1 and so the invariants of W_1 are $(r,a)=(16,4)$. This shows that W_1 is the mirror of W.

5.3.3. *The subgroups.* In the table 3 there are 4 cases where one can consider non trivial proper subgroups $\widetilde{G}=G/J_W$ of $\mathrm{SL}(W)/J_W$. Noting that $W=W^T$ and applying proposition 3 we have that $\widetilde{G^T}=(\mathrm{SL}(W)/J_W)/\widetilde{G}$. By making similar computations of the Nikulin invariants (r,a) for $W/\widetilde{G}$ and $W/\widetilde{G^T}$ one obtains that the corresponding minimal resolutions are mirror K3 surfaces, proving theorem 1.1 also in these cases. In table 4 we collect the computation for all possible subgroups in the case No. 1. The others cases are similar.

5.4. **BHCR-mirror construction for K3 surfaces of chain+fermat type or loop+fermat type.** Combining atomic types we obtain two possible potentials:

$$W_{cf}=y^{a_1}z+z^{a_2}+w^{a_3},\quad W_{lf}=y^{a_1}z+z^{a_2}y+w^{a_3}.$$

$(\widetilde{G})_+$	generators	(r,a)	$(\widetilde{G^T})_+$	generators	(r,a)
$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1)$	$(8,6)$	$\mathbb{Z}/6\mathbb{Z}$	$(1,1/6,5/6,1)$	$(12,6)$
$\mathbb{Z}/2\mathbb{Z}$	$(1,1/2,1/2,1)$	$(8,6)$	$\mathbb{Z}/6\mathbb{Z}$	$(1/2,1/3,1/6,1)$	$(12,6)$
$\mathbb{Z}/2\mathbb{Z}$	$(1/2,1,1/2,1)$	$(8,6)$	$\mathbb{Z}/6\mathbb{Z}$	$(1/2,5/6,2/3,1)$	$(12,6)$
$\mathbb{Z}/3\mathbb{Z}$	$(1,1/3,2/3,1)$	$(7,7)$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$(1/2,1/2,1,1)$, $(1,1/2,1/2,1)$	$(13,7)$

TABLE 4. The subgroups in the fermat case No. 1

Then the equation of the K3 surface W is $x^2 = W_{cf}$ or $x^2 = W_{lf}$. Observe that in the loop+fermat case we have $W = W^T$.

Proposition 9. *Let W be a potential of the form $x^2 - W_{cf}$. The automorphism group of diagonal symmetries is $\mathrm{Aut}(W) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/(a_1a_2)\mathbb{Z}\times\mathbb{Z}/a_3\mathbb{Z}$. One has*

$$|\operatorname{SL}(W_{cf})| = \frac{a_1a_2a_3}{\operatorname{lcm}\left(\frac{a_1a_2}{\gcd(a_1a_2,1-a_1)}, a_3\right)}.$$

(1) *If 2 divides $\frac{a_1a_2}{\gcd(a_1a_2,1-a_1)}$ or 2 divides a_3 then $|\operatorname{SL}(W)| = 2|\operatorname{SL}(W_{cf})|$,*
(2) *otherwise $|\operatorname{SL}(W)| = |\operatorname{SL}(W_{cf})|$.*

Proof. We have $\mathrm{Aut}(W_{cf})_+ \cong \mathbb{Z}/(a_1a_2)\mathbb{Z}\times\mathbb{Z}/a_3\mathbb{Z}$ and the elements of the groups can be written as $(\alpha/a_1a_2, \alpha(-a_1)/a_1a_2, \beta/a_3)$, with $(\alpha,\beta)\in\mathbb{Z}/(a_1a_2)\mathbb{Z}\times\mathbb{Z}/a_3\mathbb{Z}$. It corresponds to an element of determinant one if:

$$\frac{\alpha(1-a_1)}{a_1a_2} + \frac{\beta}{a_3} \in \mathbb{Z}.$$

Let $\lambda := \gcd(a_1a_2,(1-a_1))$. This is equivalent to the equation

$$\alpha\frac{(1-a_1)a_3}{\lambda} + \beta\frac{a_1a_2}{\lambda} = 0 \quad \bmod \left(\frac{a_1a_2a_3}{\lambda}\right)$$

It is now an easy exercise to count the solutions. Similarly, the elements of $\mathrm{Aut}(W)$ can be written as $(\alpha/2, \beta a_1a_2, \beta(-a_1)/a_1a_2, \gamma/a_3)$ with $(\alpha,\beta,\gamma)\in\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/(a_1a_2)\mathbb{Z}\times\mathbb{Z}/a_3\mathbb{Z}$ and are of determinant one if:

$$\frac{\alpha}{2} + \beta\frac{1-a_1}{a_1a_2} + \frac{\gamma}{a_3} \in \mathbb{Z}.$$

It is easy to see that there are $\frac{2a_1a_2a_3}{\operatorname{lcm}\left(2,\frac{a_1a_2}{\gcd(a_1a_2,1-a_1)},a_3\right)}$ solutions. This gives the result. ∎

Proposition 10. *Let W be a potential of the form $x^2 - W_{lf}$. Then automorphism group of diagonal symmetries is $\mathrm{Aut}(W) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/(a_1a_2-1)\mathbb{Z}\times\mathbb{Z}/a_3\mathbb{Z}$. One has*

$$|\operatorname{SL}(W_{lf})| = \frac{(a_1a_2-1)a_3}{\operatorname{lcm}\left(\frac{a_1a_2-1}{\gcd(a_1a_2-1,1-a_1)}, a_3\right)}$$

(1) *If 2 divides $\frac{a_1a_2-1}{\gcd(a_1a_2-1,1-a_1)}$ or 2 divides a_3 then $|\operatorname{SL}(W)| = 2|\operatorname{SL}(W_{lf})|$,*
(2) *otherwise $|\operatorname{SL}(W)| = |\operatorname{SL}(W_{lf})|$.*

Proof. Same argument as above. ■

The proof of theorem 1.1 in all these cases is similar as before. We collect the results in the tables 5 and 6. In the table 5 the group $\mathrm{SL}(W)/J_W$ is always cyclic of order two or is trivial. In the table 6 in most cases one can deduce the structure of $\mathrm{SL}(W)/J_W$ from its order and using Nikulin's list of finite abelian groups acting symplectically on a K3 surface. Only two cases are not straightforward: the first and the third. One has to look closely at the elements of $\mathrm{SL}(W)$ and J_W to deduce the structure. In the first case one can easily see that $\mathrm{SL}(W)/J_W$ contains an element of order eight and in the second case it contains an element of order four. Hence also in these cases theorem 1.1 holds.

5.4.1. *The subgroups.* In the table 6 there are 3 cases where one can consider non trivial proper subgroups $\widetilde{G} = G/J_W$ of $\mathrm{SL}(W)/J_W$. We prove theorem 1.1 in these cases as in §5.3.3.

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$No.$	(w_1,w_2,w_3,w_4)	$f(y,z,w)$	(r,a)	$\lvert\mathrm{SL}(W)\rvert$	$\lvert J_W\rvert$
$*(15)1$	$(3,1,1,1)$	$y^5z+z^6+w^6$	$(1,1)$	12	6
$*(33b)2a$	$(5,2,2,1)$	$y^4z+z^5+w^{10}$	$(6,4)$	20	10
$(39)2b$	$(5,2,2,1)$	$w^8y+y^5+z^5$	$(6,4)$	10	10
$*(18)3$	$(5,3,1,1)$	$y^3z+z^{10}+w^{10}$	$(3,1)$	20	10
$(35a)4$	$(7,3,2,2)$	$y^4z+z^7+w^7$	$(10,6)$	14	14
$(24)5$	$(7,4,2,1)$	$y^3z+z^7+w^{14}$	$(7,3)$	14	14
$*(41a)6$	$(9,4,3,2)$	$y^4w+w^9+z^6$	$(10,6)$	36	18
$*(8a)7$	$(9,5,3,1)$	$y^3z+z^6+w^{18}$	$(7,3)$	36	18
$*(7)8a$	$(9,6,2,1)$	$z^6y+y^3+w^{18}$	$(6,2)$	36	18
$(46)8b$	$(9,6,2,1)$	$w^{12}y+y^3+z^9$	$(6,2)$	18	18
$(48)8c$	$(9,6,2,1)$	$w^{16}z+z^9+y^3$	$(6,2)$	18	18
$*(1)15$	$(15,6,5,4)$	$w^6y+y^5+z^6$	$(12,6)$	60	30
$(34a)16$	$(15,7,6,2)$	$y^4w+w^{15}+z^5$	$(14,4)$	30	30
$(19)17$	$(15,8,6,1)$	$y^3z+z^5+w^{30}$	$(11,1)$	30	30
$*(3)18$	$(15,10,3,2)$	$w^{10}y+y^3+z^{10}$	$(10,4)$	60	30
$(17)19$	$(15,10,4,1)$	$z^5y+y^3+w^{30}$	$(9,1)$	30	30
$(5)24$	$(21,14,4,3)$	$z^7y+y^3+w^{14}$	$(13,3)$	42	42
$(45b)25$	$(21,14,5,2)$	$z^8w+y^3+w^{21}$	$(14,2)$	42	42
$(36)26a$	$(21,14,6,1)$	$w^{28}y+y^3+z^7$	$(10,0)$	42	42
$(47)26b$	$(21,14,6,1)$	$w^{36}z+y^3+z^7$	$(10,0)$	42	42
$(42b)29$	$(33,22,6,5)$	$w^{12}z+z^{11}+y^3$	$(18,0)$	66	66
$*(35b)30a$	$(4,2,1,1)$	$z^7w+w^8+y^4$	$(2,2)$	16	8
$*(43)30b$	$(4,2,1,1)$	$z^6y+y^4+w^8$	$(2,2)$	16	8
$*(31a)31a$	$(8,4,3,1)$	$z^4y+y^4+w^{16}$	$(6,4)$	32	16
$*(34b)31b$	$(8,4,3,1)$	$z^5w+w^{16}+y^4$	$(6,4)$	32	16
$*(45a)32$	$(8,5,2,1)$	$y^3w+w^{16}+z^8$	$(6,2)$	32	16
$*(41b)33a$	$(10,5,3,2)$	$z^6w+w^{10}+y^4$	$(8,6)$	40	20
$*(2a)33b$	$(10,5,3,2)$	$z^5y+y^4+w^{10}$	$(8,6)$	40	20
$(16)34a$	$(10,5,4,1)$	$w^{15}y+y^4+z^5$	$(6,4)$	20	20
$*(31b)34b$	$(10,5,4,1)$	$w^{16}z+z^5+y^4$	$(6,4)$	40	20
$(4)35a$	$(14,7,4,3)$	$w^7y+y^4+z^7$	$(10,6)$	28	28
$*(30a)35b$	$(14,7,4,3)$	$w^8z+z^7+y^4$	$(10,6)$	56	28
$(26a)36$	$(14,9,4,1)$	$y^3w+w^{28}+z^7$	$(10,0)$	28	28
$(2b)39$	$(20,8,7,5)$	$z^5w+y^5+w^8$	$(14,4)$	40	40
$*(6)41a$	$(6,3,2,1)$	$w^9y+y^4+z^6$	$(6,2)$	24	12
$*(33a)41b$	$(6,3,2,1)$	$w^{10}z+z^6+y^4$	$(6,2)$	24	12
$*(44)42a$	$(6,4,1,1)$	$z^8y+y^3+w^{12}$	$(2,0)$	24	12
$(29)42b$	$(6,4,1,1)$	$z^{11}w+w^{12}+y^3$	$(2,0)$	12	12
$*(30b)43$	$(12,5,4,3)$	$z^4y+y^6+w^8$	$(14,2)$	48	24
$*(42a)44$	$(12,7,3,2)$	$y^3z+z^8+w^{12}$	$(10,4)$	48	24
$*(32)45a$	$(12,8,3,1)$	$w^{16}y+y^3+z^8$	$(6,2)$	48	24
$(21)45b$	$(12,8,3,1)$	$w^{21}z+z^8+y^3$	$(6,2)$	24	24
$(8b)46$	$(18,11,4,3)$	$y^3w+w^{12}+z^9$	$(14,2)$	36	36
$(26b)47$	$(18,12,5,1)$	$z^7w+w^{36}+y^3$	$(10,0)$	36	36
$(8c)48$	$(24,16,5,3)$	$z^9w+w^{16}+y^3$	$(14,2)$	48	48

TABLE 5. The chain+fermat mirror cases

No.	(w_1, w_2, w_3, w_4)	$f(y, z, w)$	(r, a)	$\lvert \mathrm{SL}(W) \rvert$	$\lvert J_W \rvert$	$\mathrm{SL}(W)/J_W$
$*1$	$(3,1,1,1)$	$y^5z + z^5y + w^6$	$(1,1)$	48	6	$\mathbb{Z}/8\mathbb{Z}$
$*2$	$(5,2,2,1)$	$y^4z + z^4y + w^{10}$	$(6,4)$	30	10	$\mathbb{Z}/3\mathbb{Z}$
$*3$	$(5,3,1,1)$	$y^3z + z^7y + w^{10}$	$(3,1)$	40	10	$\mathbb{Z}/4\mathbb{Z}$
$*5$	$(7,4,2,1)$	$y^3z + z^5y + w^{14}$	$(7,3)$	28	14	$\mathbb{Z}/2\mathbb{Z}$
6	$(9,4,3,2)$	$y^4w + w^7y + z^6$	$(10,6)$	18	18	1
10	$(11,6,4,1)$	$y^3z + z^4y + w^{22}$	$(10,2)$	22	22	1
$*30$	$(4,2,1,1)$	$z^7w + w^7z + y^4$	$(2,2)$	48	8	$\mathbb{Z}/6\mathbb{Z}$
$*31$	$(8,4,3,1)$	$z^5w + w^{13}z + y^4$	$(6,4)$	32	16	$\mathbb{Z}/2\mathbb{Z}$
$*32$	$(8,5,2,1)$	$y^3w + w^{11}y + z^8$	$(6,2)$	32	16	$\mathbb{Z}/2\mathbb{Z}$
36	$(14,9,4,1)$	$y^3w + w^{19}y + z^7$	$(10,0)$	28	28	1
$*42$	$(6,4,1,1)$	$z^{11}w + w^{11}z + y^3$	$(2,0)$	60	12	$\mathbb{Z}/5\mathbb{Z}$
47	$(18,12,5,1)$	$z^7w + w^{31}z + y^3$	$(10,0)$	36	36	1

TABLE 6. The loop+fermat mirror cases