THÈSE

Pour l'obtention du Grade de

Docteur de l'Université de Poitiers

Faculté des Sciences Fondamentales et Appliquées

(Diplôme national - arrêté du 25 avril 2002)

Ecole Doctorale : Sciences Pour l'Ingénieur

Secteur de Recherche : Mathématiques et leurs interactions

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Caractérisation et énumération des idéaux ad-nilpotents d'une sous-algèbre parabolique d'une algèbre de Lie simple.

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Soutenue le 16 Novembre 2007

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Remerciements

C'est pour moi un plaisir de dédier cette page à toutes les personnes qui ont contribué à rendre ces trois années de thèse si enrichissantes, agréables et fructueuses, tant au point de vue Mathématiques qu'au point de vue humain.

Je ne peux que commencer par remercier Rupert, qui en plus d'avoir été un inconditionnel directeur de thèse, est devenu un ami. Je te remercie pour ta disponibilité, tes conseils toujours constructifs, ton soutien quotidien et ton amitié.

Je suis très reconnaissante envers Paolo Papi et Eric Sommers qui m'ont fait l'honneur d'être les rapporteurs de ma thèse. Je remercie également Paolo pour tous les échanges productifs que nous avons eus à Rome, qui m'ont permis d'approfondir ma thèse et d'entrevoir de nouveaux sujets de recherches.

Je voudrais également exprimer ma gratitude envers Thierry Levasseur et Patrice Tauvel qui ont accepté de participer à mon jury. Patrice, sans ta présence, tes discussions Mathématiques mais aussi tes blagues, tes coups de gueules ou tes oeillades à la fenêtre, la cafeteria du laboratoire de Poitiers ne serait pas la même.

J'en profite pour remercier toutes les personnes qui font du laboratoire un endroit où il fait bon travailler, en particulier Abderrazak Bouaziz, Alain Miranville et Pol Vanhaecke pour leurs nombreux conseils, Hervé Sabourin sans qui je n'aurais probablement pas pu assister à ces conférences à l'étranger qui ont tant enrichi ma thèse. Je remercie également tous les doctorants: Abdoulaye, Ariane, Grégory, Guilhem, Idriss, Mohamed et Marie-Eve avec qui j'ai eu le plaisir de partager mon bureau et bon nombre de discussions intéressantes. Mais il y a aussi les non-matheux: Jocelyne Attab, Nathalie Marlet, Valerie Menet, Benoît Metrot et Brigitte Brault qui contribuent tous les jours par leurs disponibilités, leurs compétences et leur bonne humeur à améliorer notre quotidien d'apprenti chercheur.

Je ne peux pas oublier les SVS-istes: Alice, Anthony, Morgan, Hélène, Camille, Paul ... avec qui j'ai partagé tant de bons moments. Un merci tout particulier à Ioan pour nos débats captivants, parfois passionnés et cet inoubliable séjour à Vienne. Et merci à toi, Anne, pour toutes nos discussions Mathématiques mais surtout pour ton soutien sans faille, quelles que soient les circonstances et la distance (Vive Internet!!). Mes années de thèse n'auraient pas été les mêmes sans ton amitié.

Et puis mon séjour en Italie est arrivé : je remercie Velleda Baldoni et le Liegritz pour m'avoir donné la possibilité de passer ces quatre mois à Rome qui ont été si enrichissants pour moi. Allora, grazie mille ad Elisabetta per tutte le notre discussioni e la meravigliosa scoperta del caffè con la crema. Grazie a Veronica e Martina, a chi desidero un bello futuro nella Matematica. E vorrei ringraziare in particolare Andrea per tutte le nostre passeggiate: senza te Roma non sarebbe stata la stessa. Vi dico a tutti, a presto...

Je ne pourrai jamais oublier le soutien des personnes qui me sont chères, et de ma petite mais merveilleuse famille. Je remercie mes amis rotomouleurs pour toutes nos prinades et ce vocabulaire que personne ne comprend. Je remercie infiniment mes parents pour leur intérêt constant et leur soutien sans condition. Je remercie mon frère sans qui, au sortir du lycée, entre les maths et la philo, j'aurais peut-être choisi la philo. Mais que voulez-vous, il m'a parlé des espaces à n dimensions, comment aurais-je pu résister? Et je voudrais réserver ma reconnaissance toute particulière à Mathieu, pour être là, tout simplement.

Contents

Introduction

Considérons $\mathfrak{g}$ une algèbre de Lie simple de dimension finie et de rang l. Choisisons $\mathfrak{h}$ une sous-algèbre de Cartan et Δ le système de racines associé. Fixons un ensemble de racines positives Δ^+ et notons $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ l'ensemble des racines simples correspondant. Si $\alpha \in \Delta$, désignons par $\mathfrak{g}_\alpha$ le sous-espace radiciel de $\mathfrak{g}$ associé à α.

Pour $I \subset \Pi$, posons $\Delta_I = \mathbb{Z}I \cap \Delta$. Fixons la sous-algèbre parabolique standard :

$$\mathfrak{p}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha \right).$$

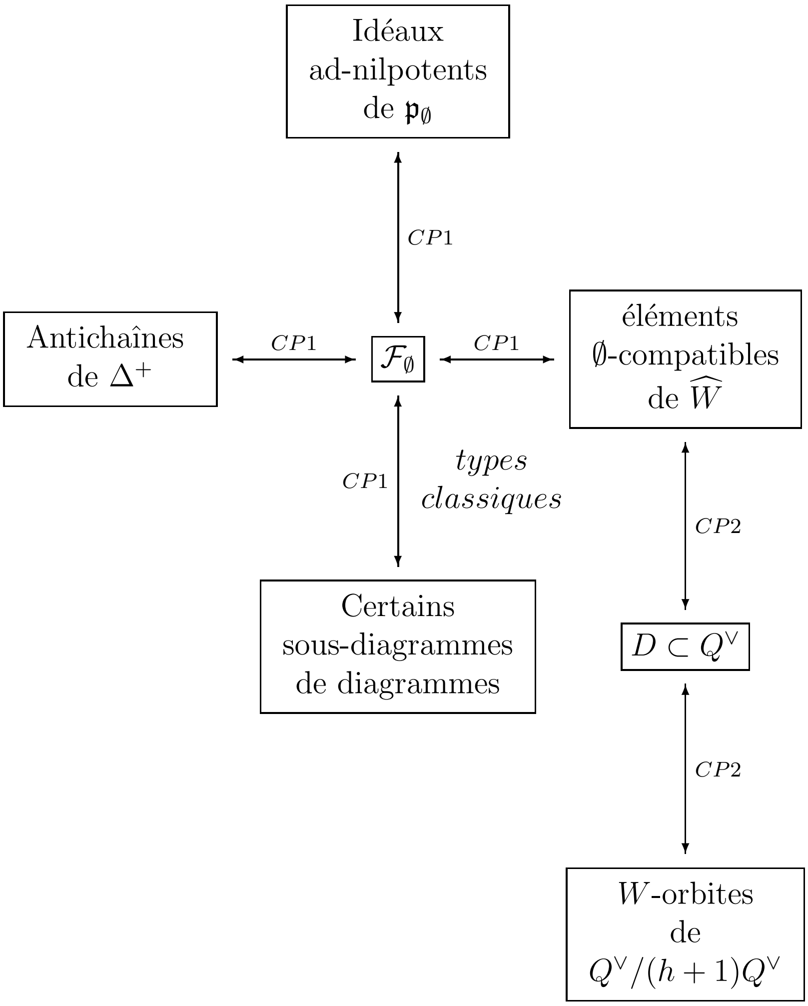
Un idéal $\mathfrak{i}$ de $\mathfrak{p}_I$ est dit ad-nilpotent si pour tout $x \in \mathfrak{i}$, $ad_{\mathfrak{p}_I} x$ est nilpotent. Puisque tout idéal de $\mathfrak{p}_I$ est $\mathfrak{h}$-stable, on en déduit facilement qu'un idéal est ad-nilpotent si et seulement s'il est nilpotent. De plus, on a $\mathfrak{i} = \bigoplus\limits_{\alpha \in \Phi} \mathfrak{g}_\alpha$, pour un certain sous-ensemble Φ de $\Delta^+ \setminus \Delta_I$.

Ces objets ont été largement étudiés lorsque $I = \emptyset$, c'est-à-dire lorsque $\mathfrak{p}_\emptyset$ est une sous-algèbre de Borel de $\mathfrak{g}$. Ces études ont été motivées par un résultat dû à Peterson :

Théorème 1 *Le nombre d'idéaux abéliens de $\mathfrak{p}_\emptyset$ est 2^l.*

Une démonstration de ce résultat se trouve dans [Ko] et utilise le groupe de Weyl affine noté $\widehat{W}$ dans la suite. Ce groupe joue un rôle important dans les différentes caractérisations des idéaux ad-nilpotents établies par Cellini et Papi dans [CP1] et [CP2]. Citons également les travaux de Panyushev, Sommers et Suter ([Pa],[So],[Sut]) entre autres, sur ce sujet.

Résumons ces différentes caractérisations dans le schéma suivant :



Précisons les notions utilisées dans ce schéma.

Soit V l'espace euclidien $\sum_{k=1}^{l} \mathbb{R}\alpha_k$ de produit scalaire $(.,.)$. Notons W le groupe de Weyl associé à Δ et h le nombre de Coxeter associé à W.

On a :

$$\mathcal{F}_\emptyset = \{\Phi \subset \Delta^+ ; \text{si } \alpha \in \Phi, \beta \in \Delta^+, \alpha + \beta \in \Delta^+, \text{alors } \alpha + \beta \in \Phi\}.$$

Rappelons la définition de l'ordre partiel suivant sur Δ^+ : $\alpha \leqslant \beta$ si $\beta - \alpha$ est une somme de racines positives. Une antichaîne est une partie de Δ^+ contenant des racines deux à deux non comparables pour $\leqslant$. Pour toute antichaîne Γ, Sommers a montré dans [So] qu'il existe un élément $w \in W$ tel que $w(\Gamma) \subset \Pi$.

Pour $\alpha \in \Delta$, notons
$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$
la coracine correspondante. Désignons par $Q^\vee$ le réseau des coracines de Δ.

Soit $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$. Etendons (voir chapitre 1) l'application bilinéaire définie plus haut sur V en une forme symétrique bilinéaire non dégénérée sur $\widehat{V}$, également notée $(.,.)$.

Notons $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$ l'ensemble des racines affines (réelles). Fixons le système de racines positives $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^*\delta)$. Soit θ la plus grande racine de Δ^+. Alors, $\widehat{\Pi} = \{\alpha_0 = -\theta + \delta, \alpha_1, \cdots, \alpha_l\}$ est l'ensemble des racines simples (affines) pour $\widehat{\Delta}^+$.

Un élément $w \in \widehat{W}$ est dit $\emptyset$-compatible s'il vérifie les deux conditions suivantes :

(1) $w^{-1}(\alpha) > 0$, pour tout $\alpha \in \Pi$.

(2) Si $w(\alpha) < 0$ pour un $\alpha \in \widehat{\Pi}$, alors $w(\alpha) = \beta - \delta$ pour un certain $\beta \in \Delta^+$.

Finalement, posons
$$D = \{\tau \in Q^\vee; (\tau, \alpha_j) \leqslant 1, j = 1, \ldots, l \text{ et } (\tau, \theta) \geqslant -2\}.$$

Expliquons brièvement les différentes correspondances du schéma.

Soit $\mathfrak{i}$ un idéal ad-nilpotent de $\mathfrak{p}_\emptyset$. Fixons
$$\Phi_\mathfrak{i} = \{\alpha \in \Delta^+ \setminus \Delta_I;\ \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}.$$
On a alors $\Phi_\mathfrak{i} \in \mathcal{F}_\emptyset$. Réciproquement, pour tout élément $\Phi \in \mathcal{F}_\emptyset$, $\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ est un idéal ad-nilpotent de $\mathfrak{p}_\emptyset$.

Soit $\Phi \in \mathcal{F}_\emptyset$. Notons
$$\Phi_{min} = \{\beta \in \Phi; \beta - \alpha \notin \Phi, \text{ pour tout } \alpha \in \Delta^+\}.$$
Alors Φ_{min} est une antichaîne de Δ^+. Réciproquement, si on considère une antichaîne Γ, alors l'ensemble des racines qui sont plus grandes qu'un élément de Γ est un élément de $\mathcal{F}_\emptyset$.

Lorsque $\mathfrak{g}$ est de type classique, on peut disposer l'ensemble des racines positives dans un diagramme de forme convenable (qui dépend du type) par rapport à l'ordre partiel de Δ^+. Les éléments de $\mathcal{F}_\emptyset$ correspondent alors à certains sous-diagrammes. Cela permet d'obtenir une énumération des idéaux ad-nilpotents de $\mathfrak{p}_\emptyset$, type par type.

Soit $w \in \widehat{W}$, $\emptyset$-compatible. Posons

$$N(w) = \{\alpha \in \widehat{\Delta}^+ ; w^{-1}(\alpha) \in \widehat{\Delta}^-\}.$$

Alors, d'après [CP1], l'ensemble

$$\Phi_w = \{\alpha \in \Delta^+ ; -\alpha + \delta \in N(w)\}$$

est un élément de $\mathcal{F}_\emptyset$.

Le groupe de Weyl affine peut s'écrire comme le produit semi-direct de W par l'ensemble des translations t_τ de vecteur $\tau \in Q^\vee$. Ceci permet d'établir une correspondance entre les éléments $\emptyset$-compatibles de $\widehat{W}$ et les éléments de D. Ces derniers sont ensuite mis en bijection avec les orbites de $Q^\vee/(h+1)Q^\vee$, ce qui permet d'obtenir une formule générale pour l'énumération des idéaux ad-nilpotents de $\mathfrak{p}_\emptyset$. Voir [CP2].

L'idée dans le chapitre 1 est de voir comment ces caractérisations peuvent être adaptées dans le cas où $I \neq \emptyset$. Le schéma des différentes caractérisations devient alors :

$$\begin{tikzcd}
\boxed{\text{Idéaux ad-nilpotents de } \mathfrak{p}_I} \arrow[d, leftrightarrow] & \\
\boxed{\mathcal{F}_I} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\textit{types classiques}"'] & \boxed{\text{éléments } I\text{-compatibles de } \widehat{W}} \\
\boxed{\text{Certains sous-diagrammes de diagrammes}} &
\end{tikzcd}$$

Il n'y a plus de correspondance directe entre les idéaux et les antichaînes, ni avec des orbites de $Q^\vee$. On a :

$$\mathcal{F}_I = \{\Phi \subset \Delta^+ \setminus \Delta_I ; \text{si } \alpha \in \Phi, \beta \in \Delta^+ \cup \Delta_I, \alpha+\beta \in \Delta^+, \text{alors } \alpha+\beta \in \Phi\}.$$

Précisons ce qu'est un élément I-compatible.

Un élément $w \in \widehat{W}$ est I-compatible si et seulement s'il est $\emptyset$-compatible et s'il vérifie

$$w^{-1}(I) \subset \widehat{\Pi}.$$

Notons

$$C = \{x \in V ; (\alpha_i, x) > 0 \text{ pour tout } \alpha_i \in \Pi\},\ A = \{x \in C ; (\theta, x) < 1\},$$

respectivement la chambre fondamentale et l'alcôve fondamentale relatives à W et $\widehat{W}$.

Posons $\widetilde{D} = \{(\tau, v) \in D \times W ; vt_\tau(A) \subset C\}$. Nous pouvons réécrire la correspondance entre les éléments $\emptyset$-compatibles et l'ensemble D de la manière suivante : l'application

$$\begin{array}{ccl} \widetilde{D} & \to & \{w \in \widehat{W}, \emptyset\text{-compatible}\} \\ (\tau, v) & \mapsto & vt_\tau \end{array}$$

est bijective.

Désignons par I_w l'unique élément maximal de $\{I \subset \Pi ; w \text{ soit } I\text{-compatible}\}$. Pour $\tau \in Q^\vee$, posons :

$$D_\tau = \begin{cases} \{\alpha \in \Pi ; (\alpha, \tau) = 0\} \cup \{-\theta\} & \text{if } (\theta, \tau) = -1, \\ \{\alpha \in \Pi ; (\alpha, \tau) = 0\} & \text{if } (\theta, \tau) \neq -1. \end{cases}$$

Nous pouvons voir la proposition suivante comme un équivalent de la correspondance entre les éléments $\emptyset$-compatibles et D.

Proposition 2 *Soit $(\tau, v) \in \widetilde{D}$, et $w = vt_\tau \in \widehat{W}$. Alors $v(D_\tau) = I_w$. En particulier, w est I-compatible si et seulement si $I \subset v(D_\tau)$.*

Dans le chapitre 2, nous nous intéressons à l'énumération des idéaux abéliens et ad-nilpotents. Papi et Cellini prouvent dans [CP1] que, à chaque idéal abélien correspond une alcôve contenue dans $2A$. Comme le volume de $2A$ est 2^l fois le volume de A, nous obtenons une démonstration du résultat de Peterson.

Nous voulons procéder de manière analogue pour énumérer les idéaux abéliens de $\mathfrak{p}_I$. Notons :

$$\mathcal{A}b_I = \{w \in \widehat{W}; \mathfrak{i}_{\Phi_w} \text{ est un idéal abélien de } \mathfrak{p}_I\}.$$

Posons $n_0 = 1$ et notons n_i, $i = 1, \ldots, l$, les entiers srtictement positifs tels que $\theta = \sum_{i=1}^{l} n_i \alpha_i$. Pour $J \subset \widehat{\Pi}$, on pose $n_J = \prod_{\alpha_j \in J} n_j$. Soit $w \in \widehat{W}$ I-compatible, d'après ce qui précède, on a $w^{-1}(I) \subset \widehat{\Pi}$, et donc $n_{w^{-1}(I)}$ a un sens. Par des considérations sur le volume des faces de l'alcôve fondamentale, nous obtenons alors le théorème suivant, qui peut être vu comme une généralisation du résultat de Peterson :

Théorème 3 *Soit $I \subset \Pi$, on a alors :*

$$\frac{1}{n_I} \sum_{w \in \mathcal{A}b_I} n_{w^{-1}(I)} = 2^{l - \sharp I}.$$

Ce qui nous permet de démontrer le théorème suivant :

Théorème 4 *Soit $I \subset \Pi$, si $\mathfrak{g}$ est de type A_l ou C_l, alors la sous-algèbre parabolique $\mathfrak{p}_I$ a exactement $2^{l-\sharp I}$ idéaux abéliens.*

Pour déterminer le nombre d'idéaux abéliens lorsque $\mathfrak{g}$ est de type B ou D, et le nombre d'idéaux ad-nilpotents dans le cas classique, nous utilisons des diagrammes, comme dans [CP1].

Définissons une relation d'équivalence sur $\Delta^+ \setminus \Delta_I$: deux racines β et γ sont dans la même classe d'équivalelnce si $\mathfrak{g}_\alpha$ et $\mathfrak{g}_\beta$ sont dans le même sous-module simple pour l'action du facteur de Levi standard de $\mathfrak{p}_I$.

Dans un premier temps, procèdons comme dans [CP1] en disposant les racines positives dans un diagramme de forme adéquate. Ensuite, regroupons dans une même case les racines qui font partie de la même classe d'équivalence. Nous obtenons alors un nouveau diagramme. Les éléments de $\mathcal{F}_I$ sont alors en bijection avec certains sous-diagrammes de ce nouveau diagramme.

Notons $\mathcal{C}_l = \dfrac{1}{l+1}\begin{pmatrix} 2l \\ l \end{pmatrix}$ le l-ième nombre de Catalan. Pour p, q deux entiers tels que $q \leqslant p$, posons :

$$\mathcal{T}'_{p,q} = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$

On a alors le théorème suivant :

Théorème 5 *Soit $I \subset \Pi$ de cardinal r. Soit X le type de $\mathfrak{g}$ et s, l_j définis comme dans 2.2.8,* (2.10) *et* (2.11).

Si $X = A_l$, alors

$$\sharp\mathcal{F}_I = \mathcal{C}_{l-r+1}.$$

Si $X = B_l$, alors

$$\sharp\mathcal{F}_I = (l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{n} \mathcal{T}'_{2(l-r)-l_j,l_j-1},$$

où $n = s-1$ si $\alpha_l \in I$, et $n = s$ sinon.

Si $X = C_l$, alors

$$\sharp\mathcal{F}_I = \begin{cases} (l-r+1)\mathcal{C}_{l-r} & si \;\; \alpha_l \notin I, \\ \dfrac{l-r+2}{2}\mathcal{C}_{l-r+1} & si \;\; \alpha_l \in I. \end{cases}$$

Si $X = D_l$, alors

$$\sharp\mathcal{F}_I = \begin{cases} \dfrac{l-r+1}{2}\mathcal{C}_{l-r} + \displaystyle\sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & si \; \sharp\{\alpha_{l-1},\alpha_l\} \cap I = 1, \\ (l-r+1)\mathcal{C}_{l-r} + \displaystyle\sum_{j=1}^{s-1} \mathcal{T}'_{2(l-r)-l_j,l_j-1} & si \; \{\alpha_{l-1},\alpha_l\} \subset I, \\ (3(l-r)-2)\mathcal{C}_{l-r-1} + & \\ \displaystyle\sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-2} + \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & sinon. \end{cases}$$

Notons $\mathcal{F}_I^{ab}$ l'ensemble des éléments de $\mathcal{F}_I$ qui correspondent à des idéaux abéliens. On a alors le théorème suivant :

Théorème 6 *Soit $I \subset \Pi$ de cardinal r. Soit X le type de $\mathfrak{g}$ et s, l_j définis comme dans 2.2.8,* (2.10) *et* (2.11).

Si $X = B_l$, alors

$$\sharp\mathcal{F}_I^{ab} = \begin{cases} 2^{l-r} + \displaystyle\sum_{j=1}^{n} 2 \binom{l-r-1}{l_j-1} & si \; \alpha_1 \notin I, \\ 2^{l-r-1} + \displaystyle\sum_{j=1}^{n} \binom{l-r-1}{l_j-1} & si \; \alpha_1 \in I, \end{cases}$$

où $n = s$ *si* $\alpha_l \notin I$ *et* $n = s-1$ *si* $\alpha_l \in I$.

Si $X = D_l$*, on pose* $t = \sharp(\{\alpha_{l-1}, \alpha_l\} \cap I)$*. Si* $\alpha_1 \in I$*, alors le cardinal de* $\mathcal{F}_I^{ab}$ *est :*

(i) $2^{l-r} - 2^{l-r-2} + \sum_{j=1}^{s} \left[2 \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix} - \begin{pmatrix} l-r-2 \\ l_j - 1 \end{pmatrix} \right]$, *si* $t = 0$,

(ii) $2^{l-r-1} + \sum_{j=1}^{s} \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix}$, *si* $t = 1$,

(iii) $2^{l-r-1} + \sum_{j=1}^{s-1} \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix}$, *si* $t = 2$.

Si $\alpha_1 \notin I$*, alors le cardinal de* $\mathcal{F}_I^{ab}$ *est :*

(iv) $2^{l-r} + \sum_{j=1}^{s} 2 \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix}$, *si* $t = 0$,

(v) $2^{l-r-1} + 2^{l-r-2} + \sum_{j=1}^{s} \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix} + \sum_{j=1}^{s-1} \begin{pmatrix} l-r-2 \\ l_j - 1 \end{pmatrix}$, *si* $t = 1$,

(vi) $2^{l-r} + 2 \sum_{j=1}^{s-1} \begin{pmatrix} l-r-1 \\ l_j - 1 \end{pmatrix}$, *si* $t = 2$.

Lorsque $\mathfrak{g}$ est de type E, F ou G, l'énumération des idéaux ad-nilpotents et abéliens est obtenue en utilisant GAP4.

Soit $\mathfrak{i}$ un idéal ad-nilpotent d'une sous-algèbre parabolique $\mathfrak{p}_I$. Notons

$$\mathfrak{i}^1 = \mathfrak{i} \text{ et } \mathfrak{i}^{k+1} = [\mathfrak{i}^k, \mathfrak{i}],$$

pour $k \geqslant 0$, la suite centrale descendante de $\mathfrak{i}$. Rappelons que l'indice de nilpotence de $\mathfrak{i}$ est le plus petit entier k tel que $\mathfrak{i}^{k+1} = \{0\}$ (i.e. le nombre de termes non nuls dans la suite centrale descendante).

Dans le chapitre 3, nous calculons le nombre d'idéaux ayant un indice de nilpotence fixé lorsque $\mathfrak{g}$ est de type A ou C. L'idée est de construire une application entre des systèmes de racines de rangs différents. Gardons les notations précédentes mais ajoutons en indice le rang du système de racines considéré. Par exemple, Δ_l^+ désigne l'ensemble des racines positives lorsque le rang est l.

Soit $I \subset \Pi_l$ de cardinal r. Supposons que $\mathfrak{g}$ soit de type A_l (resp. que $\mathfrak{g}$ soit de type C_l et que la plus longue racine ne soit pas dans I). On établit une bijection entre les idéaux ad-nilpotents de $\mathfrak{p}_{l-r,\emptyset}$ et les idéaux ad-nilpotents de $\mathfrak{p}_{l,I}$ qui conserve l'indice de nilpotence. On obtient alors d'après [KOP] :

Proposition 7 *Soit $I \subset \Pi_l$ de cardinal r. Soit X le type de $\mathfrak{g}$. Notons $\alpha_l^I(K)$ le nombre d'idéaux ad-nilpotents de $\mathfrak{p}_{l,I}$ d'indice de nilpotence K.*

Supposons que $X = A_l$, alors

$$\alpha_l^I(K) = \sum_{0=i_0<i_1<\cdots<i_K<i_{K+1}=l+1-r} \prod_{j=0}^{K-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix}.$$

Supposons que $X = C_l$ et que $\alpha_l \notin I$, alors

$$\alpha_l^I(K) = \sum_{0<i_1<\cdots<i_k<i_{k+1}=l-r} \prod_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix} \sum_{n=0}^{i_2-i_1-1} \begin{pmatrix} i_1+i_2-1 \\ n \end{pmatrix}$$

si $K = 2k$, et

$$\gamma_l^I(K) = \sum_{-i_2<i_1\leqslant 0<i_2<\cdots<i_k<i_{k+1}=l-r} 2^{i_1+i_2-1} \sum_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix}$$

si $K = 2k-1$.

Supposons que $X = C_l$ et que $\alpha_l \in I$, alors

$$\alpha_l^I(K) = \sum_{0<i_1<\cdots<i_k<i_{k+1}=l-r+1} \prod_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix} \sum_{n=0}^{i_2-i_1-1} \begin{pmatrix} i_1+i_2-2 \\ n \end{pmatrix}$$

si $K = 2k$, et

$$\alpha_l^I(K) = \sum_{1-i_2<i_1\leqslant 0<i_2<\cdots<i_k<i_{k+1}=l-r+1} 2^{i_1+i_2-2} \sum_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix}$$

si $K = 2k-1$.

En particulier, $\alpha_l^I(K)$ ne dépend que du cardinal de I.

Un chemin de Dyck de longueur $2n$ peut être vu comme un mot de $2n$ lettres u ou d, ayant le même nombre de u et de d, et tel qu'il y ait toujours plus de u que de d à gauche d'une lettre.

Soit $\Phi \in \mathcal{F}_{\emptyset}$, notons :

$$I_\Phi = \{\alpha \in \Pi; \Phi \in \mathcal{F}_{\{\alpha\}}\}.$$

C'est l'élément maximal de l'ensemble $\{I \subset \Pi; \Phi \in \mathcal{F}_I\}$.

Dans le chapitre 4, lorsque $\mathfrak{g}$ est de type A_l, en utilisant la bijection donnée dans [AKOP] entre les éléments de $\mathcal{F}_{l,\emptyset}$ et les chemins de Dyck de longueur $2l+2$, nous établissons une bijection entre les éléments $\Phi \in \mathcal{F}_{l,\emptyset}$ tels que $\sharp I_\Phi = r$ et les chemins de Dyck admettant r *udu*.

De plus, en considérant la bijection naturelle entre les éléments de $\mathcal{F}_{l,\emptyset}$ et les chemins de Dyck de longueur $2l+2$ donnée dans [Pa], nous établissons une dualité entre les éléments de $\Phi \in \mathcal{F}_{l,\emptyset}$ tels que $\sharp\Phi_{min} = p$ et ceux tels que $\sharp\Phi_{min} = l - p$. Une telle dualité a déjà été démontrée dans [Pa], mais les idéaux duaux ne sont pas les mêmes.

Chapter 1

Characterization of ad-nilpotent ideals of a parabolic subalgebra

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l. Let $\mathfrak{h}$ be a Cartan subalgebra and Δ the associated root system. We fix a system of positive roots Δ^+. Denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the corresponding set of simple roots. Let V be the Euclidian space $\sum_{k=1}^{l} \mathbb{R}\alpha_k$. For each $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ relative to α.

For $I \subset \Pi$, set $\Delta_I = \mathbb{Z}I \cap \Delta$. We fix the corresponding standard parabolic subalgebra :

$$\mathfrak{p}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha \right).$$

An ideal $\mathfrak{i}$ of $\mathfrak{p}_I$ is ad-nilpotent if and only if for all $x \in \mathfrak{i}$, $ad_{\mathfrak{p}_I} x$ is nilpotent.

In their articles [CP1] and [CP2], Cellini-Papi established different characterizations of the set $\mathcal{I}$ of ad-nilpotent ideals of a Borel subalgebra. They constructed a bijection between $\mathcal{I}$ and certain elements of the affine Weyl group $\widehat{W}$ associated to Δ, which we shall call $\emptyset$-compatible. These $\emptyset$-compatible elements are in turn characterized by elements of the coroot lattice. We extend in this chapter their theory to the case of parabolic subalgebras.

1.1 Generalities on the affine Weyl group

We shall conserve the notations given above. In this section, we shall recall some basic facts on the affine Weyl group associated to Δ. In particular, we need to recall two different realizations of this group. See [Bo], [CP1] and [Ka] for more details.

We fix a scalar product $(.,.)$ on V. For $\alpha \in \Delta$, let

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$

denote the corresponding coroot. Denote by $Q^\vee$ the coroot lattice of Δ.

Let W denote the Weyl group associated to Δ. We shall realize the affine Weyl group as a group of automorphisms of the affine root system associated to Δ. Let $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$. We extend the above bilinear form on V to a non-degenerate symmetric bilinear form on $\widehat{V}$, also denoted $(.,.)$, by setting

$$(\lambda,\lambda) = (\delta,\delta) = (\lambda, V) = (\delta, V) = 0 \text{ and } (\delta,\lambda) = 1.$$

Let $\widehat{\Delta} = \Delta + \mathbb{Z}\delta$ be the set of (real) affine roots. We fix the following positive root system $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{N}\delta) \cup (\Delta^- + \mathbb{N}^*\delta)$. We shall write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \widehat{\Delta}^+$ (resp. if $\alpha \in \widehat{\Delta}^- = -\widehat{\Delta}^+$). Let θ be the highest root of Δ, then $\widehat{\Pi} = \{\alpha_0 = -\theta + \delta, \alpha_1, \cdots, \alpha_l\}$ is the set of simple roots for $\widehat{\Delta}^+$.

Note that for any element $\beta + k\delta \in \widehat{\Delta}^+$, we have $(\beta + k\delta, \beta + k\delta) = (\beta,\beta) \neq 0$. For all $\alpha \in \widehat{\Delta}^+$, we denote by s_α the reflection of $\widehat{V}$ defined by

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha,\alpha)}\alpha$$

for $x \in \widehat{V}$. The affine Weyl group $\widehat{W}$ is the subgroup of $\mathrm{Aut}(\widehat{V})$ generated by $\{s_\alpha;\ \alpha \in \widehat{\Pi}\}$. Observe that $w(\delta) = \delta$ for all $w \in \widehat{W}$, $s_\alpha(\lambda) = \lambda$, for all $\alpha \in \Pi$ and $s_{\alpha_0}(\lambda) = \lambda - \frac{2}{\|\theta\|^2}\alpha_0$, where $\|\theta\| = \sqrt{(\theta,\theta)}$.

Let $\tau \in Q^\vee$, we define the endomorphism t_τ of $\widehat{V}$ by :

$$t_\tau(x + a\delta + b\lambda) \quad = \quad x + a\delta + b\lambda + b\tau + (\frac{b}{2}(\tau,\tau) - (x,\tau))\delta \qquad (1.1)$$

for $x \in V$ and $a, b \in \mathbb{R}$. Let $S = \{t_\tau; \tau \in Q^\vee\}$, then the group $\widehat{W}$ is the semi-direct product of S by W.

Consider the $\widehat{W}$-invariant affine subspace

$$E = \{x \in \widehat{V};\ (x, \delta) = 1\} = V \oplus \mathbb{R}\delta + \lambda.$$

Let $\pi\ :\ E \to V$ be the projection $ax + b\delta + \lambda \mapsto ax$ and

$$\begin{array}{rcccl} i & : & V & \to & E \\ & & v & \mapsto & v + \lambda \end{array}$$

For $w \in \widehat{W}$, we set $\overline{w} = \pi \circ w_{|_E} \circ i$. The map $w \mapsto \overline{w}$ defines an injective morphism of groups from $\widehat{W}$ to $\mathrm{Aut}(V)$. We shall identify $\widehat{W}$ with its image W_{aff} under this map.

For $\alpha \in \Delta$, $\overline{s_\alpha}$ is the reflection s_α on V associated to α, and for $\tau \in Q^\vee$, $\overline{t_\tau}$ is the translation T_τ by the vector τ on V. For $\alpha \in \Delta^+$, $k \geqslant 0$, $x \in V$, we obtain that

$$\begin{aligned} \overline{s_{-\alpha+k\delta}}(x) &= x - ((x, \alpha) - k)\alpha^\vee = T_{k\alpha^\vee} \circ s_\alpha(x) \\ \overline{s_{\alpha+k\delta}}(x) &= x - ((x, \alpha) + k)\alpha^\vee = T_{-k\alpha^\vee} \circ s_\alpha(x). \end{aligned}$$

Thus $\overline{s_{-\alpha+k\delta}}$ and $\overline{s_{\alpha+k\delta}}$ are the orthogonal reflections with respect to $H_{\alpha,k} = \{x \in V;\ (x, \alpha) = k\}$ and $H_{\alpha,-k}$ respectively. It follows that W_{aff} is the semi-direct product of W by the group of translations T_τ, $\tau \in Q^\vee$.

Observe that for $v \in W$, $\tau \in Q^\vee$, $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we have

$$\overline{vt_\tau}(H_{\alpha,k}) = H_{v(\alpha),k+(\tau,\alpha)}.$$

Recall that the connected components of the complement of $\bigcup_{\alpha\in\Delta,k\in\mathbb{Z}} H_{\alpha,k}$ in V are called alcoves. The group W_{aff} acts simply transitively on the set of alcoves. We denote

$$C = \{x \in V; (\alpha_i, x) > 0 \text{ for all } \alpha_i \in \Pi\},\ A = \{x \in C; (\theta, x) < 1\}$$

respectively the fundamental chamber and the fundamental alcove with respect to Π and $\widehat{\Pi}$.

We shall end this section by recording the following results :

Proposition 1.1.1 *For $w \in \widehat{W}$, let $N(w) = \{\beta \in \widehat{\Delta}^+;\ w^{-1}(\beta) < 0\}$ and denote by $\ell(w)$ the length of any reduced expression of w.*

(a) *We fix a reduced expression of $w = s_{\beta_1} \circ \cdots \circ s_{\beta_k}$ with $\beta_i \in \widehat{\Pi}$, then $N(w) = \{s_{\beta_1} \circ \cdots \circ s_{\beta_{p-1}}(\beta_p); 1 \leq p \leq k\}$. In particular, $N(w)$ contains a simple root.*

(b) *Let* $w_1,\ w_2 \in \widehat{W}$, *then* $N(w_1) \subseteq N(w_2)$ *if and only if, there exists* $u \in \widehat{W}$ *such that* $w_2 = w_1 u$, *and* $\ell(w_2) = \ell(w_1) + \ell(u)$. *In particular,* w *is uniquely determined by* $N(w)$.

(c) *If* $N(w) \cap \Delta^+ \neq \emptyset$, *then* $N(w) \cap \Pi \neq \emptyset$.

Proof. For parts (a) and (b), see for example [CP1]. Let us prove (c). The case $\widetilde{A_1}$ is clear. In the others cases, this is a direct consequence of the fact that $N(w)$ is a "compatible" set, by theorem 1.3 from [CP1]. □

1.2 I-compatible elements in $\widehat{W}$

Let $I \subset \Pi$.

Lemma 1.2.1 *An ideal* $\mathfrak{i}$ *of* $\mathfrak{p}_I$ *is ad-nilpotent if and only if it is nilpotent.*

Proof. Since $\mathfrak{i}$ is $\mathfrak{h}$-stable, we have

$$\mathfrak{i} = \mathfrak{h}' \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big),$$

where $\mathfrak{h}' \subset \mathfrak{h}$ and $\Phi \subset \Delta^+ \cup \Delta_I$.

Assume that there exists $h \in \mathfrak{h}' \setminus \{0\}$, then there exists $\alpha \in \Delta^+$ such that $\alpha(h) \neq 0$. Consequently for $x \in \mathfrak{g}_\alpha \setminus \{0\}$, we have $(ad_{\mathfrak{p}_I} h)^n(x) = \alpha(h)^n x \neq 0$. So $x \in \mathfrak{i}$ and $\mathfrak{i}$ is neither nilpotent nor ad-nilpotent. Therefore we may assume that $\mathfrak{h}' = 0$.

Assume that there exists $\alpha \in \Delta_I \cap \Phi$. Then, $\mathfrak{g}_\alpha \in \mathfrak{i}$ and hence $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset [\mathfrak{g}_\alpha, \mathfrak{p}_I] \subset \mathfrak{i}$. This is absurd because $\mathfrak{h}' = \{0\}$.

It follows that $\Phi \subset \Delta^+ \setminus \Delta_I$ and such an ideal is nilpotent and ad-nilpotent. □

Let $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_I$. We set

$$\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \setminus \Delta_I;\ \mathfrak{g}_\alpha \subseteq \mathfrak{i}\}.$$

Then by the proof of Lemma 1.2.1, we have $\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$ and if $\alpha \in \Phi_{\mathfrak{i}}$, $\beta \in \Delta^+ \cup \Delta_I$ are such that $\alpha + \beta \in \Delta^+$, then $\alpha + \beta \in \Phi_{\mathfrak{i}}$.

Conversely, set

$$\mathcal{F}_I = \{\Phi \subset \Delta^+ \setminus \Delta_I; \text{if } \alpha \in \Phi, \beta \in \Delta^+ \cup \Delta_I, \alpha + \beta \in \Delta^+, \text{then } \alpha + \beta \in \Phi\}.$$

Then for $\Phi \in \mathcal{F}_I$, $\mathfrak{i}_\Phi = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{p}_I$.

We obtain therefore a bijection

$$\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \to \mathcal{F}_I, \ \mathfrak{i} \mapsto \Phi_\mathfrak{i}.$$

For $\Phi \in \mathcal{F}_I$, we define $\Phi^1 = \Phi$, $\Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta$, for $k \geq 2$ and

$$L_\Phi = \bigcup_{k\in\mathbb{N}^*} (-\Phi^k + k\delta).$$

Since any ad-nilpotent ideal of $\mathfrak{p}_I$ is an ad-nilpotent ideal of the Borel subalgebra $\mathfrak{p}_\emptyset = \mathfrak{b}$, we have by [CP1] the following proposition :

Proposition 1.2.2 *Let $\Phi \in \mathcal{F}_I$, then there exists a unique $w_\Phi \in \widehat{W}$ such that $L_\Phi = N(w_\Phi)$.*

Thus we have the following injective map :

$$\begin{array}{rcl}\{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} & \to & \widehat{W} \\ \mathfrak{i} & \mapsto & w_{\Phi_\mathfrak{i}}\end{array}$$

Recall from [CP1] the following characterization of the image of the above map when $I = \emptyset$.

Proposition 1.2.3 *Let $w \in \widehat{W}$, then there exists an ideal $\mathfrak{i}$ of $\mathfrak{b}$ such that $N(w) = L_{\Phi_\mathfrak{i}}$ if and only if*

(a) *$w^{-1}(\alpha) > 0$, for all $\alpha \in \Pi$.*

(b) *If $w(\alpha) < 0$ for some $\alpha \in \widehat{\Pi}$, then $w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$.*

If these conditions are verified, we say that w is **Borel-compatible** *or* **$\emptyset$-compatible**.

For $w \in \widehat{W}$, let $\Phi_w = \{\alpha \in \Delta^+; -\alpha + \delta \in N(w)\}$.

Theorem 1.2.4 *Let $w \in \widehat{W}$ be Borel-compatible and $I \subset \Pi$. The following conditions are equivalent :*

(a) *$\mathfrak{i}_{\Phi_w}$ is an ad-nilpotent ideal of $\mathfrak{p}_I$.*

(b) *$s_\alpha(\Phi_w) = \Phi_w$, for all $\alpha \in I$.*

(c) *$s_\alpha(L_{\Phi_w}) = L_{\Phi_w}$, for all $\alpha \in I$.*

(d) *$N(s_\alpha w) = N(w) \cup \{\alpha\}$, for all $\alpha \in I$.*

(e) *$w^{-1}(\alpha) \in \widehat{\Pi}$, for all $\alpha \in I$.*

If the hypothesis and these equivalent conditions are verified, we say that w is **I-compatible**.

Proof. $(a) \Rightarrow (b)$ By assumption, we have $\Phi_w \in \mathcal{F}_I$. Let $\beta \in \Phi_w$, then $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, hence $s_\alpha(\Phi_w) \subset \Phi_w$, for all $\alpha \in I$. Moreover, since s_α is an involution, we obtain that $s_\alpha(\Phi_w) = \Phi_w$.

$(b) \Rightarrow (c)$ Since $w(\delta) = \delta$, for all $w \in \widehat{W}$, this is clear (by induction on k or just remark that $\Phi_w^k \in \mathcal{F}_I$).

$(c) \Rightarrow (d)$ Let $\alpha \in I$, by assumption, we have $s_\alpha(N(w)) = N(w)$, hence for $\beta \in N(w)$, we have $s_\alpha(\beta) \in N(w)$. So $w^{-1}s_\alpha(\beta) < 0$ and $\beta \in N(s_\alpha w)$. We have proved that $N(w) \subset N(s_\alpha w)$. Since $\sharp N(w) = \ell(w)$ and $\ell(s_\alpha w) = \ell(w) \pm 1$, by Proposition 1.1.1, we obtain that $\sharp N(s_\alpha w) = \sharp N(w) + 1$. Moreover we have $(s_\alpha w)^{-1}(\alpha) = w^{-1}(-\alpha) < 0$, hence $N(s_\alpha w) = N(w) \cup \{\alpha\}$.

$(d) \Rightarrow (e)$ Let $\alpha \in I$. By assumption, we have $N(w) \subset N(s_\alpha w)$, hence by Proposition 1.1.1, there exists $\beta \in \widehat{\Pi}$ such that,

$$N(s_\alpha w) = N(ws_\beta) = N(w) \cup \{w(\beta)\} = N(w) \cup \{\alpha\}.$$

Consequently, we have $w^{-1}(\alpha) = \beta \in \widehat{\Pi}$.

$(e) \Rightarrow (a)$ By assumption, Φ_w is an element of $\mathcal{F}_\emptyset$, so to prove that Φ_w belongs to $\mathcal{F}_I$, it suffices to prove that $\Phi_w \in \mathcal{F}_\alpha$, for all $\alpha \in I$.

Let $\alpha \in I$ and assume that $w^{-1}(\alpha) \in \widehat{\Pi}$. Let $\beta \in \Phi_w$ be such that $\beta - \alpha \in \Delta^+$. We have

$$w^{-1}(-(\beta - \alpha) + \delta) = w^{-1}(-\beta + \delta) + w^{-1}(\alpha) \in (\widehat{\Delta}^- + \widehat{\Pi}) \cap \widehat{\Delta}.$$

It follows that $w^{-1}(-(\beta-\alpha)+\delta) < 0$. Moreover, $w^{-1}(-\alpha+\delta) = w^{-1}(-\alpha) + \delta > 0$ hence $\alpha \notin \Phi_{\mathfrak{i}_w}$. We obtain that $\Phi_w \in \mathcal{F}_{\alpha_i}$, for all $\alpha_i \in I$, hence $\mathfrak{i}_{\Phi_w}$ is an ad-nilpotent ideal of $\mathfrak{p}_I$. □

Another characterization of ad-nilpotent ideals in $\mathfrak{b}$ is given in [CP2] via the set $D = \{\tau \in Q^\vee; (\tau, \alpha_j) \leq 1, j = 1, \dots, l \text{ and } (\tau, \theta) \geq -2\}$. Let $\widetilde{D} = \{(\tau, v) \in D \times W; vt_\tau(A) \subset C\}$. We can state this characterization in the following way:

Proposition 1.2.5 *The following map is bijective :*

$$\begin{array}{rcl} \widetilde{D} & \rightarrow & \{w \in \widehat{W}, \emptyset\text{-}compatible\} \\ (\tau, v) & \mapsto & vt_\tau. \end{array}$$

Remark 1.2.6 *In [CP2], the above correspondence is not viewed in the same way since the elements of $\widehat{W}$ are written $t_\tau v = vt_{v^{-1}(\tau)}$ instead of vt_τ, for $w \in W$ and $\tau \in Q^\vee$.*

Let $w \in \widehat{W}$ be Borel-compatible, then $I_w = \{\alpha \in \Pi; w^{-1}(\alpha) \in \widehat{\Pi}\}$ is the unique maximal element of $\{I \subset \Pi;\ w \text{ is } I\text{-compatible}\}$. For $\tau \in Q^\vee$, set

$$D_\tau = \begin{cases} \{\alpha \in \Pi; (\alpha, \tau) = 0\} \cup \{-\theta\} & \text{if } (\theta, \tau) = -1, \\ \{\alpha \in \Pi; (\alpha, \tau) = 0\} & \text{if } (\theta, \tau) \neq -1. \end{cases}$$

Proposition 1.2.7 *Let $(\tau, v) \in \widetilde{D}$, and $w = vt_\tau \in \widehat{W}$. Then $v(D_\tau) = I_w$. In particular, w is I-compatible if and only if $I \subset v(D_\tau)$.*

Proof. Let $\alpha \in I_w$, then

$$w^{-1}(\alpha) = t_{-\tau} v^{-1}(\alpha) = v^{-1}(\alpha) + (v^{-1}(\alpha), \tau)\delta \in \widehat{\Pi}.$$

If $w^{-1}(\alpha) \in \Pi$, then we have $v^{-1}(\alpha) \in \Pi$ and $(v^{-1}(\alpha), \tau) = 0$, hence $v^{-1}(\alpha) \in D_\tau$. If $w^{-1}(\alpha) = \alpha_0$, then we have $v^{-1}(\alpha) = -\theta$ and $(\theta, \tau) = -1$, hence $-\theta = v^{-1}(\alpha) \in D_\tau$.

Conversely, let $\alpha \in D_\tau \cap \Pi$, then $vt_\tau(\alpha) = v(\alpha) \in \Delta^+$, because w is Borel-compatible. Then we have $N(ws_\alpha) = N(w) \cup \{w(\alpha)\}$, and by part (3) of proposition 1.1.1 , there exists a simple root $\beta \in \Pi$ such that $\beta \in N(ws_\alpha)$. Since $N(w) \cap \Delta^+ = \emptyset$, we obtain that $w(\alpha) = \beta$ and $v(\alpha) \in I_w$.

Assume now that $-\theta \in D_\tau$. Since w is Borel-compatible, $vt_\tau(\alpha_0) = -v(\theta) \in \Delta^+$. As above we have $N(ws_{\alpha_0}) = N(w) \cup \{w(\alpha_0)\}$, and by part (3) of proposition 1.1.1, there exists a simple root $\beta \in \Pi$ such that $\beta \in N(ws_{\alpha_0})$. Since $N(w) \cap \Delta^+ = \emptyset$, we obtain that $w(\alpha_0) = \beta$ and $v(-\theta) \in I_w$.

We have therefore proved that $v(D_\tau) = I_w$, which concludes the proof. $\square$

Let us denote $H_\alpha = H_{\alpha,0}$ for $\alpha \in \Pi$, and $H_{\alpha_0} = H_{\theta,1}$. Let $\{\omega_1, \ldots, \omega_l\}$ be elements of V such that $(\omega_i, \alpha_j) = \delta_{ij}$. Set $n_0 = 1$ and let n_i, $i = 1, \ldots, l$, be the strictly positive integers such that $\theta = \sum_{i=1}^l n_i \alpha_i$. Let $\overline{\omega}_i = \omega_i / n_i$, $i = 1 \ldots l$, and $\overline{\omega}_0 = 0$. Then the closure $\overline{A}$ of A is the convex hull $\mathrm{Conv}(\overline{\omega}_0, \overline{\omega}_1, \ldots, \overline{\omega}_l)$ of $\overline{\omega}_0, \ldots, \overline{\omega}_l$. For $k \in \mathbb{N}^*$, the convex hull (resp. the image by $\overline{w} \in W_{\mathit{aff}}$ of the convex hull) of $(k+1)$ points in $\{\overline{\omega}_0, \overline{\omega}_1, \ldots, \overline{\omega}_l\}$ is called a k-face of $\overline{A}$ (resp. of $\overline{w}(\overline{A})$). For example, $H_{\alpha_i} \cap \overline{A} = \mathrm{Conv}(\overline{\omega}_0, \ldots, \overline{\omega}_{i-1}, \overline{\omega}_{i+1}, \ldots, \overline{\omega}_l)$ is an $(l-1)$-face of $\overline{A}$.

We shall give yet another characterization of ad-nilpotent ideals of $\mathfrak{p}_I$ which shall be useful in enumerating abelian ideals when $\mathfrak{g}$ is of type A or C.

Proposition 1.2.8 *Let $w \in \widehat{W}$ be Borel-compatible and $I \subset \Pi$. Then, $\mathfrak{i}_{\Phi_w}$ is an ideal of $\mathfrak{p}_I$ if and only if for all $\alpha \in I$, $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.*

Proof.

Assume that $w \in \widehat{W}$ is I-compatible. Let $(\tau, v) \in \widetilde{D}$ be such that $w = vt_\tau$. By proposition 1.2.7, $I \subset v(D_\tau)$, and $v^{-1}(\alpha) \in D_\tau$, for all $\alpha \in I$. Let $\alpha \in I$, we distinguish two cases :

If $v^{-1}(\alpha) = \beta \in \Pi$, then $(\beta, \tau) = 0$. We obtain that

$$\overline{vt_\tau}(H_\beta) = H_{v(\beta),(\tau,\beta)} = H_\alpha.$$

Hence $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.

If $v^{-1}(\alpha) = -\theta$, then $(\theta, \tau) = -1$. We obtain that

$$\overline{vt_\tau}(H_{\alpha_0}) = H_{v(\theta),(\tau,\theta)+1} = H_\alpha.$$

Hence, $\overline{w}(\overline{A}) \cap H_\alpha$ is an $(l-1)$-face of $\overline{w}(\overline{A})$.

Conversely, let $v \in W$, $\tau \in Q^\vee$ be such that $w = vt_\tau \in \widehat{W}$ is Borel-compatible. By assumption, for all $\alpha \in I$, there exists $\beta \in \widehat{\Pi}$ such that $\overline{w}(H_\beta) = H_\alpha$.

If $\beta \in \Pi$, then

$$\overline{vt_\tau}(H_\beta) = H_{v(\beta),(\tau,\beta)} = H_\alpha$$

hence $(\tau, \beta) = 0$, and $w^{-1}(\alpha) = \pm\beta$. Since w is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$.

If $\beta = \alpha_0$, then

$$\overline{vt_\tau}(H_{\alpha_0}) = H_{v(\theta),(\tau,\theta)+1} = H_\alpha$$

hence $(\tau, \theta) = -1$, and $w^{-1}(\alpha) = \pm(\theta - \delta)$. Since w is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$. We have proved that $I \subset v(D_\tau)$, and by proposition 1.2.7, w is I-compatible. $\square$

Let $H_\emptyset = V$. For $J \subset \widehat{\Pi}$ non empty, denote $H_J = \bigcap_{\alpha \in J} H_\alpha$. By the proposition above, if w is I-compatible, then we have $\overline{w}(\overline{A}) \cap H_I = \overline{w}(\overline{A} \cap H_{w^{-1}(I)})$.

1.3 Relation with antichains

Recall the following partial order on Δ^+ : $\alpha \leqslant \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}_\emptyset$ if and only if for all $\alpha \in \Phi, \beta \in \Delta^+$, such that $\alpha \leqslant \beta$, we have $\beta \in \Phi$. We can now define, for $\Phi \in \mathcal{F}_\emptyset$:

$$\Phi_{min} = \{\beta \in \Phi; \beta - \alpha \notin \Phi, \text{ for all } \alpha \in \Delta^+\}$$

the set of minimal roots of Φ for $\leqslant$.

A set of positive roots A is called an antichain of $(\Delta^+, \leqslant)$ if all the roots in A are pairwise non comparable with respect to $\leqslant$. It is clear that Φ_{min} is an antichain. Conversely, let A be an antichain. Set $\Phi = \{\beta \in \Delta^+;$ there exists $\alpha \in A$ such that $\alpha \leqslant \beta\}$, then $\Phi \in \mathcal{F}_\emptyset$. So each antichain corresponds to an element of $\mathcal{F}_\emptyset$. By a similar proof as in [Pa], we obtain the following proposition:

Proposition 1.3.1 *Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$, then we have $\sharp\Phi_{min} \leqslant l - r$.*

Proof. Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$. Set $\Gamma = \Phi_{min} \cup I = \{\gamma_1, \dots, \gamma_t\}$. Let $\gamma_i, \gamma_j \in \Gamma$, then $\gamma_i - \gamma_j \notin \Delta$ by the definition of Φ_{min} and the fact that $\Phi \in \mathcal{F}_I$. Thus the angle between any pair of distinct elements of Γ is non acute and since all the γ_i's lie in an open half space of V, they are linearly independent. Consequently, we have $\sharp\Gamma \leqslant r$, and hence $\sharp\Phi_{min} \leqslant l-r$. $\square$

Remarks 1.3.2 (i) *Recall from [CP1], that an antichain $\Gamma \subset \Delta^+$ is of cardinality l if and only if $\Gamma = \Pi$. This result has no equivalence in the general parabolic case. For example, in B_2, the set $\Phi = \{\alpha_1 + 2\alpha_2\}$ is an ad-nilpotent ideal of $\mathfrak{p}_{\alpha_1}$ such that $\Phi_{min} = \Phi$ and $\sharp\Phi_{min} = 1$.*

(ii) *Let $\mathfrak{g}$ be of type A_l. Let $I \subset \Pi$ be of cardinality r and $\Phi \in \mathcal{F}_I$, then it is possible to show that $\sharp\Phi_{min} = l - r$ if and only if $\Phi_{min} = \Pi \setminus I$.*

Chapter 2

Enumeration of ad-nilpotent and abelian ideals

2.1 Enumeration via the volume of the faces of the fundamental alcove

Recall from [CP1] and [Ko], that for $w \in \widehat{W}$ the set $\mathfrak{i}_{\Phi_w}$ is an abelian ideal of $\mathfrak{b}$ if and only if $\overline{w}(A) \subset 2A$. As a consequence, we have the following remarkable result of Peterson : the number of abelian ideals of $\mathfrak{b}$ is 2^l. Observe that the above result says that the number of abelian ideals in $\mathfrak{b}$ depends only on the rank of $\mathfrak{g}$. In the case of parabolic algebras, we shall see in this section to what extent this result can be extended.

For $J \subset \widehat{\Pi}$, let $F_J = \overline{A} \cap H_J = \mathrm{Conv}(\varpi_j;\ \alpha_j \notin J)$. Observe that the F_J are the faces of $\overline{A}$. Let $w \in \widehat{W}$, if $\overline{w}(\overline{A}) \cap H_J$ is an $(l - \sharp J)$-face of $\overline{w}(\overline{A})$, then we shall call $\overline{w}(\overline{A}) \cap H_J$ an $(l - \sharp J)$-**alcove** of H_J.

Proposition 2.1.1 (a) *Let $w \in \widehat{W}$ and $I \subset \Pi$, if $\overline{w}(A) \subset 2A$ and $\overline{w}(\overline{A}) \cap H_I$ is an $(l - \sharp I)$-alcove of H_I, then w is I-compatible.*

(b) *Let $I \subset \Pi$ and $w, w' \in \widehat{W}$ be I-compatible. If $\overline{w}(A) \subset 2A$, $\overline{w'}(A) \subset 2A$ and $\overline{w}(\overline{A}) \cap H_I = \overline{w'}(\overline{A}) \cap H_I$, then $w = w'$.*

Proof. (a) Let $w \in \widehat{W}$ and $I \subset \Pi$ be of cardinality r. If $\overline{w}(A) \subset 2A$, then w is Borel-compatible and the ideal $\mathfrak{i}_{\Phi_w}$ is abelian.

Set $N = l - r + 1$. Since $\overline{w}(\overline{A}) \cap H_I$ is an $(l - r)$-alcove of H_I, there exist N vertices $\varpi_{i_1}, \ldots, \varpi_{i_N}$ of $\overline{A}$ such that $\overline{w}(\varpi_{i_1}), \ldots, \overline{w}(\varpi_{i_N})$ belong to $\overline{w}(\overline{A}) \cap H_I$.

There exist r distinct reflecting affine hyperplanes $H'_1, \ldots, H'_r$ of the form H_α, for $\alpha \in \widehat{\Pi}$, such that $\bigcap_{j=1}^r H'_j$ contains $\varpi_{i_1}, \ldots, \varpi_{i_N}$. For $j = 1, \ldots, r$, $H_I \cap \overline{w}(H'_j)$ contains $\overline{w}(\varpi_{i_1}), \ldots, \overline{w}(\varpi_{i_N})$. Since the dimension of H_I is $N-1$, it follows that $H_I \subset \overline{w}(H'_j)$.

The hyperplane H_I is defined by the equations $(x, \alpha) = 0$ for all $\alpha \in I$, it follows that $\overline{w}(H'_j)$ is an hyperplane of the form $H_{\beta,0}$, where β is a linear combination of elements of I.

Assume that $\beta \notin I$. Then, the intersection of $H_{\beta,0}$ with the closure of the fundamental chamber C is of dimension at most $l-2$. Since by construction $H_{\beta,0}$ contains an $(l-1)$-face of $\overline{w}(\overline{A})$, and $\overline{w}(\overline{A}) \subset \overline{C}$, we obtain a contradiction. It follows that $\beta \in I$.

Set $w = vt_\tau$. We then have that for each $\beta \in I$:

$$w^{-1}(H_{\beta,0}) = H_{v^{-1}(\beta),(\tau,v^{-1}(\beta))} = H_\alpha$$

for some $\alpha \in \widehat{\Pi}$. If $\alpha \in \Pi$, then $v^{-1}(\beta) = \pm\alpha$ and $(\tau, v^{-1}(\beta)) = 0$. Since w is Borel-compatible, we obtain that $w^{-1}(\beta) = \alpha$.

If $\alpha = \alpha_0$, we obtain that $v^{-1}(\beta) = \pm\theta$ and $(\tau, v^{-1}(\beta)) = \pm 1$. Since w is Borel-compatible, we finally obtain that $w^{-1}(\beta) = \alpha_0$. Thus, $w^{-1}(I) \subset \widehat{\Pi}$, and w is I-compatible as required.

(b) Let $I \subset \Pi$ and $w,\ w' \in \widehat{W}$ be I-compatible. Let $\alpha \in I$, then w is $I \setminus \{\alpha\}$-compatible. It follows by proposition 1.2.8 that $\overline{w}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ is an $(l - \sharp I + 1)$-alcove of $H_{I\setminus\{\alpha\}}$ and it is the convex hull of $\overline{w}(\overline{A}) \cap H_I$ and a vertex of $H_{I\setminus\{\alpha\}} \cap \overline{w}(\overline{A})$, which is not in $H_I \cap \overline{w}(\overline{A})$. In the same way $\overline{w'}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ is an $(l - \sharp I + 1)$-alcove of $H_{I\setminus\{\alpha\}}$ and it is the convex hull of $\overline{w'}(\overline{A}) \cap H_I$ and a vertex of $H_{I\setminus\{\alpha\}} \cap \overline{w'}(\overline{A})$, which is not in $H_I \cap \overline{w}(\overline{A})$. Since $\overline{w}(\overline{A}) \subset 2\overline{A}$, there is a unique vertex in $H_{I\setminus\{\alpha\}}$ satisfying these conditions. So, $\overline{w}(\overline{A}) \cap H_{I\setminus\{\alpha\}} = \overline{w'}(\overline{A}) \cap H_{I\setminus\{\alpha\}}$ and by induction, we have $\overline{w}(\overline{A}) = \overline{w'}(\overline{A})$. Hence $w = w'$. $\square$

Let $F'_J = \overline{2A} \cap H_J = \mathrm{Conv}(2\varpi_j;\ \alpha_j \notin J)$. It is clear that F'_J is a union of $(l - \sharp J)$-alcoves of H_J. Let

$$\mathcal{A}b_I = \{w \in \widehat{W}; \mathfrak{i}_{\Phi_w} \text{ is an abelian ideal of } \mathfrak{p}_I\}.$$

By the above proposition and by proposition 1.2.8, we obtain the following result :

Theorem 2.1.2 *Let $I \subset \Pi$, then the map $w \mapsto \overline{w}(\overline{A}) \cap H_I$ is a bijection between $\mathcal{A}b_I$ and the set of all the $(l - \sharp I)$-alcoves of F'_I.*

Remark 2.1.3 *The above theorem can be viewed as a generalization of Peterson's result.*

In order to determine $\sharp\mathcal{A}b_I$, we are reduced to computing the volume of the $(l-\sharp I)$-alcoves of F'_I. Furthermore, to compute the volume of the $(l-\sharp I)$-alcoves of F'_I, it suffices to compute the volume of the $(l-\sharp I)$-faces of $\overline{A}$.

Let $d(x, H_\alpha)$ denote the distance from $x \in V$ to the affine hyperplane H_α, for $\alpha \in \widehat{\Pi}$. For B a k-alcove, let $\mathrm{Vol}_k(B)$ be the k-volume of B. By [Be], the volume of the fundamental alcove is

$$\mathrm{Vol}_l(A) = \frac{1}{l} \times d(0, H_{\alpha_0}) \times \mathrm{Vol}_{l-1}(F_{\alpha_0}).$$

Since the projection of 0 on H_{α_0} is $\frac{\theta}{\|\theta\|^2}$, we have $d(0, H_{\alpha_0}) = \frac{1}{\|\theta\|}$. We obtain that $\mathrm{Vol}_l(A) = \frac{1}{l\|\theta\|}\mathrm{Vol}_{l-1}(F_{\alpha_0})$. Moreover, by [CLO],

$$\mathrm{Vol}_l(A) = \frac{1}{l!}\left|\overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_l\right|.$$

Let $D = |\overline{\omega}_1 \wedge \cdots \wedge \overline{\omega}_l|$, then

$$\mathrm{Vol}_{l-1}(F_{\alpha_0}) = \frac{D}{(l-1)!} n_0 \|\theta\|. \tag{2.1}$$

To compute the $(l-1)$-volume of the faces F_{α_i}, $i = 1, \cdots, l$, we compute the l-volume of the convex hull of $(\{\overline{\omega}_1, \ldots, \overline{\omega}_l\} \setminus \{\overline{\omega}_i\}) \cup \{\frac{\alpha_i}{\|\alpha_i\|}\}$. Thus, we have :

$$\mathrm{Vol}_{l-1}(F_{\alpha_i}) = \frac{1}{(l-1)!}\left|\overline{\omega}_1 \wedge \cdots \wedge \frac{\alpha_i}{\|\alpha_i\|} \wedge \cdots \wedge \overline{\omega}_l\right|.$$

Since $\alpha_i = \sum_{k=1}^{l} (\alpha_i, \alpha_k)\omega_k$,

$$\mathrm{Vol}_{l-1}(F_{\alpha_i}) = \frac{D}{(l-1)!} n_i \|\alpha_i\|. \tag{2.2}$$

We have therefore computed the $(l-1)$-volume of the $(l-1)$-faces of $\overline{A}$. In particular, we have :

Lemma 2.1.4 *Let α_i, $\alpha_j \in \widehat{\Pi}$, be such that $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$, then :*

$$n_i \mathrm{Vol}_{l-1}(F_j) = n_j \mathrm{Vol}_{l-1}(F_i).$$

This Lemma also appears as Proposition 26 in [Sut]. We shall generalize this result. For $I \subset \widehat{\Pi}$, let $n_I = 1$ if $I = \emptyset$, and $n_I = \prod_{\alpha_i \in I} n_i$ otherwise. We shall prove the following result :

Proposition 2.1.5 *Let $I \subset \Pi$ and $w \in \widehat{W}$ be such that $w^{-1}(I) = J \subset \widehat{\Pi}$. Then, we have :*

$$n_I \mathrm{Vol}_{l-\sharp J}(F_J) \quad = \quad n_J \mathrm{Vol}_{l-\sharp I}(F_I)$$

To prove this proposition, we need the following technical lemma :

Lemma 2.1.6 *Let $I \subset \Pi$ be such that $\sharp I \leqslant l-1$. Let $w \in \widehat{W}$ be such that $w^{-1}(I) = J \subset \widehat{\Pi}$. Let α_j be any element of J if $\alpha_0 \notin J$, and $\alpha_j = \alpha_0$ if $\alpha_0 \in J$. Set $\alpha_i = w(\alpha_j)$. Then we have :*

$$n_i d(\overline{\omega}_i, H_I) = n_j d(\overline{\omega}_j, H_J).$$

Proof. The result is clear if $J = \emptyset$. We may therefore assume that $1 \leqslant \sharp J \leqslant l-1$.

Step 1 : Assume that $\alpha_0 \in J$. We shall determine the distance $d(\overline{\omega}_0, H_J)$. Let J_0 be the connected component of J containing α_0. Set $r = \sharp J_0$.

If $J_0 = \{\alpha_0\}$, then the projection of 0 on H_J is $\frac{\theta}{\|\theta\|^2}$. Therefore, the distance $d(\overline{\omega}_0, H_J)$ is $\frac{1}{\|\theta\|}$.

Now assume that $J_0 \neq \{\alpha_0\}$. Then, $J_0 \setminus \{\alpha_0\}$ contains one or two roots β such that $(\beta, \theta) \neq 0$. Set $J_0 = \{\beta_1, \ldots, \beta_r\}$, $\alpha_0 = \beta_k$ and $V_{J_0} = \bigoplus_{\beta_i \in J_0 \setminus \{\alpha_0\}} \mathbb{R}\beta_i$.

First of all, assume that $J_0 \setminus \{\alpha_0\}$ contains only one root β_t such that $(\beta_t, \theta) \neq 0$. Let $\gamma_t \in V_{J_0}$ be such that $(\gamma_t, \beta_t) = 1$ and $(\gamma_t, \beta_i) = 0$ for all $\beta_i \in J_0 \setminus \{\beta_t, \beta_k\}$. Let $\mu_t = (\|\theta\|^2(1 - \frac{(\gamma_t,\theta)}{2}))^{-1}$ and $\beta = \mu_t(\theta - (\theta, \beta_t)\gamma_t)$. Then, we have $(\beta, \alpha) = 0$ for all $\alpha \in J_0 \setminus \{\alpha_0\}$ and

$$(\beta, \theta) = \mu_t[\|\theta\|^2 - (\theta, \beta_t)(\gamma_t, \theta)] = \mu_t\|\theta\|^2[1 - \frac{(\gamma_t, \theta)}{2}] = 1.$$

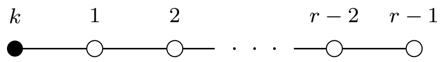
For all $x \in H_J$, we have $(\gamma_t, x) = 0$, and so

$$\begin{aligned}(\beta - x, \beta) &= \mu_t(\theta - (\theta, \beta_t)\gamma_t, \beta - x) \\ &= \mu_t[(\theta, \beta) - (\theta, x)] \\ &= 0.\end{aligned}$$

We have proved that β is the projection of $\overline{\omega}_0$ in H_J. It follows that by taking any $x \in H_J$, we have $d(\overline{\omega}_0, H_J)^2 = \|\beta\|^2 = (x, \beta) = \mu_t(x, \theta) = \mu_t$.

Since $I \subset \Pi$ and $J = w^{-1}(I)$ and I have the same Dynkin diagram, we have by a case by case consideration that J_0 is of type A_r, C_r, or D_r.

If J_0 is of type A_r, then by renumbering the roots β_i, the Dynkin diagram of J_0 is of the form :



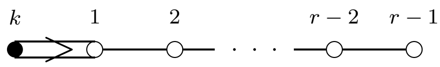
Then $t = 1$, and take

$$\gamma_t = \frac{2}{r\|\beta_1\|^2}((r-1)\beta_1 + (r-2)\beta_2 + \cdots + \beta_{r-1}).$$

So $(\gamma_t, \theta) = \frac{r-1}{r}$, and we have

$$\mu_t = \frac{2r}{(r+1)\|\theta\|^2}. \tag{2.3}$$

If J_0 is of type C_r, then the Dynkin diagram of J_0 is of the form :



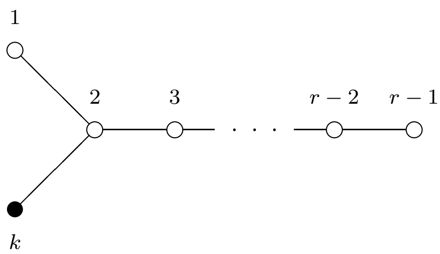
Again $t = 1$, and take

$$\gamma_t = \frac{2}{r\|\beta_1\|^2}((r-1)\beta_1 + (r-2)\beta_2 + \cdots + \beta_{r-1}).$$

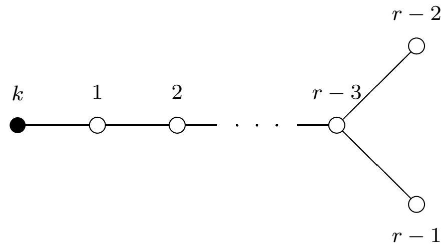
So $(\gamma_t, \theta) = \frac{2(r-1)}{r}$, and we have

$$\mu_t = \frac{r}{(r+1)\|\theta\|^2}. \tag{2.4}$$

If J_0 is of type D_r, then the Dynkin diagram of J_0 is of the form :



or of the form :



In the first case we have $t = 2$, and we take

$$\gamma_t = \frac{2}{r\|\beta_2\|^2}((r-2)\beta_1 + 2(r-2)\beta_2 + 2(r-3)\beta_3 + \cdots + 2\beta_{r-1}).$$

Thus $(\gamma_t, \theta) = \frac{2(r-2)}{r}$, and we have

$$\mu_t = \frac{r}{2\|\theta\|^2}. \tag{2.5}$$

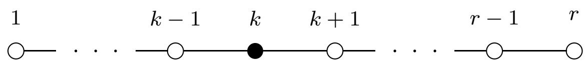
In the second case, we have $t = 1$ and we take

$$\gamma_t = \frac{1}{\|\beta_1\|^2}(2\beta_1 + 2\beta_2 + \cdots + 2\beta_{r-3} + \beta_{r-2} + \beta_{r-1}).$$

Thus we have

$$\mu_t = \frac{2}{\|\theta\|^2}. \tag{2.6}$$

Assume now that J_0 contains two roots α such that $(\alpha, \theta) \neq 0$. Then the Dynkin diagram of J_0 is of type A_r and these two roots are β_{k-1}, β_{k+1} :



Let $\eta, \eta' \in V_{J_0}$ be such that $(\eta, \beta_{k-1}) = 1 = (\eta', \beta_{k+1})$ and $(\eta, \beta_i) = 0$ (resp. $(\eta', \beta_i) = 0$) for all $\beta_i \in J_0 \setminus \{\beta_{k-1}, \beta_k\}$ (resp. $\beta_i \in J_0 \setminus \{\beta_k, \beta_{k+1}\}$). Let $\mu = (\|\theta\|^2(1 - \frac{(\eta+\eta',\theta)}{2}))^{-1}$ and $\beta = \mu(\theta - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta'))$. Then we have $(\beta, \alpha) = 0$ for all $\alpha \in J_0 \setminus \{\alpha_0\}$ and

$$(\beta, \theta) = \mu[\|\theta\|^2 - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta', \theta)] = 1$$
$$(\beta - x, \beta) = \mu(\theta - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k-1})\eta'), \beta - x) = 0$$

for all $x \in H_J$. We obtain that β is the projection of 0 on H_J. Take

$$\begin{aligned} \eta &= \frac{2}{k\|\beta_{k-1}\|^2}((k-1)\beta_{k-1} + (k-2)\beta_{k-2} + \cdots + \beta_1) \\ \eta' &= \frac{2}{(r-k+1)\|\beta_{k+1}\|^2}(\beta_r + 2\beta_{r-1} + \cdots + (r-k)\beta_{k+1}) \end{aligned}$$

then $(\eta + \eta', \theta) = \frac{k-1}{k} + \frac{r-k}{r-k+1}$. We obtain that :

$$d^2(\overline{\omega}_0, H_J) = \|\beta\|^2 = \mu = \frac{2k(r-k+1)}{n_0^2(r+1)\|\theta\|^2}. \tag{2.7}$$

Observe that the formulas (2.3) and (2.7) generalize the formula obtained when $J_0 = \{\alpha_0\}$. Let k be the position of $\alpha_0 \in J_0$, then we can sum up the above results in the following table, when $1 \leqslant \sharp J \leqslant l-1$:

J_0	A_r	C_r	D_r $t=2$	D_r $t=1$
$d(\overline{\omega}_0, H_J)^2$	$\frac{2k(r-k+1)}{n_0^2(r+1)\|\theta\|^2}$	$\frac{r}{n_0^2(r+1)\|\theta\|^2}$	$\frac{r}{2n_0^2\|\theta\|^2}$	$\frac{2}{n_0^2\|\theta\|^2}$

Table 2.1:

Step 2 : Assume that $J \subset \Pi$. Let $\alpha_j \in J$. We shall determine the distance $d(\overline{\omega}_j, H_J)$.

We have $H_J = \mathrm{Vect}(\overline{\omega}_t; t \text{ such that } \alpha_t \notin J) \subset H_{J\setminus\{\alpha_j\}} \subset V$. Let $H_J^\perp = \{x \in V;\ (x, \overline{\omega}_t) = 0 \text{ for all } t \text{ such that } \alpha_t \notin J\}$, then $H_J^\perp = \mathrm{Vect}(\alpha_t;\ \alpha_t \in J)$, and $\dim(H_J^\perp \cap H_{J\setminus\{\alpha_j\}}) = 1$. Since

$$H_{J\setminus\{\alpha_j\}} \cap H_J^\perp = \left\{ x = \sum_{\alpha_t \in J} \tau_t \alpha_t;\ (x, \beta) = 0 \text{ for all } \beta \in J \setminus \{\alpha_j\} \right\},$$

there exists $\gamma \in V$ such that $H_{J\setminus\{\alpha_j\}} \cap H_J^\perp = \mathrm{Vect}(\gamma)$, and $(\gamma, \alpha_j) \neq 0$. Thus, we have $H_{J\setminus\{\alpha_j\}} = H_J \oplus \mathbb{C}\gamma$. It follows that there exists $\mu \in \mathbb{C}^*$ such that $\overline{\omega}_j + \mu\gamma \in H_J$.

Let J_j be the connected component of J which contains α_j. Set $J_j = \{\beta_1, \ldots, \beta_r\}$, with $\alpha_j = \beta_k$. Set $V_{J_j} = \bigoplus_{\beta_j \in J_j} \mathbb{R}\beta_j$. We may choose γ such that $(\gamma,\alpha_j)=1$ and $(\gamma,\alpha)=0$ for all $\alpha\in J_j\setminus\{\alpha_j\}$. Since $(\overline{\omega}_j+\mu\gamma,\alpha_j)=0$, we obtain that $\mu=-\frac{1}{n_j}$, and hence

$$d(\overline{\omega}_j,H_J)=\frac{\|\gamma\|}{n_j}, \tag{2.8}$$

where γ depends only on the position of α_j in the Dynkin diagram of J_j.

Finally, we need to compute explicitely $d(\overline{\omega}_j,H_J)$ in some particular cases. We use the numbering of [TY, chapter 18].

If $J_j=A_r$, take

$$\gamma=\frac{2}{(r+1)\|\beta_k\|^2}[((r-k+1)\beta_1+2(r-k+1)\beta_2+\cdots$$
$$+(k-1)(r-k+1)\beta_{k-1}+k(r-k+1)\beta_k+k(r-k)\beta_{k+1}+\cdots+k\beta_r].$$

If $J_j=C_r$, and $k=r$, take

$$\gamma=\frac{2}{\|\beta_r\|^2}(\beta_1+2\beta_2+\cdots+(r-1)\beta_{r-1}+\frac{r}{2}\beta_r).$$

If $J_j=D_r$, take

$$\begin{aligned}
\gamma &= \frac{1}{\|\beta_r\|^2}[\beta_1+2\beta_2+\cdots+(r-2)\beta_{r-2} \\
&\quad +\frac{1}{2}[(r-2)\beta_{r-1}+r\beta_r]] && \text{if } k=r,\\
\gamma &= \frac{1}{\|\beta_r\|^2}[\beta_1+2\beta_2+\cdots+(r-2)\beta_{r-2} \\
&\quad +\frac{1}{2}[r\beta_{r-1}+(r-2)\beta_r]] && \text{if } k=r-1,\\
\gamma &= \frac{1}{\|\beta_1\|^2}[2\beta_1+2\beta_2+\cdots+2\beta_{r-2} \\
&\quad +\beta_{r-1}+\beta_r] && \text{if } k=1.
\end{aligned}$$

In these particular cases, we obtain the following result :

Final step : We are now in a position to prove the lemma. Let I_i be the connected component of I containing α_i. If $J\subset\Pi$, then we have the result by (2.8), since α_j and α_i have the same position in the Dynkin diagram of J_j and I_i respectively.

If $\alpha_0\in J$, then the connected component J_0 of J containing α_0 is of the type A_r, C_r, or D_r. Again since $w^{-1}(\alpha_0)$ and α_0 have the same position in the respective Dynkin diagram, we obtain the result by inspecting the correspondence between tables 2.1 and 2.2. $\square$

J_j	A_r	C_r $k=r$	D_r $k=r-1,r$	D_r $k=1$
$d(\overline{\omega}_j,H_J)^2$	$\frac{2k(r-k+1)}{n_j^2(r+1)\Vert\alpha_j\Vert^2}$	$\frac{r}{n_j^2(r+1)\Vert\alpha_j\Vert^2}$	$\frac{r}{2n_j^2\Vert\alpha_j\Vert^2}$	$\frac{2}{n_j^2\Vert\alpha_j\Vert^2}$

Table 2.2:

Proof. [Proof of proposition 2.1.5] The case $\sharp I=0$ is trivial since in this case, $F_I=F_J=\overline{A}$.

Let $I\subset\Pi$. Let us proceed by induction on $\sharp I$. If $\sharp I=1$, the result is proved in lemma 2.1.4. Assume that $l>\sharp I>1$ and that the claim is true for $\sharp I-1$. Let α_j be any element of J if $\alpha_0\notin J$, and $\alpha_j=\alpha_0$ if $\alpha_0\in J$. Set $\alpha_i=w(\alpha_j)$. Then, we have by lemma 2.1.6,

$$\begin{aligned}
n_J\mathrm{Vol}_{l-\sharp I}(F_I) &= n_J(l-\sharp I+1)\mathrm{Vol}_{l-\sharp I+1}(F_{I\setminus\{\alpha_i\}})\times\frac{1}{d(\overline{\omega}_i,H_I)}\\
&= n_j(l-\sharp I+1)\frac{n_I}{n_i}\mathrm{Vol}_{l-\sharp I+1}(F_{J\setminus\{\alpha_j\}})\times\frac{n_i}{n_jd(\overline{\omega}_j,H_J)}\\
&= n_I(l-\sharp I+1)\mathrm{Vol}_{l-\sharp I+1}(F_{J\setminus\{\alpha_j\}})\times\frac{1}{d(\overline{\omega}_j,H_J)}\\
&= n_I\mathrm{Vol}_{l-\sharp I}(F_J).
\end{aligned}$$

Finally, the result is clear if $\sharp I=l$ since in this case F_I (resp. F_J) is a single point. □

Observe that for $I\subset\Pi$, $F'_I=2F_I$, so

$$\mathrm{Vol}_{l-\sharp I}(F'_I)=2^{l-\sharp I}\mathrm{Vol}_{l-\sharp I}(F_I). \tag{2.9}$$

We obtain a generalization of Peterson's result :

Theorem 2.1.7 *Let $I\subset\Pi$, then*

$$\frac{1}{n_I}\sum_{w\in\mathcal{A}b_I}n_{w^{-1}(I)}=2^{l-\sharp I}.$$

Proof. Let $I \subset \Pi$ and $w \in \widehat{W}$. By propositions 1.2.8 and 2.1.1, then

$$\sum_{w \in \mathcal{A}b_I} \mathrm{Vol}_{l-\sharp I}(\overline{w}(\overline{A}) \cap H_I) = \mathrm{Vol}_{l-\sharp I}(F_I').$$

Observe that for any element $w \in \mathcal{A}b_I$, we have $\overline{w}(\overline{A}) \cap H_I = \overline{w}(F_{w^{-1}(I)})$, and $\mathrm{Vol}_{l-\sharp I}(\overline{w}(F_{w^{-1}(I)})) = \mathrm{Vol}_{l-\sharp I}(F_{w^{-1}(I)})$. So by proposition 2.1.5 and by (2.9), we obtain that

$$\sum_{w \in \mathcal{A}b_I} \frac{n_{w^{-1}(I)}}{n_I} \mathrm{Vol}_{l-\sharp I}(F_I) = 2^{l-\sharp I} \mathrm{Vol}_{l-\sharp I}(F_I)$$

Thus, we have the result. □

Theorem 2.1.8 *Let $I \subset \Pi$, if $\mathfrak{g}$ is of type A_l or C_l, then the parabolic subalgebras $\mathfrak{p}_I$ have exactly $2^{l-\sharp I}$ abelian ideals.*

Proof. If $\mathfrak{g}$ is of type A_l or C_l, the numbers n_i, for $i = 0, \ldots, l$, depends only on the length of α_i. It follows that for any $w \in \mathcal{A}b_I$, $n_I = n_{w^{-1}(I)}$. So by theorem 2.1.7, we obtain the result. □

Remark 2.1.9 *The fact that the numbers n_i, $i = 0, \ldots, l$, depends only on the length of α_i is false when $\mathfrak{g}$ is not of type A or C. Indeed, theorem 2.1.8 is false in general. For example, in B_3, the parabolic subalgebra $\mathfrak{p}_{\{\alpha_1\}}$ has only 3 abelian ideals. We shall see in the next section another way to count the number of abelian ideals in cases B and D.*

2.2 Enumeration via diagrams

In this section, we shall determine, via diagram enumeration, the number of ad-nilpotent (resp. abelian) ideals of $\mathfrak{p}_I$, for $I \subset \Pi$, when $\mathfrak{g}$ is simple and of classical type. We shall use the numbering of simple roots of [TY, chap.18].

Recall the following partial order on Δ^+ : $\alpha \leqslant \beta$ if $\beta - \alpha$ is a sum of positive roots. Then it is easy to see that $\Phi \in \mathcal{F}_\emptyset$ if and only if for all $\alpha \in \Phi, \beta \in \Delta^+$, such that $\alpha \leqslant \beta$, then $\beta \in \Phi$. When $\mathfrak{g}$ is of type A, B, C or D, we can display the positive roots into a diagram of suitable shape, as in [CP1]. Then, they established a bijection between elements of $\mathcal{F}_\emptyset$ and certain subdiagrams.

Let $I \subset \Pi$. In order to adapt this construction in the parabolic case $\mathfrak{p}_I$, we shall use a similar construction, but our diagram will depend not only on the type of $\mathfrak{g}$, but also on I.

Let $I \subset \Pi$ and $\gamma, \beta \in \Delta^+$. We say that $\beta \xrightarrow{I} \gamma$ if there exists $\eta \in I$ such that $\beta + \eta = \gamma$. Define an equivalence relation on $\Delta^+ \setminus \Delta_I$: for $I \subset \Pi$, $\gamma \sim_I \beta$ if there exist $\beta_1, \dots, \beta_s \in \Delta^+ \setminus \Delta_I$ such that

(i) $\beta = \beta_1, \ \gamma = \beta_s$,

(ii) either $\beta_i \xrightarrow{I} \beta_{i+1}$ or $\beta_{i+1} \xrightarrow{I} \beta_i$, for $i = 1, \dots, s-1$.

As the standard Levi factor of $\mathfrak{p}_I$ acts in a reductive way on the nilpotent radical, the fact that two roots β, γ are $\sim_I$ equivalent means that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are in the same simple submodule.

Let X be the type of $\mathfrak{g}$. The idea is to start by displaying the positive roots Δ^+ in a diagram T_X of a suitable shape as in [Sh] (or as reformulated in [CP1]) : that is, we assign to each box labelled (i,j) in T_X, a positive root $t_{i,j}$. The shape and the filling of T_X are chosen such that we obtain a bijection between elements of $\mathcal{F}_\emptyset$ and the northwest flushed subdiagrams, henceforth nw-diagrams, of T_X (in type D, we need to include also nw-diagrams modulo a permutation of certain columns). Then, for $I \subset \Pi$, we delete the boxes containing elements of Δ_I. Observe that the set of boxes of the same equivalent class is connected. Therefore, we can regroup into a big box all the roots of the same equivalent class. We obtain a new diagram denoted by T_X^I. Then, we count the nw-diagrams of T_X^I (again in type D, we need to count also nw-diagrams modulo a permutation of certain columns), which are clearly in bijection with the elements of $\mathcal{F}_I$.

2.2.1 Type A_l

If $\mathfrak{g}$ is of type A_l, then T_{A_l} is a diagram of shape $[l, l-1, \dots, 1]$. The label (i,j) means a box in the i-th row and the j-th column. The boxes (i,j) of T_{A_l} are filled by the positive roots $t_{i,j} = \alpha_i + \dots + \alpha_{l-j+1}$, $1 \leqslant i, j \leqslant l$. For example, for $l = 5$, we have :

$$\begin{array}{l}\overbrace{\hphantom{t_{1,1}t_{1,2}t_{1,3}t_{1,4}t_{1,5}}}^{l}\\ \begin{array}{|c|c|c|c|c|}\hline t_{1,1}&t_{1,2}&t_{1,3}&t_{1,4}&t_{1,5}\\ \hline t_{2,1}&t_{2,2}&t_{2,3}&t_{2,4}\\ \cline{1-4} t_{3,1}&t_{3,2}&t_{3,3}\\ \cline{1-3} t_{4,1}&t_{4,2}\\ \cline{1-2} t_{5,1}\\ \cline{1-1}\end{array}\end{array}$$

Let $I \subset \Pi$. We first delete the boxes containing elements of Δ_I. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root : for each $\alpha_i \in I$, we regroup the $(l-i+1)$-th and the $(l-i+2)$-th columns if $i \neq 1$, and the rows i, $i+1$ if $i \neq l$, on T_{A_l}. At the end, we obtain that $T^I_{A_l}$ is a diagram of shape $[l - \sharp I, l - \sharp I - 1, \ldots, 1]$. For example, for A_5 and $I = \{\alpha_2, \alpha_3\}$, we have :

$$\begin{array}{l}\overbrace{\hphantom{t_{1,1}t_{1,2}t_{1,3}t_{1,4}t_{1,5}}}^{l}\\ \begin{array}{|c|c|c|}\hline t_{1,1}&t_{1,2}&t_{1,3}t_{1,4}t_{1,5}\\ \hline t_{2,1}&t_{2,2}\\ \cline{1-2} t_{3,1}&t_{3,2}\\ \cline{1-2} t_{4,1}&t_{4,2}\\ \cline{1-2} t_{5,1}\\ \cline{1-1}\end{array}\end{array}$$

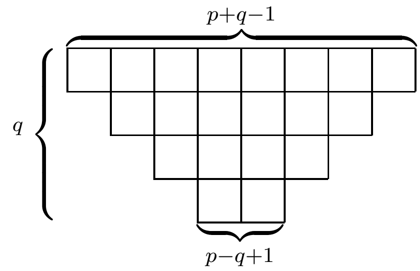
Let $\mathcal{C}_l = \dfrac{1}{l+1}\begin{pmatrix} 2l \\ l \end{pmatrix}$ denote the l-th Catalan number.

Proposition 2.2.1 *Let $I \subset \Pi$. Let $\mathcal{S}^I_{A_l}$ be the set of all nw-diagrams of $T^I_{A_l}$. Then, the cardinality of $\mathcal{S}^I_{A_l}$ is $\mathcal{C}_{l-\sharp I+1}$.*

Proof. Let $I \subset \Pi$, then $T^I_{A_l}$ if of shape $[l - \sharp I, l - \sharp I - 1, \ldots, 1]$, so by [St, 8,6.19 vv.], we obtain that the cardinality of the set of nw-diagrams of $T^I_{A_l}$ is $\mathcal{C}_{l-\sharp I+1}$. $\square$

2.2.2 Type C_l

Definition 2.2.2 *Let p, q be two integers such that $q \leqslant p$. Let $T_{p,q}$ be the (shifted) diagram of shape $[p+q-1, p+q-3, \ldots, p-q+1]$ arranged in the* *following way :*



If $\mathfrak{g}$ is of type C_l, then T_{C_l} is the diagram $T_{l,l}$, and the boxes (i,j) of T_{C_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l, & 1 \leqslant j \leqslant l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

Let $I \subset \Pi$, we first delete the boxes containing elements of Δ_I. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root : for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup column $2l-i$ and column $2l-i+1$ if $i \neq 1$, then we regroup the i-th and $(i+1)$-th columns and also the i-th and $(i+1)$-th rows on T_{C_l}. If $\alpha_l \in I$, we regroup also the columns l and $l+1$. We obtain at the end that $T_{C_l}^I$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}$ if $\alpha_l \notin I$ and of shape $T_{l-\sharp I+1, l-\sharp I}$, if $\alpha_l \in I$.

By [Pr], we obtain directly that the number of nw-diagram of $T_{p,q}$ is $\begin{pmatrix} p+q \\ p \end{pmatrix}$. Consequently, we have the following proposition :

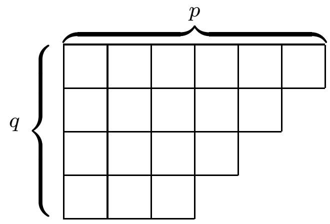
Proposition 2.2.3 *Let $I \subset \Pi$. Let $\mathcal{S}_{C_l}^I$ be the set of all nw-diagrams of $T_{C_l}^I$. Then, the cardinality of $\mathcal{S}_{C_l}^I$ is*

$$(l - \sharp I + 1)\mathcal{C}_{l-\sharp I} \quad \textit{if} \quad \alpha_l \notin I, \quad \textit{and} \quad \frac{l - \sharp I + 2}{2}\mathcal{C}_{l-\sharp I+1} \quad \textit{if} \quad \alpha_l \in I.$$

2.2.3 Type B_l and D_l

Let $I \subset \Pi$. Assume that $\mathfrak{g}$ is of type $X = B_l$ or D_l. Then the shape of T_X^I is more complicated than in the case A or C, so we need more combinatorial results on diagrams.

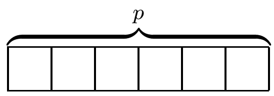
Definition 2.2.4 *Let p, q be two integers such that $q \leqslant p$. Let $T'_{p,q}$ be the diagram of q rows of the shape $[p, p-1, \ldots, p-q+1]$ arranged in the following way:*



Proposition 2.2.5 *Let p, q be two integers such that $q \leqslant p$. Then, the number of nw-diagrams of $T'_{p,q}$ is*

$$\mathcal{T}'_{p,q} = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$

Proof. Let $D_{p,q}$ be the set of nw-diagrams of $T'_{p,q}$. We shall proceed by induction on q. If $q = 1$, then $T'_{p,1}$ is



so we have

$$\sharp D_{p,q} = \mathcal{T}'_{p,q} = p + 1 = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$

Assume that $q > 1$ and the claim is true for $q-1$. For $1 \leqslant k \leqslant p-q+1$, let

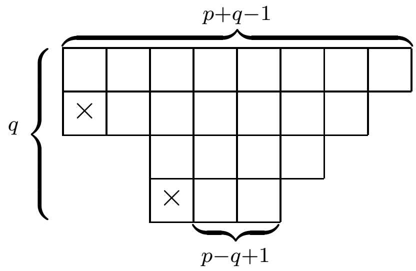
$$S_k = \{S \in D_{p,q}; (q,k) \in S \text{ and } (q,k+1) \notin S\}$$

Then, $\sharp S_k = \mathcal{T}'_{p-k,q-1}$ and $L = \bigcup_{k=1}^{p-q+1} S_k$ is the set of nw-diagrams containing at least a box in the last row of $T'_{p,q}$. Since $D_{p,q}$ is the disjoint union of $D_{p,q-1}$ and L, we obtain that :

$$\begin{aligned}
\mathcal{T}'_{p,q} &= \mathcal{T}'_{p,q-1} + \sharp L = \sum_{i=0}^{p-q+1} \mathcal{T}'_{p-i,q-1} \\
&= \sum_{k=q-1}^{p} \mathcal{T}'_{k,q-1} = \sum_{k=q-1}^{p} \frac{(k+q)!(k-q+3)}{(q-1)!(k+2)!} \\
&= \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}
\end{aligned}$$

where the last equality is a simple induction on $p \geqslant q$. □

Definition 2.2.6 *Let $p \geqslant q$ be two positive integers and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant q+1$ be some other integers. Denote by $T_{p,q}(l_1, l_2, \ldots, l_s)$ the new diagram obtained by adding to $T_{p,q}$ the boxes $(l_i, l_i - 1)$, for $1 \leqslant i \leqslant s$. For example, $T_{5,4}(2,4)$ is:*



where the added boxes are marked with a $\times$.

Proposition 2.2.7 *Let $p \geqslant q$ be two positive integers and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant q+1$ be some other integers, then the number of nw-diagrams of $T_{p,q}(l_1, l_2, \ldots, l_s)$ is*

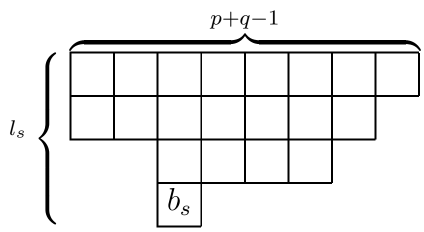
$$\binom{p+q}{p} + \sum_{j=1}^{s} \mathcal{T}'_{p+q-l_j, l_j-1}.$$

Proof. Let $D_{p,q}(l_1, \ldots, l_s)$ be the set of nw-diagrams of $T_{p,q}(l_1, \ldots, l_s)$ and $\mathcal{D}_{p,q}(l_1, \ldots, l_s)$ be its cardinality. Let $b_s = (l_s, l_s - 1)$. Set

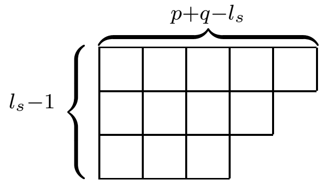
$$\begin{aligned} E &= \{S \in D_{p,q}(l_1, \ldots, l_s); b_s \notin S\}, \\ F &= \{S \in D_{p,q}(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{b_s\} \in E\}, \\ G &= \{S \in D_{p,q}(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{b_s\} \notin E\}. \end{aligned}$$

Then, we have clearly $\mathcal{D}_{p,q}(l_1, \ldots, l_s) = \sharp E + \sharp F + \sharp G$.

If $S \in F$, then S contains all the boxes north-west of b_s and the other boxes of S are strictly north-east of b_s, so there exists a bijection between F and the set of nw-diagrams of T'_{p+q-l_s, l_s-1}. For the example in definition 2.2.6, if $S \in F$, S is a nw-diagram of :



containing b_s. Hence it suffices to count the nw-diagrams of the subdiagram strictly north-east of b_s :



So by proposition 2.2.5, the cardinality of F is $\mathcal{T}'_{p+q-l_s,l_s-1}$.

If $S \in G$, then $S \setminus \{b_s\}$ is a nw-diagram of T where $T = T_{p,q}$ if $s = 1$ and $T = T_{p,q}(l_1, \ldots, l_{s-1})$ if $s > 1$. So the cardinality of G is the cardinality of the set of nw-diagrams in T minus the cardinality of the set H of nw-diagrams having at most $l_s - 1$ rows. Observe that the elements of H correspond to those of E. Hence, (by [Pr])

$$\sharp G = \begin{cases} \mathcal{D}_{p,q}(l_1, \ldots, l_{s-1}) - \sharp E & \text{if } s > 1, \\ \begin{pmatrix} p+q \\ p \end{pmatrix} - \sharp E & \text{if } s = 1. \end{cases}$$

The result now follows easily by induction on s. $\square$

Notations 2.2.8 *Fix $I \subset \Pi$. Let $I_1, \ldots, I_s$ be the connected components of I of cardinality $r_1, \ldots, r_s$ respectively. For each connected component I_j, set $m_j = \min\{i; \alpha_i \in I_j\}$. Without loss of generality, we shall assume that $m_1 < m_2 < \cdots < m_s$.*

If $\mathfrak{g}$ is of type B_l, then T_{B_l} is $T_{l,l}$ and the boxes (i,j) of T_{B_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_l), & 1 \leqslant j \leqslant l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

As before, for $I \subset \Pi$, we delete the boxes containing elements of Δ_I. For $j = 1, \ldots, s$, set

$$l_j = m_j - \sum_{k=1}^{j-1} r_k. \tag{2.10}$$

Regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root : for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup rows i and $i+1$ and if $i \neq 1$, we regroup column $2l-i$ and column $2l-i+1$, then the columns $i-1$ and i. If $\alpha_l \in I$, we also regroup the columns $l-1$, l and $l+1$. We obtain that $T_{B_l}^I$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}(l_1,\ldots,l_n)$, where the l_i are defined as above and, $n = s-1$ if $\alpha_l \in I$ and $n = s$ if $\alpha_l \notin I$. For example, for B_5 and $I = \{\alpha_2, \alpha_3, \alpha_5\}$, we have :

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} & t_{1,7} & t_{1,8} & t_{1,9} \\
\hline
 & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} & & & \\
 & & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} & & & \\
 & & & t_{4,4} & t_{4,5} & t_{4,6} & & & \\
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|}
\hline
 & & \\
\hline
b_1 & & \\
\hline
\end{array}
$$

It follows from proposition 2.2.7 that :

Proposition 2.2.9 *Let $I \subset \Pi$ be of cardinality r. Let $\mathcal{S}_{B_l}^I$ be the set of all nw-diagrams of $T_{B_l}^I$. Then, the cardinality of $\mathcal{S}_{B_l}^I$ is*

$$(l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{n} \mathcal{T}'_{2(l-r)-l_j, l_j-1}$$

where $n = s-1$ if $\alpha_l \in I$, and $n = s$ otherwise.

If $\mathfrak{g}$ is of type D_l, then T_{D_l} is $T_{l,l-1}$, and the boxes (i,j) of T_{D_l} are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l, & 1 \leqslant j \leqslant l-2, \\ \alpha_i + \cdots + \alpha_{l-2} + \alpha_l, & j = l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leqslant j \leqslant 2l-1. \end{cases}$$

For $I \subset \Pi$, we first delete the boxes containing elements of Δ_I. For $j = 1,\ldots,s$, set

$$l_j = \begin{cases} m_j - \displaystyle\sum_{k=1}^{j-1} r_k & \text{if } j \neq s \text{ or } I_s \neq \{\alpha_l\}, \\ m_j - \displaystyle\sum_{k=1}^{j-1} r_k - 1 & \text{if } j = s \text{ and } I_s = \{\alpha_l\}. \end{cases} \tag{2.11}$$

Regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root : for each $\alpha_i \in I \setminus \{\alpha_{l-1}, \alpha_l\}$, we first regroup the rows i and $i+1$ and if $i \neq 1$, we regroup column $2l-i-1$ and column $2l-i$, and then the columns $i-1$ and i.

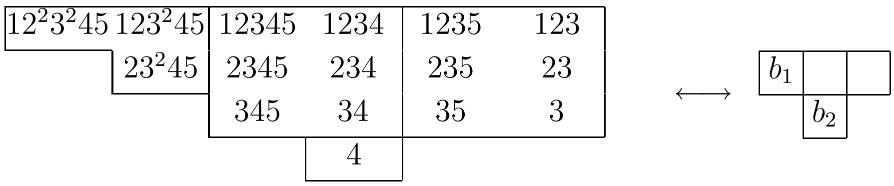
If $\alpha_{l-1} \in I$, but $\alpha_l \notin I$, then we regroup the columns $l-2, l-1$ and columns $l, l+1$.

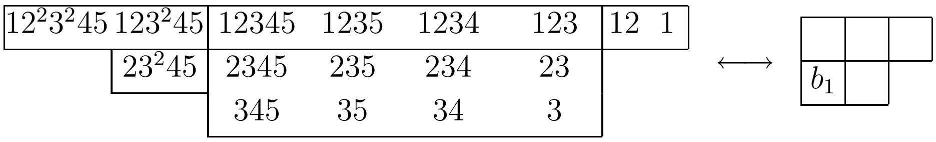
If $\alpha_l \in I$, but $\alpha_{l-1} \notin I$, then we first reverse the columns $l-1$ and l, and then we regroup the (new) columns $l-2, l-1$ and columns $l, l+1$.

If $\{\alpha_{l-1}, \alpha_l\} \subset I$, then we regroup the four columns $l-2, l-1$, l and $l+1$.

We obtain that if $\{\alpha_{l-1}, \alpha_l\} \not\subset I$, then with the l_i as defined above, $T^I_{D_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I-1}(l_1, \ldots, l_s)$. If $\{\alpha_{l-1}, \alpha_l\} \subset I$, then $T^I_{D_l}$ is a diagram of shape $T_{l-\sharp I, l-\sharp I}(l_1, \ldots, l_{s-1})$.

In the following examples, we denote by i the simple root α_i and by i^2 the element $2\alpha_i$ and for example 12^234 indicates $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. We consider, $X = D_5$ and I is respectively $\{\alpha_1, \alpha_2, \alpha_5\}$ and $\{\alpha_2, \alpha_4, \alpha_5\}$:





Definition 2.2.10 *For a subdiagram L of $T^I_{D_l}$, we shall denote by $L^\bullet$ the set of boxes of L obtained from L by exchanging columns $l-r-1$ and $l-r$ (resp. $l-r$ and $l-r+1$) if $\alpha_1 \notin I$ (resp. if $\alpha_1 \in I$).*

If $L^\bullet$ is a nw-diagram of $T^I_{D_l}$, then we say that L is a $\bullet$-nw-diagram of $T^I_{D_l}$.

Proposition 2.2.11 *Let $I \subset \Pi$ be of cardinality r. Let $\mathcal{S}^I_{D_l}$ be the set of nw-diagrams of $T^I_{D_l}$ if $\{\alpha_{l-1}, \alpha_l\} \cap I \neq \emptyset$, and be the union of the set of nw-diagrams and the set of $\bullet$-nw-diagrams of $T^I_{D_l}$ if $\{\alpha_{l-1}, \alpha_l\} \cap I = \emptyset$. Then, the cardinality of $\mathcal{S}^I_{D_l}$ is*

(i) $(3(l-r)-2)\mathcal{C}_{l-r-1} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1, l_j-2} + \mathcal{T}'_{2(l-r)-l_j-1, l_j-1}$, *if $t = 0$,*

(ii) $\displaystyle \frac{l-r+1}{2}\mathcal{C}_{l-r} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-1}$, *if* $t=1$,

(iii) $\displaystyle (l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{s-1} \mathcal{T}'_{2(l-r)-l_j,l_j-1}$, *if* $t=2$,

where $t = \sharp(\{\alpha_{l-1},\alpha_l\}\cap I)$.

Proof. Assume first that $\{\alpha_{l-1},\alpha_l\}\cap I=\emptyset$. Note that, the elements of $\mathcal{F}_I$ are in bijection with the subdiagrams S of $T^I_{D_l} = T_{l-r,l-r-1}(l_1,\dots,l_s)$ such that either S or $S^\bullet$ is a nw-diagram. Let

$$\begin{aligned} E_1 &= \text{ the set of nw-diagrams of } T^I_{D_l}, \\ E_2 &= (\text{the set of } \bullet\text{-nw-diagrams of } T^I_{D_l}) \setminus E_1. \end{aligned}$$

So $S^I_{D_l} = E_1 \cup E_2$ (disjoint union). By proposition 2.2.7, we have :

$$\sharp E_1 = \frac{l-r+1}{2}\mathcal{C}_{l-r} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-1}.$$

On the other hand, the number of elements of E_2 is $\sharp E_1 - \sharp F$, where F is the set of elements of E_1 having columns $l-r-1$ and $l-r$ (resp. $l-r$ and $l-r+1$) of the same length if $\alpha_1 \notin I$ (resp. if $\alpha_1 \in I$).

Clearly, the number of elements of F is exactly the number of nw-diagrams of the diagram obtained from $T^I_{D_l}$ by removing the $(l-r)$-th (resp. $(l-r+1)$-th) column if $\alpha_1 \notin I$ (resp. if $\alpha_1 \in I$). So, by proposition 2.2.7,

$$\sharp F = (l-r)\mathcal{C}_{l-r-1} + \sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-2,l_j-1}.$$

We obtain therefore the result since we have the equality :

$$\mathcal{T}'_{2(l-r)-l_j-1,l_j-1} - \mathcal{T}'_{2(l-r)-l_j-2,l_j-1} = \mathcal{T}'_{2(l-r)-l_j-1,l_j-2}.$$

If α_{l-1} or $\alpha_l \in I$, then there is no column reversing. Then the result follows from proposition 2.2.7 according to the shape of $T^I_{D_l}$. $\square$

As in [CP1], we have clearly a bijection between $\mathcal{F}_I$ and $\mathcal{S}^I_X$. It follows from propositions 2.2.1, 2.2.3, 2.2.9, 2.2.11, that we have :

Theorem 2.2.12 *Let $I \subset \Pi$ of cardinality r, X be the type of $\mathfrak{g}$ and s, l_j as defined in 2.2.8,* (2.10) *and* (2.11).

If $X = A_l$, then

$$\sharp\mathcal{F}_I = \mathcal{C}_{l-r+1}.$$

If $X = B_l$, then

$$\sharp\mathcal{F}_I = (l-r+1)\mathcal{C}_{l-r} + \sum_{j=1}^{n} \mathcal{T}'_{2(l-r)-l_j,l_j-1},$$

where $n = s-1$ if $\alpha_l \in I$, and $n = s$ otherwise.

If $X = C_l$, then

$$\sharp\mathcal{F}_I = \begin{cases} (l-r+1)\mathcal{C}_{l-r} & \text{if } \alpha_l \notin I, \\ \dfrac{l-r+2}{2}\mathcal{C}_{l-r+1} & \text{if } \alpha_l \in I. \end{cases}$$

If $X = D_l$, then

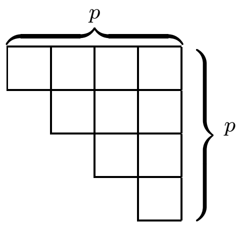
$$\sharp\mathcal{F}_I = \begin{cases} \dfrac{l-r+1}{2}\mathcal{C}_{l-r} + \displaystyle\sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & \text{if } \sharp\{\alpha_{l-1},\alpha_l\} \cap I = 1 \\ (l-r+1)\mathcal{C}_{l-r} + \displaystyle\sum_{j=1}^{s-1} \mathcal{T}'_{2(l-r)-l_j,l_j-1} & \text{if } \{\alpha_{l-1},\alpha_l\} \subset I, \\ (3(l-r)-2)\mathcal{C}_{l-r-1} + & \\ \displaystyle\sum_{j=1}^{s} \mathcal{T}'_{2(l-r)-l_j-1,l_j-2} + \mathcal{T}'_{2(l-r)-l_j-1,l_j-1} & \text{otherwise.} \end{cases}$$

2.2.4 Abelian ideals

We have already determined in theorem 2.1.8 the number of abelian ideals for type A and C. We shall now enumerate the abelian ideals of $\mathfrak{p}_I$ using diagrams when $\mathfrak{g}$ is of type B or D. Observe that a similar argument could be used to enumerate abelian ideals in type A and C.

Definition 2.2.13 *Let p be a positive integer and R_p be the diagram of shape*

$[p, p-1, \ldots, 1]$ *arranged in the following way :*



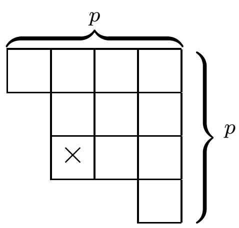
Proposition 2.2.14 *The number of nw-diagrams of R_p is 2^p.*

Proof. We shall procced by induction on p. If $p = 1$, the result is clear. Assume that $p > 1$ and the claim is true for $p - 1$. Let b be the box $(1, p)$ and

$$\begin{aligned} E &= \text{the set of nw-diagrams of } R_p \text{ which do not contain } b, \\ F &= \text{the set of nw-diagrams of } R_p \text{ which contain } b. \end{aligned}$$

Then, the number of nw-diagrams of R_p is $\sharp E + \sharp F$. Furthermore, by the induction hypothesis, we have $\sharp E = 2^{p-1} = \sharp F$, and we obtain the result. □

Definition 2.2.15 *Let p be a positive integer and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant p + 1$ be some other integers. Denote by $R_p(l_1, l_2, \ldots, l_s)$ the new diagram obtained by adding to R_p the boxes $(l_i, l_i - 1)$, for $1 \leqslant i \leqslant s$. For example, $R_4(3)$:*



Proposition 2.2.16 *Let p be a positive integer and $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant p + 1$ be some other integers, then the number of nw-diagrams of $R_p(l_1, l_2, \ldots, l_s)$ is*

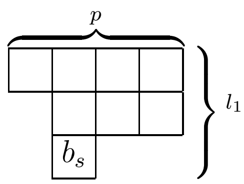
$$2^p + \sum_{j=1}^{s} \binom{p}{l_j - 1}.$$

Proof. Let $D_p(l_1, \ldots, l_s)$ be the set of nw-diagrams of $R_p(l_1, \ldots, l_s)$ and denote by $\mathcal{D}_p(l_1, \ldots, l_s)$ its cardinality. Let $b_s = (l_s, l_s - 1)$. Set

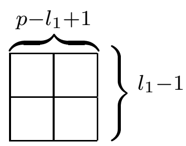
$$\begin{array}{l} E = \{S \in D_p(l_1, \ldots, l_s); b_s \notin S\} \\ F = \{S \in D_p(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{b_s\} \in E\} \\ G = \{S \in D_p(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{b_s\} \notin E\} \end{array}$$

Then we have clearly $\mathcal{D}_p(l_1, \ldots, l_s) = \sharp E + \sharp F + \sharp G$.

If $S \in F$, then S contains all the boxes north-west of b_s and the other boxes of S are strictly north-east of b_s, so there exists a bijection between F and the set of nw-diagrams of T where T is a diagram whose shape is a rectangle containing $p - l_j + 1$ columns and $l_j - 1$ rows. For the example in definition 2.2.15, if $S \in F$, S is a nw-diagram of :



containing b_s. Hence it suffices to count the nw-diagrams of the rectangular subdiagram strictly north-east of b_s :



So by [Pr] the cardinality of F is $\begin{pmatrix} p \\ l_j - 1 \end{pmatrix}$.

If $S \in G$, then $S \setminus \{b_s\}$ is a nw-diagram of L where $L = R_p$ if $s = 1$ and $L = R_p(l_1, \ldots, l_{s-1})$ if $s > 1$. So the cardinality of G is the cardinality of the set of nw-diagrams in L minus the cardinality of the set H of nw-diagrams having at most $l_s - 1$ rows. Observe that the elements of H correspond to those of E. Hence, by proposition 2.2.14

$$\sharp G = \begin{cases} \mathcal{D}_{p,q}(l_1, \ldots, l_{s-1}) - \sharp E & \text{if } s > 1, \\ 2^p - \sharp E & \text{if } s = 1. \end{cases}$$

The result now follows easily by induction on s. $\square$

Let $\mathcal{F}_I^{ab} = \{\Phi \in \mathcal{F}_I; \mathfrak{i}_\Phi \text{ is abelian }\}$. If S is a subdiagram of a diagram, let

$$\tau_h^S = \max\{k; (h,k) \in S\},$$

so (h, τ_h^S) is the right most box in the h-th row of S.

Proposition 2.2.17 *Assume that $\mathfrak{g}$ is of type B_l. Let $I \subset \Pi$ be of cardinality r. Consider $\Phi \in \mathcal{F}_I$ and S its corresponding nw-diagram in $T_{B_l}^I$. Then $\Phi \in \mathcal{F}_I^{ab}$ if and only if*
(a) $\tau_1^S \leq l - r$ *if* $\alpha_1 \in I$,
(b) $\tau_1^S + \tau_2^S \leq 2(l-r) - 1$ *if* $\alpha_1 \notin I$.

Proof. Let S_0 be the corresponding nw-diagram of Φ in $T_{B_l}^\emptyset$, then by [CP1], we have $\Phi \in \mathcal{F}_\emptyset^{ab}$ if and only if $\tau_1^{S_0} + \tau_2^{S_0} \leq 2l - 1$.

If $\alpha_1 \in I$, then $\tau_1^{S_0} = \tau_2^{S_0}$. The regrouping process reduces the number of columns on the left of column l of $T_{B_l}^\emptyset$ by one for each simple root in $I \setminus \{\alpha_1\}$. It follows that $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_1^S \leq l - r$.

The argument is similar for the case $\alpha_1 \notin I$. □

Proposition 2.2.18 *Assume that $\mathfrak{g}$ is of type B_l. Let $I \subset \Pi$ be of cardinality r, and $l_1, \ldots, l_s$ be as defined in* (2.10)*. Then we have :*

$$\sharp\mathcal{F}_I^{ab} = \begin{cases} 2^{l-r} + \displaystyle\sum_{j=1}^{n} 2\binom{l-r-1}{l_j - 1} & \text{if } \alpha_1 \notin I, \\ 2^{l-r-1} + \displaystyle\sum_{j=1}^{n} \binom{l-r-1}{l_j - 1} & \text{if } \alpha_1 \in I, \end{cases}$$

where $n = s$ if $\alpha_l \notin I$ and $n = s - 1$ if $\alpha_l \in I$.

Proof. Recall that $T_{B_l}^I$ is of shape $T_{l-r,l-r}(l_1, \ldots, l_n)$, where $n = s$ if $\alpha_l \notin I$ and $n = s - 1$ if $\alpha_l \in I$.

Let $\Phi \in \mathcal{F}_I$ and S be the nw-diagram of $T_{B_l}^I$ corresponding to Φ.

Assume that $\alpha_1 \in I$, then $l_1 = 1$. By proposition 2.2.17, S is in the left hand half of $T_{B_l}^I$, so it is a nw-diagram of $R_{l-r-1}(1, \ldots, l_n)$. We then obtain the result by proposition 2.2.16.

Assume that $\alpha_1 \notin I$. Let E be the set of nw-diagrams of $T_{B_l}^I$ associated to elements of $\mathcal{F}_I^{ab}$. Set

$$\begin{aligned} P &= \{S \in E; \tau_1^S \leqslant l - r - 1\}, \\ Q &= \{S \in E; \tau_1^S > l - r - 1\}. \end{aligned}$$

Then, we have $\sharp E = \sharp P + \sharp Q$.

If $S \in P$, then S is included in the left hand half of $T_{B_l}^I$, so

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j - 1}$$

by proposition 2.2.16.

For $i = l-r, \ldots, 2(l-r)-1$, let $Q_i = \{S \in Q; \tau_1^S = i\}$ and $P_i = \{S \in P; \tau_1^S = 2(l-r)-1-i\}$. We then have :

$$Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i.$$

For $i = l-r, \ldots, 2(l-r)-1$, we have an obvious bijection between P_i and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \ldots, (1, i)$. Therefore $\sharp P = \sharp Q$ and the result follows. □

Proposition 2.2.19 *Assume that $\mathfrak{g}$ is of type D_l. Let $I \subset \Pi$ be of cardinality r. Consider $\Phi \subset \mathcal{F}_I$ and S_Φ its corresponding subdiagram in $T_{D_l}^I$. Set $S = S_\Phi$ if S_Φ is a nw-diagram and $S = S_\Phi^\bullet$ if $S_\Phi^\bullet$ is a nw-diagram. Then $\Phi \in \mathcal{F}_I^{ab}$ if and only if*

(a) $\tau_1^S \leq l-r$ *if* $\alpha_1 \in I$,

(b) $\tau_1^S + \tau_2^S \leq 2(l-r)-2$ *if* $\alpha_1 \notin I$.

Proof. If $I = \emptyset$, set $S = S_0$, then by [CP1], we have $\Phi \in \mathcal{F}_\emptyset^{ab}$ if and only if $\tau_1^{S_0} + \tau_2^{S_0} \leq 2l-2$.

Assume that $\alpha_1 \in I$, then $\tau_1^{S_0} = \tau_2^{S_0}$. The regrouping process reduces the number of columns of the left of column l of $T_{D_l}^\emptyset$ by one for each simple root in $I \setminus \{\alpha_1\}$. It follows that $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_1^S \leq l-r$.

The argument is similar for the case $\alpha_1 \notin I$. □

Proposition 2.2.20 *Assume that $\mathfrak{g}$ is of type D_l. Let $I \subset \Pi$ be of cardinality r and $l_1, \ldots, l_s$ be as defined in* (2.11)*. Set $t = \sharp(\{\alpha_{l-1}, \alpha_l\} \cap I)$. If $\alpha_1 \in I$, then the cardinality of $\mathcal{F}_I^{ab}$ is :*

(i) $2^{l-r} - 2^{l-r-2} + \sum_{j=1}^{s} \left[2\binom{l-r-1}{l_j-1} - \binom{l-r-2}{l_j-1} \right]$, *if* $t = 0$,

(ii) $2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1}$, *if* $t = 1$,

(iii) $2^{l-r-1}+\sum_{j=1}^{s-1}\begin{pmatrix} l-r-1 \\ l_j-1 \end{pmatrix}$, *if* $t=2$.

If $\alpha_1 \notin I$*, then the cardinality of* $\mathcal{F}_I^{ab}$ *is :*

(iv) $2^{l-r}+\sum_{j=1}^{s}2\begin{pmatrix} l-r-1 \\ l_j-1 \end{pmatrix}$, *if* $t=0$,

(v) $2^{l-r-1}+2^{l-r-2}+\sum_{j=1}^{s}\begin{pmatrix} l-r-1 \\ l_j-1 \end{pmatrix}+\sum_{j=1}^{s-1}\begin{pmatrix} l-r-2 \\ l_j-1 \end{pmatrix}$, *if* $t=1$,

(vi) $2^{l-r}+2\sum_{j=1}^{s-1}\begin{pmatrix} l-r-1 \\ l_j-1 \end{pmatrix}$, *if* $t=2$.

Proof. We proceed as in the case of type B_l but here, we need to take into account column reversing.

Recall that if $t=0$ or 1, $T_{D_l}^I$ is of shape $T_{l-r,l-r-1}(l_1,\dots,l_s)$ and if $t=2$, $T_{D_l}^I$ is of shape $T_{l-r,l-r}(l_1,\dots,l_{s-1})$.

Let $\mathcal{S}_I^{ab}$ be the set of subdiagrams of $T_{D_l}^I$ corresponding to elements of $\mathcal{F}_I^{ab}$. The shape of elements of $\mathcal{S}_I^{ab}$ is conditioned by proposition 2.2.19. Let

$$\begin{aligned} E_1 &= \text{the set of nw-diagrams in } \mathcal{S}_I^{ab}, \\ E_2 &= (\text{the set of } \bullet\text{-nw-diagrams in } \mathcal{S}_I^{ab}) \setminus E_1. \end{aligned}$$

Consider $\Phi \in \mathcal{F}_I^{ab}$ and S its corresponding subdiagram in $\mathcal{S}_I^{ab}$.

First assume that $\alpha_1 \in I$, then $l_1=1$. If $S \in E_1$, by proposition 2.2.19, S is in the left hand half of $T_{D_l}^I$, so it is a nw-diagram of $R_{l-r-1}(1,\dots,l_n)$, where $n=s$ if $t=0,1$ and $n=s-1$ if $t=2$. Hence, by proposition 2.2.16, we have

$$\sharp E_1 = 2^{l-r-1}+\sum_{j=1}^{n}\begin{pmatrix} l-r-1 \\ l_j-1 \end{pmatrix}.$$

If $t\neq 0$, there is no column reversing, so $E_2=\emptyset$. If $t=0$, the number of elements of E_2 is $\sharp E_1-\sharp(F\cap E_1)$, where F is the set of nw-diagrams of $T_{D_l}^I$ having columns $l-r$ and $l-r+1$ of the same length.

Clearly, the number of elements of F is exactly the number of nw-diagrams of the diagram obtained from $T_{D_l}^I$ by removing the $(l-r+1)$-th column. So, by proposition 2.2.19, the set of elements which are in $F\cap E_1$ is in bijection with the set of nw-diagrams of $R_{l-r-2}(1,\dots,l_s)$. So by proposition 2.2.16, we obtain :

$$\sharp F = 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-2}{l_j-1}.$$

We obtain therefore the result.

Now assume that $\alpha_1 \notin I$. Set

$$\begin{aligned} P &= \{S \in E_1; \tau_1^S \leqslant l-r-1\}, \\ \widetilde{P} &= \{S \in E_1; \tau_1^S \leqslant l-r-2\}, \\ Q &= \{S \in E_1; \tau_1^S > l-r-1\}. \end{aligned}$$

Then, we have $\sharp E_1 = \sharp P + \sharp Q$.

First assume that $t=0$ or 1. If $S \in P$, then S is included in the left hand half of $T^I_{D_l}$, so by proposition 2.2.16, we have

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1}$$

For $i = l-r, \ldots, 2(l-r)-2$, let $Q_i = \{S \in Q; \tau_1^S = i\}$ and $\widetilde{P}_i = \{S \in P; \tau_1^S = 2(l-r)-2-i\}$. We then have :

$$Q = \bigcup_{i=l-r}^{2(l-r)-2} Q_i \quad \text{and} \quad \widetilde{P} = \bigcup_{i=l-r}^{2(l-r)-2} \widetilde{P}_i.$$

For $i = l-r, \ldots, 2(l-r)-2$, we have an obvious bijection between $\widetilde{P}_i$ and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \ldots, (1, i)$. Therefore $\sharp\widetilde{P} = \sharp Q$ and by proposition 2.2.16, we have

$$\sharp \widetilde{P} = 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-2}{l_j-1}.$$

If $t=1$, there is no column reversing, so we have the result. If $t=0$, then the number of elements of E_2 is $\sharp E_1 - \sharp F$, where $F = E_1 \cap \{\bullet\text{-nw-diagrams of } \mathcal{S}_I^{ab}\}$. By proposition 2.2.19, we have $F = Q \cup \widetilde{P}$, so by the consideration above, we have $\sharp F = 2\sharp Q$. It follows that $\sharp E_1 + \sharp E_2 = 2\sharp P$.

For the last case $t=2$, the shape of $T^I_{D_l}$ is $T_{l-r,l-r}(l_1, \ldots, l_{s-1})$. If $S \in P$, then S is included in the left hand half of $T^I_{D_l}$, so by proposition 2.2.16, we have

$$\sharp P = 2^{l-r-1} + \sum_{j=1}^{s-1} \binom{l-r-1}{l_j-1}.$$

We have $E_2 = \emptyset$, and Q_i is defined for $i = l-r, \ldots, 2(l-r)-1$. Set $P_i = \{S \in P; \tau_1^S = 2(l-r)-1-i\}$. We then have :

$$Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i.$$

As above, for $i = l-r, \ldots, 2(l-r)-1$, we have an obvious bijection between P_i and Q_i given by the adding or removing of boxes $(1, 2(l-r)-i), \ldots, (1, i)$. Therefore $\sharp P = \sharp Q$ and the result follows. $\square$

Remark 2.2.21 *All the results above depend on the numbering of simple roots.*

2.3 Enumeration for exceptional types

In this section, we determine the number of ad-nilpotent and abelian ideals of a parabolic subalgebra when $\mathfrak{g}$ is of exceptional types E, F and G. This is done by using GAP4.

First of all, we determine the set of antichains in Δ^+. Then, by section 1.3, we obtain from the antichains all the elements of $\mathcal{F}_\emptyset$.

Next, to check if an element $\Phi \in \mathcal{F}_\emptyset$ is an element of $\Phi \in \mathcal{F}_{\{\alpha\}}$, for $\alpha \in \Pi$, it is enough to check that $\{\beta-\alpha; \alpha \in \Phi\} \cap (\Delta^+ \cup \{0\}) \subset \Phi$. Thus, for $I \subset \Pi$, we obtain that the elements of $\mathcal{F}_I$ are those of $\mathcal{F}_\emptyset$ satisfying the condition above for each $\alpha \in I$.

Let $\Phi \in \mathcal{F}_I$, then Φ corresponds to an abelian ideal if and only if $\Phi^2 = \emptyset$. Since the roots in Φ^2 corresponds to those which are in the derived subalgebra of the ideal which corresponds to Φ, it is also an ideal. So $\Phi^2 \neq \emptyset$ if and only if the highest root θ is an element of Φ^2. Then, to check if Φ is abelian, we need only to check that $\theta \notin \Phi^2$.

Let $\mathcal{N}_I$ be the set of ad-nilpotent ideals of $\mathfrak{p}_I$ and let $\mathcal{A}b_I$ be the set of abelian ideals of $\mathfrak{p}_I$. The following tables give the cardinality of $\mathcal{N}_I$ and $\mathcal{A}b_I$ for each type E, F and G. The subset I of Π is described by the symbol $\bullet$ in the Dynkin diagram without arrow which corresponds to the type we consider.

For example, if we consider E_6, the diagram which corresponds to $I = \{\alpha_2, \alpha_5\}$ is

$$\begin{smallmatrix} & & \bullet & & \\ \circ & \circ & \circ & \bullet & \circ \end{smallmatrix}$$

where we use the numberings of [TY].

The orientations for the diagram in type F and G are those in [TY].

Table E_6

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
○ ○○○○○	833	64	○ ●○○○○	197	21
● ○○○○○	201	40	○ ○●○○○	255	40
○ ○○●○○	323	41	○ ○○○●○	255	40
○ ○○○○●	197	21	● ●○○○○	60	16
○ ●●○○○	51	10	○ ●○●○○	81	16
○ ●○○●○	68	16	○ ●○○○●	56	8
● ○●○○○	82	24	● ○○●○○	60	22
● ○○○●○	82	24	● ○○○○●	60	16
○ ○●●○○	80	22	○ ○●○●○	90	24
○ ○●○○●	68	16	○ ○○●●○	80	22
○ ○○●○●	81	16	○ ○○○●●	51	10
● ●●○○○	21	8	● ●○●○○	23	11
● ●○○●○	26	11	● ●○○○●	20	7
○ ●●●○○	16	6	○ ●●○●○	21	8
○ ●●○○●	18	5	○ ●○●●○	27	11
○ ●○●○●	24	6	○ ●○○●●	18	5
● ○●●○○	19	10	● ○●○●○	36	15
● ○●○○●	26	11	● ○○●●○	19	10
● ○○●○●	23	11	● ○○○●●	21	8
○ ○●●●○	20	10	○ ○●●○●	27	11
○ ○●○●●	21	8	○ ○○●●●	16	6
● ●●●○○	6	4	● ●●○●○	10	6
● ●●○○●	8	4	● ●○●●○	9	6
● ●○●○●	9	5	● ●○○●●	8	4
○ ●●●●○	6	4	○ ●●●○●	7	3
○ ●●○●●	8	4	○ ●○●●●	7	3
● ○●●●○	5	4	● ○●●○●	9	6
● ○●○●●	10	6	● ○○●●●	6	4
○ ○●●●●	6	4	● ●●●●○	2	2
● ●●●○●	3	2	● ●●○●●	4	3
● ●○●●●	3	2	○ ●●●●●	3	2
● ○●●●●	2	2	● ●●●●●	1	1

Table E_7

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
$\circ\,\circ\,\overset{\circ}{\circ}\,\circ\,\circ\,\circ$	4160	128	$\bullet\,\circ\,\overset{\circ}{\circ}\,\circ\,\circ\,\circ$	837	70
$\circ\,\circ\,\overset{\bullet}{\circ}\,\circ\,\circ\,\circ$	980	78	$\circ\,\bullet\,\overset{\circ}{\circ}\,\circ\,\circ\,\circ$	1251	73
$\circ\,\circ\,\overset{\circ}{\bullet}\,\circ\,\circ\,\circ$	1600	84	$\circ\,\circ\,\overset{\circ}{\circ}\,\bullet\,\circ\,\circ$	1385	73
$\circ\,\circ\,\overset{\circ}{\circ}\,\circ\,\bullet\,\circ$	1076	70	$\circ\,\circ\,\overset{\circ}{\circ}\,\circ\,\circ\,\bullet$	879	32
$\bullet\,\circ\,\overset{\bullet}{\circ}\,\circ\,\circ\,\circ$	261	42	$\bullet\,\bullet\,\overset{\circ}{\circ}\,\circ\,\circ\,\circ$	202	35
$\bullet\,\circ\,\overset{\circ}{\bullet}\,\circ\,\circ\,\circ$	358	47	$\bullet\,\circ\,\overset{\circ}{\circ}\,\bullet\,\circ\,\circ$	314	45
$\bullet\,\circ\,\overset{\circ}{\circ}\,\circ\,\bullet\,\circ$	261	42	$\bullet\,\circ\,\overset{\circ}{\circ}\,\circ\,\circ\,\bullet$	215	24
$\circ\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\circ\,\circ$	391	39	$\circ\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\circ$	298	46
$\circ\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\circ$	374	39	$\circ\,\circ\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\circ$	315	42
$\circ\,\circ\,\overset{\bullet}{\circ}\,\circ\,\circ\,\bullet$	231	23	$\circ\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\circ\,\circ$	373	41
$\circ\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\circ\,\circ$	456	48	$\circ\,\bullet\,\overset{\circ}{\circ}\,\circ\,\bullet\,\circ$	377	45
$\circ\,\bullet\,\overset{\circ}{\circ}\,\circ\,\circ\,\bullet$	285	21	$\circ\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\circ$	415	41
$\circ\,\circ\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\circ$	467	47	$\circ\,\circ\,\overset{\circ}{\bullet}\,\circ\,\circ\,\bullet$	357	25
$\circ\,\circ\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\circ$	291	35	$\circ\,\circ\,\overset{\circ}{\circ}\,\bullet\,\circ\,\bullet$	288	22
$\circ\,\circ\,\overset{\circ}{\circ}\,\circ\,\bullet\,\bullet$	203	13	$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\circ\,\circ$	83	18
$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\circ$	101	25	$\bullet\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\circ$	110	25
$\bullet\,\circ\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\circ$	95	26	$\bullet\,\circ\,\overset{\bullet}{\circ}\,\circ\,\circ\,\bullet$	71	16
$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\circ\,\circ$	64	20	$\bullet\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\circ\,\circ$	88	25
$\bullet\,\bullet\,\overset{\circ}{\circ}\,\circ\,\bullet\,\circ$	77	23	$\bullet\,\bullet\,\overset{\circ}{\circ}\,\circ\,\circ\,\bullet$	62	15
$\bullet\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\circ$	123	26	$\bullet\,\circ\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\circ$	122	30
$\bullet\,\circ\,\overset{\circ}{\bullet}\,\circ\,\circ\,\bullet$	94	18	$\bullet\,\circ\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\circ$	89	23
$\bullet\,\circ\,\overset{\circ}{\circ}\,\bullet\,\circ\,\bullet$	81	17	$\bullet\,\circ\,\overset{\circ}{\circ}\,\circ\,\bullet\,\bullet$	61	11
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\circ$	83	18	$\circ\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\circ$	160	24
$\circ\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\circ$	138	25	$\circ\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\circ\,\bullet$	98	13
$\circ\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\circ$	80	18	$\circ\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\circ$	113	25
$\circ\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\bullet$	80	17	$\circ\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\circ$	94	18
$\circ\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\bullet$	105	16	$\circ\,\circ\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\bullet$	64	10
$\circ\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\circ$	95	22	$\circ\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\circ$	137	26
$\circ\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\circ\,\bullet$	100	15	$\circ\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\circ$	115	25
$\circ\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\circ\,\bullet$	111	14	$\circ\,\bullet\,\overset{\circ}{\circ}\,\circ\,\bullet\,\bullet$	73	10
$\circ\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\circ$	93	20	$\circ\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\bullet$	101	15
$\circ\,\circ\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\bullet$	86	11	$\circ\,\circ\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\bullet$	54	7

Table E_7 (cont'd)

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\circ$	21	8	$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\circ$	39	13
$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\circ$	34	13	$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\circ\,\bullet$	26	8
$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\circ$	34	12	$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\circ$	41	16
$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\bullet$	30	11	$\bullet\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\circ$	34	13
$\bullet\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\bullet$	35	12	$\bullet\,\circ\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\bullet$	23	8
$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\circ$	23	11	$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\circ$	29	14
$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\circ\,\bullet$	23	10	$\bullet\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\circ$	33	14
$\bullet\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\circ\,\bullet$	28	10	$\bullet\,\bullet\,\overset{\circ}{\circ}\,\circ\,\bullet\,\bullet$	22	8
$\bullet\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\circ$	35	14	$\bullet\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\bullet$	35	11
$\bullet\,\circ\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\bullet$	28	9	$\bullet\,\circ\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\bullet$	20	6
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\circ$	20	8	$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\circ$	37	12
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\bullet$	27	8	$\circ\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\circ$	45	13
$\circ\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\bullet$	49	9	$\circ\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\bullet$	29	7
$\circ\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\circ$	21	8	$\circ\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\bullet$	32	10
$\circ\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\bullet$	26	8	$\circ\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\bullet$	24	6
$\circ\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\circ$	27	11	$\circ\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\bullet$	31	8
$\circ\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\bullet$	31	8	$\circ\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\bullet$	23	5
$\circ\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\bullet$	18	5	$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\circ$	6	4
$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\circ$	10	6	$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\circ\,\bullet$	8	4
$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\circ$	15	8	$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\circ\,\bullet$	14	6
$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\circ\,\bullet\,\bullet$	10	5	$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\circ$	10	6
$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\bullet$	14	7	$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\bullet$	11	6
$\bullet\,\circ\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\bullet$	10	5	$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\circ$	9	6
$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\circ\,\bullet$	9	5	$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\circ\,\bullet\,\bullet$	10	6
$\bullet\,\bullet\,\overset{\circ}{\circ}\,\bullet\,\bullet\,\bullet$	9	4	$\bullet\,\circ\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\bullet$	8	4
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\circ$	6	4	$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\bullet$	10	4
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\bullet$	11	5	$\circ\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\bullet$	12	4
$\circ\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\bullet$	8	4	$\circ\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\bullet$	7	3
$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\circ$	2	2	$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\circ\,\bullet$	3	2
$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\circ\,\bullet\,\bullet$	4	3	$\bullet\,\bullet\,\overset{\bullet}{\circ}\,\bullet\,\bullet\,\bullet$	5	3
$\bullet\,\circ\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\bullet$	4	3	$\bullet\,\bullet\,\overset{\circ}{\bullet}\,\bullet\,\bullet\,\bullet$	3	2
$\circ\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\bullet$	3	2	$\bullet\,\bullet\,\overset{\bullet}{\bullet}\,\bullet\,\bullet\,\bullet$	1	1

Table E_8

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
○ ○ ○ ○ ○ ○ ○ ○	25080	256	○ ● ○ ○ ○ ○ ○ ○	4554	143
● ○ ○ ○ ○ ○ ○ ○	6188	124	○ ○ ● ○ ○ ○ ○ ○	7397	153
○ ○ ○ ● ○ ○ ○ ○	9541	163	○ ○ ○ ○ ● ○ ○ ○	8691	153
○ ○ ○ ○ ○ ● ○ ○	7169	149	○ ○ ○ ○ ○ ○ ● ○	5512	111
○ ○ ○ ○ ○ ○ ○ ●	4452	121	● ● ○ ○ ○ ○ ○ ○	1422	58
○ ● ● ○ ○ ○ ○ ○	1121	79	○ ● ○ ● ○ ○ ○ ○	1981	85
○ ● ○ ○ ● ○ ○ ○	1818	80	○ ● ○ ○ ○ ● ○ ○	1554	85
○ ● ○ ○ ○ ○ ● ○	1260	65	○ ● ○ ○ ○ ○ ○ ●	961	71
● ○ ● ○ ○ ○ ○ ○	2183	63	● ○ ○ ● ○ ○ ○ ○	1898	71
● ○ ○ ○ ● ○ ○ ○	2183	86	● ○ ○ ○ ○ ● ○ ○	2047	81
● ○ ○ ○ ○ ○ ● ○	1583	61	● ○ ○ ○ ○ ○ ○ ●	1139	71
○ ○ ● ● ○ ○ ○ ○	2150	88	○ ○ ● ○ ● ○ ○ ○	2786	88
○ ○ ● ○ ○ ● ○ ○	2448	93	○ ○ ● ○ ○ ○ ● ○	1890	73
○ ○ ● ○ ○ ○ ○ ●	1411	79	○ ○ ○ ● ● ○ ○ ○	2502	84
○ ○ ○ ● ○ ● ○ ○	3039	99	○ ○ ○ ● ○ ○ ● ○	2365	79
○ ○ ○ ● ○ ○ ○ ●	1787	85	○ ○ ○ ○ ● ● ○ ○	1906	74
○ ○ ○ ○ ● ○ ● ○	2091	74	○ ○ ○ ○ ● ○ ○ ●	1568	80
○ ○ ○ ○ ○ ● ● ○	1279	55	○ ○ ○ ○ ○ ● ○ ●	1244	71
○ ○ ○ ○ ○ ○ ● ●	924	52	● ● ● ○ ○ ○ ○ ○	392	26
● ● ○ ● ○ ○ ○ ○	521	30	● ● ○ ○ ● ○ ○ ○	599	41
● ● ○ ○ ○ ● ○ ○	549	41	● ● ○ ○ ○ ○ ● ○	439	31
● ● ○ ○ ○ ○ ○ ●	317	37	○ ● ● ● ○ ○ ○ ○	361	42
○ ● ● ○ ● ○ ○ ○	484	40	○ ● ● ○ ○ ● ○ ○	446	48
○ ● ● ○ ○ ○ ● ○	378	40	○ ● ● ○ ○ ○ ○ ●	267	43
○ ● ○ ● ● ○ ○ ○	658	41	○ ● ○ ● ○ ● ○ ○	741	53
○ ● ○ ● ○ ○ ● ○	598	45	○ ● ○ ● ○ ○ ○ ●	433	48
○ ● ○ ○ ● ● ○ ○	514	39	○ ● ○ ○ ● ○ ● ○	541	43
○ ● ○ ○ ● ○ ○ ●	391	46	○ ● ○ ○ ○ ● ● ○	356	32
○ ● ○ ○ ○ ● ○ ●	336	43	○ ● ○ ○ ○ ○ ● ●	239	30
● ○ ● ● ○ ○ ○ ○	430	29	● ○ ● ○ ● ○ ○ ○	873	45
● ○ ● ○ ○ ● ○ ○	814	45	● ○ ● ○ ○ ○ ● ○	636	35
● ○ ● ○ ○ ○ ○ ●	461	41	● ○ ○ ● ● ○ ○ ○	461	39

Table E_8 (cont'd)

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
$\circ\circ\overset{\bullet}{\bullet}\circ\bullet\circ\circ$	689	49	$\circ\circ\overset{\bullet}{\bullet}\circ\circ\bullet\circ$	580	39
$\circ\circ\overset{\bullet}{\bullet}\circ\circ\circ\bullet$	388	45	$\circ\circ\overset{\bullet}{\circ}\bullet\bullet\circ\circ$	553	49
$\circ\circ\overset{\bullet}{\circ}\bullet\circ\bullet\circ$	681	39	$\circ\circ\overset{\bullet}{\circ}\bullet\circ\circ\bullet$	475	49
$\circ\circ\overset{\bullet}{\circ}\circ\bullet\bullet\circ$	412	34	$\circ\circ\overset{\bullet}{\circ}\circ\bullet\circ\bullet$	412	44
$\circ\circ\overset{\bullet}{\circ}\circ\circ\bullet\bullet$	260	34	$\circ\bullet\overset{\circ}{\bullet}\bullet\circ\circ\circ$	566	43
$\circ\bullet\overset{\circ}{\bullet}\circ\bullet\circ\circ$	815	55	$\circ\bullet\overset{\circ}{\bullet}\circ\circ\bullet\circ$	668	47
$\circ\bullet\overset{\circ}{\bullet}\circ\circ\circ\bullet$	463	50	$\circ\bullet\overset{\circ}{\circ}\bullet\bullet\circ\circ$	726	45
$\circ\bullet\overset{\circ}{\circ}\bullet\circ\bullet\circ$	788	49	$\circ\bullet\overset{\circ}{\circ}\bullet\circ\circ\bullet$	560	52
$\circ\bullet\overset{\circ}{\circ}\circ\bullet\bullet\circ$	498	38	$\circ\bullet\overset{\circ}{\circ}\circ\bullet\circ\bullet$	476	49
$\circ\bullet\overset{\circ}{\circ}\circ\circ\bullet\bullet$	315	36	$\circ\circ\overset{\circ}{\bullet}\bullet\bullet\circ\circ$	577	42
$\circ\circ\overset{\circ}{\bullet}\bullet\circ\bullet\circ$	736	46	$\circ\circ\overset{\circ}{\bullet}\bullet\circ\circ\bullet$	512	49
$\circ\circ\overset{\circ}{\bullet}\circ\bullet\bullet\circ$	596	41	$\circ\circ\overset{\circ}{\bullet}\circ\bullet\circ\bullet$	580	52
$\circ\circ\overset{\circ}{\bullet}\circ\circ\bullet\bullet$	390	39	$\circ\circ\overset{\circ}{\circ}\bullet\bullet\bullet\circ$	363	30
$\circ\circ\overset{\circ}{\circ}\bullet\bullet\circ\bullet$	377	41	$\circ\circ\overset{\circ}{\circ}\bullet\circ\bullet\bullet$	319	36
$\circ\circ\overset{\circ}{\circ}\circ\bullet\bullet\bullet$	224	28	$\bullet\bullet\overset{\bullet}{\bullet}\circ\circ\circ\circ$	87	11
$\bullet\bullet\overset{\bullet}{\circ}\bullet\circ\circ\circ$	186	18	$\bullet\bullet\overset{\bullet}{\circ}\circ\bullet\circ\circ$	170	20
$\bullet\bullet\overset{\bullet}{\circ}\circ\circ\bullet\circ$	141	16	$\bullet\bullet\overset{\bullet}{\circ}\circ\circ\circ\bullet$	105	19
$\bullet\circ\overset{\bullet}{\bullet}\bullet\circ\circ\circ$	160	17	$\bullet\circ\overset{\bullet}{\bullet}\circ\bullet\circ\circ$	218	23
$\bullet\circ\overset{\bullet}{\bullet}\circ\circ\bullet\circ$	182	19	$\bullet\circ\overset{\bullet}{\bullet}\circ\circ\circ\bullet$	130	22
$\bullet\circ\overset{\bullet}{\circ}\bullet\bullet\circ\circ$	188	25	$\bullet\circ\overset{\bullet}{\circ}\bullet\circ\bullet\circ$	214	21
$\bullet\circ\overset{\bullet}{\circ}\bullet\circ\circ\bullet$	150	27	$\bullet\circ\overset{\bullet}{\circ}\circ\bullet\bullet\circ$	140	19
$\bullet\circ\overset{\bullet}{\circ}\circ\bullet\circ\bullet$	133	25	$\bullet\circ\overset{\bullet}{\circ}\circ\circ\bullet\bullet$	86	19
$\bullet\bullet\overset{\circ}{\bullet}\bullet\circ\circ\circ$	112	18	$\bullet\bullet\overset{\circ}{\bullet}\circ\bullet\circ\circ$	161	26
$\bullet\bullet\overset{\circ}{\bullet}\circ\circ\bullet\circ$	143	25	$\bullet\bullet\overset{\circ}{\bullet}\circ\circ\circ\bullet$	98	26
$\bullet\bullet\overset{\circ}{\circ}\bullet\bullet\circ\circ$	163	20	$\bullet\bullet\overset{\circ}{\circ}\bullet\circ\bullet\circ$	175	25
$\bullet\bullet\overset{\circ}{\circ}\bullet\circ\circ\bullet$	124	26	$\bullet\bullet\overset{\circ}{\circ}\circ\bullet\bullet\circ$	123	20
$\bullet\bullet\overset{\circ}{\circ}\circ\bullet\circ\bullet$	117	27	$\bullet\bullet\overset{\circ}{\circ}\circ\circ\bullet\bullet$	76	19
$\bullet\circ\overset{\circ}{\bullet}\bullet\bullet\circ\circ$	186	21	$\bullet\circ\overset{\circ}{\bullet}\bullet\circ\bullet\circ$	225	26
$\bullet\circ\overset{\circ}{\bullet}\bullet\circ\circ\bullet$	161	27	$\bullet\circ\overset{\circ}{\bullet}\circ\bullet\bullet\circ$	184	24
$\bullet\circ\overset{\circ}{\bullet}\circ\bullet\circ\bullet$	170	31	$\bullet\circ\overset{\circ}{\bullet}\circ\circ\bullet\bullet$	112	23
$\bullet\circ\overset{\circ}{\circ}\bullet\bullet\bullet\circ$	119	17	$\bullet\circ\overset{\circ}{\circ}\bullet\bullet\circ\bullet$	127	24
$\bullet\circ\overset{\circ}{\circ}\bullet\circ\bullet\bullet$	99	22	$\bullet\circ\overset{\circ}{\circ}\circ\bullet\bullet\bullet$	71	16

Table E_8 (cont'd)

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
● ○●●●○○○	105	16	● ○●●○●○○	187	22
● ○●●○○●○	161	18	● ○●●○○○●	112	21
● ○●○●●○○	255	28	● ○●○●○●○	301	24
● ○●○●○○●	211	30	● ○●○○●●○	192	22
● ○●○○●○●	186	28	● ○●○○○●●	116	22
● ○○●●●○○	121	23	● ○○●●○●○	184	19
● ○○●●○○●	124	25	● ○○●○●●○	165	23
● ○○●○●○●	170	29	● ○○●○○●●	99	23
● ○○○●●●○	134	17	● ○○○●●○●	146	27
● ○○○●○●●	122	21	● ○○○○●●●	77	18
○ ○●●●●○○	154	22	○ ○●●●○●○	206	27
○ ○●●●○○●	141	28	○ ○●●○●●○	200	25
○ ○●●○●○●	195	32	○ ○●●○○●●	119	24
○ ○●○●●●○	153	21	○ ○●○●●○●	169	28
○ ○●○●○●●	133	26	○ ○●○○●●●	87	20
○ ○○●●●●○	127	19	○ ○○●●●○●	139	26
○ ○○●●○●●	121	24	○ ○○●○●●●	101	21
○ ○○○●●●●	66	17	● ●●●●○○○	24	6
● ●●●○●○○	42	9	● ●●●○○●○	37	8
● ●●●○○○●	29	9	● ●●○●●○○	70	12
● ●●○●○●○	75	11	● ●●○●○○●	55	14
● ●●○○●●○	54	11	● ●●○○●○●	50	14
● ●●○○○●●	34	11	● ●○●●●○○	48	11
● ●○●●○●○	69	10	● ●○●●○○●	50	13
● ●○●○●●○	65	13	● ●○●○●○●	62	16
● ●○●○○●●	40	13	● ●○○●●●○	53	10
● ●○○●●○●	57	16	● ●○○●○●●	45	13
● ●○○○●●●	30	11	○ ●●●●●○○	36	9
○ ●●●●○●○	49	13	○ ●●●●○○●	37	13
○ ●●●○●●○	52	13	○ ●●●○●○●	50	17
○ ●●●○○●●	32	13	○ ●●○●●●○	44	10
○ ●●○●●○●	51	14	○ ●●○●○●●	39	14

Table E_8 (cont'd)

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
$\bullet\bullet\overset{\circ}{\circ}\circ\bullet\bullet\bullet$	27	10	$\bullet\circ\overset{\circ}{\bullet}\bullet\bullet\bullet\circ$	48	11
$\bullet\circ\overset{\circ}{\bullet}\bullet\bullet\circ\bullet$	54	15	$\bullet\circ\overset{\circ}{\bullet}\bullet\circ\bullet\bullet$	47	15
$\bullet\circ\overset{\circ}{\bullet}\circ\bullet\bullet\bullet$	36	13	$\bullet\circ\overset{\circ}{\circ}\bullet\bullet\bullet\bullet$	26	10
$\circ\bullet\overset{\bullet}{\bullet}\bullet\bullet\circ\circ$	32	10	$\circ\bullet\overset{\bullet}{\bullet}\bullet\circ\bullet\circ$	49	9
$\circ\bullet\overset{\bullet}{\bullet}\bullet\circ\circ\bullet$	35	12	$\circ\bullet\overset{\bullet}{\bullet}\circ\bullet\bullet\circ$	58	12
$\circ\bullet\overset{\bullet}{\bullet}\circ\bullet\circ\bullet$	58	15	$\circ\bullet\overset{\bullet}{\bullet}\circ\circ\bullet\bullet$	36	12
$\circ\bullet\overset{\bullet}{\circ}\bullet\bullet\bullet\circ$	68	12	$\circ\bullet\overset{\bullet}{\circ}\bullet\bullet\circ\bullet$	75	18
$\circ\bullet\overset{\bullet}{\circ}\bullet\circ\bullet\bullet$	61	15	$\circ\bullet\overset{\bullet}{\circ}\circ\bullet\bullet\bullet$	38	13
$\circ\circ\overset{\bullet}{\bullet}\bullet\bullet\bullet\circ$	34	8	$\circ\circ\overset{\bullet}{\bullet}\bullet\bullet\circ\bullet$	43	14
$\circ\circ\overset{\bullet}{\bullet}\bullet\circ\bullet\bullet$	40	11	$\circ\circ\overset{\bullet}{\bullet}\circ\bullet\bullet\bullet$	34	12
$\circ\circ\overset{\bullet}{\circ}\bullet\bullet\bullet\bullet$	29	9	$\circ\bullet\overset{\circ}{\bullet}\bullet\bullet\bullet\circ$	39	11
$\circ\bullet\overset{\circ}{\bullet}\bullet\bullet\circ\bullet$	50	15	$\circ\bullet\overset{\circ}{\bullet}\bullet\circ\bullet\bullet$	43	15
$\circ\bullet\overset{\circ}{\bullet}\circ\bullet\bullet\bullet$	39	13	$\circ\bullet\overset{\circ}{\circ}\bullet\bullet\bullet\bullet$	31	12
$\circ\circ\overset{\circ}{\bullet}\bullet\bullet\bullet\bullet$	25	11	$\bullet\bullet\overset{\bullet}{\bullet}\bullet\bullet\circ\circ$	8	4
$\bullet\bullet\overset{\bullet}{\bullet}\bullet\circ\bullet\circ$	12	4	$\bullet\bullet\overset{\bullet}{\bullet}\bullet\circ\circ\bullet$	10	5
$\bullet\bullet\overset{\bullet}{\bullet}\circ\bullet\bullet\circ$	17	6	$\bullet\bullet\overset{\bullet}{\bullet}\circ\bullet\circ\bullet$	16	7
$\bullet\bullet\overset{\bullet}{\bullet}\circ\circ\bullet\bullet$	12	6	$\bullet\bullet\overset{\bullet}{\circ}\bullet\bullet\bullet\circ$	23	6
$\bullet\bullet\overset{\bullet}{\circ}\bullet\bullet\circ\bullet$	25	9	$\bullet\bullet\overset{\bullet}{\circ}\bullet\circ\bullet\bullet$	20	8
$\bullet\bullet\overset{\bullet}{\circ}\circ\bullet\bullet\bullet$	14	7	$\bullet\circ\overset{\bullet}{\bullet}\bullet\bullet\bullet\circ$	16	5
$\bullet\circ\overset{\bullet}{\bullet}\bullet\bullet\circ\bullet$	19	8	$\bullet\circ\overset{\bullet}{\bullet}\bullet\circ\bullet\bullet$	19	7
$\bullet\circ\overset{\bullet}{\bullet}\circ\bullet\bullet\bullet$	16	8	$\bullet\circ\overset{\bullet}{\circ}\bullet\bullet\bullet\bullet$	13	6
$\bullet\bullet\overset{\circ}{\bullet}\bullet\bullet\bullet\circ$	11	5	$\bullet\bullet\overset{\circ}{\bullet}\bullet\bullet\circ\bullet$	15	7
$\bullet\bullet\overset{\circ}{\bullet}\bullet\circ\bullet\bullet$	15	8	$\bullet\bullet\overset{\circ}{\bullet}\circ\bullet\bullet\bullet$	13	7
$\bullet\bullet\overset{\circ}{\circ}\bullet\bullet\bullet\bullet$	12	6	$\bullet\circ\overset{\circ}{\bullet}\bullet\bullet\bullet\bullet$	12	7
$\circ\bullet\overset{\bullet}{\bullet}\bullet\bullet\bullet\circ$	10	4	$\circ\bullet\overset{\bullet}{\bullet}\bullet\bullet\circ\bullet$	14	7
$\circ\bullet\overset{\bullet}{\bullet}\bullet\circ\bullet\bullet$	14	6	$\circ\bullet\overset{\bullet}{\bullet}\circ\bullet\bullet\bullet$	15	7
$\circ\bullet\overset{\bullet}{\circ}\bullet\bullet\bullet\bullet$	16	7	$\circ\circ\overset{\bullet}{\bullet}\bullet\bullet\bullet\bullet$	9	4
$\circ\bullet\overset{\circ}{\bullet}\bullet\bullet\bullet\bullet$	10	6	$\bullet\bullet\overset{\bullet}{\bullet}\bullet\bullet\bullet\circ$	3	2
$\bullet\bullet\overset{\bullet}{\bullet}\bullet\bullet\circ\bullet$	4	3	$\bullet\bullet\overset{\bullet}{\bullet}\bullet\circ\bullet\bullet$	5	3
$\bullet\bullet\overset{\bullet}{\bullet}\circ\bullet\bullet\bullet$	6	4	$\bullet\bullet\overset{\bullet}{\circ}\bullet\bullet\bullet\bullet$	7	4
$\bullet\circ\overset{\bullet}{\bullet}\bullet\bullet\bullet\bullet$	5	3	$\bullet\bullet\overset{\circ}{\bullet}\bullet\bullet\bullet\bullet$	4	3
$\circ\bullet\overset{\bullet}{\bullet}\bullet\bullet\bullet\bullet$	3	2	$\bullet\bullet\overset{\bullet}{\bullet}\bullet\bullet\bullet\bullet$	1	1

Table F_4

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
○ ○ ○ ○	105	16	● ○ ○ ○	24	6
○ ● ○ ○	35	12	○ ○ ● ○	32	10
○ ○ ○ ●	49	9	● ● ○ ○	10	5
● ○ ● ○	8	4	● ○ ○ ●	12	4
○ ● ● ○	14	7	○ ● ○ ●	14	6
○ ○ ● ●	10	4	● ● ● ○	4	3
● ● ○ ●	5	3	● ○ ● ●	3	2
○ ● ● ●	3	2	● ● ● ●	1	1

where we use the following orientation for the Dynkin diagram of F_4 :

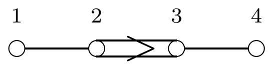
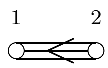


Table G_2

I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$	I	$\sharp\mathcal{N}_I$	$\sharp\mathcal{A}b_I$
○ ○	8	4	● ○	3	2
○ ●	4	3	● ●	1	1

where we use the following orientation for the Dynkin diagram of G_2 :



Chapter 3

Ad-nilpotent ideals having a fixed index of nilpotence

Let $\mathfrak{i}$ be an ad-nilpotent ideal of a parabolic subalgebra $\mathfrak{p}_I$. Let

$$\mathfrak{i}^1 = \mathfrak{i} \text{ and } \mathfrak{i}^{k+1} = [\mathfrak{i}^k, \mathfrak{i}],$$

for $k \geqslant 0$, be its central descending series. Recall that the index of nilpotence of $\mathfrak{i}$ is the smallest k such that $i^{k+1} = \{0\}$ (i.e. the number of nonzero terms in the central descending series). For example, we have $n(\{0\}) = 0$ and the abelian ideals are the ideals whose index of nilpotence is smaller than or equal to one. Let $\Phi_{\mathfrak{i}}$ be the element of $\mathcal{F}_I$ which corresponds to $\mathfrak{i}$ as in 1.2. Then $n(\mathfrak{i})$ is the smallest integer such that $\Phi^{n(\mathfrak{i})+1} = \emptyset$. We set $n(\Phi_{\mathfrak{i}}) = n(\mathfrak{i})$.

Let $I \subset \Pi$. In this chapter, we enumerate the number of ad-nilpotent ideals of a parabolic subalgebra $\mathfrak{p}_I$ having a fixed index of nilpotence when $\mathfrak{g}$ is of type A or C. The results depend essentially on the cardinality of I. The idea is to construct a suitable map between root systems of different ranks. The method we use does not work in the other cases.

We shall conserve the notations used in previous chapters. Since we consider root systems of the same type but of different ranks, we shall add as an indice its rank to distinguish them. For example, Δ_l^+ will denote the set of positive roots when the rank is l. All the numberings used for the simple roots are those of [TY].

3.1 Type A

Assume that $\Delta_\bullet$ is of type $A_\bullet$. Let $I \subset \Pi_l$ be of cardinality r. We shall construct a bijection between $\mathcal{F}_{l,I}$ and $\mathcal{F}_{l-r,\emptyset}$. By abuse of notation, if $n < l$, a simple root $\alpha_j \in \Delta_n^+$ is still denoted by α_j in Δ_l^+.

For $1 \leqslant p \leqslant l$, set :

$$\begin{array}{rcrcl} \chi_p^l & : & \Delta_{l-1}^+ & \to & \Delta_l^+ \\ & & \alpha_i + \cdots + \alpha_j & \mapsto & \begin{cases} \alpha_i + \cdots + \alpha_j & \text{if } j \leqslant p-1, \\ \alpha_{i+1} + \cdots + \alpha_{j+1} & \text{if } i \geqslant p, \\ \alpha_i + \cdots + \alpha_{j+1} & \text{if } i < p \text{ and } j \geqslant p. \end{cases} \end{array}$$

Let $I = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Pi_l$ be such that $i_1 < i_2 < \cdots < i_r$, we set $\chi_I^l = \chi_{i_r}^l \circ \chi_{i_{r-1}}^{l-1} \circ \cdots \circ \chi_{i_1}^{l-r+1}$.

Example 3.1.1 *Take $l = 4$, $I = \{\alpha_1, \alpha_3\}$. Then we have :*

$$\begin{array}{ccccccc} \chi_I^l & : & \Delta_2^+ & \xrightarrow{\chi_1^3} & \Delta_3^+ & \xrightarrow{\chi_3^4} & \Delta_4^+ \\ & & \alpha_1 + \alpha_2 & \mapsto & \alpha_2 + \alpha_3 & \mapsto & \alpha_2 + \alpha_3 + \alpha_4 \\ & & \alpha_1 & \mapsto & \alpha_2 & \mapsto & \alpha_2 \\ & & \alpha_2 & \mapsto & \alpha_3 & \mapsto & \alpha_4 \end{array}$$

Let R_I^l be the set of elements of Δ_l^+, which do not begin or end with an element of $\Delta_{l,I}$.

Lemma 3.1.2 *The image of χ_I^{l-r} is R_I^l.*

Proof. We shall proceed by induction on r. If $I = \{\alpha_p\}$, by the definition of χ_p^l, the result is clear.

Assume that the result is true for $r-1$, $r \geqslant 2$. Let $I = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Pi_l$ be such that $i_1 < i_2 < \cdots < i_r$. The image of $\chi_{I \setminus \{\alpha_{i_r}\}}^{l-1}$ is then, by the induction hypothesis, $R_{I \setminus \{\alpha_{i_r}\}}^{l-1}$.

Let $\alpha = \alpha_i + \cdots + \alpha_j$ be an element of $R_{I \setminus \{\alpha_{i_r}\}}^{l-1}$ and set $\beta = \chi_{i_r}^l(\alpha)$. If $j < i_r$, then $\beta = \alpha$, so $\beta \in R_I^l$.

If $i \leqslant i_r$, then $\beta = \alpha_{i+1} + \cdots + \alpha_{j+1} \in R_I^l$. Since $i_1 < i_2 < \cdots < i_r$, we also have that $\beta \in R_I^l$.

If $i < i_r$ and $j \geqslant i_r$, then $\beta = \alpha_i + \cdots + \alpha_{j+1}$. With the same argument as before, we also have that $\beta \in R_I^l$.

By the definition of χ_p^l, it is injective, hence the map χ_I^{l-r} is also injective. Since

$$\sharp R_I^l = \frac{l(l+1)}{2} - rl + \frac{r(r-1)}{2} = \frac{(l-r)(l-r+1)}{2} = \sharp\Delta_{l-r}^+,$$

we obtain that the image of χ_I^{l-r} is R_I^l. $\square$

Let $\alpha \in \Delta_l^+ \setminus \Delta_{l,I}$, we denote by $eq_I(\alpha)$ the set of roots $\beta \in \Delta_l^+$, such that $\beta \sim_I \alpha$ (see 2.2). And for an equivalence class $eq_I(\alpha)$, we denote by α_{min} the unique minimal root of this equivalence class for the partial order $\leqslant$. By the previous lemma, it is clear that we have :

$$\Delta_l^+ \setminus \Delta_{l,I} = \bigcup_{\alpha\in\Delta_{l-r}^+} eq_I(\chi_I^l(\alpha)).$$

and R_I^l, the image of χ_I^l, is exactly the set of minimal roots of each of these equivalence classes.

In Example 3.1.1, we have $\chi_I^l(\Delta_2^+) = \{\alpha_2+\alpha_3+\alpha_4, \alpha_2, \alpha_4\} = R_I^4$ and

$$\begin{aligned}\bigcup_{\alpha\in\Delta_{l-r}^+} eq_I(\chi_I^l(\alpha)) \quad &= \quad \{\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4\} \\ &\quad\ \ \cup\{\alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2, \alpha_2\} \\ &\quad\ \ \cup\{\alpha_3+\alpha_4, \alpha_4\} \\ &= \quad \Delta_4^+ \setminus \Delta_{4,I}.\end{aligned}$$

Lemma 3.1.3 *Let $\alpha, \beta \in \Delta_{l-r}^+$ be such that $\alpha+\beta \in \Delta_{l-r}^+$. Then $\chi_I^l(\alpha+\beta) \in eq_I(\chi_I^l(\alpha)) + \chi_I^l(\beta)$.*

Proof. We shall proceed by induction on r. For $i \leqslant j$, denote by $\alpha_{i,j}$ the root $\alpha_i + \cdots + \alpha_j \in \Delta_{l-r}^+$. Set $\alpha = \alpha_{i,j}$ and $\beta = \alpha_{j+1,k}$, where $i \leqslant j < k$.

Assume that $r = 1$ and that $I = \{\alpha_p\}$. By a simple case by case verification, we have that if $j \neq p-1$, then $\chi_I^l(\alpha+\beta) = \chi_I^l(\alpha) + \chi_I^l(\beta)$. If $j = p-1$, then

$$\chi_I^l(\alpha+\beta) = \alpha_{i,p-1} + \alpha_p + \alpha_{p+1,k+1} = \chi_I^l(\alpha) + \alpha_p + \chi_I^l(\beta)$$

and $\alpha_{i,p-1} + \alpha_p \in eq_I(\chi_I^l(\alpha))$. The induction follows easily. $\square$

Let $\overline{\chi}_I^l : R_I^l \to \Delta_{l-r}^+$ be the map such that $\overline{\chi}_I^l \circ \chi_I^l(\alpha) = \alpha$ for all $\alpha \in \Delta_{l-r}^+$. Similar to lemma 3.1.3, we also have by a straightforward verification the following lemma :

Lemma 3.1.4 *Let $\gamma \in R_I^l$ be the sum of two roots α and $\beta \in \Delta_l^+$. Then, we have $\overline{\chi}_I^l(\alpha+\beta) = \overline{\chi}_I^l(\alpha_{min}) + \overline{\chi}_I^l(\beta_{min})$.*

In Example 3.1.1, if we take $\gamma = \alpha_2 + \alpha_3 + \alpha_4$, $\alpha = \alpha_2 + \alpha_3$ and $\beta = \alpha_4$, we have $\alpha_{min} = \alpha_2$, $\beta_{min} = \alpha_4$ and :

$$\overline{\chi}_I^4(\gamma) = \alpha_1 + \alpha_2 = \overline{\chi}_I^4(\alpha_{min}) + \overline{\chi}_I^4(\beta_{min}).$$

Observe that we obtain the same result if we take $\alpha = \alpha_2$ and $\beta = \alpha_3 + \alpha_4$.

For $\Phi \in \mathcal{F}_{l-r,\emptyset}$, set

$$b(\Phi) = \bigcup_{\alpha \in \Phi} eq_I(\chi_I^l(\alpha)).$$

Proposition 3.1.5 *The map b is a bijection between the set $\mathcal{F}_{l-r,\emptyset}$ of ad-nilpotent ideals of the Borel subalgebra $\mathfrak{p}_{l-r,\emptyset}$ and the set $\mathcal{F}_{l,I}$ of ad-nilpotent ideals of the parabolic subalgebra $\mathfrak{p}_{l,I}$. Moreover, it preserves the index of nilpotence.*

Proof. We shall first prove that the image of b is the set of ad-nilpotent ideals of $\mathfrak{p}_{l,I}$.

Let $\Phi \in \mathcal{F}_{l-r,\emptyset}$. By Lemma 3.1.2, for $\alpha \in \Phi$, $\chi_I^l(\alpha) \in R_I^l$, and we have

$$eq_I(\chi_I^l(\alpha)) \subset \Delta_l^+ \setminus \Delta_{l,I}.$$

Let $\gamma \in b(\Phi)$, then there exists $\alpha \in \Phi$ such that $\gamma \in eq_I(\chi_I^l(\alpha))$. Let $\beta \in \Delta_l^+ \cup \Delta_{l,I}$ be such that $\gamma + \beta \in \Delta_l^+$. We shall prove that $\eta = \gamma + \beta \in b(\Phi)$.

First assume that $\beta \in \Delta_{l,I}$, then $eq_I(\gamma+\beta) = eq_I(\gamma)$ and $\gamma + \beta \in eq_I(\chi_I^l(\alpha))$.

Assume that $\beta \in \Delta_l^+ \setminus \Delta_{l,I}$. Replacing β and γ by elements in the same equivalence class, we may assume that $\eta = \eta_{min}$ (this is possible because we are in type A). Exchanging β and γ if necessary, we may further assume that γ does not begin by a root in $\Delta_{l,I}$ and β does not end by a root in $\Delta_{l,I}$.

We then have by Lemma 3.1.4 that :

$$\overline{\chi}_I^l(\eta) = \overline{\chi}_I^l(\gamma+\beta) = \overline{\chi}_I^l(\gamma_{min}) + \overline{\chi}_I^l(\beta_{min}) = \alpha + \beta'.$$

We obtain that $\gamma + \beta = \chi_I^l(\alpha + \beta')$, where $\beta' \in \Delta_{l-r}^+$. Since $\Phi \in \mathcal{F}_{l-r,\emptyset}$, we have that $\alpha + \beta' \in \Phi$, which allows us to conclude that $\eta \in b(\Phi)$, hence we have $b(\Phi) \in \mathcal{F}_{l,I}$.

Conversely for $\Psi \in \mathcal{F}_{l,I}$ set $\overline{b}(\Psi) = \{\overline{\chi}_I^l(\alpha_{min}); \alpha \in \Psi\} \subset \Delta_{l-r}^+$. We shall prove that $\overline{b}(\Psi) \in \mathcal{F}_{l-r,\emptyset}$.

Let $\beta = \overline{\chi}_I^l(\alpha_{min}) \in \overline{b}(\Psi)$ and $\gamma \in \Delta_{l-r}^+$ be such that $\beta + \gamma \in \Delta_{l-r}^+$. By Lemma 3.1.3 and since $\chi_I^l(\beta) = \alpha_{min}$, we have :

$$\chi_I^l(\beta + \gamma) \in eq_I(\alpha_{min}) + \chi_I^l(\gamma).$$

Set $\chi_I^l(\beta + \gamma) = \eta = \alpha + \gamma'$ where $\alpha \in eq_I(\alpha_{min})$ and $\gamma' = \chi_I^l(\gamma)$. Since $\Psi \in \mathcal{F}_{l,I}$, we have $\alpha \in \Psi$ and therefore $\eta \in \Psi$. It follows that $\gamma + \beta = \overline{\chi}_I^l(\eta_{min}) \in \overline{b}(\Psi)$.

Now, an easy verification shows that b and $\overline{b}$ are mutually inverse, which proves that b is a bijection.

It remains to prove that b preserves the index of nilpotence. Let $\Phi \in \mathcal{F}_{l-r,\emptyset}$, $\Psi = b(\Phi)$ and $\alpha, \beta \in \Psi$ be such that $\gamma = \alpha + \beta \in b(\Phi)$. Since γ and γ_{min} are in the same equivalence class, we can assume without loss of generality that $\gamma = \gamma_{min} \in R_I^l$. By Lemma 3.1.4, we have :

$$\overline{\chi}_I^l(\alpha + \beta) = \overline{\chi}_I^l(\alpha_{min}) + \overline{\chi}_I^l(\beta_{min}) = \alpha' + \beta'.$$

It follows that $\alpha + \beta = \chi_I^l(\alpha' + \beta') \in \chi_I^l(\Phi^2)$. We obtain that $n(\Psi) \leqslant n(\Phi)$.

Conversely, let $\alpha, \beta \in \Phi$ be such that $\gamma = \alpha + \beta \in \Phi$, then by Lemma 3.1.3, we have :

$$\chi_I^l(\alpha + \beta) \in eq_I(\chi_I^l(\alpha)) + \chi_I^l(\beta) \subset \Psi^2.$$

So, we have $n(\Phi) \leqslant n(\Psi)$, which concludes the proof. □

Let $I \subset \Pi_l$ be of cardinality r. The above bijection allows us to apply all the results known for the ad-nilpotent ideals of the Borel subalgebra $\mathfrak{p}_{l-r,\emptyset}$, to the ad-nilpotent ideals of $\mathfrak{p}_{l,I}$. We deduce from [OP, AKOP], the following corollary :

Corollary 3.1.6 *Let $\mathfrak{g}$ be of type A_l and $I \subset \Pi_l$ be of cardinality r. Let $\alpha_l^I(K)$ denote the number of ideals of $\mathcal{F}_{l,I}$ whose index of nilpotence is K. Then*

$$\alpha_l^I(K) = \sum_{0=i_0<i_1<\cdots<i_K<i_{K+1}=l+1-r} \prod_{j=0}^{K-1} \binom{i_{j+2} - i_j - 1}{i_{j+1} - i_j}.$$

In particular, $\alpha_l^I(K)$ depends only on the cardinality of I.

Example 3.1.7 *If we consider A_3, we have :*

$\sharp I$	$\alpha_l^I(0)$	$\alpha_l^I(1)$	$\alpha_l^I(2)$	$\alpha_l^I(3)$
0	1	7	5	1
1	1	3	1	0
2	1	1	0	0

3.2 Type C

In this section, we shall determine the number of ad-nilpotent ideals of a parabolic subalgebra having a fixed class of nilpotence when $\mathfrak{g}$ is of type C. We shall consider the following two cases separately : if I contains the long simple root α_l or not. If I does not contain the long simple root α_l, the methods used are similar to those used for type A in the previous section. If I contains α_l, we need extra tools.

3.2.1 The index of nilpotence when $\alpha_l \notin I$.

In this section, $\Delta_\bullet$ is of type $C_\bullet$.

Let $I \subset \Pi_l \setminus \{\alpha_l\}$ be of cardinality r. We shall construct a bijection between $\mathcal{F}_{l,I}$ and $\mathcal{F}_{l-r,\emptyset}$.

For $1 \leqslant p < l$, set :

$$\begin{array}{rccl} \chi_p^l & : & \Delta_{l-1}^+ & \rightarrow & \Delta_l^+ \\ & & \alpha_i + \cdots + \alpha_j & \mapsto & \left\{ \begin{array}{ll} \alpha_i + \cdots + \alpha_j & \text{if } j \leqslant p-1, \\ \alpha_{i+1} + \cdots + \alpha_{j+1} & \text{if } i \geqslant p, \\ \alpha_i + \cdots + \alpha_{j+1} & \text{if } i < p \text{ and } j \geqslant p. \end{array} \right. \end{array}$$

and

$$\chi_p^l(\alpha_i + \cdots + 2\alpha_{q+1} + \cdots + 2\alpha_{l-2} + \alpha_{l-1}) =$$

$$\left\{ \begin{array}{ll} \chi_p^l(\alpha_i + \cdots + \alpha_{l-1}) + \chi_p^l(\alpha_{q+1} + \cdots + \alpha_{l-2}) & \text{if } p \neq l-1, \\ \chi_p^l(\alpha_i + \cdots + \alpha_{l-1}) + \chi_p^l(\alpha_{q+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} & \text{if } p = l-1. \end{array} \right.$$

Let $I = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Pi_l \setminus \{\alpha_l\}$ be such that $i_1 < i_2 < \cdots < i_r$, and set $\chi_I^l = \chi_{i_r}^l \circ \chi_{i_{r-1}}^{l-1} \circ \cdots \circ \chi_{i_1}^{l-r+1}$.

Example 3.2.1 *Take $l = 4$, $I = \{\alpha_1, \alpha_3\}$. Then we have :*

$$\begin{array}{cccccc} \chi_I^l & : & \Delta_2^+ & \xrightarrow{\chi_1^3} & \Delta_3^+ & \xrightarrow{\chi_3^4} & \Delta_4^+ \\ & & 2\alpha_1+\alpha_2 & \mapsto & 2\alpha_2+\alpha_3 & \mapsto & 2\alpha_2+2\alpha_3+\alpha_4 \\ & & \alpha_1+\alpha_2 & \mapsto & \alpha_2+\alpha_3 & \mapsto & \alpha_2+\alpha_3+\alpha_4 \\ & & \alpha_1 & \mapsto & \alpha_2 & \mapsto & \alpha_2 \\ & & \alpha_2 & \mapsto & \alpha_3 & \mapsto & \alpha_4 \end{array}$$

Observe that the definition of χ_p^l is similar to the definition given in section 3.1. Using similar arguments, we have the following proposition :

Proposition 3.2.2 *For $\Phi \in \mathcal{F}_{l-r,\emptyset}$ and $I \subset \Pi_l \setminus \{\alpha_l\}$, set*

$$b(\Phi) = \bigcup_{\alpha \in \Phi} eq_I(\chi_I^l(\alpha)).$$

The map b is a bijection between the set $\mathcal{F}_{l-r,\emptyset}$ of ad-nilpotent ideals of $\mathfrak{p}_{l-r,\emptyset}$ and the set $\mathcal{F}_{l,I}$ of ad-nilpotent ideals of the parabolic subalgebra $\mathfrak{p}_{l,I}$ which preserves the index of nilpotence.

Then, by [KOP], we have the following corollary :

Corollary 3.2.3 *Let $\mathfrak{g}$ be of type C_l and $I \subset \Pi_l \setminus \{\alpha_l\}$ be of cardinality r. Let $\alpha_l^I(K)$ denote the number of ideals of $\mathcal{F}_{l,I}$ whose index of nilpotence is K. Then*

$$\alpha_l^I(K) = \sum_{0<i_1<\cdots<i_k<i_{k+1}=l-r} \prod_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix} \sum_{n=0}^{i_2-i_1-1} \begin{pmatrix} i_1+i_2-1 \\ n \end{pmatrix}$$

if $K = 2k$, and

$$\alpha_l^I(K) = \sum_{-i_2<i_1\leqslant 0<i_2<\cdots<i_k<i_{k+1}=l-r} 2^{i_1+i_2-1} \sum_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix}$$

if $K = 2k-1$.

In particular, $\alpha_l^I(K)$ depends only on the cardinality of I.

Example 3.2.4 *If we consider C_3 and $I \subset \Pi$ which does not contain α_3, we have :*

$\sharp I$	$\alpha_l^I(0)$	$\alpha_l^I(1)$	$\alpha_l^I(2)$	$\alpha_l^I(3)$	$\alpha_l^I(4)$	$\alpha_l^I(5)$
0	1	7	5	5	1	1
1	1	3	1	1	0	0
2	1	1	0	0	0	0

3.2.2 AKOP algorithm

Let us recall the algorithm given in [AKOP, KOP] for computing the index of nilpotence in type A.

An l-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{N}^l$ is a partition if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$.

Definition 3.2.5 *Let T_l be the diagram of shape $[l, l-1, \ldots, 1]$ (see 2.2.1). We say that λ is contained in T_l if $\lambda_i \leqslant i$ for $i = 1, \ldots, l$. We shall represent λ as a subdiagram F_λ of T_l, called Ferrers diagram of λ, above the line $x+y = l+1$.*

For example, for $l = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$:

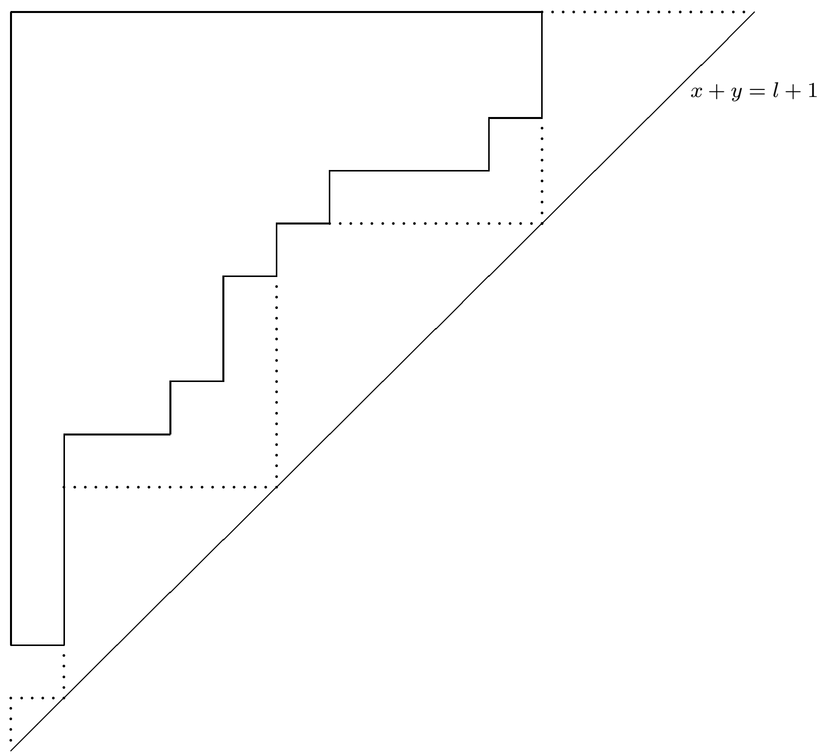


Figure 3.1: F_λ

We shall draw a dotted line associated to λ. We start at the top of the line $x+y = l+1$. We go left until we meet F_λ. Then, we continue downwards until we reach $x+y=l+1$. Then we iterate the procedure until we reach the bottom. See Figure 3.1.

Let $n(\lambda)$ be the number of points of the dotted line on $x+y=l+1$, which are not at the top or bottom. For example, we have $n((0,\dots,0))=0$, and for the partition λ of figure 3.1, we have $n(\lambda)=3$.

We have easily from this iterative construction that :

Proposition 3.2.6 *[AKOP] Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_l)$ be a partition contained in T_l. Then :*

$$n(\lambda_1,\lambda_2,\dots,\lambda_l)=\begin{cases} n(\lambda_{l+2-\lambda_1},\dots,\lambda_l)+1 & \text{if } \lambda_1>1,\\ 1 & \text{if } \lambda_1=1,\\ 0 & \text{if } \lambda_1=0.\end{cases}$$

This provides an easy algorithm to compute $n(\lambda)$.

For example, for λ as in Figure 3.1, we have :

$$\begin{aligned} n(\lambda) &= n(10,10,9,6,5,4,4,3,1,1,1,1,0)\\ &= n(5,4,4,3,1,1,1,1,0)+1\\ &= n(1,1,1,0)+2\\ &= 3. \end{aligned}$$

Recall from 2.2.1, that if $\mathfrak{g}$ is of type A_l, we can display the positive roots Δ_l^+ in T_l. Then the ad-nilpotent ideals of the Borel subalgebra are in bijection with the nw-diagrams of T_l. Let $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_{l,\emptyset}$. Let T be the nw-diagram corresponding to $\mathfrak{i}$ in T_l. Then it is clear that T is the Ferrers diagram of a partition contained in T_l. We say that λ is the partition corresponding to $\mathfrak{i}$.

By [AKOP] we have the following proposition:

Proposition 3.2.7 *Let $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_{l,\emptyset}$ and λ be the corresponding partition. Then :*

$$n(\mathfrak{i})=n(\lambda).$$

Let $J\subset\{1,\dots,l\}$. We decompose J into connected components $J_1\cup J_2\cup\cdots\cup J_s$. For $i\in J$ contained in the connected component J_p, set

$$k_{p,i}=\sharp\{k\in J_p; k>i\}.$$

Let $I\subset\Pi_l$ and $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{p}_{l,\emptyset}$. Let λ be the partition correponding to $\mathfrak{i}$ and F its Ferrers diagram. We have that $\mathfrak{i}$ is an ideal of

$\mathfrak{p}_{l,I}$ if the cells of F don't contain elements of $\Delta_{l,I}$ and if F contains the equivalence classes of its roots. In terms of the partition λ, it means that for $J = \{j; \alpha_j \in I\}$, we have :

$$\begin{cases} (j, l-j+1-k) \notin F & \text{for } j \in J_p, p = 1, \ldots, s, \text{ and } k = 0, \ldots, k_{p,j}, \\ \lambda_p \neq l-j+1 & \text{for all } j \in J \setminus \{1\} \text{ and } k = 1, \ldots, l, \\ \lambda_j = \lambda_{j+1} & \text{for all } j \in J \setminus \{l\}. \end{cases}$$

If these conditions are satisfied, we say that λ is J-compatible. The partitions which are J-compatible correspond exactly to the ad-nilpotent ideals of $\mathfrak{p}_{l,I}$.

Let $J \subset \{1, \ldots, l\}$ be of cardinality r. For a partition λ which is J-compatible, we set $p(\lambda)$ to be the partition in T_{l-r} obtained from λ by regrouping, for $i \in J$, the $(l-i+1)$-th and the $(l-i+2)$-th columns if $i \neq 1$, and the rows $i, i+1$ if $i \neq l$ on its Ferrers diagram. Then we have by 2.2.1 and Proposition 3.1.5 that $p(\lambda)$ is the partition corresponding to the ad-nilpotent ideal $\overline{b}(\mathfrak{i})$ of $\mathfrak{p}_{l-r,\emptyset}$. Since by Proposition 3.1.5, we have $n(\mathfrak{i}) = n(\overline{b}(\mathfrak{i}))$. By Proposition 3.2.7, we obtain the following proposition :

Proposition 3.2.8 *Let $J \subset \{1, \ldots, l\}$ be of cardinality r. Let λ be a J-compatible partition and $\lambda' = p(\lambda)$, then :*

$$n(\lambda) = n(\lambda').$$

3.2.3 The index of nilpotence when I contains α_l.

Let $\mathfrak{g}$ be of type C_l. We fill the diagram T_{2l-1} with the positive roots of C_l by associating the cells of the "upper half" of T_{2l-1} with positive roots as in 2.2.2 and by associating the cells in the rest of T_{2l-1} in a symmetric fashion in such a way that cell (i, j) gets associated the same root as cell (j, i), as in [AKOP].

For example, for $l = 3$, we obtain the following diagram :

1^22^23	12^23	123	12	1
12^23	2^23	23	2	
123	23	3		
12	2			
1				

where we use the notation of 2.2.3.

Now, for $\Phi \in \mathcal{F}_{l,\emptyset}$, let $\widetilde{F}$ be the collection of cells which contain a root $\alpha \in \Phi$. A subdiagram of T_{2l-1} is called self-conjugate if when it contains a cell (i,j), it contains also the cell (j,i). Then, $\widetilde{F}$ is clearly self-conjugate. For example, according to this description, $\Phi = \{\theta, \alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ corresponds to the self-conjugate partition $(3,2,1,0,0)$. We say that this partition λ is the partition corresponding to Φ.

Let λ be a partition corresponding $\Phi \in \mathcal{F}_{l,\emptyset}$, then for $n(\lambda)$ defined as in proposition 3.2.6, we have by [KOP] that :

$$n(\lambda) = n(\Phi). \tag{3.1}$$

Let $I \subset \Pi_l$ be of cardinality r and such that $\alpha_l \in I$. Let $\Phi \in \mathcal{F}_{l,I}$. Let $\widetilde{F}$ be the diagram associated to Φ included in T_{2l-1} constructed as above and let λ be the corresponding partition. Since $\widetilde{F}$ is self-conjugate, we have that λ is J-compatible, where $J = \{j, 2l-j; \alpha_j \in I\}$. Observe that J is of cardinality $2r-1$.

Now by Proposition 3.2.8, we have that $\lambda' = p(\lambda) \in T_{2(l-r)}$ and

$$n(\lambda) = n(\lambda').$$

By (3.1), to calculate the index of nilpotence of Φ, it is enough to calculate $n(\lambda')$. Hence, to calculate the number of ideals of $\mathcal{F}_{l,I}$ with index of nilpotence K, we have to calculate the number of self-conjugate partitions $\lambda' \in T_{2(l-r)}$ satisfying $n(\lambda') = K$.

In [KOP], they computed the number of self-conjugate partitions $\lambda' \in T_n$, for n odd. We can obtain the result for n even by a simple shift of indice.

Proposition 3.2.9 *Let $\mathfrak{g}$ be of type C_l and $I \subset \Pi_l$ containing α_l, be of cardinality r. Let $\alpha_l^I(K)$ denote the number of ideals of $\mathcal{F}_{l,I}$ whose index of nilpotence is K. Then*

$$\alpha_l^I(K) = \sum_{0<i_1<\cdots<i_k<i_{k+1}=l-r+1} \prod_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix} \sum_{n=0}^{i_2-i_1-1} \begin{pmatrix} i_1+i_2-2 \\ n \end{pmatrix}$$

if $K = 2k$, and

$$\alpha_l^I(K) = \sum_{1-i_2<i_1\leqslant 0<i_2<\cdots<i_k<i_{k+1}=l-r+1} 2^{i_1+i_2-2} \sum_{j=1}^{k-1} \begin{pmatrix} i_{j+2}-i_j-1 \\ i_{j+1}-i_j \end{pmatrix}$$

if $K = 2k - 1$.

In particular, $\alpha_l^I(K)$ *depends only on the cardinality of* I.

For example, if we consider C_3 and $I \subset \Pi$ which contains α_3, we have :

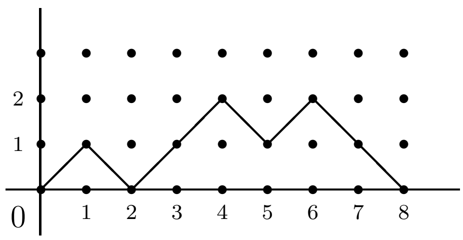
$\sharp I$	$\alpha_l^I(0)$	$\alpha_l^I(1)$	$\alpha_l^I(2)$	$\alpha_l^I(3)$	$\alpha_l^I(4)$
1	1	3	4	1	1
2	1	1	1	0	0

Chapter 4

Application of Dyck paths in the enumeration of ad-nilpotent ideals

A Dyck path of length $2n$ is a path which begins at the origin $(0,0)$, ends at $(2n,0)$ and consists of diagonal lines of direction $(1,1)$ and $(1,-1)$, such that the path stays above the line $x=0$. We can encode each $(1,1)$ by the letter u (for up), each $(1,-1)$ by the letter d (for down), thus obtaining the encoding of a Dyck path by a word, called a Dyck word. Observe that there is a correspondence between the set of Dyck paths and the set of words using the letters u and d, having the same number of u and d and such that there is always more u's than d's to the left of a letter. We call a peak of a Dyck path, an occurrence of ud in the corresponding Dyck word. The height of a Dyck path is the maximum ordinate of its peaks.

For example, the Dyck path of length 8 and of height 2 which corresponds to the Dyck word *uduududd* is :



In this chapter, we assume that $\mathfrak{g}$ is of type A. We describe two different ways, from [Pa] and [AKOP], of associating a Dyck path to an ad-nilpotent ideal of $\mathfrak{p}_\emptyset$. Then, we use the AKOP correspondence to link the number of occurrence of udu in a Dyck word D and the cardinality of the maximal subset $I \subset \Pi$ such that the ideal which corresponds to D is an ideal of $\mathfrak{p}_I$.

In the last section, we use these correspondences to establish a duality between the elements $\Phi \in \mathcal{F}_{l,\emptyset}$ such that $\sharp\Phi_{min} = k$ and those such that $\sharp\Phi_{min} = l - k$. This duality is different from the one given in [Pa].

4.1 Partition and Dyck path

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition whose Ferrers diagram F is included in T_l. We draw a dotted horizontal line from the top of the line $x+y = l+1$ to F and a dotted vertical line from F to the bottom of the line $x+y = l+1$. For example, when $\lambda = (5, 3, 1, 1, 1, 0, 0)$, we have :

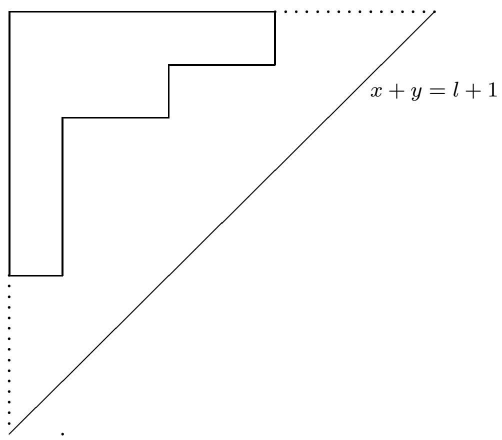
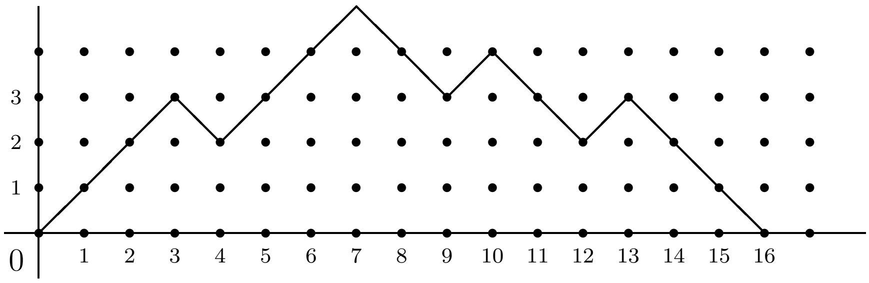


Figure 4.1:

If we rotate the figure clockwise by 45 degrees, we can easily see that we obtain a Dyck path of length $2l+2$ called $P(\lambda)$ as in [Pa]. This construction defines clearly a bijection $P : \lambda \mapsto P(\lambda)$ between partitions whose Ferrers diagram are included in T_l and Dyck paths of length $2l + 2$. In the above example, the Dyck path $P(\lambda)$ is :



4.2 AKOP bijection

In this section, we shall see how to generate another Dyck path from a partition. The construction is taken from [AKOP].

Let λ be a partition whose Ferrers diagram is included in T_l. We can construct the dotted line as in the section 3.2.2. We shall describe the construction of this line in a more formal way.

Let $k = n(\lambda)$. Set $i_n = l+1$ for all $n > k$, $i_k = \lambda_1$, $i_{k-1} = \lambda_{l-i_k+2}$, $i_{k-2} = \lambda_{l-i_{k-1}+2}$, $\ldots$, $i_1 = \lambda_{l-i_2+2}$ and $i_p = 0$ for all $p \leqslant 0$. We have $0 < i_1 < \cdots < i_k < l+1$. The dotted line describes the shape of a partition

$$\lambda^M = (i_k^{l-i_k+1}, i_{k-1}^{i_k-i_{k-1}}, \ldots, i_1^{i_2-i_1}, 0^{i_1-1}). \tag{4.1}$$

Any partition λ whose associated dotted line gives the partition λ^M must necessarily contain the cells

$$(1, i_k), (l-i_k+2, i_{k-1}), (l-i_{k-1}+2, i_{k-2}), \ldots, (l-i_2+2, i_1).$$

The "minimal" partition in the sense of inclusion of diagram that contains these cells is :

$$\lambda^m = (i_k, i_{k-1}^{l-i_k+1}, i_{k-2}^{i_k-i_{k-1}}, \ldots, i_1^{i_3-i_2}, 0^{i_2-2}). \tag{4.2}$$

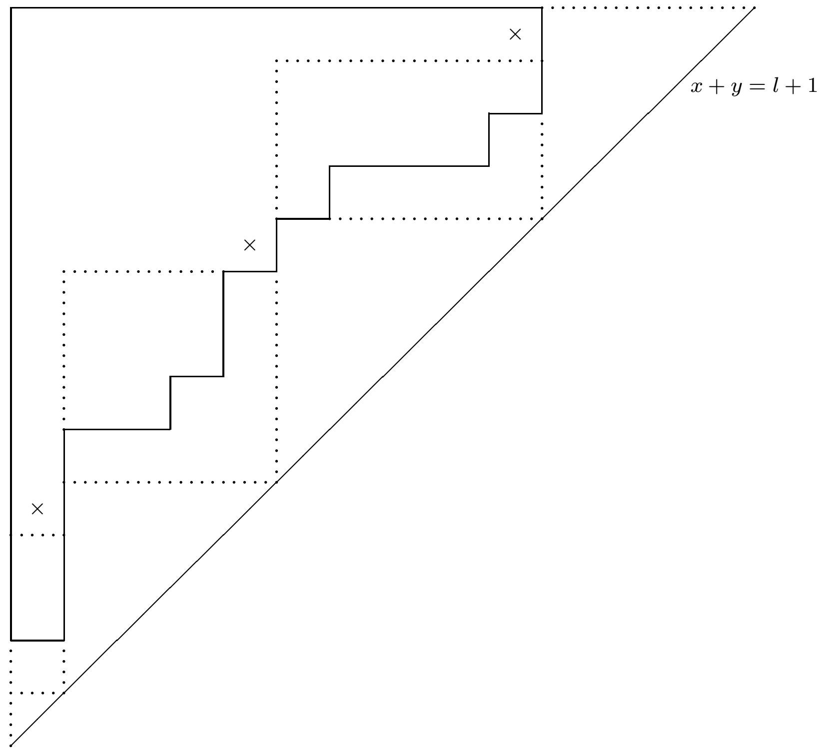
For example, take $l = 13$ and $\lambda = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0)$, as in 3.2.2, we have $n(\lambda) = k = 3$, $i_3 = 10, i_2 = 5, i_1 = 1$. The three distinguished cells above are :

$$(1, 10), (5, 5), (10, 1).$$

So we have :

$$\lambda^M = (10, 10, 10, 10, 5, 5, 5, 5, 5, 1, 1, 1, 1), \text{ and}$$
$$\lambda^m = (10, 5, 5, 5, 5, 1, 1, 1, 1, 1, 0, 0, 0).$$

These partitions are illustrated in the figure below, where the distinguished cells are marked with $\times$, and λ^M is the partition corresponding to the dotted line outside λ, while λ^m is the one which corresponds to the dotted line inside λ.



Observe that the difference $\lambda^M \setminus \lambda^m$ is a disjoint union of k rectangles, denoted by $R_k, \ldots, R_1$ from the top to the bottom. More precisely,

$$R_j = \{(s,t); l - i_{p+1} + 2 < s < l - i_p + 2 \text{ and } i_{p-1} < t \leqslant i_p\}.$$

Inside each rectangle R_j, the shape of λ could be described by a word M_j, whose letters are d and l, where d indicates a down step and l indicates a left step.

Let h_j be the number of d in M_j, which is at most the height of R_j and let l_j be the number of l in M_j, which is the length of R_j. Then we have :

$$h_j = i_{j+1} - i_j - 1 \text{ if } j \neq 1, \text{ and } h_j \leqslant i_{j+1} - i_j - 1 \text{ if } j = 1,$$
$$l_j = i_j - i_{j-1},$$

so $h_j \leqslant l_{j+1} - 1$ and the equality holds if $j \neq 1$. Furthermore the shape of

M_j is $l^{a_{j,0}}dl^{a_{j,1}}d\dots dl^{a_{j,h_j}}$, where $a_{j,i} \in \mathbb{N}$, $0 \leqslant i \leqslant h_j$. We then have :

$$l_j = \sum_{i=0}^{h_j} a_{j,i}. \tag{4.3}$$

In the above example, we have $M_3 = dldl^3dl$, $M_2 = lddldl^2d$ and $M_1 = ddl$.

We shall now generate a Dyck path step by step from the M_j.

First, let D_{k+1} be the Dyck path of length $2(l+1-i_k)$ containing $l+1-i_k$ peaks. Next, we have $M_k = l^{a_{k,0}}dl^{a_{k,1}}d\dots dl^{a_{k,h_k}}$. We insert $a_{k,0}$ peaks on the first peak of the already existing Dyck path D_{k+1}, then $a_{k,1}$ peaks on the second peak, and so on. We call D_k the new Dyck path obtained. Observed that the highest peaks of D_k are exactly those newly inserted, so there are exactly l_k. Since $h_{k-1} \leqslant l_k - 1$, the procedure can then be iterated by inserting peaks only on highest peaks. Each intermediate Dyck path obtained after using the word M_j is denoted by D_j. At the end, we obtain a Dyck path D_λ of length $2l+2$.

For example, if we consider $l = 7$ and $\lambda = (5, 3, 1, 1, 1, 0, 0)$:

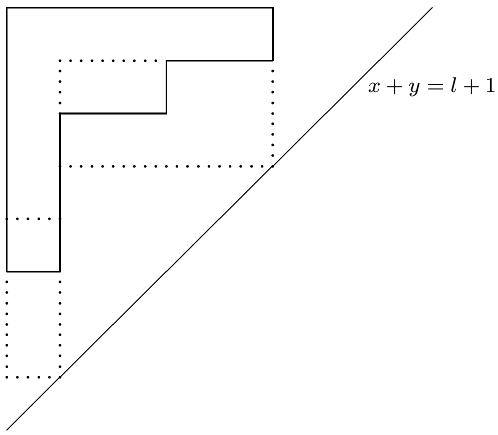


Figure 4.2:

We have $n(\lambda) = k = 2$, $i_2 = 5$ and $i_1 = 1$. Then D_3 is the Dyck path :

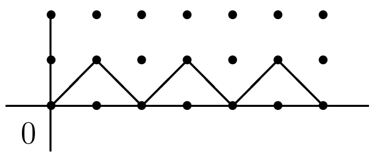


Figure 4.3: D_3

We have $M_2 = l^2dl^2d$, so we first insert 2 peaks on the first peak of D_3, then again two peaks on the second one. We obtain D_2:

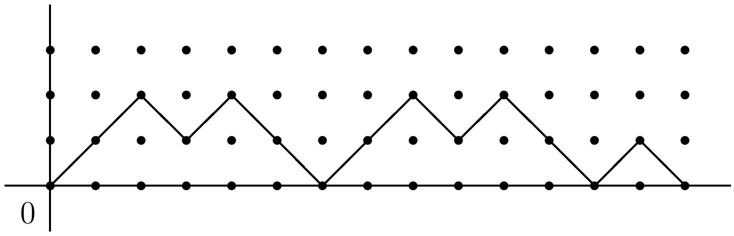


Figure 4.4: D_2

Finally, $M_1 = dl$ so we insert $a_{1,0} = 0$ peak on the first highest peak of D_2 and $a_{1,1} = 1$ peak on the second highest peak. We obtain D_λ:

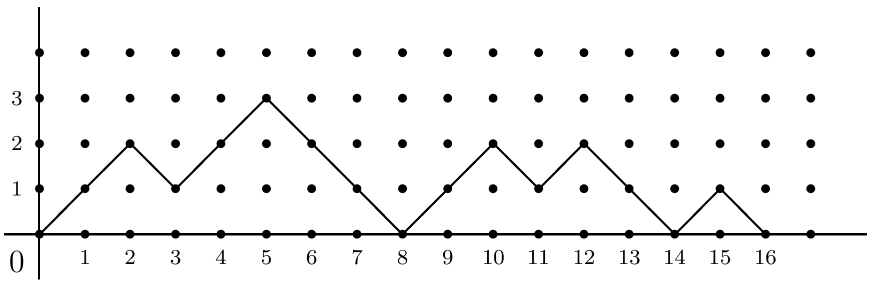


Figure 4.5: D_λ

By [AKOP], we have the following proposition :

Proposition 4.2.1 *The map $D : \lambda \mapsto D_\lambda$ defines a bijection between the set of partitions contained in T_l and the set of Dyck paths of length $2l+2$.*

4.3 Dyck path and number of *udu*

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition such that $n(\lambda) = k$. Let D_λ be the Dyck path obtained from λ as described in section 4.2. We shall see how to count the number of *udu* contained in D_λ.

Recall that a peak is an occurrence ud in the Dyck word. This peak could be followed by an u, a d or nothing in the Dyck word. If it is followed by a u, we call it a u-peak. Each u-peak will give an udu and vice versa.

Let $1 \leqslant j \leqslant k+1$. Let u_j be the number of u-peaks in the Dyck path D_j. For example, D_{k+1} consists in $l - \lambda_1 + 1 = l - i_k + 1$ peaks, so it is easy to see that $u_{k+1} = l - \lambda_1$.

To construct D_{j-1} from D_j, we add some peaks on the highest peaks of D_j. Then, one must understand how the insertion of p peaks on a highest peak modifies the number of udu. Consider a peak P of maximal height on a Dyck path. If we add p peaks, the part of the Dyck word which corresponds to P (which was ud) becomes $uudud \ldots udd$ (with p ud), so we obtain $p-1$ udu. If P is a u-peak, then we also "destroy" the udu given by P. So at the end, we only add $p-2$ udu. For example, in the following Dyck path which contains 2 udu :

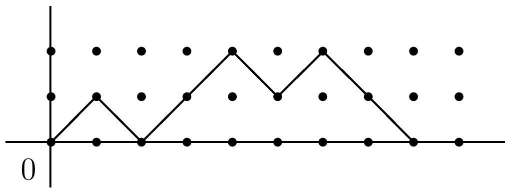
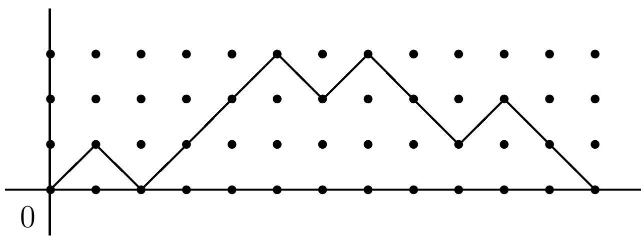
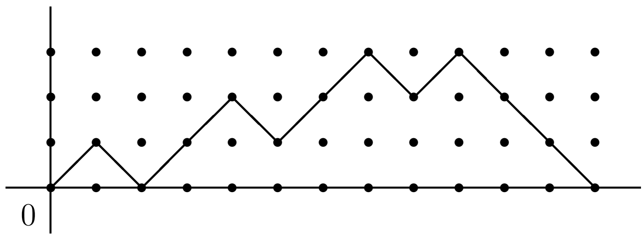


Figure 4.6:

if we add 2 peaks on the first highest peak, we add $2-2=0$ udu, so we obtain the following Dyck path with still 2 udu:



If P is not a u-peak, then we do not "destroy" a udu, so we effectly add $p-1$ udu. For example, if we add 2 peaks on the second highest peak of Figure 4.6, we add $2-1=1$ udu, so we obtain 3 udu at the end:



Set $a_{k+1,0} = l - i_k + 1$, $M_{k+1} = l^{a_{k+1,0}}$, and $h_{k+1} = 0$. We have seen that each word M_j is in the form $l^{a_{j,0}} d l^{a_{j,1}} d \dots d l^{a_{j,h_j}}$. Let

$$\mathcal{A}_j = \{(j,t); t \in \{0,\dots,h_j\}; a_{j,t} \neq 0\},$$

$$\mathcal{A} = \bigcup_{j=1}^{k} \mathcal{A}_j.$$

Recall from the construction that the number of highest peaks in D_j is :

$$\sum_{t=0}^{h_j} a_{j,i} = l_j. \tag{4.4}$$

Observe that a highest peak is a u-peak if it is not the last one of a consecutive group of highest peak. Hence, the q-th peak of D_j is not a u-peak if and only if there exists $r \in \{0,\dots,h_j\}$ such that $q = \sum_{s=0}^{r} a_{j,s}$. Set

$$\mathcal{L}_p = \left\{ (p,t); \text{there exists } \ 0 \leqslant r \leqslant h_{p+1}; t+1 = \sum_{q=0}^{r} a_{p+1,q} \right\},$$

$$\mathcal{U}_p = \mathcal{A}_p \setminus \mathcal{L}_p, \quad \mathcal{L} = \bigcup_{p=1}^{k} \mathcal{L}_p, \quad \mathcal{U} = \bigcup_{p=1}^{k} \mathcal{U}_p.$$

Thus $\mathcal{L}_j$ corresponds exactly to the set of highest peaks in D_j which are not u-peaks and where we insert new peaks. It follows that :

$$u_{j-1} = u_j + \sum_{(j-1,t)\in\mathcal{U}_{j-1}} (a_{j-1,t} - 2) + \sum_{(j-1,t)\in\mathcal{L}_{j-1}} (a_{j-1,t} - 1).$$

At the end of the construction, the number of udu in D_λ is u_1. By induction, we have

$$\begin{aligned} u_1 &= \sum_{p=2}^{k+1} u_p + \sum_{(1,t)\in\mathcal{U}_1} (a_{1,t}-2) + \sum_{(1,t)\in\mathcal{L}_1} (a_{1,t}-1) \\ &= l-\lambda_1 + \sum_{(j,t)\in\mathcal{U}} (a_{j,t}-2) + \sum_{(j,t)\in\mathcal{L}} (a_{j,t}-1). \end{aligned}$$

Since $\sum_{(j,t)\in\mathcal{A}} a_{j,t} = \lambda_1$, we obtain the following proposition :

Proposition 4.3.1 *Let λ be a partition contained in T_l. Then, the number of udu in D_λ is $l - 2\sharp\mathcal{U} - \sharp\mathcal{L}$.*

To illustrate this, we could follow again the construction of the Dyck path which corresponds to $\lambda = (5,3,1,1,1,0,0)$. We first have the Dyck path D_3 in Figure 4.3, with $n-\lambda_1+1 = 3$ peaks, and $u_3 = 2$. Then we use the word $M_2 = l^2dl^2d = l^{a_{2,0}}dl^{a_{2,1}}d$, where $a_{2,0}, a_{2,1} \in \mathcal{L}_2$, so we add $a_{2,0}-2+a_{2,1}-2 = 0$ peak. So $u_2 = 2$. Then we use the word $M_1 = dl = l^{a_{1,0}}dl^{a_{1,1}}$, where $a_{1,1} \in \mathcal{U}_1$, so we add $a_{1,1}-1 = 0$ peak. Hence, $u_1 = 2$.

4.4 Ad-nilpotent ideals of a parabolic subalgebra and Dyck path

In this section, we assume that $\mathfrak{g}$ is of type A_l.

Let $\Phi \in \mathcal{F}_\emptyset$ which corresponds to an ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{p}_\emptyset$. By 2.2.1, Φ corresponds to a nw-diagram T of T_l. Let λ be the partition whose Ferrers diagram is T and which corresponds to $\mathfrak{i}$ as in section 3.2.2. By [AKOP], the construction of 4.2 which to a partition λ associate the Dyck path D_λ is a bijection between the set of ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{p}_\emptyset$ having index of nilpotence k and the set of Dyck path from $(0,0)$ to $(2l+2,0)$ with height $k+1$.

For $\Phi \in \mathcal{F}_\emptyset$, set

$$I_\Phi = \{\alpha \in \Pi; \Phi \in \mathcal{F}_{\{\alpha\}}\}.$$

It is the maximal element of $\{I \subset \Pi; \Phi \in \mathcal{F}_I\}$. (denoted by I_w in 1.2.7) We shall see how to link the number of udu of the Dyck path which corresponds to Φ and I_Φ.

Set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, for all $1 \leqslant i \leqslant j \leqslant l$. Since $\mathfrak{g}$ is of type A_l, we have easily the following lemma :

Lemma 4.4.1 *An element $\Phi \in \mathcal{F}_\emptyset$ is an element of $\mathcal{F}_I$, for $I \subset \Pi$, if and only if for all $\alpha_{i,j} \in \Phi_{min}$, we have $\alpha_i, \alpha_j \notin I$.*

It follows from Lemma 4.4.1 that :

$$I_\Phi = \Pi \setminus \{\alpha_i \in \Pi; \text{there exists } \alpha_{i,j} \text{ or } \alpha_{l,i} \in \Phi_{min}\}.$$

The problem is not to count the same root twice. For example, in A_7, for $\Phi_{min} = \{\alpha_{1,3}, \alpha_{2,5}, \alpha_{5,7}\}$, we have $\Pi \setminus I_\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ but we find α_5 in the beginning or in the end of the support of two roots in Φ_{min}. So if we set :

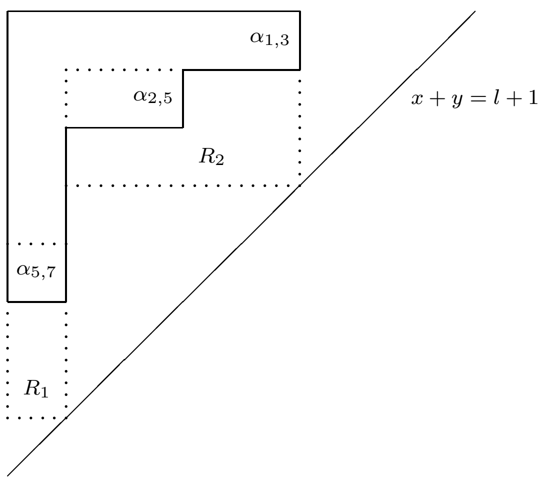
$$L = \{\alpha_{i,j} \in \Phi_{min};\ \text{there exists a root of shape } \alpha_{p,i} \in \Phi_{min}\},$$

$$U = \Phi_{min} \setminus L,$$

we obtain that

$$\sharp I_\Phi = l - 2\sharp U - \sharp L. \tag{4.5}$$

Let λ be the partition which corresponds to Φ and let F be the Ferrers diagram of λ. Let D_λ be the Dyck path which corresponds to λ. Let $\alpha_{i,j} \in \Phi_{min}$. Then the cell $(i, l+1-j) = (i, \lambda_i)$ of $\alpha_{i,j}$ in F is a south-east corner of the diagram and two cases are possible : there exists a rectangle R_p such that $(i, \lambda_i) \in R_p$ or (i, λ_i) is not in any rectangle. If the last case occurs, then $(i, l+1-j)$ is above a rectangle R_p. For example, if $\lambda = (5,3,1,1,1,0,0)$, we have that $\alpha_{2,5}, \alpha_{5,7}$ are in the first case and $\alpha_{1,3}$ is in the second case.



If $\alpha_{i,j}$ is in the rectangle R_p, then the cell $(i, \lambda_i) = (i, l-j+1)$ which corresponds to $\alpha_{i,j}$ in F satisfies :

$$l - i_{p+1} + 2 < i < l - i_p + 2, \tag{4.6}$$

$$i_{p-1} < \lambda_i \leqslant i_p, \tag{4.7}$$

and so we have :

$$l - i_p + 1 \leqslant j < l - i_{p-1} + 1. \tag{4.8}$$

If $\alpha_{i,j}$ is above the rectangle R_p, then the cell $(i, l-j+1)$ which corresponds to $\alpha_{i,j}$ in F satisfies :

$$(i, l-j+1) = (l - i_{p+1} + 2, i_p). \tag{4.9}$$

Define the map r from Φ_{min} to $\{1, \dots, k\}$ which to $\alpha_{i,j}$ associate the integer $r(\alpha_{i,j}) = p$ such that $\alpha_{i,j}$ is in or immediately above the rectangle R_p.

Let $\alpha_{i,j} \in \Phi_{min}$ and $p = r(\alpha_{i,j})$. Since the cell $(i, l-j+1)$ which contains $\alpha_{i,j}$ in T_l is a south-east corner, there is a horizontal line under this cell. If $c = (i, l-j+1)$ is in the rectangle R_p, then it is at the row $q = i-(l-i_{p+1}+2)$ of R_p and the line under c correspond to the part $l^{a_{p,q}}$ in M_p. Furthermore $(p, q) \in \mathcal{A}_p$.

If c is immediately above the rectangle R_p, then the line under c corresponds to $l^{a_{p,0}}$ in M_p and $(p, 0) \in \mathcal{A}_p$. Since in this case, by (4.9) we have $(i, l-j+1) = (l - i_{p+1} + 2, i_p)$, we obtain that $i - (l - i_{p+1} + 2) = 0$. We can define in any case the map s from Φ_{min} to $\mathbb{N}$ by :

$$s(\alpha_{i,j}) = i - (l - i_{r(\alpha_{i,j})+1} + 2). \tag{4.10}$$

Furthermore, in both cases, the line under the cell which contains $\alpha_{i,j}$ is the part $l^{a_{r(\alpha_{i,j}),s(\alpha_{i,j})}}$ in $M_{r(\alpha_{i,j})}$ and $(r(\alpha_{i,j}), s(\alpha_{i,j})) \in \mathcal{A}_{r(\alpha_{i,j})}$.

Conversely, let $(p, q) \in \mathcal{A}_p$. Then, there is a horizontal line under the row $i = q - l - i_{p+1} + 2$ of F which is under a south-east corner of F. This south-east corner is a cell (i, λ_i) which corresponds to a root $\alpha_{i,j}$, where $l - j + 1 = \lambda_i$. So we have a bijection :

$$\begin{array}{cccc} \Psi & : & \Phi_{min} & \rightarrow & \mathcal{A} \\ & & \alpha_{i,j} & \mapsto & (r(\alpha_{i,j}), s(\alpha_{i,j})) \end{array}$$

Lemma 4.4.2 *We have $\Psi(U) = \mathcal{U}$ and $\Psi(L) = \mathcal{L}$.*

Proof. Since $L = \Phi_{min} \setminus U$ and $\mathcal{L} = \mathcal{A} \setminus \mathcal{U}$, it suffices to prove that $\Psi(L) = \mathcal{L}$.

Let $\alpha_{i,j} \in L$. Set $p = r(\alpha_{i,j})$, $q = s(\alpha_{i,j})$ and let $c = (i, \lambda_i)$ be the cell which corresponds to $\alpha_{i,j}$ in F.

First assume that $i = j$. Then, we have $c = (i, l - i + 1)$. If $c \in R_p$, then by (4.6) and (4.8), we have :

$$i = l - i_p + 1,$$

so by (4.10), we have that $q = i_{p+1} - i_p - 1$ so by (4.3), $a_{p,q} \in \mathcal{L}_p$.

If c is above R_p, then by (4.9), we have $c = (i, l-i+1) = (l-i_{p+1}+2, i_p)$, so $q = 0$ and $i_{p+1} - i_p = 1$, hence by (4.3) we also have $a_{p,q} \in \mathcal{L}_p$.

Now assume that $i \neq j$ and there exists a root of shape $\alpha_{m,i} \in \Phi_{min}$. Set $t = r(\alpha_{m,i})$. Let $(m, \lambda_m) = (m, l - i + 1)$ be the cell which corresponds to $\alpha_{m,i}$ in λ. If $c \in R_p$, then by (4.6), we have

$$i_p \leqslant \lambda_m \leqslant i_{p+1} - 2.$$

So either $(m, \lambda_m) \in R_{p+1}$ or $(m, \lambda_m) = (l - i_{p+1} + 2, i_p)$.

If $(m, \lambda_m) \in R_{p+1}$, then between the columns i_{p+1} and $\lambda_m = l - i + 1$, we have $i_{p+1} - (l - i + 1)$ columns, so there exists n such that $\sum_{u=0}^{n} a_{p+1,u} = i_{p+1} - (l - i + 1)$. Furthermore, by (4.10), we have $q = i - (l - i_{p+1} + 2)$, hence $a_{p,q} \in \mathcal{L}_p$.

If $(m, \lambda_m) = (l - i_{p+1} + 2, i_p)$, then $i = l - i_p + 1$ and by (4.10), we have that :

$$q = (l - i_p + 1) - (l - i_{p+1} + 2) = i_{p+1} - i_p - 1.$$

Hence, by (4.3), we have $a_{p,q} \in \mathcal{L}_p$.

Conversely, let $a_{p,q} \in \mathcal{L}_p$, then there exists $0 \leqslant t \leqslant h_{p+1}$ such that $q + 1 = \sum_{f=0}^{t} a_{p+1,f}$. There also exists $\alpha_{i,j} \in \Phi_{min}$ such that $r(\alpha_{i,j}) = p$ and $s(\alpha_{i,j}) = q$. By (4.10), we have that :

$$q = i - (l - i_{p+1} + 2).$$

Observe that for all $0 \leqslant j \leqslant h_{p+1}$, there exists a south-east corner (n_j, λ_{n_j}) in or above the rectangle R_{p+1} such that :

$$\lambda_{n_j} = i_{p+1} - \sum_{f=0}^{j} a_{p+1,f}.$$

So there exists a south-east corner (n_j, λ_{n_j}) such that :

$$\lambda_{n_j} = i_{p+1} - (q+1) = l - i + 1.$$

The element of Φ_{min} which corresponds to the cell (n_j, λ_{n_j}) is $\alpha_{n_j,i}$, so we have $\alpha_{i,j} \in L$. □

It follows by Proposition 4.3.1 and equation (4.5) that we have the following theorem :

Theorem 4.4.3 *Let $\mathfrak{g}$ be of type A_l. There is a bijection between the elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$ and the Dyck paths of length $2l+2$ having r udu.*

Since the number of Dyck paths having a fixed number of *udu* is calculated in [Sun], we have the following corollary :

Corollary 4.4.4 *Let $\mathfrak{g}$ be of type A_l. The number of elements of $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$ is*

$$\begin{pmatrix} l \\ r \end{pmatrix} \sum_{k=0}^{[l-r]/2} \begin{pmatrix} l-r \\ 2k \end{pmatrix} \mathcal{C}_k$$

where $\mathcal{C}_k$ denotes the k-th Catalan number.

Example 4.4.5 *Let N_r^l be the number of elements $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$. We have by Corollary 4.4.4 :*

r	N_r^1	N_r^2	N_r^3	N_r^4	N_r^5
0	1	2	4	9	21
1	1	2	6	16	45
2		1	3	12	40
3			1	4	20
4				1	5
5					1

Remark 4.4.6 *Using GAP4 and the methods described in section 2.3, we computed the number N_r^X of elements of $\Phi \in \mathcal{F}_\emptyset$ such that $\sharp I_\Phi = r$, when $\mathfrak{g}$*

is of exceptional type X :

r	$N_r^{G_2}$	$N_r^{F_4}$	$N_r^{E_6}$	$N_r^{E_7}$	$N_r^{E_8}$
0	2	19	111	432	2033
1	5	45	217	1213	6351
2	1	29	253	1296	7729
3		11	136	794	5433
4		1	44	323	2510
5			11	84	819
6			1	17	175
7				1	29
8					1

4.5 Duality

In this section, we assume that $\mathfrak{g}$ is of type A_l. We shall construct a duality between the elements of $\mathcal{F}_{l,\emptyset}$ such that $\sharp\Phi_{min} = p$ and those such that $\sharp\Phi_{min} = l - p$.

Proposition 4.5.1 *Let $\Phi \in \mathcal{F}_{l,\emptyset}$. Let λ be the corresponding partition in T_l and D_λ the Dych path which corresponds to λ. Let N be the number of peaks in D_λ, then we have :*

$$\sharp\Phi_{min} = l - (N - 1).$$

Proof. Recall that the construction of D_λ is iterative. At each step, when we add $a_{p,q}$ peaks to a highest peak, for $(p, q) \in \mathcal{A}_p$, we also "destroy" the initial highest peak. So, we add only $a_{p,q} - 1$ peaks. At the end of the construction we have :

$$l - \lambda_1 + 1 + \sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} (a_{p,q} - 1)$$

peaks. Since $\sum_{p=1}^{k} \sum_{(p,q)\in\mathcal{A}_p} a_{p,q} = \sum_{(p,q)\in\mathcal{A}} a_{p,q} = \lambda_1$ and $\mathcal{A}$ is in bijection with Φ_{min} by section 4.4, we obtain the result. □

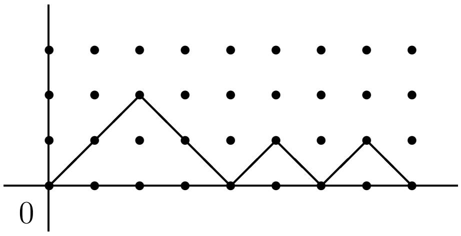
Proposition 4.5.2 *Let $\Phi \in \mathcal{F}_{l,\emptyset}$ and λ be its corresponding partition. Let P be the number of peaks in $P(\lambda)$, then we have :*

$$\sharp\Phi_{min} = P - 1.$$

Proof. The result is clear by the construction of $P(\lambda)$ defined in 4.1. $\square$

Theorem 4.5.3 *The map $P^{-1} \circ D$ (where P is defined in section 4.1 and D in section 4.2), induces a bijection from $\mathcal{F}_{l,\emptyset}$ to $\mathcal{F}_{l,\emptyset}$ which sends $\Phi \in \mathcal{F}_{l,\emptyset}$ such that $\sharp\Phi_{min} = p$ to $\Psi \in \mathcal{F}_{l,\emptyset}$ such that $\sharp\Psi_{min} = l - p$.*

For example, in A_3, the element $\Phi = \{\theta\} \in \mathcal{F}_{\emptyset}$ corresponds to the partition $\lambda = (1, 0, 0)$, and the Dyck path D_λ is :



Then, $P^{-1}(D_\lambda) = (3, 2, 0)$ which is the partition which corresponds to Ψ such that $\Psi_{min} = \{\alpha_1, \alpha_2\}$.

Remark 4.5.4 *It was proved in [Pa] that in type A and C, the number of elements $\Phi \in \mathcal{F}_{l,\emptyset}$ such that $\sharp\Phi_{min} = p$ is the same as the number of elements $\Phi \in \mathcal{F}_{l,\emptyset}$ such that $\sharp\Phi_{min} = l - p$. But the duality of [Pa] is not the same as the one defined above. For example, in A_3, if we consider $\Phi = \{\theta\}$ like above, the dual ideal defined by [Pa] is Ψ where $\Psi_{min} = \{\alpha_1 + \alpha_2, \alpha_3\}$.*

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Résumé

Dale Peterson a démontré que le nombre d'idéaux abéliens d'une sous-algèbre de Borel d'une algèbre de Lie simple de rang l était 2^l. Ce résultat a motivé l'étude des idéaux ad-nilpotents d'une sous-algèbre de Borel par de nombreux chercheurs. L'idée directrice de cette thèse a été de généraliser les caractérisations obtenues par Paola Cellini et Paolo Papi sur ces idéaux aux idéaux ad-nilpotents d'une sous-algèbre parabolique.

Soit $\mathfrak{g}$ une algèbre de Lie simple complexe. A tout idéal ad-nilpotent d'une sous-algèbre parabolique standard de $\mathfrak{g}$, nous associons naturellement un sous-ensemble du système de racines positives. Ensuite nous établissons une correspondance entre ces sous-ensembles et certains éléments du groupe de Weyl affine. Par des considérations sur le volume des faces de l'alcôve fondamentale, nous établissons alors une généralisation du théorème de Peterson.

Parallèlement, grâce à un dénombrement de tableaux, nous obtenons des formules explicites pour les nombres d'idéaux ad-nilpotents et abéliens d'une sous-algèbre parabolique dans le cas classique. Lorsque $\mathfrak{g}$ est de type exceptionnel, l'énumération de ces idéaux est obtenue en utilisant $GAP4$.

Ensuite, nous utilisons la bijection construite par George Andrews, Christian Krattenthaler, Luigi Orsina et Paolo Papi entre les tableaux de Young classiques T_l et les chemins de Dyck de longueur $2l+2$. Cette bijection nous permet d'obtenir le nombre d'idéaux ad-nilpotents d'une sous-algèbre parabolique d'indice de nilpotence K, en type A et C. D'autre part, lorsque $\mathfrak{g}$ est de type A, elle nous permet d'établir un lien entre la plus grande sous-algèbre parabolique contenant un idéal et le nombre de séquences "udu" contenues dans le chemin de Dyck associé à cet idéal.

Mots clés

Algèbre de Lie, théorie des représentations, idéaux ad-nilpotents, énumération de diagrammes, combinatoire algébrique, chemins de Dyck.