Céline Righi
Le Petit Fief Clairet
86280 Saint-Benoit, France
Tel : 00 33 (0)5 49 50 11 46
e-mail : righi@math.univ-poitiers.fr

Research project

My research areas are the theory of Lie algebras and their representations. More precisely, I study the following problems related to ad-nilpotent ideals of a parabolic subalgebra of a complex simple Lie algebra :

- Characterization and enumeration of ad-nilpotent and abelian ideals.
- Combinatorial objects related to these ideals, for example certain labeled diagrams, Dyck paths, antichains.
- Index of the quotient of a Borel subalgebra by an ad-nilpotent ideal.

Characterizations of ad-nilpotent ideals

The study of ad-nilpotent ideals of a Borel subalgebra have been motivated by a result of D. Peterson. He stated that the number of abelian ideals of a Borel subalgebra of a simple Lie algebra $\mathfrak{g}$ of rank l is 2^l. A proof of this result can be found in [6] and involves the affine Weyl group associated to $\mathfrak{g}$. In that paper, Kostant indicated connections of these abelians ideals with discrete series and made a link with his old (1965) results on the structure of $\wedge\mathfrak{g}$ in terms of commutative subalgebras of $\mathfrak{g}$.

The affine Weyl group plays an important role in the different characterizations of the ad-nilpotent ideals established by P. Cellini and P. Papi in [3] and [4]. Moreover, in [4], the ad-nilpotent ideals are parametrized by certain points of a canonical simplex. This description allowed them to find a formula for the number of ad-nilpotent ideals which depends on the Coxeter number, the cardinality and the exponents of the Weyl group. This formula also counts several other objects like real hyperplan arrangements ([2]), or the number of clusters which correspond to an irreducible root system ([5]).

On the other hand, when $\mathfrak{g}$ is of classical type, the ad-nilpotent ideals can be encoded by certain subdiagrams of a (suitably shifted) Young diagram (see [15] and [3]). Others combinatorial objects are linked with ad-nilpotent ideals, like Dyck paths, Chebyshev polynomial, or Catalan and Motzkin numbers. Let us also mention the works of Panyushev, Sommers, and Suter ([9],[16],[17]) among others on this subject.

In [12], I generalized the characterization obtained by Paola Cellini and Paolo Papi on ad-nilpotent ideals of a Borel subalgebra to parabolic subalgebras.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Δ the associated root system. Fix a system of positive roots Δ^+ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the corresponding set of simple roots. Let $I \subset \Pi$. Denote by $\mathfrak{p}_I$ the corresponding standard parabolic subalgebra of $\mathfrak{g}$. Observe that when $I = \emptyset$, then $\mathfrak{p}_\emptyset = \mathfrak{b}$ is a Borel subalgebra. Since ad-nilpotent ideals of a parabolic subalgebra are Borel-stable, we have the encoding of these ideals via certain elements of the affine Weyl group $\widehat{W}$ (see [3]) called *Borel-compatible*. If an element of $\widehat{W}$ corresponds to an ad-nilpotent ideal of a parabolic subalgebra $\mathfrak{p}_I$, we say that it is I*-compatible*. I gave in [12] a complete characterization of these ideals. Let $\widehat{\Pi} = \{\alpha_0\} \cup \Pi$ be the set of affine simple roots. In particular, an element in $w \in \widehat{W}$ is I-compatible if and only if it is Borel-compatible and satisfies :

$$w^{-1}(I) \subset \widehat{\Pi}.$$

I established another characterization of ad-nilpotent ideals of a parabolic subalgebra in terms of faces in the alcove of the corresponding element in the affine Weyl group. Consequently, I obtained a generalization of Peterson's result. Let

$$\mathcal{A}b_I = \{w \in \widehat{W}; w \text{ corresponds to an abelian ideal of } \mathfrak{p}_I\}.$$

Set $n_0 = 1$ and let n_i, $i = 1, \dots, l$, be the strictly positive integers such that the highest positive root $\theta = \sum_{i=1}^{l} n_i \alpha_i$. For $J \subset \widehat{\Pi}$, set $n_J = \prod_{\alpha_j \in J} n_j$. Let $w \in \widehat{W}$ be I-compatible, by the above consideration, $w^{-1}(I) \subset \widehat{\Pi}$, and so $n_{w^{-1}(I)}$ has a sense. By considering the volume of the faces of the fundamental alcove, I proved the following theorem :

Theorem [12]**.** *Let* $I \subset \Pi$*, then*

$$\frac{1}{n_I} \sum_{w \in \mathcal{A}b_I} n_{w^{-1}(I)} = 2^{l - \sharp I}.$$

Since in type A and C the integers n_i, $i = 0, \dots, l$ depend only on the length of the α_i, one obtains as a corollary :

Theorem [12]**.** *Let* $I \subset \Pi$*, if* $\mathfrak{g}$ *is of type* A_l *or* C_l*, then the parabolic subalgebra* $\mathfrak{p}_I$ *has exactly* $2^{l-\sharp I}$ *abelian ideals.*

Combinatorial objects related to ad-nilpotent ideals

Young diagrams

To determine the number of abelian ideals when $\mathfrak{g}$ is of type B or D, and the number of ad-nilpotent ideals in the classical case, I adapted the case by case encoding of these ideals by diagrams as in [15] and reformulated in [3]. The idea is to display the set of positive roots in a diagram of suitable shape. The shape and the filling of the diagrams in each type are chosen such that we obtain a bijection between ad-nilpotent ideals and north-west flushed subdiagrams. In the exceptional cases, the enumeration is obtained using *GAP* (see [12], [13]). Part of my current research project is to find a uniform formula like in the Borel case.

Dyck paths

A *Dyck path* of length $2n$ can be defined as a word of $2n$ letters u or d, having the same number of u and d, and such that there is always more u's than d's to the left of a letter. When $\mathfrak{g}$ is of type A_l, there is a natural bijection between ad-nilpotent ideals of a Borel subalgebra and Dyck paths of length $2l+2$.

Let $\mathfrak{i}$ be an ad-nilpotent ideal of a parabolic subalgebra $\mathfrak{p}_I$. Let

$$\mathfrak{i}^1 = \mathfrak{i} \text{ and } \mathfrak{i}^{k+1} = [\mathfrak{i}^k, \mathfrak{i}],$$

for $k \geqslant 0$, be its central descending series. Recall that the index of nilpotence of $\mathfrak{i}$ is the smallest integer k such that $\mathfrak{i}^{k+1} = \{0\}$ (i.e. the number of non zero terms in the central descending series). In [11, 13], I enumerated the number of ad-nilpotent ideals of $\mathfrak{p}_I$ having a fixed index of nilpotence when $\mathfrak{g}$ is of type A or C. The idea is to construct a map between root systems of different ranks and use the tools introduced in [1] related to Dyck paths.

Another interesting invariant of an ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{p}_\emptyset$ is the maximal element $I_\mathfrak{i}$ of the set $\{I \subset \Pi; \mathfrak{i} \text{ is an ad-nilpotent ideal of } \mathfrak{p}_I\}$. In type A_l, I established a bijection between the ad-nilpotent ideals $\mathfrak{i}$ such that $\sharp I_\mathfrak{i} = r$ and Dyck paths of length $2l+2$ having r *udu*. These combinatorial results are published in [11].

Looking at some calculation with GAP on the other classical types, it seems that the number of ad-nilpotent ideals $\mathfrak{i}$ of $\mathfrak{p}_\emptyset$ such that $\sharp I_\mathfrak{i} = r$ are the same in type B and C. It could mean that this number depends on the exponents of the Weyl group associated to $\mathfrak{g}$. I am studying this question currently.

Antichains

Recall the definition of the following partial order on the set of positive roots $\Delta^+ : \alpha \leqslant \beta$ if $\beta - \alpha$ is a sum of positive roots. An *antichain* is a subset of Δ^+ containing roots which are pairwise non comparable for $\leqslant$.

There is a natural bijection between ad-nilpotent ideals of a Borel subalgebra and antichains (see [9],[16], [4]). Panyushev established in [9] a duality between antichains of cardinality p and of cardinality $l-p$, in types A_l and C_l. By considering the natural bijection between ad-nilpotent ideals of a Borel subalgebra and the Dyck paths of length $2l+2$ in type A given in [9] and the bijection of [1], I established in [12] another duality. Part of my current work is to understand more deeply the sense of these dualities.

For any antichain Γ, let $\mathcal{I}(\Gamma)$ be the set of positive roots which are bigger than an element of Γ. In [10], D. Panyushev defines the map $\mathfrak{X}$ between the set of antichains of Δ^+ : for an antichain Γ, $\mathfrak{X}(\Gamma)$ is the set of maximal elements for $\leqslant$ of $\Delta^+ \setminus \mathcal{I}(\Gamma)$. He stated that $\mathfrak{X}$ conserves his duality and established several conjectures on $\mathfrak{X}$ and on the $\mathfrak{X}$-orbits. Since the definition of $\mathfrak{X}$ is quite natural, one can expect that the properties of $\mathfrak{X}$ are close to some properties of Δ^+. Part of my research project is to study these conjectures using the tools developped from the study of ad-nilpotent ideals or using more combinatorial tools like Dyck paths.

Index of the quotient of a Borel subalgebra by an ad-nilpotent ideal

Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra. For $f \in \mathfrak{g}^*$, we denote by $\mathfrak{g}^f = \{X \in \mathfrak{g}; f([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}$, the annihilator of f for the coadjoint representation of $\mathfrak{g}$. The index of $\mathfrak{g}$, denoted by $\chi(\mathfrak{g})$ is defined to be

$$\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f.$$

It is well known that if $\mathfrak{g}$ is an algebraic Lie algebra and G denotes its algebraic adjoint group, then $\chi(\mathfrak{g})$ is the transcendence degree of the field of G-invariant rational functions on $\mathfrak{g}^*$. For example, the index of a semisimple Lie algebra $\mathfrak{g}$ is equal the rank of $\mathfrak{g}$. This can be obtained easily from the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form.

Recently, the index of the quotient of a Borel subalgebra $\mathfrak{b}$ by an ad-nilpotent ideal is considered in the works of P. Damianou, H. Sabourin and P. Vanhaecke on problems related to Toda-lattices and integrable systems. More precisely, let $\mathfrak{g}$ be a simple complex Lie algebra. Denote by $\mathfrak{g}_\alpha$ the root space of $\mathfrak{g}$ relative to a root α. Let $\Phi \subset \Delta^+$ be such that $\Pi \subset \Phi$. Then one can deduce from Φ an Hamiltonian system if and only if $\bigoplus_{\alpha \in \Delta^+ \setminus \Phi} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{b}$. The problem is to determine whether this system is integrable.

For any ad-nilpotent ideal $\mathfrak{i}$ of $\mathfrak{b}$, $(\mathfrak{b}/\mathfrak{i})^*$ is a Poisson submanifold of $\mathfrak{b}^*$. Its Poisson rank L is equal to the dimension of $\mathfrak{b}/\mathfrak{i}$ minus the index of $\mathfrak{b}/\mathfrak{i}$. Since the number of equations required for the previous Hamiltonian system to be integrable is $\dim((\mathfrak{b}/\mathfrak{i})^*) - L/2$, the calculation of the index of $\mathfrak{b}/\mathfrak{i}$ is involved in this problem.

I am currently working with R.Yu on the determination of the index of the quotient of a Borel subalgebra $\mathfrak{b}$ by an ad-nilpotent ideal $\mathfrak{i}$. Inspired from the results of A.Panov [8] and of P.Tauvel and R.Yu [18], we have obtained an upper bound for the index of $\mathfrak{b}/\mathfrak{i}$ which is excat in certain cases. However, we have an example in type C where the bound is not exact. Our present task is to find conditions for the bound to be exact. We believe to have a proof that the bound is exact in type A.

References

[1] G. E. Andrews, C. Krattenthaler, L. Orsina and P. Papi. *Ad-nilpotent $\mathfrak{b}$-ideals in $sl(n)$ having a fixed class of nilpotence : combinatorics and enumeration.* Trans. Amer. Math. Soc. 354, 3835-3853.

[2] C.A. Athanasiadis. *On noncrossing and nonnesting partitions for classical reflection groups*. Electronic J. of Comb. 5 (1998) ♯R42
[3] P. Cellini, P. Papi. *Ad-nilpotent ideals of a Borel subalgebra*. J. Algebra 225 (2000), 130-140.
[4] P. Cellini, P. Papi. *Ad-nilpotent ideals of a Borel subalgebra II*. J. Algebra 258 (2002), 112-121.
[5] S. Fomin, A. Zelevinsky. *Y-systems and generalized associahedra*. Ann. of Math. 158 (2003), 977-1018.
[6] B. Kostant. *The set of Abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations*. Int. Math. Res. Not. 5 (1998), 225-252.
[7] B. Kostant. *Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra*. Invent. Math. 158 (2004), 181-226.
[8] A.N. Panov. *On index of certain nilpotent Lie algebra*. arXiv:0801.3025v1
[9] D.I. Panyushev. *Ad-nilpotent ideals of a Borel subalgebra : generators and duality*. J. Algebra 274 (2004), 822-846.
[10] D.I. Panyushev. *On orbits of antichains of positive roots*. arXiv:0711.3353v1
[11] C. Righi. *Number of "udu" of a Dyck path and ad-nilpotent ideals of parabolics subalgebra of $sl_{l+1}(\mathbb{C})$*. Séminaire Lotharingien de Combinatoire 59 (2008), article B59c.
[12] C. Righi. *Ad-nilptent ideals of a parabolic subalgebra*. J. Algebra 319 (2008) 1555-2584.
[13] C. Righi. *Characterization and enumeration of ad-nilpotent ideals of a parabolic subalgebra of a simple Lie algebra*. PhD thesis. (2007)
[14] C. Righi.*Enumeration of ad-nilpotent ideals of parabolic subalgebras for exceptional types*, available on arXiv math.RT/0804.2404v1 (2008).
[15] J.Y. Shi. *The number of $\oplus$-sign types*. Quart. J. Math. Oxford 48 (1997), 93-105.
[16] E. Sommers. *B-Stable Ideals in the Nilradical of a Borel Subalgebra*. Canadian Mathematical Bulletin 48 (2005), 460-472.
[17] R. Suter. *Abelian ideals in a Borel subalgebra of a complex simple Lie algebra*. Invent. Math. 156 (2004), 175-221.
[18] P. Tauvel, R.W.T.Yu. *Sur l'indice de certaines algbres de Lie*. Ann. Inst. Fourier (Grenoble) 54 (2004), no. 6, 1793-1810 (2005).