Ad-nilpotent ideals of a parabolic subalgebra

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Abstract

We extend the results of Cellini and Papi [P. Cellini, P. Papi, Ad-nilpotent ideals of a Borel subalgebra, J. Algebra 225 (2000) 130–140; P. Cellini, P. Papi, Ad-nilpotent ideals of a Borel subalgebra II, J. Algebra 258 (2002) 112–121] on the characterizations of ad-nilpotent and abelian ideals of a Borel subalgebra to parabolic subalgebras of a simple Lie algebra. These characterizations are given in terms of elements of the affine Weyl group and faces of alcoves. In the case of a parabolic subalgebra of a classical simple Lie algebra, we give formulas for the number of these ideals.

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1. Introduction

Let \( g \) be a complex simple Lie algebra of rank \( l \). Let \( h \) be a Cartan subalgebra and \( \Delta \) the associated root system. We fix a system of positive roots \( \Delta^+ \). Denote by \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) the corresponding set of simple roots. Let \( V \) be the Euclidian space \( \sum_{k=1}^l \mathbb{R}\alpha_k \). For each \( \alpha \in \Delta \), let \( g_\alpha \) be the root space of \( g \) relative to \( \alpha \).

For \( I \subset \Pi \), set \( \Delta_I = \mathbb{Z}I \cap \Delta \). We fix the corresponding standard parabolic subalgebra:

\[
p_I = h \oplus \bigoplus_{\alpha \in \Delta_I \cup \Delta^+} g_\alpha.
\]
An ideal $i$ of $p_I$ is ad-nilpotent if and only if for all $x \in i$, $\text{ad}_{p_I}x$ is nilpotent. Since any ideal of $p_I$ is $\theta$-stable, we can deduce easily that an ideal is ad-nilpotent if and only if it is nilpotent. Moreover, we have $i = \bigoplus_{\alpha \in \Phi} g_\alpha$, for some subset $\Phi \subset \Delta^+ \setminus \Delta_I$.

The purpose of this paper is to characterize and to enumerate ad-nilpotent and abelian ideals of a parabolic subalgebra.

When $I = \emptyset$, $p_\emptyset$ is a Borel subalgebra of $g$. Peterson proved that the number of abelian ideals of $p_\emptyset$ is $2^l$. Motived by this result, Cellini and Papi, Kostant, Panyushev, Sommers and Suter among others studied ad-nilpotent and abelian ideals of a Borel subalgebra.

In their articles [CP1] and [CP2], Cellini and Papi established different characterizations of the set $I$ of ad-nilpotent ideals of a Borel subalgebra. They constructed a bijection between $I$ and certain elements of the affine Weyl group $\tilde{W}$ associated to $\Delta$, which we shall call $\varnothing$-compatible. These $\varnothing$-compatible elements are in turn characterized by elements of the coroot lattice. They established also, when $g$ is of classical type, a correspondence between ad-nilpotent ideals of $g$ and some diagrams. We extend here their theory to the case of parabolic subalgebras.

Fix $I \subset \Pi$, we establish a bijection between ad-nilpotent ideals of $p_I$ and what we call $I$-compatible elements of the affine Weyl group $\tilde{W}$. We identify $\tilde{W}$ with the group of affine transformations $W_{\text{aff}}$ defined in [Bo] and we give a characterization of the $I$-compatible elements via the dimension of the intersection of the image of the fundamental alcove associated to $W_{\text{aff}}$ with some affine hyperplanes of $V$.

Using this result, we obtain an identity (Theorem 4.7) which generalizes the result of Peterson. This identity links the number of abelian ideals and the coefficients of the simple roots in the highest root of $\Delta$. This allows us to conclude that if $g$ is of type $A$ or $C$, the number of abelian ideals of $p_I$ is $2^l - \#I$. It also explains why this result does not hold in general.

On the other hand, the enumeration of ad-nilpotent and abelian ideals of $p_I$, when $g$ is of classical type, is obtained using the diagrams given in [CP1], modified, by deleting some rows and columns and grouping together some boxes, according to the type of $g$. The formulas obtained depend on the decomposition in connected components of $I$. Note that the formulas obtained when $g$ is of type $A$ or $C$ are nicer than the ones obtained when $g$ is of type $B$ or $D$ (Theorems 4.7, 5.12 and Propositions 5.18, 5.20).

This paper is organized as follows: in Section 2, we recall some results on the affine Weyl group. In Section 3, we give different characterizations of $I$-compatible elements of $\tilde{W}$. The study, in Section 4, of the volume of the intersection of some affine hyperplanes on $V$ gives the results stated above on abelian ideals. Section 5 deals with the enumeration of both ad-nilpotent and abelian ideals when $g$ is of classical type, using diagrams. We give some remarks concerning the exceptional cases and the relations with antichains in Section 6.

2. Generalities on the affine Weyl group

We shall conserve the notations given in the introduction. In this section, we shall recall some basic facts on the affine Weyl group associated to $\Delta$. In particular, we need to recall two different realizations of this group. See [Bo,CP1,K] for more details.

We fix a scalar product $\langle \cdot, \cdot \rangle$ on $V$. For $\alpha \in \Delta$, let

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

denote the corresponding coroot. Denote by $Q^\vee$ the coroot lattice of $\Delta$. 
Let $W$ denote the Weyl group associated to $\Delta$. We shall realize the affine Weyl group as a group of automorphisms of the affine root system associated to $\Delta$. Let $\widehat{V} = V \oplus \mathbb{R} \delta \oplus \mathbb{R} \lambda$. We extend the above bilinear form on $V$ to a non-degenerate symmetric bilinear form on $\widehat{V}$, also denoted $(\cdot,\cdot)$, by setting:

$$(\lambda, \lambda) = (\delta, \delta) = (\lambda, V) = (\delta, V) = 0 \quad \text{and} \quad (\delta, \lambda) = 1.$$ 

Let $\Delta = \Delta + \mathbb{Z} \delta$ be the set of (real) affine roots. We fix the following positive root system $\Delta^+ = (\Delta^+ + \mathbb{N} \delta) \cup (\Delta^- + \mathbb{N}^* \delta)$. We shall write $\alpha > 0$ (respectively $\alpha < 0$) if $\alpha \in \widehat{\Delta}^+$ (respectively if $\alpha \in \widehat{\Delta}^- = -\widehat{\Delta}^+$). Let $\theta$ be the highest root of $\Delta$, then $\widehat{\Pi} = \{\alpha_0 = -\theta + \delta, \alpha_1, \ldots, \alpha_l\}$ is the set of simple roots for $\widehat{\Delta}^+$.

Note that for any element $\beta + k\delta \in \widehat{\Delta}^+$, we have $(\beta + k\delta, \beta + k\delta) = (\beta, \beta) \neq 0$. For all $\alpha \in \widehat{\Delta}^+$, we denote by $s_\alpha$ the reflection of $\widehat{V}$ defined by

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)} \alpha$$

for $x \in \widehat{V}$. The affine Weyl group $\widehat{W}$ is the subgroup of $\text{Aut}(\widehat{V})$ generated by $\{s_\alpha; \alpha \in \widehat{\Pi}\}$. Observe that $w(\delta) = \delta$ for all $w \in \widehat{W}$, $s_\alpha(\lambda) = \lambda$, for all $\alpha \in \Pi$ and $s_{\alpha_0}(\lambda) = \lambda - \frac{1}{2\|\theta\|} \alpha_0$, where $\|\theta\| = \sqrt{(\theta, \theta)}$.

Let $\tau \in Q^\vee$, we define the endomorphism $t_\tau$ of $\widehat{V}$ by:

$$t_\tau(x + a\delta + b\lambda) = x + a\delta + b\lambda + b\tau + \left(\frac{b}{2}(\tau, \tau) - (x, \tau)\right) \delta$$

(1)

for $x \in V$ and $a, b \in \mathbb{R}$. Let $S = \{t_\tau; \tau \in Q^\vee\}$, then the group $\widehat{W}$ is the semi-direct product of $S$ by $W$.

Consider the $\widehat{W}$-invariant affine subspace

$$E = \{x \in \widehat{V}; (x, \delta) = 1\} = V \oplus \mathbb{R} \delta + \lambda.$$

Let $\pi : E \rightarrow V$ be the projection $ax + b\delta + \lambda \mapsto ax$ and

$$i : V \rightarrow E,$n

$$v \mapsto v + \lambda.$$

For $w \in \widehat{W}$, we set $\overline{w} = \pi \circ w|_E \circ i$. The map $w \mapsto \overline{w}$ defines an injective morphism of groups from $\widehat{W}$ to $\text{Aut}(V)$. We shall identify $\widehat{W}$ with its image $W_{\text{aff}}$ under this map.

For $\alpha \in \Delta$, $s_\alpha$ is the reflection $s_\alpha$ on $V$ associated to $\alpha$, and for $\tau \in Q^\vee$, $\overline{\tau}$ is the translation $T_\tau$ by the vector $\tau$ on $V$. For $\alpha \in \Delta^+, k \geq 0, x \in V$, we obtain that

$$\overline{s_{\alpha + k\delta}}(x) = x - ((x, \alpha) - k) \alpha^\vee = T_{k\alpha^\vee} \circ s_\alpha(x),$$

$$\overline{s_{\alpha + k\delta}}(x) = x - ((x, \alpha) + k) \alpha^\vee = T_{-k\alpha^\vee} \circ s_\alpha(x).$$

Thus $\overline{s_{\alpha + k\delta}}$ and $\overline{s_{\alpha + k\delta}}$ are the orthogonal reflections with respect to $H_{\alpha,k} = \{x \in V; (x, \alpha) = k\}$ and $H_{\alpha,-k}$ respectively. It follows that $W_{\text{aff}}$ is the semi-direct product of $W$ by the group of translations $T_\tau, \tau \in Q^\vee$. 
Observe that for \( v \in W, \tau \in Q^\vee, \alpha \in \Delta \) and \( k \in \mathbb{Z} \), we have
\[
\overline{\tau}(H_{\alpha,k}) = H_{v(\alpha),k+(\tau,\alpha)}.
\]
Recall that the connected components of the complement in \( V \) of \( \bigcup_{\alpha \in \Delta, k \in \mathbb{Z}} H_{\alpha,k} \) are called alcoves. The group \( W_{\text{aff}} \) acts simply transitively on the set of alcoves. We denote
\[
C = \{ x \in V; (\alpha_i, x) > 0 \text{ for all } \alpha_i \in \Pi \}, \quad A = \{ x \in C; (\theta, x) < 1 \}
\]
respectively the fundamental chamber and the fundamental alcove with respect to \( \Pi \) and \( \hat{\Pi} \).

We shall end this section by recording the following results:

**Proposition 2.1.** For \( w \in \hat{W} \), let \( N(w) = \{ \beta \in \hat{\Delta}^+; w^{-1}(\beta) < 0 \} \) and denote by \( \ell(w) \) the length of any reduced expression of \( w \).

(a) We fix a reduced expression of \( w = s_{\beta_1} \circ \cdots \circ s_{\beta_k} \) with \( \beta_i \in \hat{\Pi} \), then \( N(w) = \{ s_{\beta_1} \circ \cdots \circ s_{\beta_p-1}(\beta_p); 1 \leq p \leq k \} \). In particular, \( N(w) \) contains a simple root.

(b) Let \( w_1, w_2 \in \hat{W} \), then \( N(w_1) \subseteq N(w_2) \) if and only if, there exists \( u \in \hat{W} \) such that \( w_2 = w_1 u \), and \( \ell(w_2) = \ell(w_1) + \ell(u) \). In particular, \( w \) is uniquely determined by \( N(w) \).

(c) If \( N(w) \cap \Delta^+ \neq \emptyset \), then \( N(w) \cap \hat{\Pi} \neq \emptyset \).

**Proof.** For parts (a) and (b), see for example [CP1]. Let us prove (c). The case \( \tilde{A}_1 \) is clear. In the others cases, this is a direct consequence of the fact that \( N(w) \) is a “compatible” set, by Theorem 1.3 from [CP1]. \( \square \)

### 3. \( I \)-compatible elements in \( \hat{W} \)

Let \( I \subset \Pi \) and \( i \) be an ad-nilpotent ideal of \( p_I \). We set
\[
\Phi_i = \{ \alpha \in \Delta_+ \setminus \Delta_I; g_\alpha \subseteq i \}.
\]
Then \( i = \bigoplus_{\alpha \in \Phi_i} g_\alpha \) and if \( \alpha \in \Phi_i, \beta \in \Delta_+ \cup \Delta_I \) are such that \( \alpha + \beta \in \Delta^+ \), then \( \alpha + \beta \in \Phi_i \).

Conversely, set
\[
\mathcal{F}_I = \{ \Phi \subset \Delta^+ \setminus \Delta_I; \text{ if } \alpha \in \Phi, \beta \in \Delta^+ \cup \Delta_I, \alpha + \beta \in \Delta^+, \text{ then } \alpha + \beta \in \Phi \}.
\]
Then for \( \Phi \in \mathcal{F}_I \), \( i_\Phi = \bigoplus_{\alpha \in \Phi} g_\alpha \) is an ad-nilpotent ideal of \( p_I \).

We obtain therefore a bijection
\[
\{ \text{ad-nilpotent ideals of } p_I \} \rightarrow \mathcal{F}_I, \quad i \mapsto \Phi_i.
\]
For \( \Phi \in \mathcal{F}_I \), we define \( \Phi^1 = \Phi, \Phi^k = (\Phi^{k-1} + \Phi) \cap \Delta, \text{ for } k \geq 2 \) and
\[
L_\Phi = \bigcup_{k \in \mathbb{N}^*} (-\Phi^k + k\delta).
\]
Since any ad-nilpotent ideal of $p_I$ is an ad-nilpotent ideal of the Borel subalgebra $p_\emptyset = b$, we have by [CP1] the following proposition:

**Proposition 3.1.** Let $\Phi \in F_I$, then there exists a unique $w_\Phi \in \hat{W}$ such that $L_\Phi = N(w_\Phi)$.

Thus we have the following injective map:

$$\{\text{ad-nilpotent ideals of } p_I\} \rightarrow \hat{W},$$

$$i \mapsto w_{\Phi_i}.$$

Recall from [CP1] the following characterization of the image of the above map when $I = \emptyset$.

**Proposition 3.2.** Let $w \in \hat{W}$, then there exists an ideal $i$ of $b$ such that $N(w) = L_\Phi_i$ if and only if

(a) $w^{-1}(\alpha) > 0$, for all $\alpha \in \Pi$.

(b) If $w(\alpha) < 0$ for some $\alpha \in \hat{\Pi}$, then $w(\alpha) = \beta - \delta$ for some $\beta \in \Delta^+$.

If these conditions are verified, we say that $w$ is Borel-compatible or $\emptyset$-compatible.

For $w \in \hat{W}$, let $\Phi_w = \{\alpha \in \Delta; -\alpha + \delta \in N(w)\}$. It follows that if $w$ is $\emptyset$-compatible, then $\Phi_w \subset \Delta^+.$

**Theorem 3.3.** Let $w \in \hat{W}$ be Borel-compatible and $I \subset \Pi$. The following conditions are equivalent:

(a) $i_{\Phi_w}$ is an ad-nilpotent ideal of $p_I$.

(b) $s_\alpha(\Phi_w) = \Phi_w$, for all $\alpha \in I$.

(c) $s_\alpha(L_{\Phi_w}) = L_{\Phi_w}$, for all $\alpha \in I$.

(d) $N(s_\alpha w) = N(w) \cup \{\alpha\}$, for all $\alpha \in I$.

(e) $w^{-1}(\alpha) \in \hat{\Pi}$, for all $\alpha \in I$.

If the hypothesis and these conditions are verified, we say that $w$ is $I$-compatible.

**Proof.** (a) $\Rightarrow$ (b). By assumption, we have $\Phi_w \in F_I$. Let $\beta \in \Phi_w$, then $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, hence $s_\alpha(\Phi_w) \subset \Phi_w$, for all $\alpha \in I$. Moreover, since $s_\alpha$ is an involution, we obtain that $s_\alpha(\Phi_w) = \Phi_w$.

(b) $\Rightarrow$ (c). Since $w(\delta) = \delta$, for all $w \in \hat{W}$, this is clear (by induction on $k$ or just remark that $\Phi_w^k \in F_I$).

(c) $\Rightarrow$ (d). Let $\alpha \in I$, by assumption, we have $s_\alpha(N(w)) = N(w)$, hence for $\beta \in N(w)$, we have $s_\alpha(\beta) \in N(w)$. So $w^{-1}s_\alpha(\beta) < 0$ and $\beta \in N(s_\alpha w)$. We have proved that $N(w) \subset N(s_\alpha w)$. Since $\sharp N(w) = \ell(w)$ and $\ell(s_\alpha w) = \ell(w) + 1$, by Proposition 2.1, we obtain that $\sharp N(s_\alpha w) = \sharp N(w) + 1$. Moreover we have $(s_\alpha w)^{-1}(\alpha) = w^{-1}(-\alpha) < 0$, hence $N(s_\alpha w) = N(w) \cup \{\alpha\}$.

(d) $\Rightarrow$ (e). Let $\alpha \in I$. By assumption, we have $N(w) \subset N(s_\alpha w)$, hence by Proposition 2.1, there exists $\beta \in \hat{\Pi}$ such that,

$$N(s_\alpha w) = N(ws_\beta) = N(w) \cup \{w(\beta)\} = N(w) \cup \{\alpha\}.$$

Consequently, we have $w^{-1}(\alpha) = \beta \in \hat{\Pi}$. 

(e) ⇒ (a). Let α ∈ I and assume that $w^{-1}(α) ∈ \hat{\Pi}$. Let β ∈ Φ_w be such that $β - α ∈ Δ^+$. We have

$$w^{-1}(-(β - α) + δ) = w^{-1}(-β + δ) + w^{-1}(α) ∈ (\hat{Δ}^- + \hat{\Pi}) \cap \hat{\Delta}.$$ 

It follows that $w^{-1}(-(β - α) + δ) < 0$. Moreover, $w^{-1}(-α + δ) = w^{-1}(-α) + δ > 0$ hence $α ∉ Φ_w$. We obtain that $Φ_w ∈ \mathcal{F}_I$, for all $α_i ∈ I$, hence $Φ_w$ belongs to $\mathcal{F}_I$ and $iΦ_w$ is an ideal of $p_I$. □

Another characterization of ad-nilpotent ideals in $b$ is given in [CP2] via the set $D = \{τ ∈ Q^\vee; (τ, α_j) ≤ 1, j = 1, \ldots, l \text{ and } (τ, θ) ≥ -2\}$. Let $\hat{D} = \{(τ, v) ∈ D × W; vtτ(A) ⊂ C\}$. We can state this characterization in the following way:

**Proposition 3.4.** The following map is bijective:

$$\tilde{D} \rightarrow \{w ∈ \hat{W}, θ\text{-compatible}\},$$

$$(τ, v) \mapsto vtτ.$$ 

**Remark 3.5.** In [CP2], the above correspondence is not viewed in the same way since the elements of $\hat{W}$ are written $t_v = vt_{w^{-1}(τ)}$ instead of $vtτ$, for $w ∈ W$ and $τ ∈ Q^\vee$.

Let $w ∈ \hat{W}$ be Borel-compatible, then $I_w = \{α ∈ Π; w^{-1}(α) ∈ \hat{\Pi}\}$ is the unique maximal element of $\{I ⊂ Π; w \text{ is } I\text{-compatible}\}$. For $τ ∈ Q^\vee$, set

$$D_τ = \begin{cases} \{α ∈ Π; (α, τ) = 0\} \cup \{-θ\} & \text{if } (θ, τ) = -1, \\ \{α ∈ Π; (α, τ) = 0\} & \text{if } (θ, τ) ≠ -1. \end{cases}$$

**Proposition 3.6.** Let $(τ, v) ∈ \tilde{D}$, and $w = vtτ ∈ \hat{W}$. Then $v(D_τ) = I_w$. In particular, $w$ is $I$-compatible if and only if $I ⊂ v(D_τ)$.

**Proof.** Let $α ∈ I_w$, then

$$w^{-1}(α) = t_{-τ}v^{-1}(α) = v^{-1}(α) + (v^{-1}(α), τ)δ ∈ \hat{\Pi}.$$ 

If $w^{-1}(α) ∈ Π$, then we have $v^{-1}(α) ∈ Π$ and $(v^{-1}(α), τ) = 0$, hence $v^{-1}(α) ∈ D_τ$. If $w^{-1}(α) = α_0$, then we have $v^{-1}(α) = -θ$ and $(θ, τ) = -1$, hence $-θ = v^{-1}(α) ∈ D_τ$.

Conversely, let $α ∈ D_τ ∩ Π$, then $vtτ(α) = v(α) ∈ Δ^+$, because $w$ is Borel-compatible. Then we have $N(ws_α) = N(w) ∪ \{w(α)\}$, and by part (3) of Proposition 2.1, there exists a simple root $β ∈ Π$ such that $β ∈ N(ws_α)$. Since $N(w) ∩ Δ^+ = ∅$, we obtain that $w(α) = β$ and $v(α) ∈ I_w$.

Assume now that $-θ ∈ D_τ$. Since $w$ is Borel-compatible, $vtτ(α_0) = -v(θ) ∈ Δ^+$. As above we have $N(ws_α) = N(w) ∪ \{w(α_0)\}$, and by part (3) of Proposition 2.1, there exists a simple root $β ∈ Π$ such that $β ∈ N(ws_α)$. Since $N(w) ∩ Δ^+ = ∅$, we obtain that $w(α_0) = β$ and $v(-θ) ∈ I_w$.

We have therefore proved that $v(D_τ) = I_w$, which concludes the proof. □

Let us denote $H_α = H_{α,0}$ for $α ∈ Π$, and $H_{α_0} = H_{θ,1}$. Let $\{ω_1, \ldots, ω_l\}$ be elements of $V$ such that $(ω_i, α_j) = δ_{ij}$. Set $n_0 = 1$ and let $n_i, i = 1, \ldots, l$, be the strictly positive integers such
that $\theta = \sum_{i=1}^{l} n_i \alpha_i$. Let $\omega_i = \omega_i / n_i$, $i = 1, \ldots, l$, and $\omega_0 = 0$. Then the closure $\overline{A}$ of $A$ is the convex hull $\text{Conv}(\omega_0, \omega_1, \ldots, \omega_l)$ of $\omega_0, \ldots, \omega_l$. For $k \in \mathbb{N}^*$, the convex hull (respectively the image by $\overline{w} \in W_{\text{aff}}$ of the convex hull) of $(k + 1)$ points in $[\omega_0, \omega_1, \ldots, \omega_l]$ is called a $k$-face of $\overline{A}$ (respectively of $\overline{w}(\overline{A})$). For example, $H_{\alpha_i} \cap \overline{A} = \text{Conv}(\omega_0, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_l)$ is an $(l - 1)$-face of $\overline{A}$.

We shall give yet another characterization of ad-nilpotent ideals of $p$ which shall be useful in enumerating abelian ideals when $g$ is of type $A$ or $C$.

**Proposition 3.7.** Let $w \in \hat{W}$ be Borel-compatible and $I \subset \Pi$. Then, $i_{\Phi w}$ is an ideal of $p$ if and only if for all $\alpha \in I$, $\overline{w}(A) \cap H_\alpha$ is an $(l - 1)$-face of $\overline{w}(\overline{A})$.

**Proof.** Assume that $w \in \hat{W}$ is $I$-compatible. Let $(\tau, v) \in \tilde{D}$ be such that $w = vt\tau$. By Proposition 3.6, $I \subset v(D_\tau)$, and $v^{-1}(\alpha) \in D_\tau$, for all $\alpha \in I$. Let $\alpha \in I$, we distinguish two cases:

If $v^{-1}(\alpha) = \beta \in \hat{\Pi}$, then $(\beta, \tau) = 0$. We obtain that

$$\overline{w}(H_\beta) = H_{v(\beta), (\tau, \beta)} = H_\alpha.$$ Hence, $\overline{w}(A) \cap H_\alpha$ is an $(l - 1)$-face of $\overline{w}(\overline{A})$.

If $v^{-1}(\alpha) = -\theta$, then $(\theta, \tau) = -1$. We obtain that

$$\overline{w}(H_\theta) = H_{v(\theta), (\tau, \theta) + 1} = H_\alpha.$$ Hence, $\overline{w}(A) \cap H_\alpha$ is an $(l - 1)$-face of $\overline{w}(\overline{A})$.

Conversely, let $v \in W$, $\tau \in Q^\vee$ be such that $w = vt\tau \in \hat{W}$ is Borel-compatible. By assumption, for all $\alpha \in I$, there exists $\beta \in \hat{\Pi}$ such that $\overline{w}(H_\beta) = H_\alpha$.

If $\beta \in \Pi$, then

$$\overline{w}(H_\beta) = H_{v(\beta), (\tau, \beta)} = H_\alpha$$ hence $(\tau, \beta) = 0$, and $w^{-1}(\alpha) = \pm \beta$. Since $w$ is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$.

If $\beta = \alpha_0$, then

$$\overline{w}(H_{\alpha_0}) = H_{v(\theta), (\tau, \theta) + 1} = H_\alpha$$ hence $(\tau, \theta) = -1$, and $w^{-1}(\alpha) = \pm (\theta - \delta)$. Since $w$ is Borel-compatible, we have necessarily $w^{-1}(\alpha) > 0$, and so $\alpha \in v(D_\tau)$. We have proved that $I \subset v(D_\tau)$, and by Proposition 3.6, $w$ is $I$-compatible. □

Let $H_\emptyset = V$. For $J \subset \hat{\Pi}$ non-empty, denote $H_J = \bigcap_{\alpha \in J} H_\alpha$. By the proposition above, if $w$ is $I$-compatible, then we have $\overline{w}(\overline{A}) \cap H_I = \overline{w}(\overline{A} \cap H_{w^{-1}(J)})$.

### 4. Volume of the faces of the fundamental alcove

Recall from [CP1] and [Ko], that $w \in \hat{W}$ is Borel-compatible and the ideal $i_{\Phi w}$ of $b$ is abelian if and only if $\overline{w}(A) \subset 2A$. As a consequence, we have the following remarkable result of Peterson: the number of abelian ideals of $b$ is $2^l$. Observe that the above result says that the number
of abelian ideals in $b$ depends only on the rank of $g$. In the case of parabolic algebras, we shall see in this section to what extent this result can be extended.

For $J \subset \widehat{\Pi}$, let $F_J = A \cap H_J = \text{Conv}(\omega_j ; \alpha_j \notin J)$. Observe that the $F_J$ are the faces of $\tilde{A}$. Let $w \in \widehat{W}$, if $\omega(\tilde{A}) \cap H_J$ is an $(l - J)$-face of $\omega(\tilde{A})$, then we shall call $\omega(\tilde{A}) \cap H_J$ an $(l - J)$-alcove of $H_J$.

**Proposition 4.1.**

(a) Let $w \in \widehat{W}$ and $I \subset \Pi$, if $\omega(A) \subset 2A$ and $\omega(\tilde{A}) \cap H_I$ is an $(l - I)$-alcove of $H_I$, then $w$ is $I$-compatible.

(b) Let $I \subset \Pi$ and $w, w' \in \widehat{W}$ be $I$-compatible. If $\omega(A) \subset 2A$, $\omega'(A) \subset 2A$ and $\omega(\tilde{A}) \cap H_I = \omega'(\tilde{A}) \cap H_I$, then $w = w'$.

**Proof.** (a) Let $w \in \widehat{W}$ and $I \subset \Pi$ be of cardinality $r$. If $\omega(A) \subset 2A$, then $w$ is Borel-compatible and the ideal $i_{\phi_w}$ is abelian.

Set $N = l - r + 1$. Since $\omega(\tilde{A}) \cap H_I$ is an $(l - r)$-alcove of $H_I$, there exist $N$ vertices $\tilde{\omega}_1, \ldots, \tilde{\omega}_N$ of $\tilde{A}$ such that $\omega(\tilde{\omega}_1), \ldots, \omega(\tilde{\omega}_N)$ belong to $\omega(\tilde{A}) \cap H_I$.

There exist $r$ distinct reflecting affine hyperplanes $H'_1, \ldots, H'_r$ of the form $H_\alpha$, for $\alpha \in \widehat{\Pi}$, such that $\bigcap_{j=1}^r H'_j$ contains $\tilde{\omega}_1, \ldots, \tilde{\omega}_N$. For $j = 1, \ldots, r$, $H_I \cap \omega(H'_j)$ contains $\omega(\tilde{\omega}_1), \ldots, \omega(\tilde{\omega}_N)$. Since the dimension of $H_I$ is $N - 1$, it follows that $H_I \subset \omega(H'_j)$.

The hyperplane $H_I$ is defined by the equations $(x, \alpha) = 0$ for all $\alpha \in I$, it follows that $\omega(H'_j)$ is an hyperplane of the form $H_{\beta,0}$, where $\beta$ is a linear combination of elements of $I$.

Assume that $\beta \notin I$. Then, the intersection of $H_{\beta,0}$ with the closure of the fundamental chamber $C$ is of dimension at most $l - 2$. Since by construction $H_{\beta,0}$ contains an $(l - 1)$-face of $\omega(\tilde{A})$, and $\omega(\tilde{A}) \subset \tilde{C}$, we obtain a contradiction. It follows that $\beta \in I$.

Set $w = v^\tau$. We then have that for each $\beta \in I$:

$$w^{-1}(H_{\beta,0}) = H_{v^{-1}(\beta),(\tau, v^{-1}(\beta))} = H_\alpha$$

for some $\alpha \in \widehat{\Pi}$. If $\alpha \in \Pi$, then $v^{-1}(\beta) = \pm \alpha$ and $(\tau, v^{-1}(\beta)) = 0$. Since $w$ is Borel-compatible, we obtain that $w^{-1}(\beta) = \alpha$.

If $\alpha = \alpha_0$, we obtain that $v^{-1}(\beta) = \pm \theta$ and $(\tau, v^{-1}(\beta)) = \pm 1$. Since $w$ is Borel-compatible, we finally obtain that $w^{-1}(\beta) = \alpha_0$. Thus, $w^{-1}(I) \subset \widehat{\Pi}$, and $w$ is $I$-compatible as required.

(b) Let $I \subset \Pi$ and $w, w' \in \widehat{W}$ be $I$-compatible. Let $\alpha \in I$, then $w$ is $I \setminus \{\alpha\}$-compatible. It follows by Proposition 3.7 that $\omega(\tilde{A}) \cap H_I \setminus \{\alpha\}$ is an $(l - \#I + 1)$-alcove of $H_I \setminus \{\alpha\}$ and it is the convex hull of $\omega(\tilde{A}) \cap H_I$ and a vertex of $H_I \setminus \{\alpha\} \cap \omega(\tilde{A})$, which is not in $H_I \cap \omega(\tilde{A})$. In the same way $\omega'(\tilde{A}) \cap H_I \setminus \{\alpha\}$ is an $(l - \#I + 1)$-alcove of $H_I \setminus \{\alpha\}$ and it is the convex hull of $\omega'(\tilde{A}) \cap H_I$ and a vertex of $H_I \setminus \{\alpha\} \cap \omega'(\tilde{A})$, which is not in $H_I \cap \omega'(\tilde{A})$. Since $\omega(\tilde{A}) \subset 2\tilde{A}$, there is a unique vertex in $H_I \setminus \{\alpha\}$ satisfying these conditions. So, $\omega(\tilde{A}) \cap H_I \setminus \{\alpha\} = \omega'(\tilde{A}) \cap H_I \setminus \{\alpha\}$ and by induction, we have $\omega(\tilde{A}) = \omega'(\tilde{A})$. Hence $w = w'$. □

Let $F'_J = 2\tilde{A} \cap H_J = \text{Conv}(2\omega_j ; \alpha_j \notin J)$. It is clear that $F'_J$ is a union of $(l - J)$-alcoves of $H_J$. Let

$$\text{Ab}_I = \{w \in \widehat{W} ; i_{\phi_w} \text{ is an abelian ideal of } p_I\}.$$
By the above proposition and by Proposition 3.7, we obtain the following result:

**Theorem 4.2.** Let \( I \subset \Pi \), then the map \( w \mapsto \overline{w}(\bar{A}) \cap H_I \) is a bijection between \( Ab_I \) and the set of all the \( (l - \sharp I) \)-alcoves of \( F'_I \).

**Remark 4.3.** The above theorem can be viewed as a generalization of Peterson’s result.

In order to determine \( \sharp Ab_I \), we are reduced to computing the volume of the \( (l - \sharp I) \)-alcoves of \( F'_I \). Furthermore, to compute the volume of the \( (l - \sharp I) \)-alcoves of \( F'_I \), it suffices to compute the volume of the \( (l - \sharp I) \)-faces of \( \bar{A} \).

Let \( d(x, H_\alpha) \) denote the distance from \( x \in V \) to the affine hyperplane \( H_\alpha \), for \( \alpha \in \hat{\Pi} \). For a \( k \)-alcove, let \( \text{Vol}_k(B) \) be the \( k \)-volume of \( B \). By [Be], the volume of the fundamental alcove is

\[
\text{Vol}(A) = \frac{1}{l} \times \text{Vol}_0(\mathcal{F}_0) = \frac{1}{l! |\omega_1 \wedge \ldots \wedge \omega_l|}.
\]

Since the projection of 0 on \( H_{\alpha_0} \) is \( \frac{\theta}{\|\theta\|} \), we have \( d(0, H_{\alpha_0}) = \frac{1}{\|\theta\|} \). We obtain that \( \text{Vol}(A) = \frac{1}{l!} \text{Vol}_0(\mathcal{F}_0) \). Moreover, by [CLO],

\[
\text{Vol}(A) = \frac{1}{l!} |\bar{\omega}_1 \wedge \ldots \wedge \bar{\omega}_l|.
\]

Let \( D = |\bar{\omega}_1 \wedge \ldots \wedge \bar{\omega}_l| \), then

\[
\text{Vol}(F_{\alpha_0}) = \frac{D}{(l - 1)!} n_0 \|\theta\|.
\] (2)

To compute the \( (l - 1) \)-volume of the faces \( F_{\alpha_i}, i = 1, \ldots, l \), we compute the \( l \)-volume of the convex hull of \( (\{\bar{\omega}_1, \ldots, \bar{\omega}_l\} \setminus \{\bar{\alpha}_i\}) \cup \{\frac{\alpha_i}{\|\alpha_i\|}\} \). Thus, we have:

\[
\text{Vol}_{l-1}(F_{\alpha_i}) = \frac{1}{(l - 1)!} |\bar{\omega}_1 \wedge \ldots \wedge \frac{\alpha_i}{\|\alpha_i\|} \wedge \ldots \wedge \bar{\omega}_l|.
\]

Since \( \alpha_i = \sum_{k=1}^l (\alpha_i, \alpha_k) \alpha_k \),

\[
\text{Vol}_{l-1}(F_{\alpha_i}) = \frac{D}{(l - 1)!} n_i \|\alpha_i\|.
\] (3)

We have therefore computed the \( (l - 1) \)-volume of the \( (l - 1) \)-faces of \( \bar{A} \). In particular, we have:

**Lemma 4.4.** Let \( \alpha_i, \alpha_j \in \hat{\Pi} \), be such that \( (\alpha_i, \alpha_i) = (\alpha_j, \alpha_j) \), then:

\[
n_i \text{Vol}_{l-1}(F_j) = n_j \text{Vol}_{l-1}(F_i).
\]

This lemma also appears as Proposition 26 in [Sut]. We shall generalize this result. For \( I \subset \hat{\Pi} \), let \( n_I = 1 \) if \( I = \emptyset \), and \( n_I = \prod_{\alpha_i \in I} n_i \) otherwise. We shall prove the following result:
Proposition 4.5. Let $I \subset \Pi$ and $w \in \hat{W}$ be such that $w^{-1}(I) = J \subset \hat{\Pi}$. Then, we have:

$$n_J \text{Vol}_{1-\tilde{g}_J}(F_J) = n_J \text{Vol}_{1-\tilde{g}_J}(F_I).$$

To prove this proposition, we need the following technical lemma:

Lemma 4.6. Let $I \subset \Pi$ be such that $\sharp I \leq l - 1$. Let $w \in \hat{W}$ be such that $w^{-1}(I) = J \subset \hat{\Pi}$. Let $\alpha_j$ be any element of $J$ if $\alpha_0 \notin J$, and $\alpha_j = \alpha_0$ if $\alpha_0 \in J$. Set $\alpha_i = w(\alpha_j)$. Then we have:

$$n_j \text{d}(\widehat{\alpha}_i, H_I) = n_j \text{d}(\widehat{\alpha}_j, H_J).$$

Proof. The result is clear if $J = \emptyset$. We may therefore assume that $1 \leq \sharp J \leq l - 1$.

Step 1: Assume that $\alpha_0 \in J$. We shall determine the distance $d(\widehat{\alpha}_0, H_J)$.

Let $J_0$ be the connected component of $J$ containing $\alpha_0$. Set $r = 1, J_0$.

If $J_0 = \{\alpha_0\}$, then the projection of $0$ on $H_J$ is $0 = \frac{\alpha_0}{\|\alpha_0\|^2}$. Therefore, the distance $d(\widehat{\alpha}_0, H_J)$ is $\frac{1}{\|\alpha_0\|^2}$.

Now assume that $J_0 \neq \{\alpha_0\}$. Then, $J_0 \setminus \{\alpha_0\}$ contains one or two roots $\beta$ such that $(\beta, \theta) \neq 0$.

Set $J_0 = \{\beta_1, \ldots, \beta_r\}$, $\alpha_0 = \beta_k$ and $V_{J_0} = \bigoplus_{\beta_i \in J_0 \setminus \{\alpha_0\}} \mathbb{R}_\beta_i$.

First of all, assume that $J_0 \setminus \{\alpha_0\}$ contains only one root $\beta_i$ such that $(\beta_i, \theta) \neq 0$. Let $\gamma_t \in V_{J_0}$ be such that $(\gamma_t, \beta_i) = 1$ and $(\gamma_t, \beta_i) = 0$ for all $\beta_i \in J_0 \setminus \{\beta_i, \beta_k\}$. Let $\mu_t = (\|\theta\|^2(1 - \frac{\gamma_t, \theta}{2}))^{-1}$ and $\beta = \mu_t(\theta - (\beta_i, \gamma_t)\gamma_t)$. Then, we have $(\beta, \alpha) = 0$ for all $\alpha \in J_0 \setminus \{\alpha_0\}$ and

$$(\beta, \theta) = \mu_t\left[\|\theta\|^2 - (\theta, \beta_i)(\gamma_t, \theta)\right] = \mu_t\|\theta\|^2\left[1 - \frac{(\gamma_t, \theta)}{2}\right] = 1.$$

For all $x \in H_J$, we have $(\gamma_t, x) = 0$, and so

$$(\beta - x, \beta) = \mu_t(\theta - (\theta, \beta_i)(\gamma_t, \beta) - (\theta, x)) = \mu_t[(\theta, \beta) - (\theta, x)] = 0.$$

We have proved that $\beta$ is the projection of $\widehat{\alpha}_0$ in $H_J$. It follows that by taking any $x \in H_J$, we have $d(\widehat{\alpha}_0, H_J)^2 = \|\beta\|^2 = (x, \beta) = \mu_t(x, \theta) = \mu_t$.

Since $I \subset \Pi$ and $J = w^{-1}(I)$ and $I$ have the same Dynkin diagram, we have by a case by case consideration that $J_0$ is of type $A_r$, $C_r$, or $D_r$.

If $J_0$ is of type $A_r$, then by renumbering the roots $\beta_i$, the Dynkin diagram of $J_0$ is of the form:

```
   k
  ↓ 1 2 3 4 5
  - - - - -
```

Then $t = 1$, and take

$$\gamma_t = \frac{2}{r\|\beta_1\|^2}(r - 1)\beta_1 + (r - 2)\beta_2 + \cdots + \beta_{r-1}).$$

So $$(\gamma_t, \theta) = \frac{r - 1}{r},$$ and we have

$$\mu_t = \frac{2r}{(r + 1)\|\theta\|^2}. \quad (4)$$
If $J_0$ is of type $C_r$, then the Dynkin diagram of $J_0$ is of the form:

$$
\begin{array}{ccccccc}
& & & & & & \\
& 1 & 2 & r-2 & r-1 \\
& & & & & & \\
& & & k & & & \\
\end{array}
$$

Again $t = 1$, and take

$$
\gamma_t = \frac{2}{r \| \beta_1 \|^2} \left( (r-1)\beta_1 + (r-2)\beta_2 + \cdots + \beta_{r-1} \right).
$$

So $(\gamma_t, \theta) = \frac{2(r-1)}{r}$, and we have

$$
\mu_t = \frac{r}{(r+1)\|\theta\|^2}. \quad (5)
$$

If $J_0$ is of type $D_r$, then the Dynkin diagram of $J_0$ is of the form:

or of the form:

$$
\begin{array}{ccccccc}
& & & & & & \\
& 1 & & & & & \\
& & & 2 & 3 & r-2 & r-1 \\
& & & & & & \\
\end{array}
$$

In the first case we have $t = 2$, and we take

$$
\gamma_t = \frac{2}{r \| \beta_2 \|^2} \left( (r-2)\beta_1 + 2(r-2)\beta_2 + 2(r-3)\beta_3 + \cdots + 2\beta_{r-1} \right).
$$

Thus $(\gamma_t, \theta) = \frac{2(r-2)}{r}$, and we have

$$
\mu_t = \frac{r}{2\|\theta\|^2}. \quad (6)
$$

In the second case, we have $t = 1$ and we take

$$
\gamma_t = \frac{1}{\| \beta_1 \|^2} \left( 2\beta_1 + 2\beta_2 + \cdots + 2\beta_{r-3} + \beta_{r-2} + \beta_{r-1} \right).
$$
Thus we have

\[ \mu_t = \frac{2}{\|\theta\|^2}. \]  \hspace{1cm} (7)

Assume now that \( J_0 \) contains two roots \( \alpha \) such that \((\alpha, \theta) \neq 0\). Then the Dynkin diagram of \( J_0 \) is of type \( A_r \) and these two roots are \( \beta_{k-1}, \beta_{k+1} \):

\[
\begin{array}{ccccccc}
1 & \cdots & k-1 & k & k+1 & r-1 & r \\
\end{array}
\]

Let \( \eta, \eta' \in V_{J_0} \) be such that \((\eta, \beta_{k-1}) = 1 = (\eta', \beta_{k+1}) \) and \((\eta, \beta_i) = 0 \) (respectively \((\eta', \beta_i) = 0 \) for all \( \beta_i \in J_0 \setminus \{\beta_{k-1}, \beta_{k+1}\} \)). Let \( \mu = (\|\theta\|^2(1 - \frac{(\eta + \eta', \theta)}{2}))^{-1} \) and \( \beta = \mu(\theta - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k+1})\eta')) \). Then we have \((\beta, \alpha) = 0 \) for all \( \alpha \in J_0 \setminus \{\alpha_0\} \) and

\[
(\beta, \theta) = \mu[\|\theta\|^2 - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k+1})\eta', \theta)] = 1,
\]

\[
(\beta - x, \beta) = \mu(\theta - ((\theta, \beta_{k-1})\eta + (\theta, \beta_{k+1})\eta'), \beta - x) = 0
\]

for all \( x \in H_J \). We obtain that \( \beta \) is the projection of 0 on \( H_J \). Take

\[
\eta = \frac{2}{k\|\beta_{k-1}\|^2}(k-1)\beta_{k-1} + (k-2)\beta_{k-2} + \cdots + \beta_1,
\]

\[
\eta' = \frac{2}{(r-k+1)\|\beta_{k+1}\|^2}(\beta_r + 2\beta_{r-1} + \cdots + (r-k)\beta_{k+1})
\]

then \((\eta + \eta', \theta) = \frac{k-1}{k} + \frac{r-k}{r-k+1} \). We obtain that:

\[
d^2(\omega_0, H_J) = \|\beta\|^2 = \mu = \frac{2k(r-k+1)}{n_0^2(r+1)\|\theta\|^2}. \hspace{1cm} (8)
\]

Observe that the formulas (4) and (8) generalize the formula obtained when \( J_0 = \{\alpha_0\} \). Let \( k \) be the position of \( \alpha_0 \in J_0 \), then we can sum up the above results in the following table, when \( 1 \leq \sharp J \leq l - 1 \):

Table 1

<table>
<thead>
<tr>
<th>( J_0 )</th>
<th>( A_r )</th>
<th>( C_r )</th>
<th>( D_r ) ( t = 2 )</th>
<th>( D_r ) ( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d(\omega_0, H_J)^2 )</td>
<td>( \frac{2k(r-k+1)}{n_0^2(r+1)|\theta|^2} )</td>
<td>( \frac{r}{n_0^2(r+1)|\theta|^2} )</td>
<td>( \frac{r}{n_0^2|\theta|^2} )</td>
<td>( \frac{2}{n_0^2|\theta|^2} )</td>
</tr>
</tbody>
</table>

Step 2: Assume that \( J \subset \Pi \). Let \( \alpha_j \in J \). We shall determine the distance \( d(\omega_j, H_J) \).

We have \( H_J = \text{Vect}(\omega_j; t \text{ such that } \alpha_t \notin J) \subset H_{J\setminus\{\alpha_j\}} \subset V \). Let \( H_J^\perp = \{x \in V; (x, \omega_j) = 0 \) for all \( t \) such that \( \alpha_t \notin J \}, \) then \( H_J^\perp = \text{Vect}(\alpha_j; \alpha_t \in J) \), and \( \dim(H_J^\perp \cap H_{J\setminus\{\alpha_j\}}) = 1 \). Since

\[
H_J\setminus\{\alpha_j\} \cap H_J^\perp = \left\{ x = \sum_{\alpha_t \in J} t_\alpha \alpha_t; (x, \beta) = 0 \text{ for all } \beta \in J \setminus \{\alpha_j\} \right\},
\]
there exists $\gamma \in V$ such that $H_{J \setminus \{\alpha_j\}} \cap H_J^\perp = \text{Vect}(\gamma)$, and $(\gamma, \alpha_j) \neq 0$. Thus, we have $H_{J \setminus \{\alpha_j\}} = H_J \oplus \mathbb{C}\gamma$. It follows that there exists $\mu \in \mathbb{C}^*$ such that $\omega_j + \mu \gamma \in H_J$.

Let $J_j$ be the connected component of $J$ which contains $\alpha_j$. Set $J_j = \{ \beta_1, \ldots, \beta_r \}$, with $\alpha_j = \beta_k$. Set $V_{J_j} = \bigoplus_{\beta_j \in J_j} \mathbb{R}\beta_j$. We may choose $\gamma$ such that $(\gamma, \alpha_j) = 1$ and $(\gamma, \alpha) = 0$ for all $\alpha \in J_j \setminus \{\alpha_j\}$. Since $(\omega_j + \mu \gamma, \alpha_j) = 0$, we obtain that $\mu = -\frac{1}{n_j}$, and hence

$$d(\omega_j, H_J) = \frac{\|\gamma\|}{n_j}, \quad (9)$$

where $\gamma$ depends only on the position of $\alpha_j$ in the Dynkin diagram of $J_j$.

Finally, we need to compute explicitly $d(\omega_j, H_J)$ in some particular cases. We use the numbering of [TY, Chapter 18].

If $J_j = A_r$, take

$$\gamma = \frac{2}{(r + 1)\|\beta_k\|^2} \left[ (r - k + 1)\beta_1 + 2(r - k + 1)\beta_2 + \ldots + (k - 1)(r - k + 1)\beta_{k-1} + k(r - k + 1)\beta_k + k(r - k)\beta_{k+1} + \ldots + k\beta_r \right].$$

If $J_j = C_r$, and $k = r$, take

$$\gamma = \frac{2}{\|\beta_r\|^2} \left( \beta_1 + 2\beta_2 + \ldots + (r - 1)\beta_{r-1} + \frac{r}{2}\beta_r \right).$$

If $J_j = D_r$, take

$$\gamma = \frac{1}{\|\beta_r\|^2} \left[ \beta_1 + 2\beta_2 + \ldots + (r - 2)\beta_{r-2} + \frac{1}{2} \left[ (r - 2)\beta_{r-1} + r\beta_r \right] \right] \quad \text{if } k = r,$n

$$\gamma = \frac{1}{\|\beta_r\|^2} \left[ \beta_1 + 2\beta_2 + \ldots + (r - 2)\beta_{r-2} + \frac{1}{2} \left[ r\beta_{r-1} + (r - 2)\beta_r \right] \right] \quad \text{if } k = r - 1,$n

$$\gamma = \frac{1}{\|\beta_1\|^2} \left[ 2\beta_1 + 2\beta_2 + \ldots + 2\beta_{r-2} + \beta_{r-1} + \beta_r \right] \quad \text{if } k = 1.$$n

In these particular cases, we obtain the following result:

<table>
<thead>
<tr>
<th>$J_j$</th>
<th>$A_r$</th>
<th>$C_r$</th>
<th>$D_r$</th>
<th>$D_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d(\omega_j, H_J)^2$</td>
<td>$\frac{2k(r-k+1)}{n_j^2(r+1)</td>
<td>\alpha_j</td>
<td>^2}$</td>
<td>$\frac{r}{n_j^2(r+1)</td>
</tr>
</tbody>
</table>

**Final step:** We are now in a position to prove the lemma. Let $I_i$ be the connected component of $I$ containing $\alpha_i$. If $J \subset \Pi$, then we have the result by (9), since $\alpha_j$ and $\alpha_i$ have the same position in the Dynkin diagram of $J_j$ and $I_i$ respectively.

If $\alpha_0 \in J$, then the connected component $J_0$ of $J$ containing $\alpha_0$ is of the type $A_r$, $C_r$, or $D_r$. Again since $w^{-1}(\alpha_0)$ and $\alpha_0$ have the same position in the respective Dynkin diagram, we obtain the result by inspecting the correspondence between Tables 1 and 2. □
Proof of Proposition 4.5. The case \( \sharp I = 0 \) is trivial since in this case, \( F_I = F_J = \overline{A} \).

Let \( I \subset \Pi \). Let us proceed by induction on \( \sharp I \). If \( \sharp I = 1 \), the result is proved in Lemma 4.4. Assume that \( l > \sharp I > 1 \) and that the claim is true for \( \sharp I - 1 \). Let \( \alpha_j \) be any element of \( J \) if \( \alpha_0 \notin J \), and \( \alpha_j = \alpha_0 \) if \( \alpha_0 \in J \). Set \( \alpha_i = w(\alpha_j) \). Then, we have by Lemma 4.6,

\[
\begin{align*}
n_J \Vol_{l-\sharp I}(F_I) &= n_J (l - \sharp I + 1) \Vol_{l-\sharp I+1}(F_I) \times \frac{1}{d(\overline{w}_i, H_I)} \\
&= n_J (l - \sharp I + 1) \frac{n_J}{n_i} \Vol_{l-\sharp I+1}(F_J) \times \frac{n_i}{n_j d(\overline{w}_j, H_J)} \\
&= n_J (l - \sharp I + 1) \Vol_{l-\sharp I+1}(F_J) \times \frac{1}{d(\overline{w}_j, H_J)} \\
&= n_J \Vol_{l-\sharp I}(F_J).
\end{align*}
\]

Finally, the result is clear if \( \sharp I = l \) since in this case \( F_I \) (respectively \( F_J \)) is a single point.

Observe that for \( I \subset \Pi \), \( F'_I = 2F_I \), so

\[
\Vol_{l-\sharp I}(F'_I) = 2^{l-\sharp I} \Vol_{l-\sharp I}(F_I).
\] (10)

We obtain a generalization of Peterson’s result:

**Theorem 4.7.** Let \( I \subset \Pi \), then

\[
\frac{1}{n_I} \sum_{w \in Ab_I} n_w^{-1}(I) = 2^{l-\sharp I}.
\]

**Proof.** Let \( I \subset \Pi \) and \( w \in \widehat{W} \). By Propositions 3.7 and 4.1, then

\[
\sum_{w \in Ab_I} \Vol_{l-\sharp I}(\overline{w}(A) \cap H_I) = \Vol_{l-\sharp I}(F'_I).
\]

Observe that for \( w \in Ab_I \) we have \( \overline{w}(A) \cap H_I = \overline{w}(F_{w^{-1}(I)}) \), and \( \Vol_{l-\sharp I}(\overline{w}(F_{w^{-1}(I)})) = \Vol_{l-\sharp I}(F_{w^{-1}(I)}) \). So by Proposition 4.5 and by (10), we obtain that

\[
\sum_{w \in Ab_I} \frac{n_w^{-1}(I)}{n_I} \Vol_{l-\sharp I}(F_I) = 2^{l-\sharp I} \Vol_{l-\sharp I}(F_I).
\]

Thus, we have the result.

**Theorem 4.8.** Let \( I \subset \Pi \), if \( g \) is of type \( A_l \) or \( C_l \), then the parabolic subalgebras \( p_I \) have exactly \( 2^{l-\sharp I} \) abelian ideals.

**Proof.** If \( g \) is of type \( A_l \) or \( C_l \), the numbers \( n_i \), for \( i = 0, \ldots, l \), depends only on the length of \( \alpha_i \). It follows that for any \( w \in Ab_I \), \( n_I = n_{w^{-1}(I)} \). So by Theorem 4.7, we obtain the result.
Remark 4.9. The fact that the integers \( n_i \), for \( i = 0, \ldots, l \), depends only on the length of \( \alpha \) is false when \( g \) is not of type \( A \) or \( C \). Indeed, Theorem 4.8 is false in general. For example, in \( B_3 \), the parabolic subalgebra \( p_{[\alpha]} \) has only 3 abelian ideals. We shall see in the next section another way to count the number of abelian ideals in cases \( B \) and \( D \).

5. Enumeration of ideals via diagrams

In this section, we shall determine, via diagram enumeration, the number of ad-nilpotent (respectively abelian) ideals of \( p_I \), for \( I \subset \Pi \), when \( g \) is simple and of classical type. We shall use the numbering of simple roots of [TY, Chapter 18].

Recall the following partial order on \( \Delta^+ \): \( \alpha \leq \beta \) if \( \beta - \alpha \) is a sum of positive roots. Then it is easy to see that \( \Phi \in F_{\emptyset} \) if and only if for all \( \alpha \in \Phi, \beta \in \Delta^+ \), such that \( \alpha \leq \beta \), then \( \beta \in \Phi \). When \( g \) is of type \( A, B, C \) or \( D \), we can display the positive roots into a diagram of suitable shape, as in [CP1]. Then, they established a bijection between elements of \( F_{\emptyset} \) and certain subdiagrams.

Let \( I \subset \Pi \). In order to adapt this construction in the parabolic case \( p_I \), we shall use a similar construction, but our diagram will depend not only on the type of \( g \), but also on \( I \).

Let \( I \subset \Pi \) and \( \gamma, \beta \in \Delta^+ \). We say that \( \beta \xrightarrow{I} \gamma \) if there exists \( \eta \in I \) such that \( \beta + \eta = \gamma \). Define an equivalence relation on \( \Delta^+ \setminus \Delta_I \): for \( I \subset \Pi \), \( \gamma \sim_I \beta \) if there exist \( \beta_1, \ldots, \beta_s \in \Delta^+ \setminus \Delta_I \) such that

(i) \( \beta = \beta_1, \gamma = \beta_s \),
(ii) either \( \beta_i \xrightarrow{I} \beta_{i+1} \) or \( \beta_{i+1} \xrightarrow{I} \beta_i \), for \( i = 1, \ldots, s - 1 \).

As the standard Levi factor of \( p_I \) acts in a reductive way on the nilpotent radical, the fact that two roots \( \beta, \gamma \) are \( \sim_I \) equivalent means that \( g_\alpha \) and \( g_\beta \) are in the same simple submod- ule.

Let \( X \) be the type of \( g \). The idea is to start by displaying the positive roots \( \Delta^+ \) in a diagram \( T_X \) of a suitable shape as in [CP1]: that is, we assign to each box labeled \((i, j)\) in \( T_X \), a positive root \( t_{i,j} \). The shape and the filling of \( T_X \) are chosen such that we obtain a bijection between elements of \( F_{\emptyset} \) and the northwest flushed subdiagrams, henceforth nw-diagrams, of \( T_X \) (in type \( D \), we need to include also nw-diagrams modulo a permutation of certain columns). Then, for \( I \subset \Pi \), we delete the boxes containing elements of \( \Delta_I \). Observe that the set of boxes of the same equivalent class is connected. Therefore, we can regroup into a big box all the roots of the same equivalent class. We obtain a new diagram denoted by \( T^I_X \). Then, we count the nw-diagrams of \( T^I_X \) (again in type \( D \), we need to count also nw-diagrams modulo a permutation of certain columns), which are clearly in bijection with the elements of \( F_I \).

5.1. Type \( A_l \)

If \( g \) is of type \( A_l \), then \( T_{A_l} \) is a diagram of shape \([l, l - 1, \ldots, 1]\). The label \((i, j)\) means a box in the \( i \)th row and the \( j \)th column. The boxes \((i, j)\) of \( T_{A_l} \) are filled by the positive roots \( t_{i,j} = \alpha_i + \cdots + \alpha_{l-j+1}, 1 \leq i, j \leq l \). For example, for \( l = 5 \), we have:
Let $I \subset \Pi$. We first delete the boxes containing elements of $\Delta_I$. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I$, we regroup the $(l-i+1)$th and the $(l-i+2)$th columns if $i \neq 1$, and the rows $i, i+1$ if $i \neq l$, on $T_{\lambda_I}$. At the end, we obtain that $T_{\lambda_I}^I$ is a diagram of shape $[l-\sharp I, l-\sharp I - 1, \ldots, 1]$. For example, for $A_5$ and $I = \{\alpha_2, \alpha_3\}$, we have:

\[
\begin{array}{cccccc}
I_1 & I_2 & I_3 & I_4 & I_5 & I_6 \\
I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} & I_{1,5} & \\
I_{2,1} & I_{2,2} & I_{2,3} & I_{2,4} & \\
I_{3,1} & I_{3,2} & I_{3,3} & \\
I_{4,1} & I_{4,2} & \\
I_{5,1} & \\
\end{array}
\]

Let $C_l = \frac{1}{l+1} \binom{2l}{l}$ denote the $l$th Catalan number.

**Proposition 5.1.** Let $I \subset \Pi$. Let $S_{\lambda_I}^I$ be the set of all nw-diagrams of $T_{\lambda_I}^I$. Then, the cardinality of $S_{\lambda_I}^I$ is $C_{l-\sharp I + 1}$.

**Proof.** Let $I \subset \Pi$, then $T_{\lambda_I}^I$ if of shape $[l-\sharp I, l-\sharp I - 1, \ldots, 1]$, so by [S, 8, 6.19 vv.], we obtain that the cardinality of the set of nw-diagrams of $T_{\lambda_I}^I$ is $C_{l-\sharp I + 1}$. \qed

5.2. Type $C_l$

**Definition 5.2.** Let $p, q$ be two integers such that $q \leq p$. Let $T_{p,q}$ be the (shifted) diagram of shape $[p+q-1, p+q-3, \ldots, p-q+1]$ arranged in the following way:

\[
\begin{array}{cccccc}
P_{p+q-1} \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
Q_{p-q+1} \\
\end{array}
\]
If $g$ is of type $C_l$, then $T_{Cl}$ is the diagram $T_{l,l}$, and the boxes $(i, j)$ of $T_{Ci}$ are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} 
\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l, & 1 \leq j \leq l-1, \\
\alpha_i + \cdots + \alpha_{2l-j}, & l \leq j \leq 2l-1.
\end{cases}$$

Let $I \subset \Pi$, we first delete the boxes containing elements of $\Delta_I$. Then, we regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup column $2l-i$ and column $2l-i+1$ if $i \neq 1$, then we regroup the $i$th and $(i+1)$th columns and also the $i$th and $(i+1)$th rows on $T_{Ci}$. If $\alpha_l \in I$, we regroup also the columns $l$ and $l+1$. We obtain at the end that $T_{Cl}^I$ is a diagram of shape $T_{l-\#I, l-\#I}$ if $\alpha_l \notin I$ and of shape $T_{l-\#I+1, l-\#I}$, if $\alpha_l \in I$.

By [Pr], we obtain directly that the number of nw-diagram of $T_{p,q}$ is $\binom{p+q}{p}$. Consequently, we have the following proposition:

**Proposition 5.3.** Let $I \subset \Pi$. Let $S_{Cl}^I$ be the set of all nw-diagrams of $T_{Cl}^I$. Then, the cardinality of $S_{Cl}^I$ is

$$(l-\#I+1)C_{l-\#I} \quad \text{if} \ \alpha_l \notin I, \quad \text{and} \quad \frac{l-\#I+2}{2}C_{l-\#I+1} \quad \text{if} \ \alpha_l \in I.$$

5.3. Type $B_l$ and $D_l$

Let $I \subset \Pi$. Assume that $g$ is of type $X = B_l$ or $D_l$. Then the shape of $T_{X}^I$ is more complicated than in the case $A$ or $C$, so we need more combinatorial results on diagrams.

**Definition 5.4.** Let $p, q$ be two integers such that $q \leq p$. Let $T_{p,q}'$ be the diagram of $q$ rows of the shape $[p, p-1, \ldots, p-q+1]$ arranged in the following way:

![Diagram](image)

**Proposition 5.5.** Let $p, q$ be two integers such that $q \leq p$. Then, the number of nw-diagrams of $T_{p,q}'$ is

$$T_{p,q}' = \frac{(p+q+1)!(p-q+2)}{q!(p+2)!}.$$
Proof. Let $D_{p,q}$ be the set of nw-diagrams of $T'_{p,q}$. We shall proceed by induction on $q$. If $q = 1$, then $T'_{p,1}$ is

$$
\begin{array}{c}
\text{p} \\
\hline
\end{array}
$$

so we have

$$
\sharp D_{p,q} = T'_{p,q} = p + 1 = \frac{(p + q + 1)! (p - q + 2)}{q!(p + 2)!}.
$$

Assume that $q > 1$ and the claim is true for $q - 1$. For $1 \leq k \leq p - q + 1$, let

$$
S_k = \{ S \in D_{p,q}; (q,k) \in S \text{ and } (q,k + 1) \notin S \}.
$$

Then, $\sharp S_k = T'_{p-k,q-1}$ and $L = \bigcup_{k=1}^{p-q+1} S_k$ is the set of nw-diagrams containing at least a box in the last row of $T'_{p,q}$. Since $D_{p,q}$ is the disjoint union of $D_{p,q-1}$ and $L$, we obtain that:

$$
T'_{p,q} = T'_{p,q-1} + \sharp L = \sum_{i=0}^{p-q+1} T'_{p-i,q-1}
$$

$$
= \sum_{k=q-1}^{p} T'_{k,q-1} = \sum_{k=q-1}^{p} \frac{(k + q)! (k - q + 3)}{(k - 1)!(k + 2)!}
$$

$$
= \frac{(p + q + 1)! (p - q + 2)}{q!(p + 2)!}
$$

where the last equality is a simple induction on $p \geq q$. $\square$

Definition 5.6. Let $p \geq q$ be two positive integers and $1 \leq l_1 < l_2 < \cdots < l_s \leq q + 1$ be some other integers. Denote by $T_{p,q}(l_1,l_2,\ldots,l_s)$ the new diagram obtained by adding to $T_{p,q}$ the boxes $(l_i, l_i - 1)$, for $1 \leq i \leq s$. For example, $T_{5,4}(2,4)$ is:

$$
\begin{array}{c}
\text{p+q-1} \\
q \\
\hline
\end{array}
\begin{array}{c}
\times \\
\times \\
\hline
\text{p-q+1}
\end{array}
$$

where the added boxes are marked with a $\times$. 
Proposition 5.7. Let \( p \geq q \) be two positive integers and \( 1 \leq l_1 < l_2 < \cdots < l_s \leq q + 1 \) be some other integers, then the number of nw-diagrams of \( T_{p,q}(l_1,l_2,\ldots,l_s) \) is

\[
\binom{p+q}{p} + \sum_{j=1}^{s} T'_{p+q-l_j,l_j-1}.
\]

Proof. Let \( D_{p,q}(l_1,\ldots,l_s) \) be the set of nw-diagrams of \( T_{p,q}(l_1,\ldots,l_s) \) and \( D_{p,q}(l_1,\ldots,l_s) \) be its cardinality. Let \( b_s = (l_s,l_s-1) \). Set

\[
\begin{align*}
E &= \{ S \in D_{p,q}(l_1,\ldots,l_s); b_s \notin S \}, \\
F &= \{ S \in D_{p,q}(l_1,\ldots,l_s); b_s \in S \text{ and } S \setminus \{b_s\} \in E \}, \\
G &= \{ S \in D_{p,q}(l_1,\ldots,l_s); b_s \in S \text{ and } S \setminus \{b_s\} \notin E \}.
\end{align*}
\]

Then, we have clearly \( D_{p,q}(l_1,\ldots,l_s) = \#E + \#F + \#G \).

If \( S \in F \), then \( S \) contains all the boxes north-west of \( b_s \) and the other boxes of \( S \) are strictly north-east of \( b_s \), so there exists a bijection between \( F \) and the set of nw-diagrams of \( T'_{p+q-l_s,l_s-1} \). For the example in Definition 5.6, if \( S \in F \), \( S \) is a nw-diagram of:

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\hline
b_s & & & & & & & & \\
\hline
\end{array}
\]

containing \( b_s \). Hence it suffices to count the nw-diagrams of the subdiagram strictly north-east of \( b_s \):

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\hline
\end{array}
\]

So by Proposition 5.5, the cardinality of \( F \) is \( T'_{p+q-l_s,l_s-1} \).

If \( S \in G \), then \( S \setminus \{b_s\} \) is a nw-diagram of \( T \) where \( T = T_{p,q} \) if \( s = 1 \) and \( T = T_{p,q}(l_1,\ldots,l_{s-1}) \) if \( s > 1 \). So the cardinality of \( G \) is the cardinality of the set of nw-diagrams in \( T \) minus the cardinality of the set \( H \) of nw-diagrams having at most \( l_s - 1 \) rows. Observe that the elements of \( H \) correspond to those of \( E \). Hence, (by [Pr])

\[
\#G = \begin{cases} 
D_{p,q}(l_1,\ldots,l_{s-1}) - \#E & \text{if } s > 1, \\
\binom{p+q}{p} - \#E & \text{if } s = 1.
\end{cases}
\]

The result now follows easily by induction on \( s \). \( \square \)
Notations 5.8. Fix $I \subset \Pi$. Let $I_1, \ldots, I_s$ be the connected components of $I$ of cardinality $r_1, \ldots, r_s$ respectively. For each connected component $I_j$, set $m_j = \min\{i; \alpha_i \in I_j\}$. Without loss of generality, we shall assume that $m_1 < m_2 < \cdots < m_s$.

If $g$ is of type $B_l$, then $T_{B_l}$ is $T_{l,l-1}$ and the boxes $(i, j)$ of $T_{B_l}$ are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_l), & 1 \leq j \leq l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leq j \leq 2l-1. \end{cases}$$

As before, for $I \subset \Pi$, we delete the boxes containing elements of $\Delta_I$. For $j = 1, \ldots, s$, set

$$l_j = m_j - \sum_{k=1}^{j-1} r_k. \quad (11)$$

Regroup the equivalent classes of $\sim_I$ proceeding simple root by simple root: for each $\alpha_i \in I \setminus \{\alpha_l\}$, we first regroup rows $i$ and $i + 1$ and if $i \neq 1$, we regroup column $2l - i$ and column $2l - i + 1$, then the columns $i - 1$ and $i$. If $\alpha_l \in I$, we also regroup the columns $l - 1, l$ and $l + 1$. We obtain that $T_{B_l}$ is a diagram of shape $T_{l-\#I,l-\#I(l_1, \ldots, l_n)}$, where the $l_i$ are defined as above and, $n = s - 1$ if $\alpha_l \in I$ and $n = s$ if $\alpha_l \notin I$. For example, for $B_5$ and $I = \{\alpha_2, \alpha_3, \alpha_5\}$, we have:

![Diagram](image)

It follows from Proposition 5.7 that:

**Proposition 5.9.** Let $I \subset \Pi$ be of cardinality $r$. Let $S_{B_l}^I$ be the set of all nw-diagrams of $T_{B_l}^I$. Then, the cardinality of $S_{B_l}^I$ is

$$(l - r + 1)C_{l-r} + \sum_{j=1}^{n} T'_{2(l-r)-l_j,l_j-1}$$

where $n = s - 1$ if $\alpha_l \in I$, and $n = s$ otherwise.

If $g$ is of type $D_l$, then $T_{D_l}$ is $T_{l,l-1}$, and the boxes $(i, j)$ of $T_{D_l}$ are filled by the positive roots $t_{i,j}$, where

$$t_{i,j} = \begin{cases} \alpha_i + \cdots + 2(\alpha_{j+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l, & 1 \leq j \leq l-2, \\ \alpha_i + \cdots + \alpha_{l-2} + \alpha_l, & j = l-1, \\ \alpha_i + \cdots + \alpha_{2l-j}, & l \leq j \leq 2l-1. \end{cases}$$
For $I \subset \Pi$, we first delete the boxes containing elements of $\Delta_I$. For $j = 1, \ldots, s$, set

$$I_j = \begin{cases} 
m_j - \sum_{k=1}^{j-1} r_k & \text{if } j \neq s \text{ or } I_s \neq \{\alpha_I\}, \\
m_j - \sum_{k=1}^{j-1} r_k - 1 & \text{if } j = s \text{ and } I_s \neq \{\alpha_I\}. 
\end{cases} \quad (12)$$

Regroup the equivalent classes of $\sim_l$ proceeding simple root by simple root: for each $\alpha_I \in I \setminus \{\alpha_{I-1}, \alpha_{I}\}$, we first regroup the rows $i$ and $i + 1$ and if $i \neq 1$, we regroup column $2l - i - 1$ and column $2l - i$. Then the columns $i - 1$ and $i$.

If $\alpha_{I-1} \in I$, but $\alpha_I \notin I$, then we regroup the columns $l - 2, l - 1$ and columns $l, l + 1$.

If $\alpha_I \in I$, but $\alpha_{I-1} \notin I$, then we first reverse the columns $l - 1$ and $l$, and then we regroup the (new) columns $l - 2, l - 1$ and columns $l, l + 1$.

We obtain that if $\{\alpha_{I-1}, \alpha_I\} \subset I$, then $T_{D(I)}$ is a diagram of shape $T_{l-i-1}^{-1}(l_1, \ldots, l_s)$, where the $l_i$ are defined as above. If $\{\alpha_{I-1}, \alpha_I\} \subset I$, then $T_{D(I)}^I$ is a diagram of shape $T_{l-i-1}^{-1}(l_1, \ldots, l_{s-1})$.

In the following examples, we denote by $i$ the simple root $\alpha_i$ and by $i^2$ the element $2\alpha_i$. We consider, $X = D_5$ and $I$ is respectively $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\alpha_2, \alpha_4, \alpha_5\}$:

\[
\begin{array}{cccc|c}
12^23245 & 123^245 & 12345 & 1234 & 1235 & 123 \\
23^245 & 2345 & 234 & 235 & 23 & \\
345 & 34 & 35 & 3 & \\
\hline
4 & & & &
\end{array}
\]

\[
\begin{array}{cccc|c}
12^23245 & 123^245 & 12345 & 1235 & 123 & 12 & 1 \\
23^245 & 2345 & 235 & 234 & 23 & \\
345 & 35 & 34 & 3 & \\
\hline
& & & &
\end{array}
\]

\[
\begin{array}{c}
b_1 \\
\hline
b_2
\end{array}
\]

\[
\begin{array}{c}
b_1 \\
\hline
b_2
\end{array}
\]

**Definition 5.10.** For a subdiagram $L$ of $T_{D(I)}^I$, we shall denote by $L^\bullet$ the set of boxes of $L$ obtained from $L$ by exchanging columns $l - r - 1$ and $l - r$ (respectively $l - r$ and $l - r + 1$) if $\alpha_1 \notin I$ (respectively if $\alpha_1 \in I$).

If $L^\bullet$ is a nw-diagram of $T_{D(I)}^I$, then we say that $L$ is a $\bullet$-nw-diagram of $T_{D(I)}^I$.

**Proposition 5.11.** Let $I \subset \Pi$ be of cardinality $r$. Let $S_{D(I)}^I$ be the set of nw-diagrams of $T_{D(I)}^I$ if $\{\alpha_{I-1}, \alpha_I\} \cap I \neq \emptyset$, and be the union of the set of nw-diagrams and the set of $\bullet$-nw-diagrams of $T_{D(I)}^I$ if $\{\alpha_{I-1}, \alpha_I\} \cap I = \emptyset$. Then, the cardinality of $S_{D(I)}^I$ is

(i) $$(3l - r - 2)C_{l-r-1} + \sum_{j=1}^{s} T'_{2(l-r)-l_j-1,l_j-2} + T'_{2(l-r)-l_j,l_j-1}, \text{ if } t = 0;$$

(ii) $$\frac{l - r + 1}{2} C_{l-r} + \sum_{j=1}^{s} T'_{2(l-r)-l_j-1,l_j-1}, \text{ if } t = 1.$$
(iii) \[(l - r + 1)C_{l-r} + \sum_{j=1}^{s-1} T'_{2(l-r)-l_j,l_j-1}, \text{ if } t = 2,\]

where \(t = \sharp(\{\alpha_{l-1}, \alpha_l\} \cap I).\)

**Proof.** Assume first that \(\{\alpha_{l-1}, \alpha_l\} \cap I = \emptyset.\) Note that, the elements of \(\mathcal{F}_I\) are in bijection with the subdiagrams \(S\) of \(T'_{D_l} = T_{l-r,l-r-1}(l_1, \ldots, l_s)\) such that either \(S\) or \(S^{•}\) is a nw-diagram. Let \(E_1 = \) the set of nw-diagrams of \(T'_{D_l},\)

\(E_2 = \) (the set of •-nw-diagrams of \(T'_{D_l}\)) \(\setminus E_1.\)

So \(S_{D_l} = E_1 \cup E_2\) (disjoint union). By Proposition 5.7, we have:

\[\sharp E_1 = \frac{l - r + 1}{2} C_{l-r} + \sum_{j=1}^{s} T'_{2(l-r)-l_j-1,l_j-1}.\]

On the other hand, the number of elements of \(E_2\) is \(\sharp E_1 - \sharp F,\) where \(F\) is the set of elements of \(E_1\) having columns \(l - r - 1\) and \(l - r\) (respectively \(l - r + 1\)) of the same length if \(\alpha_1 \notin I\) (respectively if \(\alpha_1 \in I\)).

Clearly, the number of elements of \(F\) is exactly the number of nw-diagrams of the diagram obtained from \(T'_{D_l}\) by removing the \((l-r)\)th (respectively \((l-r+1)\)th) column if \(\alpha_1 \notin I\) (respectively if \(\alpha_1 \in I\)). So, by Proposition 5.7,

\[\sharp F = (l - r)C_{l-r-1} + \sum_{j=1}^{s} T'_{2(l-r)-l_j-2,l_j-1}.\]

We obtain therefore the result since we have the equality:

\[T'_{2(l-r)-l_j-1,l_j-1} - T'_{2(l-r)-l_j-2,l_j-1} = T'_{2(l-r)-l_j-1,l_j-2}.\]

If \(\alpha_{l-1}\) or \(\alpha_l \in I,\) then there is no column reversing. Then the result follows from Proposition 5.7 according to the shape of \(T'_{D_l}.\)

As in [CP1], we have clearly a bijection between \(\mathcal{F}_I\) and \(S_{X}^{I}\). It follows from Propositions 5.1, 5.3, 5.9, 5.11, that we have:

**Theorem 5.12.** Let \(I \subset \Pi\) of cardinality \(r,\) \(X\) be the type of \(\mathfrak{g}\) and \(s, l_j\) as defined in 5.8, (11) and (12).

If \(X = A_l,\) then

\[\# \mathcal{F}_I = C_{l-r+1}.\]
If \( X = B_l \), then
\[
\sharp F_I = (l - r + 1)C_{l-r} + \sum_{j=1}^{n} T'_{2(l-r)-l_j,l_j-1},
\]
where \( n = s - 1 \) if \( \alpha_l \notin I \), and \( n = s \) otherwise.

If \( X = C_l \), then
\[
\sharp F_I = \begin{cases} 
\frac{l-r+1}{2}C_{l-r+1} & \text{if } \alpha_l \notin I, \\
(l - r + 1)C_{l-r} + \sum_{j=1}^{s-1} T'_{2(l-r)-l_j,l_j-1} & \text{if } \alpha_l \in I.
\end{cases}
\]

If \( X = D_l \), then
\[
\sharp F_I = \begin{cases} 
\frac{l-r+1}{2}C_{l-r} + \sum_{j=1}^{s} T'_{2(l-r)-l_j,l_j-1} & \text{if } \sharp \{\alpha_{l-1}, \alpha_l\} \cap I = 1, \\
(l - r + 1)C_{l-r} + \sum_{j=1}^{s-1} T'_{2(l-r)-l_j,l_j-1} & \text{if } \{\alpha_{l-1}, \alpha_l\} \subset I, \\
3(l - r) - 2C_{l-r-1} + \sum_{j=1}^{s} T'_{2(l-r)-l_j,l_j-2} + T'_{2(l-r)-l_j-1,l_j-1} & \text{otherwise}.
\end{cases}
\]

5.4. Abelian ideals

We have already determined in Theorem 4.8 the number of abelian ideals for type \( A \) and \( C \). We shall now enumerate the abelian ideals of \( pI \) using diagrams when \( \gamma \) is of type \( B \) or \( D \). Observe that a similar argument could be used to enumerate abelian ideals in type \( A \) and \( C \).

**Definition 5.13.** Let \( p \) be a positive integer and \( R_p \) be the diagram of shape \([p, p-1, \ldots, 1]\) arranged in the following way:

\[
\begin{array}{c}
p \\
\hline
p
\end{array}
\]

**Proposition 5.14.** The number of nw-diagrams of \( R_p \) is \( 2^p \).

**Proof.** We shall proceed by induction on \( p \). If \( p = 1 \), the result is clear. Assume that \( p > 1 \) and the claim is true for \( p - 1 \). Let \( b \) be the box \((1, p)\) and

\[
E = \text{the set of nw-diagrams of } R_p \text{ which do not contain } b,
\]
\[
F = \text{the set of nw-diagrams of } R_p \text{ which contain } b.
\]

Then, the number of nw-diagrams of \( R_p \) is \( \sharp E + \sharp F \). Furthermore, by the induction hypothesis, we have \( \sharp E = 2^{p-1} = \sharp F \), and we obtain the result. \( \square \)
Definition 5.15. Let $p$ be a positive integer and $1 \leq l_1 < l_2 < \cdots < l_s \leq p + 1$ be some other integers. Denote by $R_p(l_1, l_2, \ldots, l_s)$ the new diagram obtained by adding to $R_p$ the boxes $(l_i, l_i - 1)$, for $1 \leq i \leq s$. For example, $R_4(3)$:

\[
\begin{array}{c}
p \\
\times \\
p
\end{array}
\]

Proposition 5.16. Let $p$ be a positive integer and $1 \leq l_1 < l_2 < \cdots < l_s \leq p + 1$ be some other integers, then the number of nw-diagrams of $R_p(l_1, l_2, \ldots, l_s)$ is

\[
2^p + \sum_{j=1}^{s} \binom{p}{l_j - 1}.
\]

Proof. Let $D_p(l_1, \ldots, l_s)$ be the set of nw-diagrams of $R_p(l_1, \ldots, l_s)$ and $D_p(l_1, \ldots, l_s)$ be its cardinality. Let $b_s = (l_s, l_s - 1)$. Set

\[
\begin{align*}
E &= \{ S \in D_p(l_1, \ldots, l_s); b_s \notin S \}, \\
F &= \{ S \in D_p(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{ b_s \} \in E \}, \\
G &= \{ S \in D_p(l_1, \ldots, l_s); b_s \in S \text{ and } S \setminus \{ b_s \} \notin E \}.
\end{align*}
\]

Then we have clearly $D_p(l_1, \ldots, l_s) = \#E + \#F + \#G$.

If $S \in F$, then $S$ contains all the boxes north-west of $b_s$ and the other boxes of $S$ are strictly north-east of $b_s$, so there exists a bijection between $F$ and the set of nw-diagrams of $T$ where $T$ is a diagram whose shape is a rectangle containing $p - l_j + 1$ columns and $l_j - 1$ rows. For the example in Definition 5.15, if $S \in F$, $S$ is a nw-diagram of:

\[
\begin{array}{c}
p \\
\times \\
p
\end{array}
\]

containing $b_s$. Hence it suffices to count the nw-diagrams of the rectangular subdiagram strictly north-east of $b_s$:

\[
\begin{array}{c}
p-l_1+1 \\
\times \\
p-l_1+1
\end{array}
\]
So by [Pr] the cardinality of $F$ is $\binom{p}{j-1}$.

If $S \in G$, then $S \setminus \{b_s\}$ is a nw-diagram of $L$ where $L = R_p$ if $s = 1$ and $L = R_p(l_1, \ldots, l_{s-1})$ if $s > 1$. So the cardinality of $G$ is the cardinality of the set of nw-diagrams in $L$ minus the cardinality of the set $H$ of nw-diagrams having at most $l_s - 1$ rows. Observe that the elements of $H$ correspond to those of $E$. Hence, by Proposition 5.14

$$\sharp G = \begin{cases} D_{p,q}(l_1, \ldots, l_{s-1}) - \sharp E & \text{if } s > 1, \\ 2^p - \sharp E & \text{if } s = 1. \end{cases}$$

The result now follows easily by induction on $s$. □

Let $\mathcal{F}_I^{ab} = \{\Phi \in \mathcal{F}_I; \phi \text{ is abelian}\}$. If $S$ is a subdiagram of a diagram, let

$$\tau_S = \max\{k; (h, k) \in S\},$$

so $(h, \tau_S)$ is the right most box in the $h$th row of $S$.

**Proposition 5.17.** Assume that $g$ is of type $B_l$. Let $I \subset \Pi$ be of cardinality $r$. Consider $\Phi \in \mathcal{F}_I$ and $S$ its corresponding nw-diagram in $T_{B_l}^I$. Then $\Phi \in \mathcal{F}_I^{ab}$ if and only if

(a) $\tau_S \leq l - r$ if $\alpha_1 \in I$,
(b) $\tau_S + \tau^S_2 \leq 2(l - r) - 1$ if $\alpha_1 \notin I$.

**Proof.** Let $S_0$ be the corresponding nw-diagram of $\Phi$ in $T_{B_l}^I$, then by [CP1], we have $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_1 + \tau_2 \leq 2l - 1$.

If $\alpha_1 \in I$, then $\tau_S = \tau_2^S$. The regrouping process reduces the number of columns on the left of column $l$ of $T_{B_l}^I$ by one for each simple root in $I \setminus \{\alpha_1\}$. It follows that $\Phi \in \mathcal{F}_I^{ab}$ if and only if $\tau_S \leq l - r$.

The argument is similar for the case $\alpha_1 \notin I$. □

**Proposition 5.18.** Assume that $g$ is of type $B_l$. Let $I \subset \Pi$ be of cardinality $r$, and $l_1, \ldots, l_s$ be as defined in (11). Then we have:

$$\# \mathcal{F}_I^{ab} = \begin{cases} 2^{l-r} + \sum_{j=1}^{n} \binom{l-r-1}{l_j - 1} & \text{if } \alpha_1 \notin I, \\ 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j - 1} & \text{if } \alpha_1 \in I, \end{cases}$$

where $n = s$ if $\alpha_1 \notin I$ and $n = s - 1$ if $\alpha_1 \in I$.

**Proof.** Recall that $T_{B_l}^I$ is of shape $T_{l-r,l-r}(l_1, \ldots, l_n)$, where $n = s$ if $\alpha_1 \notin I$ and $n = s - 1$ if $\alpha_1 \in I$.

Let $\Phi \in \mathcal{F}_I$ and $S$ be the nw-diagram of $T_{B_l}^I$ corresponding to $\Phi$.

Assume that $\alpha_1 \in I$, then $l_1 = 1$. By Proposition 5.17, $S$ is in the left-hand half of $T_{B_l}^I$, so it is a nw-diagram of $R_{l-r-1}(1, \ldots, l_n)$. We then obtain the result by Proposition 5.16.

Assume that $\alpha_1 \notin I$. Let $E$ be the set of nw-diagrams of $T_{B_l}^I$ associated to elements of $\mathcal{F}_I^{ab}$. Set
\[ P = \{ S \in E; \tau_1^S \leq l - r - 1 \}, \]
\[ Q = \{ S \in E; \tau_1^S > l - r - 1 \}. \]

Then, we have \( \sharp E = \sharp P + \sharp Q \).

If \( S \in P \), then \( S \) is included in the left-hand half of \( T_{B_1}^l \), so
\[
\sharp P = 2^{l-r-1} + \sum_{j=1}^{n} \binom{l-r-1}{l_j-1}
\]
by Proposition 5.16.

For \( i = l - r, \ldots, 2(l - r) - 1 \), let \( Q_i = \{ S \in Q; \tau_1^S = i \} \) and \( P_i = \{ S \in P; \tau_1^S = 2(l - r) - 1 - i \} \). We then have:
\[
Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i.
\]

For \( i = l - r, \ldots, 2(l - r) - 1 \), we have an obvious bijection between \( P_i \) and \( Q_i \) given by the adding or removing of boxes \((1, 2(l - r) - i), \ldots, (1, i)\). Therefore \( \sharp P = \sharp Q \) and the result follows. \( \square \)

**Proposition 5.19.** Assume that \( g \) is of type \( D_l \). Let \( I \subset \Pi \) be of cardinality \( r \). Consider \( \Phi \subset F_I \) and \( S_\Phi \) its corresponding subdiagram in \( T_{D_l}^l \). Set \( S = S_\Phi \) if \( S_\Phi \) is a nw-diagram and \( S = S_{\Phi}^* \) if \( S_\Phi \) is a nw-diagram. Then \( \Phi \in F_I^{ab} \) if and only if

(a) \( \tau_1^{S} \leq l - r \) if \( \alpha_1 \in I \),
(b) \( \tau_1^{S} + \tau_2^{S} \leq 2(l - r) - 2 \) if \( \alpha_1 \notin I \).

**Proof.** If \( I = \emptyset \), set \( S = S_0 \), then by [CP1], we have \( \Phi \in F_0^{ab} \) if and only if \( \tau_1^{S_0} + \tau_2^{S_0} \leq 2l - 2 \).

Assume that \( \alpha_1 \in I \), then \( \tau_1^{S_0} = \tau_2^{S_0} \). The regrouping process reduces the number of columns of the left of column \( l \) of \( T_{D_l}^l \) by one for each simple root in \( I \setminus \{ \alpha_1 \} \). It follows that \( \Phi \in F_I^{ab} \) if and only if \( \tau_1^{S} \leq l - r \).

The argument is similar for the case \( \alpha_1 \notin I \). \( \square \)

**Proposition 5.20.** Assume that \( g \) is of type \( D_l \). Let \( I \subset \Pi \) be of cardinality \( r \) and \( l_1, \ldots, l_s \) be as defined in (12). Set \( t = \sharp(\{ \alpha_{l-1}, \alpha_l \} \cap I) \). If \( \alpha_1 \in I \), then the cardinality of \( F_I^{ab} \) is:

(i) \( 2^{l-r} - 2^{l-r-2} + \sum_{j=1}^{s} \left[ 2 \binom{l-r-1}{l_j-1} - \binom{l-r-2}{l_j-1} \right] \), if \( t = 0 \),
(ii) \( 2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1} \), if \( t = 1 \),
(iii) \( 2^{l-r-1} + \sum_{j=1}^{t-1} \binom{l-r-1}{l_j-1} \), if \( t = 2 \).
If $\alpha_1 \notin I$, then the cardinality of $\mathcal{F}_I^{ab}$ is:

(iv)  \[ 2^{l-r} + \sum_{j=1}^{s} 2 \left( \frac{l-r-1}{l_j-1} \right), \quad \text{if } t = 0, \]

(v)  \[ 2^{l-r-1} + 2^{l-r-2} + \sum_{j=1}^{s} \left( \frac{l-r-1}{l_j-1} \right) + \sum_{j=1}^{s-1} \left( \frac{l-r-2}{l_j-1} \right), \quad \text{if } t = 1, \]

(vi)  \[ 2^{l-r} + 2 \sum_{j=1}^{s-1} \left( \frac{l-r-1}{l_j-1} \right), \quad \text{if } t = 2. \]

**Proof.** We proceed as in the case of type $B_l$ but here, we need to take into account column reversing.

Recall that if $t = 0$ or 1, $T_{D_l}^I$ is of shape $T_{l-r,l-r-1}(l_1, \ldots, l_s)$ and if $t = 2$, $T_{D_l}^I$ is of shape $T_{l-r,l-r}(l_1, \ldots, l_{s-1})$.

Let $S_I^{ab}$ be the set of subdiagrams of $T_{D_l}^I$ corresponding to elements of $\mathcal{F}_I^{ab}$. The shape of elements of $S_I^{ab}$ is conditioned by Proposition 5.19. Let

\[ E_1 = \text{the set of nw-diagrams in } S_I^{ab}, \]

\[ E_2 = (\text{the set of } \bullet\text{-nw-diagrams in } S_I^{ab}) \setminus E_1. \]

Consider $\Phi \in \mathcal{F}_I^{ab}$ and $S$ its corresponding subdiagram in $S_I^{ab}$.

First assume that $\alpha_1 \in I$, then $l_1 = 1$. If $S \in E_1$, by Proposition 5.19, $S$ is in the left-hand half of $T_{D_l}^I$, so it is a nw-diagram of $R_{l-r-1}(l_1, \ldots, l_n)$, where $n = s$ if $t = 0$, 1 and $n = s - 1$ if $t = 2$. Hence, by Proposition 5.16, we have

\[ \#E_1 = 2^{l-r-1} + \sum_{j=1}^{n} \left( \frac{l-r-1}{l_j-1} \right). \]

If $t \neq 0$, there is no column reversing, so $E_2 = \emptyset$. If $t = 0$, the number of elements of $E_2$ is $\#E_1 - \#(F \cap E_1)$, where $F$ is the set of nw-diagrams of $T_{D_l}^I$ having columns $l-r$ and $l-r+1$ of the same length.

Clearly, the number of elements of $F$ is exactly the number of nw-diagrams of the diagram obtained from $T_{D_l}^I$ by removing the $(l-r+1)$th column. So, by Proposition 5.19, the set of elements which are in $F \cap E_1$ is in bijection with the set of nw-diagrams of $R_{l-r-2}(1, \ldots, l_s)$. So by Proposition 5.16, we obtain:

\[ \#F = 2^{l-r-2} + \sum_{j=1}^{s} \left( \frac{l-r-2}{l_j-1} \right). \]

We obtain therefore the result.

Now assume that $\alpha_1 \notin I$. Set
\[ P = \{ S \in E_1; \tau_1^S \leq l - r - 1 \}, \]
\[ \tilde{P} = \{ S \in E_1; \tau_1^S \leq l - r - 2 \}, \]
\[ Q = \{ S \in E_1; \tau_1^S > l - r - 1 \}. \]

Then, we have \( \sharp E_1 = \sharp P + \sharp Q \).

First assume that \( t = 0 \) or \( 1 \). If \( S \in P \), then \( S \) is included in the left-hand half of \( T_{D_l^1} \), so by Proposition 5.16, we have
\[ \sharp P = 2^{l-r-1} + \sum_{j=1}^{s} \binom{l-r-1}{l_j-1}. \]

For \( i = l - r, \ldots, 2(l - r) - 2 \), let \( Q_i = \{ S \in Q; \tau_1^S = i \} \) and \( \tilde{P}_i = \{ S \in P; \tau_1^S = 2(l - r) - 2 - i \} \). We then have:
\[ Q = \bigcup_{i=l-r}^{2(l-r)-2} Q_i \quad \text{and} \quad \tilde{P} = \bigcup_{i=l-r}^{2(l-r)-2} \tilde{P}_i. \]

For \( i = l - r, \ldots, 2(l - r) - 2 \), we have an obvious bijection between \( \tilde{P}_i \) and \( Q_i \) given by the adding or removing of boxes \((1, 2(l - r) - i), \ldots, (1, i)\). Therefore \( \sharp \tilde{P} = \sharp Q \) and by Proposition 5.16, we have
\[ \sharp \tilde{P} = 2^{l-r-2} + \sum_{j=1}^{s} \binom{l-r-2}{l_j-1}. \]

If \( t = 1 \), there is no column reversing, so we have the result. If \( t = 0 \), then the number of elements of \( E_2 \) is \( \sharp E_1 - \sharp F \), where \( F = E_1 \cap \{ \text{\bullet-nw-diagrams of } S_l^{ab} \} \). By Proposition 5.19, we have \( F = Q \cup P \), so by the consideration above, we have \( \sharp F = 2 \sharp Q \). It follows that \( \sharp E_1 + \sharp E_2 = 2 \sharp P \).

For the last case \( t = 2 \), the shape of \( T_{D_l^1} \) is \( T_{l-r,l-r}(l_1, \ldots, l_{s-1}) \). If \( S \in P \), then \( S \) is included in the left-hand half of \( T_{D_l^1} \), so by Proposition 5.16, we have
\[ \sharp P = 2^{l-r-1} + \sum_{j=1}^{s-1} \binom{l-r-1}{l_j-1}. \]

We have \( E_2 = \emptyset \), and \( Q_i \) is defined for \( i = l - r, \ldots, 2(l - r) - 1 \). Set \( P_i = \{ S \in P; \tau_1^S = 2(l - r) - 1 - i \} \). We then have:
\[ Q = \bigcup_{i=l-r}^{2(l-r)-1} Q_i \quad \text{and} \quad P = \bigcup_{i=l-r}^{2(l-r)-1} P_i. \]

As above, for \( i = l - r, \ldots, 2(l - r) - 1 \), we have an obvious bijection between \( P_i \) and \( Q_i \) given by the adding or removing of boxes \((1, 2(l - r) - i), \ldots, (1, i)\). Therefore \( \sharp P = \sharp Q \) and the result follows. \( \square \)
Remark 5.21. All the results above depend on the numbering of simple roots.

6. Remarks

6.1. Exceptional types

In the exceptional types $E, F$ and $G$, the number of ad-nilpotent and abelian ideals has been determined by using GAP 4.

The following tables give the cardinality of $\mathcal{F}_I$ and $\mathcal{A}_b I$ for the types $F_4$ and $G_2$. The subset $I$ of $\Pi$ is described by the symbol $\bullet$ in the Dynkin diagram without arrow.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$# \mathcal{F}_I$</th>
<th>$# \mathcal{A}_b I$</th>
<th>$I$</th>
<th>$# \mathcal{F}_I$</th>
<th>$# \mathcal{A}_b I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\circ \circ \circ \circ$</td>
<td>105</td>
<td>16</td>
<td>$\bullet \circ \circ \circ$</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td>$\circ \bullet \circ \circ$</td>
<td>35</td>
<td>12</td>
<td>$\circ \circ \bullet \circ$</td>
<td>32</td>
<td>10</td>
</tr>
<tr>
<td>$\circ \circ \circ \bullet$</td>
<td>49</td>
<td>9</td>
<td>$\bullet \bullet \circ \circ$</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>$\bullet \circ \bullet \circ$</td>
<td>8</td>
<td>4</td>
<td>$\bullet \circ \circ \bullet$</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>$\circ \bullet \bullet \circ$</td>
<td>14</td>
<td>7</td>
<td>$\bullet \bullet \bullet \circ$</td>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>$\circ \circ \bullet \bullet$</td>
<td>10</td>
<td>4</td>
<td>$\bullet \bullet \bullet \bullet$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$\bullet \bullet \circ \bullet$</td>
<td>5</td>
<td>3</td>
<td>$\bullet \bullet \bullet \circ$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\circ \bullet \bullet \bullet$</td>
<td>3</td>
<td>2</td>
<td>$\bullet \bullet \bullet \bullet$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where we use the following orientation for the Dynkin diagram of $F_4$:

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\circ & \bullet & \circ & \circ
\end{array}
$$

where we use the following orientation for the Dynkin diagram of $G_2$:

$$
\begin{array}{cc}
1 & 2 \\
\circ & \circ
\end{array}
$$

6.2. Relation with antichains

For $\Phi \in \mathcal{F}_\emptyset$, let

$\Phi_{\text{min}} = \{ \beta \in \Phi; \beta - \alpha \notin \Phi, \text{ for all } \alpha \in \Delta^+ \}$
be the set of minimal roots of $\Phi$, also called an antichain of $(\Delta^+, \leq)$, see [P]. It is clear that each antichain corresponds to an element of $F_0$ and vice versa.

By a similar proof as in [P], we obtain the following proposition:

**Proposition 6.1.** Let $I \subset \Pi$ be of cardinality $r$ and $\Phi \in F_I$, then we have $\sharp \Phi_{\min} \leq l - r$.

**Proof.** Let $I \subset \Pi$ be of cardinality $r$ and $\Phi \in F_I$. Set $\Gamma = \Phi_{\min} \cup I = \{\gamma_1, \ldots, \gamma_t\}$. Let $\gamma_i, \gamma_j \in \Gamma$, then $\gamma_i - \gamma_j \notin \Delta$ by the definition of $\Phi_{\min}$ and the fact that $\Phi \in F_I$. Thus the angle between any pair of distinct elements of $\Gamma$ is non-acute and since all the $\gamma_i$’s lie in an open half space of $V$, they are linearly independent. Consequently, we have $\sharp \Gamma \leq r$, and hence $\sharp \Phi_{\min} \leq l - r$. \hfill $\Box$

**Remarks 6.2.**

(i) Recall from [CP1], that an antichain $\Gamma \subset \Delta^+$ is of cardinality $l$ if and only if $\Gamma = \Pi$.

This result has no equivalence in the general parabolic case. For example, in $B_2$, the set $\Phi = \{\alpha_1 + 2\alpha_2\}$ is an ad-nilpotent ideal of $p_{\alpha_1}$ such that $\Phi_{\min} = \Phi$ and $\sharp \Phi_{\min} = 1$.

(ii) Let $g$ be of type $A_l$. Let $I \subset \Pi$ be of cardinality $r$ and $\Phi \in F_I$, then it is possible to show that $\sharp \Phi_{\min} = l - r$ if and only if $\Phi_{\min} = \Pi \setminus I$.

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**References**


