ON SOME INTEGRO-DIFFERENTIAL EQUATIONS

by Anthony Phan

What is exposed in the following is how the fairly simple difference-differential equation

$$M'(x) = M(x) + M(x - r), \qquad x \text{ real},$$

can arise from a quite peculiar mathematical problem. Here, r is a real parameter ("*r*etard" or "delay"). The matter is that this equation may be of anticipative type—a type which has not been extensively studied. Anticipative equations are truly more difficult to handle than delayed or neutral ones since the evolution of their solutions relies deeply on their future.

1. Origin: the Azéma's martingales

Let q be some real number. An Azéma's martingale of parameter q is any local martingale X satisfaying the *structure equation*:

$$\mathrm{d}[X, X]_t = \mathrm{d}t + (q - 1)X_{t-}\,\mathrm{d}X_t, \qquad t \geqslant 0.$$

The law of such a local martingale is uniquely determined by the one of its initial value and the parameter q (*see* [6, 7]). The real q being given, the set of such laws forms a Fellerian process whose generator A^q is given by

$$A^q f(x) = \begin{cases} \dfrac{1}{2} f''(0) & \text{if } x = 0, \\ \dfrac{f(qx) - f(x) - (q-1)xf'(x)}{(q-1)^2 x^2} & \text{otherwise,} \end{cases}$$

on its domain—which is large enough to fit practical needs (*see* [8]), in particular it contains all regular functions with compact support.

Those Fellerian processes are finely recurrent if and only if q is greater or equal to

$$q^* = -0.278\,464\,542\ldots$$

the unique real solution of $q + \exp(q - 1) = 0$.[1] In these cases, each process admits a essentially unique Borelian invariant measure (*see* [2]). When q is strictly less than q^*, the associated process is punctually transient since it goes to infinity in absolute value.

When q is less or equal to -1, it is not difficult to see that the process has no invariant measure. When q is strictly positive, we know that the essentially unique invariant measure is the Lebesgue's measure (this fact is a corollary of a duality

1. This real number can be expressed in terms of the Lambert's W function: $q^* = -W(-\mathrm{e})$. We may have to recall that W is the reciprocical function of $w \mapsto w\,\mathrm{e}^w$ over $[-1, +\infty[$. The Lambert's function may be used to give more explicit expressions in the rest of this paper.

relation joined to the recurrence of the process). Our problem is to find the invariant measures when q lies in the interval $]-1,0]$.

2. Naïve search for invariant measures

The naïve way consists in searching for some Radonian measure μ, that is to say a Borelian measure finite on compact sets, such that

$$\int_{\mathbb{R}} A^q f(x)\,\mu(\mathrm{d}x) = 0 \tag{1}$$

for any regular function f with compact support. The search of such a measure would be done then by means of Schwartz' distributional calculus.

The seemingly most naïve thing is the limitation to Radonian measures. *A priori*, we are missing every measure that is infinite on some compact set. In fact, at least when $q^* \leqslant q \leqslant 0$, we are missing nothing here : according to [3], when the Hunt (or Fellerian) process is recurrent and its resolvent is strongly Fellerian (this means that the 1-potential U_1 maps bounded Borelian functions into continuous ones), then the essentially unique invariant measure is Radonian over each recurrent class (here there is at most one such class which is the whole state space when recurrence holds). Checking that the resolvent of the Azéma's martingale of parameter $q \in \mathbb{R}$ is strongly Feller is not difficult. Just like the Feller's property, it may be proved by means of connectivity and fine topology arguments. But existence and regularity of invariant measures remain open problems for $-1 < q < q^*$ ("moderately" transient cases).

But anyway, even if invariant measures were Radonian, the relation (1) would of course not be a sufficient condition but might be also not a necessary one: since we are dealing with infinite (σ-finite) measures, integrability problems appear almost everywhere. Solving this equation is just a first step—and maybe not in the good direction—for looking toward invariant measures.

Writting μ as the sum of a Dirac mass at 0, a positive Radonian measure μ^+ over $]0,+\infty[$ and the image of another such measure μ^- by $-\mathrm{Id}$, the equation (1) tested on regular functions with compact support in $]0,+\infty[$ or $]-\infty,0[$, leads to the following relations:

$$(\mu^+)' = \frac{q\mu^+ - q^2(\mathrm{Id}\,/|q|)_*\mu^-}{(q-1)x} \qquad \text{and} \qquad (\mu^-)' = \frac{q\mu^- - q^2(\mathrm{Id}\,/|q|)_*\mu^+}{(q-1)x}.$$

From these relations, one deduces $\mu^+ = m^+$ and $\mu^- = m^-$ are indefinitly differentiable functions over $]0,+\infty[$, with

$$(m^+)'(x) = \frac{qm^+(x) - q^2m^-(x/|q|)}{(q-1)x} \quad \text{and} \quad (m^-)'(x) = \frac{qm^-(x) - q^2m^+(x/|q|)}{(q-1)x}.$$

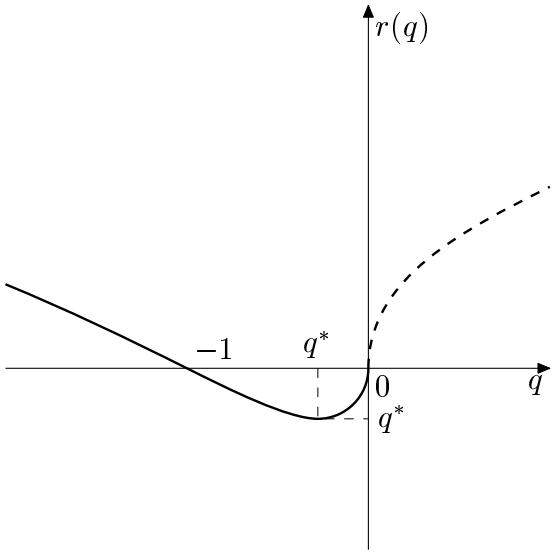
Changing of variable with respect to $x = \exp(y(q-1)/q)$, we finally get

$$(M^+)'(y) = M^+(y) + M^-(y-r) \qquad \text{and} \qquad (M^-)'(y) = M^-(y) + M^+(y-r),$$

where

$$r = \frac{q\ln|q|}{q-1} \tag{2}$$

ranges from q^* to $+\infty$ when q goes from 0 to $-\infty$.



It's easy to see that the positive solutions have all their derivatives also positive, and thus are totally positive, and then are analytic. Assuming $M = M^+ = M^-$ (this means that we're interested in symmetric invariant measures), or considering $M = (M^+ + M^-)/2$, we come to study the positive solutions of

$$M'(x) = M(x) + M(x - r), \qquad x \text{ real.} \tag{3}$$

It's easy to prove that any such solution admits a limit at $-\infty$ which is 0:

$$\lim_{x \to -\infty} M(x) = 0;$$

then to show that the following equation is fulfilled:

$$M(x) = \int_{[r,+\infty[} M(x - y) \exp(r - y)\, \mathrm{d}y, \qquad x \text{ real,} \tag{4}$$

which is a convolution equation by the exponential probability law of parameter 1 and support $[r, +\infty[$.

3. **Basic results**

When r is non-negative, finding the solutions to equation (3) is easy, but has a very few interest for us since $r \geqslant 0$ is equivalent to $q \leqslant -1$ in which case there is no invariant measure of any kind (except point masses at infinity, but these have no interest for us).

Proposition 1. — *When r is non-negative, positive solutions to* (3) *are of the form $M(x) = M(0)\exp(\alpha x)$, where α is the unique real satisfaying $\exp(-r\alpha) = \alpha - 1$; moreover α is strictly positive.*

Proof. — The easiest part is to check that such a function is a solution, which is straightforward. The no more difficult part is to prove that uniqueness holds. Suppose you're given two positive solutions equal at some real z. Consider their difference M. It is a continuous function vanishing at z and $-\infty$ satisfaying (3).

Denote by m its maximum over $]-\infty, z]$ and x a point where it is reached. The equation (4) evaluated at x uses on the right side the values of M over $]-\infty, x-r]$ which is included in $]-\infty, x]$. Thus the equality holds if and only if M is constantly equal to m on $]-\infty, x-r]$. Since M is an analytic function (weaker arguments apply), M is constant, and thus $M = 0$. This proves uniqueness. □

With this kind of proof, one can imagine the difficulties that arise when r is strictly negative: the right side of equation (4) involves values of a solution that lie in the future, and the simple argument above does not apply.

Proposition 2. — *When $q^* < r < 0$, one can find a couple of independent exponential solutions. When $q^* = r$, these is an essentially unique exponential solution. When $r < q^*$, there is no exponential solution.*

Proof. — It's just elementary calculus. □

Remarks. — *a*) We will see later that when $r < q^*$ there is no positive (non zero) solution at all.

b) One should notice that the probabilistic critical value q^* appears to be also critical in this problem. Elementary calculus shows that r defined as a function of q is always greater than q^*, and equality holds if and only if $q = q^*$.

c) Thus, the exponential found when $q^* < r < 0$ or $r = q^*$ may lead to find candidates for invariant (symmetric) measures. Calculus again, but less elementary, shows that the only one exponent should be kept and that the point measure at 0 must be the null measure. More precisely, the measure

$$\mu(\mathrm{d}x) = c|x|^\gamma \,\mathrm{d}x$$

should be the invariant measure we are searching for, if γ satisfies

$$|q|^{1-\gamma} - 1 + (1-\gamma) \times (|q| + 1) = 0$$

and is equal to 1 if and only if $q = q^*$. Note that γ starts from 0 at $q = 0$ and grows to infinity when q decreases down to -1.

4. **Summary of problems**

4.1. Probabilistic part

Problem. — *When $-1 < q < q^*$, do invariant measures exist and are Radonian?*

Remark. — It is quite natural for probabilists to simply call invariant measure invariant *probability* measure. Here, there is no hope to find any invariant probability measure. We told that equation (1) is not even a necessary condition. It falls out to be necessary and sufficient for martingale problems (this is our case) and probability measures (so well...) (*see* [4, 5]). On the other hand, some classical results (*see* [9]) give sufficient conditions for the existence of σ-finite or even Radonian invariant measures. In most cases, a compactness hypothesis is made (for instance, the existence of a recurrent compact set). In transient case for Azéma's martingales, we know that there is no recurrent compact set at all...

4.2. Analytic part

Problem. — *What can be said about uniqueness when $q \leqslant r < 0$?*

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